A NUMERICAL STUDY OF COUPLED NON-LINEAR EQUATIONS OF THERMO-VISCOUS FLUID FLOW IN CYLINDRICAL GEOMETRY

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In this paper we present numerical solutions to coupled non-linear governing equations of thermo-viscous fluid flow in cylindrical geometry using MATHEMATICA software solver. The numerical results are presented in terms of velocity, temperature and pressure distribution for various values of the material parameters such as the thermo-mechanical stress coefficient, thermal conductivity coefficient, Reiner Rivlin cross viscosity coefficient and the Prandtl number in the form of tables and graphs. Also, the solutions to governing equations for slow steady motion of a fluid have been obtained numerically and compared with the existing analytical results and are found to be in excellent agreement. The results of the present study will hopefully enable a better understanding applications of the flow under consideration.

Key words: Reiner Rivlin Cross viscosity, thermal conductivity, thermo-stress viscosity, Prandtl number.

1. Introduction

1.1. Literature

The basic equations of classical hydrodynamics are non-linear in the velocities and only a few instances have been reported in literature in which exact solutions could be obtained. The same situation applies to a higher extent in the case of non-Newtonian fluids, since even the constitutive relations qualifying these fluids are non-linear. The failure of the linear theories in predicting to a reasonable extent the mechanical behaviour of materials such as liquid polymers, fluid plastic, molten metals etc., subjected to stresses has been the motivating force behind the study of the non-linear theories for material description. The non-Newtonian nature of materials has been the subject of an extensive study for over one and half centuries. It is only in the last seven or eight decades that serious attempts have been made to extend these investigation in the realm of non-linearity. With the growing importance of non-Newtonian fluids in modern technology and industries, the investigations on such fluids are desirable. The non-linear theory reflecting the interaction/interrelation between thermal and viscous effects has been preliminarily studied by Koh and Eringen [5]. A systematic rational approach for such class of fluids has been developed by Green and Nagdhi [2, 3]. Kelly[4] examined some simple shear flows of second order thermo-viscous fluids.

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The problem of a steady flow of a second order thermo- viscous fluid over an infinite plate was studied by Rao and Ramacharyulu [11]. The steady flow of a thermo-viscous fluid through straight tubes was examined by Rao and Ramacharyulu [12]. In this paper, the velocity and temperature fields have been obtained by employing a four step recursive approach. A steady slow motion through a circular tube was studied by Rao and Ramacharyulu [13]. The effects of large and small values of the thermo physical parameters on the flow field have been discussed in this paper. Rao and Ramacharyulu [10] examined some steady state problems dealing with certain flows of thermo-viscous fluids. Anuradha [1] and Nagaratnam [9] studied the flow in plane, cylindrical and spherical geometries. Recently, Pothanna *et al.* [18] *et al* studied unsteady forced oscillations of a fluid bounded by a rigid bottom. Pothanna *et al.* [15, 16, 17] also investigated certain flows of thermo-viscous fluids in a porous slab bounded between two horizontal parallel plates and the effects of various physical parameters on the flow field have been discussed. Srinivas *et al.* [19] studied a slow steady motion of a thermo-viscous fluid between two parallel plates with constant pressure and temperature gradients.

Curved pipe/annular configurations are of immense practical importance in almost all piping systems, the human cardiovascular system and in several engineering devices such as heat and mass exchanges, chemical reactors, chromatography columns and other processing equipment. Owing to the wide range of applications, the interest in the study of flow characteristics in these configurations has grown enormously during the last decades. Green and Nagdhi [3] has given a new thermo-viscous theory for fluids. Green and Nagdhi [2] explained a dynamical theory of interacting continua. Longlois [6] examined a slow steady flow of viscoelastic fluids through non-linear tubes. Muthuraj and Srinivas [7] studied the flow of a Thermo-viscous fluid through an annular tube with constriction. The motion is assumed to be slow and the governing equations have been solved in terms of the modified Bessel functions. Srinivas and Muthuraj [20] also examined the flow of a thermo-viscous fluid in a radially non-symmetric constricted tube. Srinivas *et al.* [21], studied peristaltic transport of a thermo-viscous fluid. Hossain *et al.* [8], studied a fluctuating flow of a thermo-micropolar fluid past a vertical surface. Anuradha *et al.* [1] investigated the problem of a steady flow of a thermo-viscous fluid through a moving circular pipe .

It can be noted, from the past studies, that the governing equations of the fluid flow in cylindrical geometry are linear in nature and the closed form solutions are obtained using various analytical methods. However, in reality the flow and heat transfer problems are non-linear in nature. The basic governing equations of the present study are coupled and highly non linear in nature. Hence, it is important to study the solutions of non-dimensional non-linear partial differential equations which govern the momentum and energy. They are solved numerically with the help of *MATHEMATICA* software solvers. As far as known, this problem has not been discussed in the literature.

1.2. Basic equations

The flow of an incompressible homogeneous thermo-viscous fluid satisfies the usual conservation equations.

The equation of continuity(law of conservation of mass)

$$v_{i,i} = 0 .$$

(1.1)

The equation of momentum(law of conservation of momentum)

$$\rho \left[\frac{\partial v_i}{\partial t} + v_k v_{i,k} \right] = \rho f_i + t_{ji,j} ,$$

(1.2)

and the energy equation(law of conservation of energy)

$$\rho c \, \dot{\eta} = t_{ij} d_{ij} - q_{i,i} + \rho \gamma$$

(1.3)

where

$f_i = i^{th}$ component of external force per unit mass

$c =$ specific heat

$\rho =$ density of the fluid

$\gamma =$ energy source per unit mass

$q_i = i^{th}$ component of heat flux bivector $= \in_{ijk} h_{jk} / 2$

$d_{ij} = (u_{i,j} + u_{j,i})/2 =$ rate of deformation tensor

$b_{ij} = \in_{ijk} \eta_k =$ thermal gradient bivector

where u_i is the ith component of velocity and η is the temperature field.

As proposed by Koh and Eringen [5], the constitutive equations for the stress tensor and heat flux bivector of second order thermo-viscous fluids coupled in d and b are given by

$$t = \alpha_1 I + \alpha_3 d + \alpha_5 d^2 + \alpha_6 b^2 + \alpha_8 \left(db - bd\right) \ , \tag{1.4}$$

and

$$h = \beta_1 b + \beta_3 \left(bd + db\right) \tag{1.5}$$

where the coefficients $\alpha_i^{'s}$ and $\beta_i^{'s}$ are scalar polynomials in $tr\,d$, $tr\,d^2$, $tr\,b^2$. Explicit expressions for the constitutive coefficients $\alpha_i^{'s}$ and $\beta_i^{'s}$ for the second order theory may be obtained as

$$\alpha_1 = \alpha_{1000} + \alpha_{1010} tr\,d + \alpha_{1020} tr\,d^2 + \alpha_{1002}\ tr\,b^2 \ ,$$

$$\alpha_3 = \alpha_{3010} + \alpha_{3020} tr\,d \ ,$$

$$\alpha_5 = \alpha_{5020} \ ,$$

$$\alpha_6 = \alpha_{6002} \ ,$$

$$\alpha_8 = \alpha_{8011}$$

$$\beta_1 = \beta_{1001} + \beta_{1011} tr\,d \ ,$$

and

$$\beta_3 = \beta_{1011} \ ,$$

the secondary coefficients α_{isrt} and β_{isrt} are functions of ρ and η.

The fluids defined by the constitutive Eqs (1.4) and (1.5) with constant values for the constitutive coefficients $\alpha_i^{'s}$ and $\beta_i^{'s}$ may be called second order thermo-viscous fluids. This is the simplest model of a thermo-viscous fluid which exhibits an interaction between mechanical and thermal phenomena.

2. Mathematical analysis

Consider the steady flow of a second order thermo-viscous fluid around a circular pipe moving with a given velocity (Fig.1). Introducing the cylindrical polar coordinate system (r, θ, z) where the z-axis is along

the axis of the pipe and r is the radial distance from the centre of the circular pipe. The flow is represented by the velocity $(0,0,w(r))$ and the temperature $\eta(r)$. This choice of velocity evidently satisfies the continuity equation.
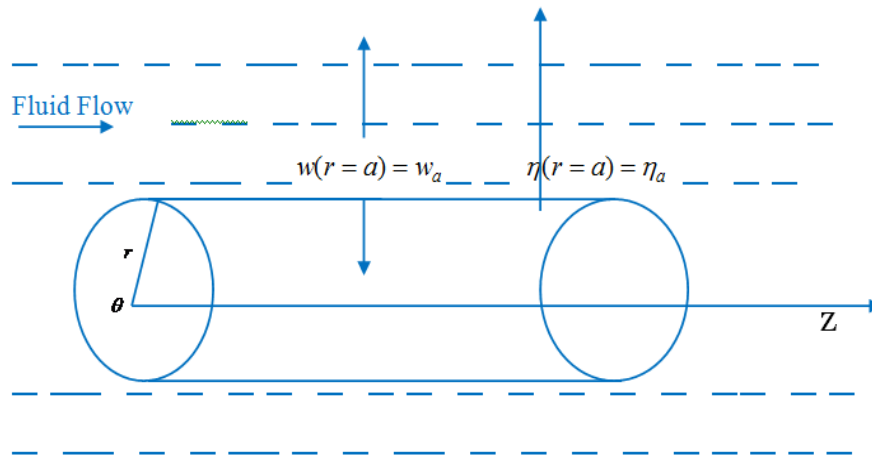
Fig.1. Schematic representation of fluid flow.

The flow is assumed to be under the action of constant temperature and pressure gradients. It is also assumed that the pipe is moving with a constant velocity w_a and η_a is the temperature prescribed on the boundary. Under these assumptions the basic equations characterizing the flow are the following:

In the radial direction

$$\frac{\partial p}{\partial r} = \mu_c \left\{ \frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2 + 2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}\right\} - \alpha_6 \frac{1}{r}\left(\frac{\partial \eta}{\partial r}\right)^2 + \rho F_r. \tag{2.1}$$

In the transverse direction

$$\rho F_\theta = \alpha_8 \left\{ \frac{\partial w}{\partial r}\nabla^2 \eta + \frac{\partial \eta}{\partial r}\nabla^2 w \right\}. \tag{2.2}$$

In the axial direction (z-direction)

$$\frac{\partial p}{\partial z} = \mu\nabla^2 u - \alpha_6 c_2 \nabla^2 \eta + \rho F_z, \tag{2.3}$$

and the energy equation

$$\rho c u \left(\frac{\partial \eta}{\partial z}\right) = \mu \frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2 - \alpha_6 \left(\frac{\partial \eta}{\partial z}\right)\left(\frac{\partial w}{\partial r}\right)\left(\frac{\partial \eta}{\partial r}\right) + k\nabla^2 \eta + \beta_3 \left(\frac{\partial \eta}{\partial z}\right)\nabla^2 u. \tag{2.4}$$

The appropriate boundary conditions of the problem are

$$w(r = a) = w_a, \qquad w(r = \infty) = \text{finite}, \qquad p(r = a) = p_a,$$

(2.5)

$$\eta(r = a) = \eta_a, \qquad \eta(r = \infty) = \text{finite}$$

where w_a, p_a and η_a are respectively the constant velocity, pressure and temperature prescribed on the boundary.

Further, it is assumed that the flow is generated with the constant pressure gradient $\dfrac{\partial p}{\partial z}(= -C_1)$ and the temperature gradient $\dfrac{\partial \eta}{\partial z}(= C_2)$. $\rho(F_r, F_\theta, F_z)$ are the components of the external force/volume. It is assumed that there is no external force down the tube length, i.e., F_r, F_θ and F_z are zero.

Introducing the following non-dimensional quantities

$$r = aR, \quad w = \frac{a^2}{\mu}W, \quad p = \frac{a^2}{\mu^2}P, \quad \eta = \eta_a T, \quad a_6 = \frac{\alpha_6 \eta_a}{a^2}, \quad b_3 = \frac{\beta_3}{\mu},$$

(2.6)

$$m_1 = \frac{\mu^4 \eta_a}{a^2}, \quad m = \frac{\eta_a c}{p_r} \text{ and } p_r = \frac{\mu c}{k} \text{ (Prandtl number).}$$

Using the above non-dimensional quantities, the equations of momentum in the radial and axial direction and the energy equation in non-dimensional form now reduce to:

In the radial direction

$$\frac{\partial P}{\partial R} = \mu_c \left\{ \frac{1}{R}\left(\frac{\partial W}{\partial R}\right)^2 + 2\frac{\partial W}{\partial R}\frac{\partial^2 W}{\partial R^2} \right\} - m_1 a_6 \frac{1}{R}\left(\frac{\partial T}{\partial R}\right)^2.$$

(2.7)

In the axial direction

$$-C_1 = \mu \nabla^2 W - a_6 C_2 \nabla^2 T,$$

(2.8)

and the energy equation

$$\rho c C_2 W = \mu \frac{1}{R}\left(\frac{\partial W}{\partial R}\right)^2 - a_6 C_2 \left(\frac{\partial W}{\partial R}\right)\left(\frac{\partial T}{\partial R}\right) + k\nabla^2 T + b_3 C_2 \nabla^2 W$$

(2.9)

where $\nabla^2 = \dfrac{\partial^2}{\partial R^2} + \dfrac{1}{R}\dfrac{\partial}{\partial R}$ and a_6 is the non-dimensional thermo-mechanical stress viscosity coefficient and b_3 is the non-dimensional strain thermal conductivity coefficient.

The boundary conditions of the problem reduce to

$$W(R = 1) = W_a, \quad W(R = \infty) = \text{finite}, \qquad P(R = 1) = P_a,$$

(2.10)

$$T(R = 1) = 1 \text{ and } T(R = \infty) = \text{finite}.$$

3. Numerical study of the problem

The governing coupled highly non-linear differential Eqs (2.7)-(2.9) together with the boundary conditions (2.10) for the velocity, temperature and pressure distributions are solved by using the R-K method of 6th order with shooting methods with the help of *MATHEMATICA* software ND solver. The boundary conditions for both the velocity and temperature as $r \to \infty$ are considered as finite values. For computation purpose, the coordinate r (radial distance) is considered as varied from 1 to 2 where the value $r = 2$ represents infinity. The convergence of the method is guaranteed by the satisfaction of the boundary conditions. The influence of various physical parameters such as the thermo-mechanical stress interaction coefficient (a_6), strain thermal conductivity coefficient (b_3), cross viscosity coefficient (μ_c) and Prandtl number (p_r), on the velocity, temperature field and pressure distribution is shown in the form of tables and graphs. The numerical results presented in Tabs 1-3 show the solutions to highly non-linear coupled governing equations of velocity, temperature and pressure distribution with the various physical parameters.

Table 1. Numerical results of velocity field with various material parameters.

R	Velocity distribution $W(R)$ with $c=1$, $\rho=1$, $c1=1$, $c2=1$, $\mu=1$, $\mu_c = 1$, $\eta_a = 1$, $p_a = 1$, $w_a = 1$, $p_r = 1$								
	$b_3 = 1$			$b_3 = 3$			$b_3 = 5$		
	$a_6 = 1$	$a_6 = 2$	$a_6 = 3$	$a_6 = 1$	$a_6 = 2$	$a_6 = 3$	$a_6 = 1$	$a_6 = 2$	$a_6 = 3$
1	1	1	1	1	1	1	1	1	1
1.1	1.520899	1.113976	1.004379	1.535279	1.303372	1.145115	1.540151	1.381759	1.253298
1.2	1.988644	1.262008	1.019939	2.019922	1.623575	1.326993	2.030555	1.763813	1.534417
1.3	2.417157	1.451608	1.055749	2.464818	1.96196	1.551572	2.481051	2.147631	1.844797
1.4	2.817069	1.691822	1.126606	2.878359	2.320094	1.82551	2.899243	2.534623	2.186063
1.5	3.19677	1.993568	1.256102	3.267178	2.699761	2.156295	3.291147	2.926167	2.56004
1.6	3.5631	2.370057	1.481134	3.636637	3.102968	2.552367	3.661616	3.323621	2.968764
1.7	3.921819	2.837303	1.858463	3.991165	3.53196	3.023259	4.01464	3.728345	3.414492
1.8	4.277934	3.414749	2.474211	4.334496	3.989233	3.57975	4.353556	4.141712	3.899723
1.9	4.63594	4.126047	3.457502	4.669838	4.477557	4.234049	4.681195	4.565119	4.427214
2	5	5	5	5	5	5	5	5	5

Table 2. Numerical results of temperature distribution with various material parameters.

R	Temperature distribution $T(R)$ with $c=1$, $\rho=1$, $c1=1$, $c2=1$, $\mu=1$, $\mu_c = 1$, $\eta_a = 1$, $p_a = 1$, $w_a = 1$, $p_r = 1$								
	$b_3 = 1$			$b_3 = 3$			$b_3 = 5$		
	$a_6 = 1$	$a_6 = 2$	$a_6 = 3$	$a_6 = 1$	$a_6 = 2$	$a_6 = 3$	$a_6 = 1$	$a_6 = 2$	$a_6 = 3$
1	1	1	1	1	1	1	1	1	1
1.1	1.470272	1.306681	1.243159	1.484652	1.38295	1.290809	1.489523	1.401379	1.328263
1.2	1.901368	1.613435	1.544661	1.932646	1.789764	1.665425	1.943279	1.794218	1.692264
1.3	2.305774	1.927136	1.831554	2.353434	2.184141	2.028452	2.369667	2.182311	2.048289
1.4	2.692999	2.25473	2.111691	2.754289	2.569085	2.383622	2.775173	2.568866	2.401871
1.5	3.070548	2.603598	2.394938	3.140956	2.947101	2.734056	3.164925	2.956695	2.758095
1.6	3.444546	2.981895	2.694728	3.518083	3.32033	3.082436	3.543062	3.348351	3.121823
1.7	3.820168	3.398895	3.030201	3.889514	3.69064	3.431136	3.912989	3.746224	3.497864
1.8	4.201937	3.86537	3.429204	4.258498	4.059697	3.78231	4.277558	4.152611	3.891114
1.9	4.593941	4.394022	3.932539	4.627838	4.429016	4.137963	4.639195	4.569777	4.306681
2	5	5	5	5	5	5	5	5	5

Table 3. Numerical results of pressure distribution with various material parameters.

R	Pressure distribution $P(R)$ with $c=1$, $\rho=1$, $c1=1$, $c2=1$, $\mu=1$, $\mu_c=1$, $\eta_a=1$, $p_a=1$, $w_a=1$, $p_r=1$ $b_3=1$						Pressure distribution $P(R)$ with $c=1$, $\rho=1$, $c1=1$, $c2=1$, $\mu=1$, $\mu_c=1$, $\eta_a=1$, $p_a=1$, $w_a=1$, $\mu_c=1$, $b_3=1$					
	$a_6=1$			$a_6=2$			$a_6=1$			$a_6=2$		
	$\mu_c=1$	$\mu_c=2$	$\mu_c=3$	$\mu_c=1$	$\mu_c=2$	$\mu_c=3$	$p_r=1$	$p_r=2$	$p_r=3$	$p_r=1$	$p_r=2$	$p_r=3$
1	1	1	1	1	1	1	1	1	1	1	1	1
1.1	-4.95445	-8.79601	-12.6376	0.034708	0.862605	1.690501	-4.95445	-4.39928	-4.11997	0.034708	-0.10358	-0.14444
1.2	-8.93598	-15.1395	-21.343	-0.2942	1.842051	3.978302	-8.93598	-7.83247	-7.2907	-0.2942	-0.66319	-0.79248
1.3	-11.6486	-19.2545	-26.8603	0.163168	4.331613	8.500055	-11.6486	-10.0222	-9.23607	0.163168	-0.48801	-0.75017
1.4	-13.5048	-21.8552	-30.2055	1.688983	8.973191	16.2574	-13.5048	-11.3738	-10.3525	1.688983	0.808521	0.396727
1.5	-14.7539	-23.3696	-31.9853	4.74091	16.7558	28.77068	-14.7539	-12.1186	-10.859	4.74091	3.906962	3.409901
1.6	-15.5489	-24.0569	-32.5649	10.02445	29.16953	48.31458	-15.5489	-12.3867	-10.8707	10.02445	9.94935	9.620935
1.7	-15.9834	-24.0706	-32.1578	18.60501	48.43877	78.27243	-15.9834	-12.2445	-10.4365	18.60501	20.81444	21.30836
1.8	-16.112	-23.4949	-30.8777	32.07745	77.87124	123.6651	-16.112	-11.7147	-9.55806	32.07745	39.56382	42.33074
1.9	-15.962	-22.3642	-28.7664	52.82066	122.3805	191.9402	-15.962	-10.7859	-8.19763	52.82066	71.16428	79.18548
2	-15.5403	-20.6752	-25.8101	84.3785	189.265	294.1515	-15.5403	-9.41701	-6.27975	84.3785	123.6547	142.782

4. Special case for slow steady motion of the fluid

4.1. Governing equations

The motion is assumed to be slow such that the non-linear terms in the equation of momentum and energy could be neglected. The equation of momentum and energy now reduce to the following:
The equation of momentum in the axial direction

$$-C_1 = \mu\nabla^2 W - a_6 C_2 \nabla^2 T , \tag{4.1}$$

and the energy equation

$$\rho c C_2 W = k\nabla^2 T + b_3 C_2 \nabla^2 W . \tag{4.2}$$

Equations (4.1) and (4.2) are coupled in terms of the velocity $W(R)$ and the temperature $T(R)$. In these two equations C_1 and C_2 represent constant pressure and temperature gradients, respectively. a_6 is the thermo-mechanical stress viscosity coefficient and b_3 is the strain thermal conductivity coefficient.
The appropriate boundary conditions for the velocity and temperature are

$$W(R=1)=W_a, \quad W(R=\infty)=\text{finite}, \quad T(R=1)=1 \text{ and } T(R=\infty)=\text{finite} . \tag{4.4}$$

4.2. Analytical and numerical solutions

According to Anuradha [1] *et al. a*nalytical solutions to the coupled linear Eqs (4.1) and (4.2) together with the boundary conditions (4.3) yield the velocity distribution

$$W(R) = \frac{K_0(mR)}{K_0(m)} W_a + \frac{d_1}{m^2}\left\{1 - \frac{K_0(mR)}{K_0(m)}\right\}, \tag{4.5}$$

and the temperature field

$$T(R) = \frac{m_1}{m^2}\left\{\frac{K_0(mR)}{K_0(m)} - 1\right\}W_a - \frac{d_1 m_1}{m^4}\left\{\frac{K_0(mR)}{K_0(m)} - 1\right\} + \frac{m_1}{4}\left\{\frac{d_1}{m^2} + b_3\right\}(R^2 - 1) + 1 \tag{4.6}$$

where $K_0( )$ is the modified Bessel function of order zero.

Here $\quad d_1 = \dfrac{1}{1 + a_6 b_3 p_r}, \quad m^2 = \dfrac{a_6 p_r}{1 + a_6 b_3 p_r}, \quad m_1 = \dfrac{d_2 p_r}{1 + a_6 b_3 p_r}, \quad d_2 = \dfrac{\rho c_1 c_2 a^4}{\mu^2 \eta_a},$

$\quad a_6 = \dfrac{\alpha_6 \rho a^2}{\mu^2} c_2{}^2, \quad b_3 = \dfrac{\beta_3}{\rho c a^2}, \quad p_r = \dfrac{\mu c}{k}.$

In order to get the analytical results of Eqs (4.4) and (4.5) for the velocity $W(R)$ and the temperature $T(R)$ for various values of physical parameters, the code of the algorithm has been executed in *MATLAB* running on a PC. The results are presented in Tabs 4-5.

In the present study, the numerical solutions of Eqs (4.1) and (4.2) with respect to the boundary conditions (4.3) are obtained numerically using the *MATHEMATICA* software package. An excellent convergence was achieved for all the results for slow steady motion of the fluid when compared to the analytical results. The numerical and analytical results for the velocity and temperature distributions are presented in Tabs 4-5. For computation purpose, the boundary conditions as $r \to \infty$ for the finite value are replaced by the value 5, the coordinate r (radial distance) is considered as ranging from 1 to 2 where the value $r = 2$ represents infinity. The numerical results obtained are compared with the analytical results of Anuradha[1] *et.al* and are found to be good in agreement. The analytical results obtained substantiate the validity and accuracy of the present numerical results.

Table 4. Comparison of velocity profiles for various material parameters.

R	Present Results for $b_3 = 1$		Results of Anuradha for $b_3 = 1$		Present Results for $b_3 = 3$		Results of Anuradha for $b_3 = 3$	
	$a_6 = 1$	$a_6 = 2$	$a_6 = 1$	$a_6 = 2$	$a_6 = 1$	$a_6 = 2$	$a_6 = 1$	$a_6 = 2$
1	1	1	1	1	1	1	1	1
1.1	1.506824	1.477614	1.506810	1.477590	1.527799	1.517688	1.527791	1.517699
1.2	1.971945	1.919882	1.971951	1.919862	2.010905	1.993084	2.010890	1.993076
1.3	2.404484	2.335827	2.404495	2.335805	2.457747	2.434504	2.457748	2.434493
1.4	2.811721	2.732729	2.811731	2.732739	2.87497	2.848521	2.87495	2.848500
1.5	3.199603	3.116622	3.199585	3.116600	3.267924	3.240445	3.267900	3.240428
1.6	3.573094	3.492626	3.573076	3.492636	3.640998	3.614647	3.640979	3.614635
1.7	3.936419	3.86519	3.936400	3.865100	3.997852	3.974790	3.997851	3.974791
1.8	4.293244	4.238265	4.293222	4.238274	4.341588	4.32399	4.341569	4.32376
1.9	4.646803	4.615431	4.646793	4.615422	4.674866	4.664941	4.674865	4.664922
2	5	5	5	5	5	5	5	5

Table 5. Comparison of temperature profiles for various material parameters.

R	Present Results for $b_3 = 1$		Results of Anuradha for $b_3 = 1$		Present Results for $b_3 = 3$		Results of Anuradha For $b_3 = 3$	
	$a_6 = 1$	$a_6 = 2$	$a_6 = 1$	$a_6 = 2$	$a_6 = 1$	$a_6 = 2$	$a_6 = 1$	
1	1	1	1	1	1	1	1	1
1.1	1.456196	1.48850	1.456177	1.488761	1.477171	1.508539	1.477190	1.508539
1.2	1.884669	1.942372	1.884681	1.942382	1.923629	1.978976	1.923630	1.978970
1.3	2.29310	2.369245	2.29291	2.369235	2.346363	2.418584	2.346333	2.418588
1.4	2.687651	2.775183	2.687662	2.775164	2.75090	2.833082	2.75092	2.833069
1.5	3.073381	3.165125	3.073393	3.165111	3.141702	3.227037	3.141700	3.227034
1.6	3.454540	3.543180	3.454522	3.543171	3.522444	3.604380	3.522445	3.604389
1.7	3.834768	3.912839	3.834749	3.912850	3.896201	3.967650	3.896208	3.967639
1.8	4.217246	4.277127	4.217245	4.277108	4.26559	4.31999	4.26567	4.31998
1.9	4.604803	4.638715	4.604784	4.638693	4.632866	4.66347	4.632860	4.66350
2	5	5	5	5	5	5	5	5

4. Results analysis and discussion

The results are illustrated graphically in Figs 2-14. To get the physical insight into the problem the velocity, temperature field, pressure distribution have been discussed by assigning numerical values to various material parameters such as the thermo-mechanical interaction coefficient (a_6), the strain thermal conductivity coefficient (b_3), cross viscosity coefficient (μ_c) and Prandtl number (p_r) which characterise the flow phenomena. The influences of these parameters on the velocity, temperature and pressure distribution have been studied and are presented graphically.

4.1. Velocity field

The impact variation of the thermo-mechanical stress interaction coefficient (a_6) on the velocity field is presented graphically in Figs 2-4 and the influence of the strain thermal conductivity coefficient (b_3) on the velocity field is shown in Fig.5. Figures 2-4 show that the velocity of the fluid decreases as the values of the thermo-mechanical stress interaction coefficient (a_6) increase and the velocity of the fluid increases with the increase of the distance from the pipe and attains maximum velocity far away from the tube boundary for $R>>1$. It is observed from Fig.5 that the velocity of the fluid increases at a slower rate for an increasing value of the thermal conductivity coefficient (b_3). This signifies that the effect of the thermal conductivity coefficient on the flow field is much smaller.
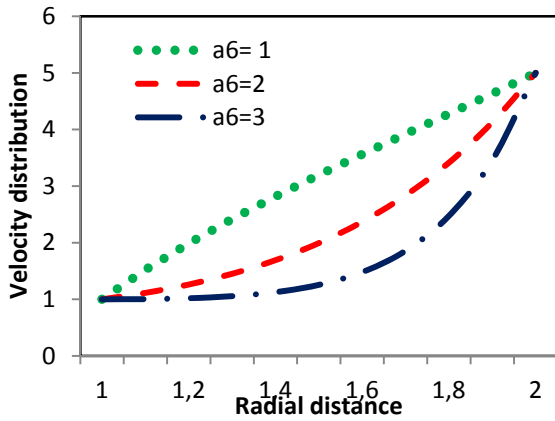
Fig.2. Variation of velocity for $b_3 = 1$.
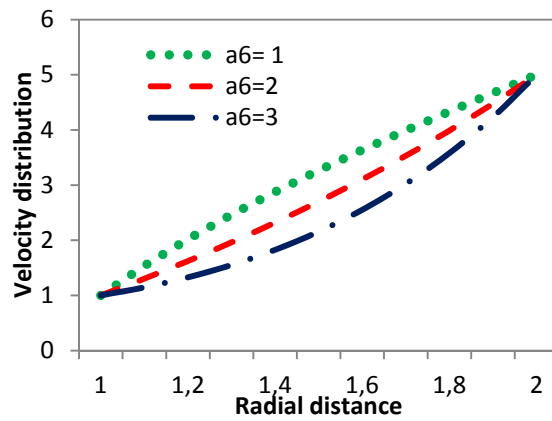
Fig.3. Variation of velocity for $b_3 = 3$.
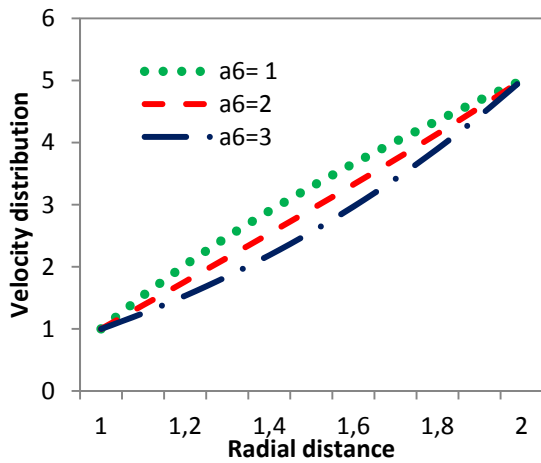
Fig.4. Variation of velocity for $b_3 = 5$.
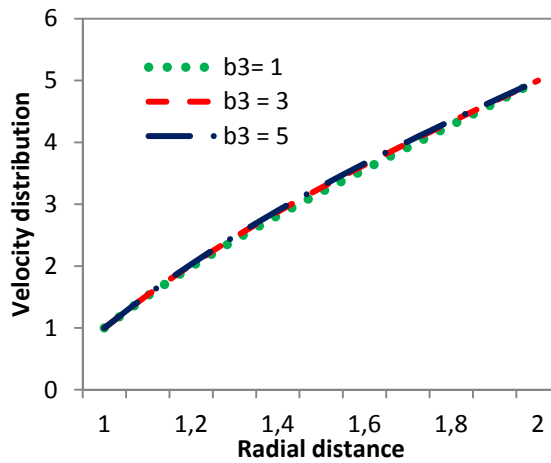
Fig.5. Variation of velocity for $a_6 = 1$.

4.2. Temperature distribution

It is observed that the material parameter thermo-mechanical stress interaction coefficient (a_6) is influenced by the temperature distribution and which is graphically shown in Figs 6-8. The effect of the strain thermal conductivity coefficient (b_3) is depicted in Fig.9. It is noticed that the temperature distribution decreases with the increasing values of thermo-mechanical stress interaction coefficient (a_6). This is due to the greater conversion of thermal energy to kinetic energy. The reverse effect is observed in the variation of temperature distribution for different values of the strain thermal conductivity coefficient (b_3). From Fig.9, it is observed that the rate of increase of temperature distribution is very slow and all the temperature profiles coincide far away from the tube boundary.
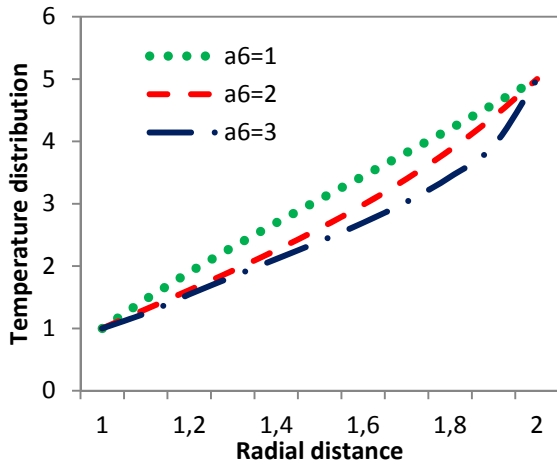
Fig.6. Variation of temperature for $b_3 = 1$.
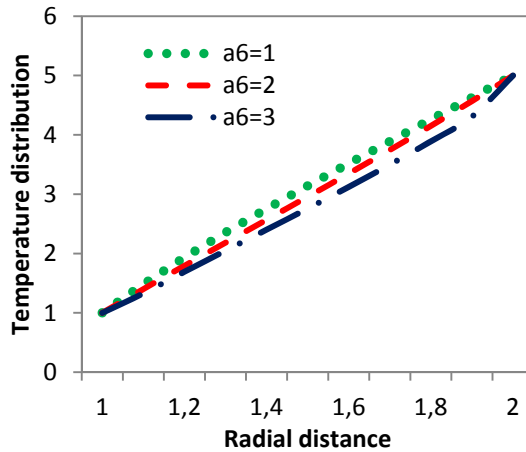
Fig.7. Variation of temperature for $b_3 = 3$.
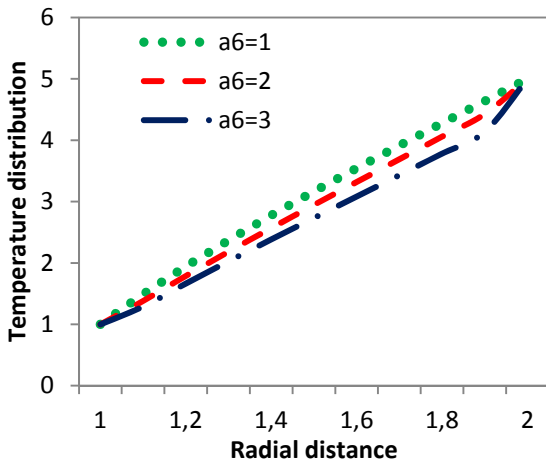
Fig.8. Variation of temperature for $b_3 = 5$.
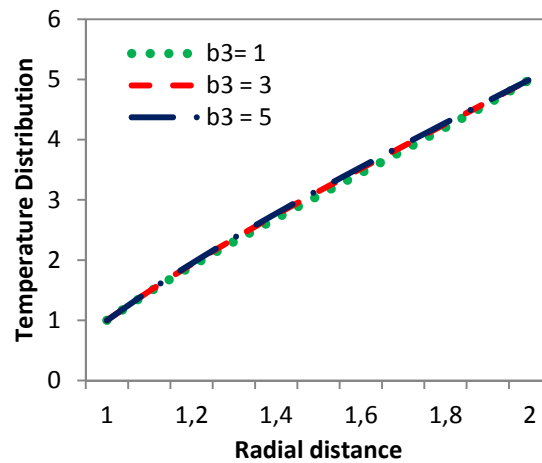
Fig.9. Variation of temperature or $a_6 = 1$.

4.3. Pressure distribution

The pressure distribution throughout the boundary is in the direction normal to the cylindrical surface and which is influenced by the Reiner Rivlin cross viscosity coefficient (μ_c). The effects of the Prandtl number (p_r) and cross viscosity coefficient (μ_c) on the pressure distribution are presented in Figs 10-13. From Fig.10, it can be noted that the pressure distribution decreases with the increase of the cross viscosity coefficient (μ_c) whereas Fig.11 shows that the flow suddenly takes a reverse direction to the positive side and increases with the increase of the cross viscosity coefficient (μ_c). This is due to the effect of the thermo-mechanical stress interaction coefficient of thermo viscous fluid flows. It is interesting to note that there is a marked departure between the flows of thermo-viscous fluids when compared to that of Newtonian fluids. The dimensionless parameter Prandtl number is the ratio between the kinematic viscosity and the thermal diffusivity, the effect of this on the pressure distribution is shown in Figs 12-13. Figure 12 presents that the pressure distribution increases with the increase of the Prandtl number (p_r) and suddenly takes in to the positive direction. This is also due to the influence of the thermo-mechanical stress interaction coefficient (a_6). The physical influence of this in heat transfer problems is the Prandtl number controls the thickness of the velocity and thermal boundary layers.
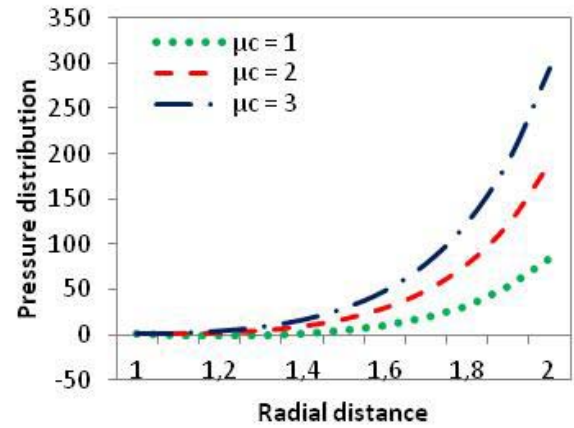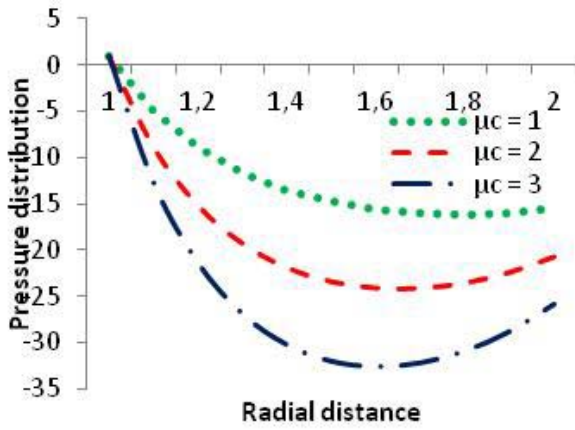
Fig.10. Variation of pressure distribution for $a_6 = 1$. Fig.11. Variation of pressure distribution for $a_6 = 2$.
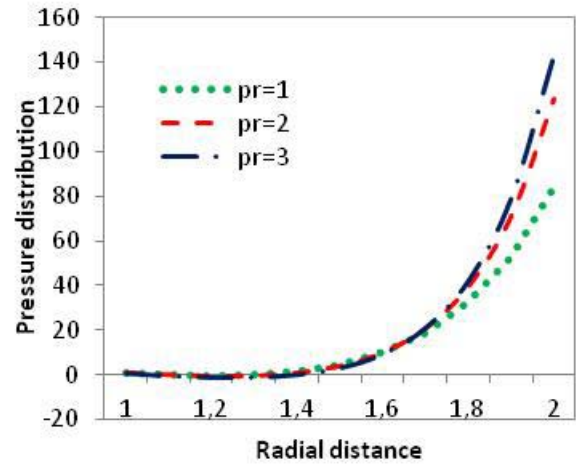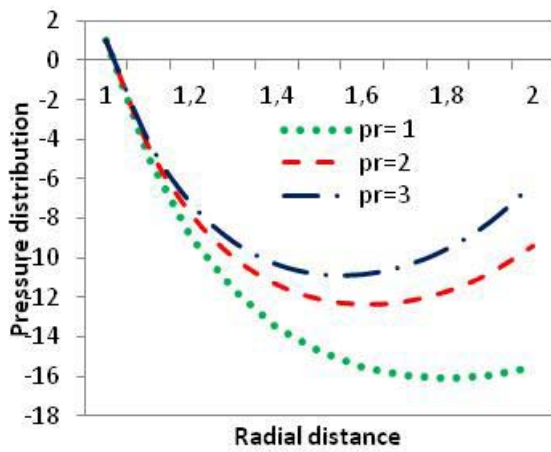
Fig.12. Variation of pressure distribution for $a_6 = 1$.   Fig.13. Variation of pressure distribution for $a_6 = 2$.
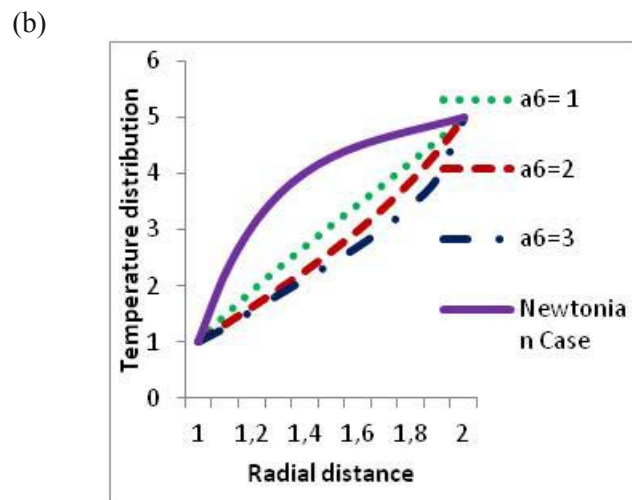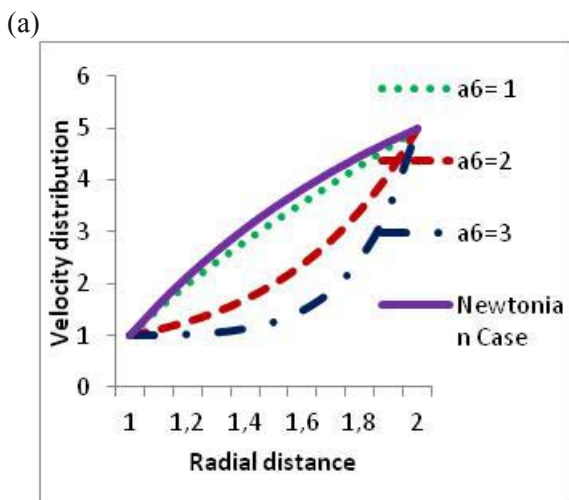

Fig.14. Comparison of (a) velocity and (b) temperature for a Newtonian fluid.

4.4. Comparison with a Newtonian case

To make a comparative study, in Figs 14a and b the variation of the velocity and temperature for a Newtonian-viscous fluid is presented graphically. For a Newtonian fluid the parabolic profile is realized for both the velocity and the temperature distributions. It can be noticed that the velocity of the thermo-viscous fluid is smaller compared to that of a classical Newtonian-viscous fluid, while the temperature of the thermo-viscous fluid is much less compared to that of classical Newtonian-viscous fluid. It is observed that the temperature profiles of thermo-viscous fluids deviate very much from the classical viscous fluids. This is due to the fact that the decoupled nature of velocity and temperature of the fluid in momentum and energy equations. Therefore the interaction between thermal and mechanical responses of fluids in motion are lost in Newtonian type of fluids.

5. CONCLUSIONS

In this paper, the steady thermo-viscous incompressible fluid flows around a circular cylinder are studied numerically. The resulting governing steady, non-linear and coupled equations are solved by using *MATHEMATICA* software ND solver . The computations are carried out for different values of a_6, b_3, μ_c, p_r and for the fixed values of other physical parameters.

1. The fluid velocity decreases with the increase of the thermo-mechanical stress interaction coefficient (a_6) while it increases as the thermal conductivity coefficient (b_3) increases.
2. The fluid temperature decreases with the increase of the thermo-mechanical stress interaction coefficient (a_6) while it increases as the strain thermal conductivity coefficient (b_3) increases.
3. The fluid pressure decreases as increase in cross-viscosity coefficient (μ_c) while increases as thermo-mechanical stress interaction coefficient (a_6) increases.
4. The fluid pressure increases as increase in Prandtl number (p_r) and thermo-mechanical stress interaction coefficient (a_6).
5. The numerical results obtained using *MATHEMATICA ND solver* are in good agreement with the analytical results obtained in terms of modified Bessel functions.

Nomenclature

a_6 – dimensionless thermo mechanical stress interaction coefficient

b_{ij} – thermal gradient bivector

b_3 – dimensionless strain thermal conductivity coefficient

C_1 – constant pressure gradient

C_2 – constant temperature gradient

c – specific heat

d_{ij} – rate of deformation tensor

f_i – i^{th} component of external force per unit mass

p_r – Prandtl number

R – dimensionless radial distance

r – radial distance

T – dimensionless temperature

t_{ji} – stress tensor

u_i – i^{th} component of velocity

q_i − i^{th} component of heat flux bivector

W − dimensionless velocity

w − velocity of the fluid

z − coordinate axis along the tube

$\alpha_1 = -p$ − fluid pressure

$\alpha_3 = 2\mu$ − coefficient of classical (Newtonian) viscosity

$\alpha_5 = 4\mu_c$ − coefficient of (Reiner-Rivlin) cross-viscosity

α_6 − thermo mechanical stress interaction coefficient

α_8 − thermo stress viscosity coefficient

$\alpha_i's$ − viscosity coefficients

$\beta_1 = k$ − Fourier thermal conductivity coefficient

β_3 − strain thermal conductivity coefficient

$\beta_i's$ − thermal conductivity coefficients

γ − energy source per unit mass

η − temperature of the fluid

$\dot{\eta}$ − temperature derivative

θ − angle in radial direction

ρ − density of the fluid

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