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A numerical test of differential equations for one- and two-loop sunrise diagrams using configuration space techniques

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Abstract We use configuration space methods to write down one-dimensional integral representations for one- and two-loop sunrise diagrams (also called Bessel moments) which we use to numerically check on the correctness of the second order differential equations for one- and two-loop sunrise diagrams that have recently been discussed in the literature.

1 Introduction

Sunrise-type diagrams have been under investigation since many years. Exact analytical results can be obtained only for special mass or kinematic configurations such as for the equal or zero mass cases or for the threshold region. For example, threshold expansions of the non-degenerate massive two-loop sunrise diagram have been studied in Refs. [1, 2]. The construction of differential equations for the corresponding correlator function provides some hope that by solving these differential equations, a general analytical solution can be obtained. Recently, mathematical methods were used to construct the coefficients of such a differential equation in a systematic way [3]. This work supplements the work of Kotikov [4] and Remiddi et al. [5–7] on the same subject. While traditionally the correlator is calculated in momentum space, configuration space techniques allow for a surprisingly simple solution for sunrise-type diagrams: The correlator in configuration space is just a product of single propagators which in turn can be expressed by modified Bessel functions of the second kind. Transforming back to momentum space, one ends up with a one-dimensional integral over Bessel functions, known as Bessel moments [8, 9]. As outlined in a series of papers [2, 10–16], the corresponding one-dimensional integral can be easily integrated numerically for an arbitrary number of propagators with different masses in any space-time dimension. Therefore, configuration space techniques can be used to numerically check the differential equations for the correlator function obtained by other means. This will be detailed in this note.

The paper is organized as follows: In Sect. 2 we introduce the configuration space techniques which will be used in Sect. 3 to check the differential equations for one-loop sunrise-type diagrams. In Sect. 4 we check the differential equations for the two-loop sunrise diagrams for the equal mass case, while in Sect. 5 we will deal with nondegenerate cases. Our conclusions can be found in Sect. 6. Even though the configuration space techniques are well suited to treat general $D\neq 4$ space-time dimensions, we will mainly deal with the case of $D=2$ space-time dimensions in this paper. For reasons of simplicity, throughout this paper we work in the Euclidean domain. The transition to the Minkowskian domain can be obtained as usual by a Wick rotation (or, equivalently, by replacing $p^2\to -p^2$).

2 Configuration space techniques

In configuration space, the n-loop n-particle irreducible correlation function

$$\Pi(x)=\langle 0|T\,\bar{j}(x)j(0)|0\rangle \tag{1}$$

connecting the space-time points 0 and x is given by the product of the propagators,

$$\Pi(x)=\prod_{i=1}^{n+1}D(x,m_i), \tag{2}$$

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where the free propagator of a particle with mass m in D-dimensional (Euclidean) space-time is given by

$$D(x,m)=\frac{1}{(2\pi)^D}\int\frac{e^{i(p\cdot x)}\,d^Dp}{p^2+m^2}=\frac{(mx)^\lambda K_\lambda(mx)}{(2\pi)^{\lambda+1}x^{2\lambda}} \tag{3}$$

($D=2\lambda+2$). $K_\lambda(z)$ is the McDonald function (modified Bessel function of the second kind). Note that p and x in the integral expression of Eq. (3) are D-dimensional Lorentz vectors, i.e. $p\cdot x=p_\mu x^\mu$, while the quantity x in the rightmost expression of Eq. (3) (and, therefore, also in the argument of the propagator) denotes the absolute value $x=\sqrt{x_\mu x^\mu}$. In the limit $mx\to 0$ at fixed x, the propagator simplifies to

$$D(x,0)=\frac{1}{(2\pi)^D}\int\frac{e^{i(p\cdot x)}\,d^Dp}{p^2}=\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}x^{2\lambda}}, \tag{4}$$

where $\Gamma(\lambda)$ is Euler's Gamma function.

In this note we write the n-particle irreducible correlator function in (Euclidean) momentum space. The momentum space n-particle irreducible correlator function is given by the Fourier transform of the n-particle irreducible correlator function $\Pi(x)$ in configuration space,

$$\tilde{\Pi}(p)=\int\Pi(x)e^{i(p\cdot x)}\,d^Dx. \tag{5}$$

As a product of propagators, $\Pi(x)$ in (5) depends only on the absolute value $x=\sqrt{x_\mu x^\mu}$. Therefore, one proceeds by first integrating the exponential factor over the $D-1$ dimensional hypersphere. We write $d^Dx=x^{D-1}\,d^D\hat{x}\,dx$ where $d^D\hat{x}$ denotes the $D-1$ dimensional integration measure over the $D-1$ dimensional hypersphere. The integration of the exponential factor over the unit sphere gives

$$\int e^{i(p\cdot x)}\,d^D\hat{x}=2\pi^{\lambda+1}\left(\frac{px}{2}\right)^{-\lambda}J_\lambda(px). \tag{6}$$

$J_\lambda(z)$ is the Bessel function of the first kind. As before, p and x on the right hand side of Eq. (6) stand for the absolute values $p=\sqrt{p_\mu p^\mu}$ and $x=\sqrt{x_\mu x^\mu}$. Therefore, the correlator in momentum space depends only on the absolute value of the momentum,

$$\tilde{\Pi}(p)=2\pi^{\lambda+1}\int_0^\infty\left(\frac{px}{2}\right)^{-\lambda}J_\lambda(px)\Pi(x)x^{2\lambda+1}\,dx. \tag{7}$$

This is the central formula for our numerical verification of the correctness of the differential equations.

3 The one-loop case

In Ref. [5], Remiddi explains how to obtain the differential equation for the one-loop sunrise-type diagram with arbitrary masses and dimensions. By applying the integration-by-parts technique to the correlator in momentum space, recurrence relations can be obtained. Finally, Euler's theorem for homogeneous functions connects the loose ends of the iterative steps involving partial derivative with respect to p^2. We have numerically checked all these steps and have found numerical consistency—up to Stokes' contributions due to surface terms in integer space-times dimensions.

To be more precise, the integral

$$\int\frac{d^Dk}{(2\pi)^D}\frac{\partial}{\partial k_\mu}\left(\frac{v_\mu}{(k^2+m_1^2)((p-k)^2+m_2^2)}\right) \tag{8}$$

for $v=k,p$ (or a linear combination of both) leads to a surface term which can be assumed to vanish (up to Stokes' contributions). The integration-by-parts technique consists in calculating the integral explicitly and one then expresses the result in terms of scalar integrals

$$S(\alpha_1,\alpha_2):=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{(k^2+m_1^2)^{\alpha_1}((p-k)^2+m_2^2)^{\alpha_2}}. \tag{9}$$

The (two) resulting recurrence relations read

$$\begin{aligned}0&=DS(\alpha_1,\alpha_2)+2\alpha_1\big(m_1^2S(\alpha_1+1,\alpha_2)-S(\alpha_1,\alpha_2)\big)\\&\quad+\alpha_2\big(\big(p^2+m_1^2+m_2^2\big)S(\alpha_1,\alpha_2+1)\\&\quad-S(\alpha_1-1,\alpha_2+1)-S(\alpha_1,\alpha_2)\big),\end{aligned} \tag{10}$$

$$\begin{aligned}0&=-\alpha_1\big(\big(p^2-m_1^2+m_2^2\big)S(\alpha_1+1,\alpha_2)\\&\quad-S(\alpha_1+1,\alpha_2-1)+S(\alpha_1,\alpha_2)\big)\\&\quad+\alpha_2\big(\big(p^2+m_1^2-m_2^2\big)S(\alpha_1,\alpha_2+1)\\&\quad-S(\alpha_1-1,\alpha_2+1)+S(\alpha_1,\alpha_2)\big).\end{aligned} \tag{11}$$

Equation (11) can be replaced by Eq. (10) with the two lines interchanged,

$$\begin{aligned}0&=DS(\alpha_1,\alpha_2)+2\alpha_2\big(m_2^2S(\alpha_1,\alpha_2+1)-S(\alpha_1,\alpha_2)\big)\\&\quad+\alpha_1\big(\big(p^2+m_1^2+m_2^2\big)S(\alpha_1+1,\alpha_2)\\&\quad-S(\alpha_1+1,\alpha_2-1)-S(\alpha_1,\alpha_2)\big).\end{aligned} \tag{12}$$

It is obvious that Eq. (11) is reproduced as difference of Eq. (10) and Eq. (12). Therefore, one has to check only Eq. (10). We will perform this numerical check for the parameter choice $\alpha_1=\alpha_2=1$ which is relevant for the differential equation, and for $D=2$ space-time dimensions. As mentioned in Ref. [3], even though other dimensions are feasible, this choice avoids singular contributions and serves for the simplest integrand. The equation to be checked is

$$\begin{aligned}&2m_1^2S(2,1)+\big(p^2+m_1^2+m_2^2\big)S(1,2)\\&\quad=S(1,1)+S(0,2).\end{aligned} \tag{13}$$

Starting from

$$\begin{aligned}S(1,1)&=\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+m_1^2)((p-k)^2+m_2^2)}\\&=\frac{1}{2\pi}\int_0^\infty J_0(px)K_0(m_1x)K_0(m_2x)\,dx,\end{aligned} \tag{14}$$

the integrals $S(\alpha_1,\alpha_2)$ with higher (integer) values of α_i can be obtained as partial derivatives with respect to the masses,

$$\frac{-1}{2m_1}\frac{\partial}{\partial m_1}S(1,1)$$
$$=-\frac{\partial}{\partial m_1^2}S(1,1)$$
$$=-\frac{\partial}{\partial m_1^2}\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+m_1^2)((p-k)^2+m_2^2)}$$
$$=\int\frac{d^k}{(2\pi)^2}\frac{1}{(k^2+m_1^2)^2((p-k)^2+m_2^2)}=S(2,1) \qquad (15)$$

and accordingly

$$\frac{-1}{2m_2}\frac{\partial}{\partial m_2}S(1,1)=S(1,2). \qquad (16)$$

In addition one has

$$S(2,0)=\int\frac{d^2k}{(2\pi)^2}\frac{1}{(k^2+m_1^2)^2}=\frac{1}{4\pi m_1^2},$$
$$S(0,2)=\int\frac{d^2k}{(2\pi)^2}\frac{1}{((p-k)^2+m_2^2)^2}=\frac{1}{4\pi m_2^2}. \qquad (17)$$

The derivative can be expressed by $K_0'(z)=-K_1(z)$. The higher order integrals in the configuration space representation are given by

$$S(2,1)=\frac{-1}{2m_1}\frac{\partial}{\partial m_1}S(1,1)$$
$$=\frac{1}{4\pi m_1^2}\int_0^\infty J_0(px)(m_1x)K_1(m_1x)K_0(m_2x)x\,dx,$$
$$S(1,2)=\frac{-1}{2m_2}\frac{\partial}{\partial m_2}S(1,1)$$
$$=\frac{1}{4\pi m_2^2}\int_0^\infty J_0(px)K_0(m_1x)(m_2x)K_1(m_2x)x\,dx. \qquad (18)$$

Therefore, Eq. (13) in the configuration space representation reads

$$1=\int_0^\infty J_0(px)\big[2(m_2x)^2(m_1x)K_1(m_1x)K_0(m_2x)$$
$$+\big((px)^2+(m_1x)^2+(m_2x)^2\big)$$
$$\times K_0(m_1x)(m_2x)K_1(m_2x)$$
$$-2(m_2x)^2K_0(m_1x)K_0(m_2x)\big]\frac{dx}{x}. \qquad (19)$$

We were able to check this equation numerically for different values of p as function of m_1 and m_2. The 3D-plot in MATHEMATICA shows stochastic fluctuations around the expected value of 1 of the order of 10^{-9}.

Euler's theorem of homogeneous functions leads to the differential equation

$$\left(p^2\frac{\partial}{\partial p^2}+m_1^2\frac{\partial}{\partial m_1^2}+m_2^2\frac{\partial}{\partial m_2^2}+1\right)S(1,1)=0. \qquad (20)$$

Because of $J_0'(z)=-J_1(z)$, Eq. (20) can be translated to

$$0=\int_0^\infty(px)^2\big[(px)J_1(px)K_0(m_1x)K_0(m_2x)$$
$$+J_0(px)(m_1x)K_1(m_1x)K_0(m_2x)$$
$$+J_0(px)K_0(m_1x)(m_2x)K_1(m_2x)$$
$$-2J_0(px)K_0(m_1x)K_0(m_2x)\big]\frac{dx}{x}. \qquad (21)$$

We checked on the latter relation with an even better precision of the order of 10^{-13}.

The differential equation

$$\big(p^2+(m_1+m_2)^2\big)\big(p^2+(m_1-m_2)^2\big)\frac{\partial}{\partial p^2}S(1,1)$$
$$=-\big(p^2+m_1^2+m_2^2\big)S(1,1) \qquad (22)$$

in Ref. [5] is obtained by inserting the recurrence relations into Euler's differential equation (20). Using the configuration space representation, Eq. (22) reads

$$1=\int_0^\infty\Big[\frac{-1}{2(px)^2}\big((px)^2+(m_1x+m_2x)^2\big)$$
$$\times\big((px)^2+(m_1x-m_2x)^2\big)(px)J_1(px)$$
$$+\big((px)^2+(m_1x)^2+(m_2x)^2\big)J_0(px)\Big]$$
$$\times K_0(m_1x)K_0(m_2x)\frac{dx}{x}. \qquad (23)$$

This equation could be checked with a precision of the order of 10^{-8}.

While Euler's differential equation can be derived from general principles also for the configuration space representation, the recurrence relations can be derived only via the momentum space representation. If one does not use this technique, it remains unclear why such integral identities exist for general parameters p, m_1 and m_2. In order to check whether one can derive further relations by using integral identities in configuration space, we have used the general ansatz

$$1=\int_0^\infty J_0(px)\big[A_0(px,m_1x,m_2x)K_0(m_1x)K_0(m_2x)$$
$$+A_1(px,m_1x,m_2x)(m_1x)K_1(m_1x)K_0(m_2x)$$
$$+A_2(px,m_1x,m_2x)K_0(m_1x)(m_2x)K_1(m_2x)\big]\frac{dx}{x}, \qquad (24)$$

where $A_i(p,m_1,m_2)=a_{i0}p^2+a_{i1}m_1^2+a_{i2}m_2^2$ $(i=0,1,2)$. By choosing random values for p, m_1 and m_2, and solving the resulting system of equations, one obtains

$$a_{00}=0,\quad a_{01}=-2a,\quad a_{02}=-2+2a,$$
$$a_{10}=a,\quad a_{11}=a,\quad a_{12}=2-a, \qquad (25)$$
$$a_{20}=1-a,\quad a_{21}=1+a,\quad a_{22}=1-a,$$

where a is an arbitrary parameter. This leads to Eq. (19) and

$$0=\int_0^\infty J_0(px)\big[2\big((m_2x)^2-(m_1x)^2\big)K_0(m_1x)K_0(m_2x) + \big((px)^2+(m_1x)^2-(m_2x)^2\big)(m_1x) \times K_1(m_1x)K_0(m_2x) - \big((px)^2-(m_1x)^2+(m_2x)^2\big) \times K_0(m_1x)(m_2x)K_1(m_2x)\big]\frac{dx}{x}, \tag{26}$$

which is the difference of Eq. (19) and the same equation with m_1 and m_2 interchanged. We conclude that no more recurrence relations can be found that go beyond Eq. (19).

4 The two-loop case with equal masses

The differential equation for the two-loop degenerate sunrise diagram has been given in Ref. [7]. It reads

$$\bigg(2p^2\big(p^2+m^2\big)\big(p^2+9m^2\big)\bigg(\frac{d}{dp^2}\bigg)^2 + \big(3(4-D)p^4+10(6-D)m^2p^2+9Dm^4\big)\frac{d}{dp^2} + (D-3)\big((D-4)p^2-(D+4)m^2\big)\bigg)S(1,1,1) = \frac{3}{(D-4)^2\pi^2}, \tag{27}$$

which simplifies to

$$\bigg(p^2\big(p^2+m^2\big)\big(p^2+9m^2\big)\bigg(\frac{d}{dp^2}\bigg)^2 + \big(3p^4+20m^2p^2+9m^4\big)\frac{d}{dp^2} + \big(p^2+3m^2\big)\bigg)S(1,1,1) = \frac{3}{8\pi^2} \tag{28}$$

in $D=2$ space-time dimensions. We write $S(1,1,1)$ in a form which is easily adapted to the non-degenerate mass case to be discussed later on. One has

$$S(1,1,1) = \frac{1}{(2\pi)^2}\int_0^\infty J_0(px)K_0(m_1x)K_0(m_2x)K_0(m_3x)x\,dx. \tag{29}$$

Differentiation of Eq. (29) gives

$$\frac{d}{dp^2}S(1,1,1) = \frac{1}{2p}\frac{d}{dp}S(1,1,1) = \frac{-1}{2(2\pi)^2p^2}\int_0^\infty (px)J_1(px)K_0(m_1x)K_0(m_2x) \times K_0(m_3x)x\,dx \tag{30}$$

and

$$\bigg(\frac{d}{dp^2}\bigg)^2 S(1,1,1) = \frac{1}{4(2\pi)^2p^4}\int_0^\infty\bigg[(px)J_1(px) + \frac{1}{2}(px)^2\big(J_2(px)-J_0(px)\big)\bigg] \times K_0(m_1x)K_0(m_2x)K_0(m_3x)x\,dx. \tag{31}$$

Returning to the degenerate mass case the differential equation (28) can be translated to

$$\frac{3}{2}=\frac{1}{4p^2}\int_0^\infty\bigg[\big(p^2+m^2\big)\big(p^2+9m^2\big) \times\bigg((px)J_1(px)+\frac{1}{2}(px)^2\big(J_2(px)-J_0(px)\big)\bigg) - 2\big(3p^4+20m^2p^2+9m^4\big)(px)J_1(px) + 4p^2\big(p^2+3m^2\big)J_0(px)\bigg]K_0(mx)^3x\,dx, \tag{32}$$

where $J_0''(z)=-J_1'(z)=(J_2(z)-J_0(z))/2$ is used. Because the result contains the second derivative of the Bessel function, one can use Bessel's differential equation

$$z^2J_\lambda''(z)+zJ_\lambda'(z)+\big(z^2-\lambda^2\big)J_\lambda(z)=0 \tag{33}$$

for $\lambda=0$ to compactify the result,

$$\frac{3}{2}=\frac{1}{4p^2}\int_0^\infty\big[\big(4p^2\big(p^2+3m^2\big) - \big(p^2+m^2\big)\big(p^2+9m^2\big)(px)^2\big)J_0(px) - 2p^2\big(p^2+5m^2\big)J_1(px)\big]K_0(mx)^3x\,dx. \tag{34}$$

These results have been checked with a precision of the order of 10^{-8}.

5 The two-loop case with arbitrary masses

The final section of this paper is devoted to the second order differential equation, derived for the two-loop sunrise diagram with arbitrary masses in Ref. [3]. After adjusting the normalization, the differential equation can be written as

$$\bigg[p_0\big(-p^2\big)\bigg(\frac{d}{dp^2}\bigg)^2+p_1\big(-p^2\big)\frac{d}{dp^2}+p_2\big(-p^2\big)\bigg]S(1,1,1) = \frac{p_3(-p^2)}{4(2\pi)^2}, \tag{35}$$

where the coefficients $p_i(t)$ $(i=0,1,2,3)$ are given by

$$\begin{aligned} p_0(t) &= t\big(t-(m_1+m_2+m_3)^2\big)\big(t-(-m_1+m_2+m_3)^2\big) \\ &\quad \times \big(t-(m_1-m_2+m_3)^2\big)\big(t-(m_1+m_2-m_3)^2\big) \\ &\quad \times \big(3t^2-2M_{100}t-M_{200}+2M_{110}\big), \end{aligned} \tag{36}$$

$$\begin{aligned} p_1(t) &= 9t^6-32M_{100}t^5+(37M_{200}+70M_{110})t^4 \\ &\quad -(8M_{300}+56M_{210}+144M_{111})t^3 \\ &\quad -(13M_{400}-36M_{310}+46M_{220}-124M_{211})t^2 \\ &\quad -(-8M_{500}+24M_{410}-16M_{320}-96M_{311} \\ &\quad +144M_{221})t-(M_{600}-6M_{510}+15M_{420} \\ &\quad -20M_{330}+18M_{411}-12M_{321}-6M_{222}), \end{aligned} \tag{37}$$

$$\begin{aligned} p_2(t) &= 3t^5-7M_{100}t^4+(2M_{200}+16M_{110})t^3 \\ &\quad +(6M_{300}-14M_{210})t^2-(5M_{400} \\ &\quad -8M_{310}+6M_{220}-8M_{211})t+(M_{500}-3M_{410} \\ &\quad +2M_{320}+8M_{311}-10M_{221}), \end{aligned} \tag{38}$$

$$\begin{aligned} p_3(t) &= -18t^4+24M_{100}t^3+(4M_{200}-40M_{110})t^2 \\ &\quad +(-8M_{300}+8M_{210}+48M_{111})t \\ &\quad +(-2M_{400}+8M_{310}-12M_{220}-8M_{211}) \\ &\quad +2c(t,m_1,m_2,m_3)\ln\big(m_1^2/\mu^2\big) \\ &\quad +2c(t,m_2,m_3,m_1)\ln\big(m_2^2/\mu^2\big) \\ &\quad +2c(t,m_3,m_1,m_2)\ln\big(m_3^2/\mu^2\big), \end{aligned} \tag{39}$$

and where

$$M_{\lambda_1\lambda_2\lambda_3} = \sum_\sigma \big(m_1^2\big)^{\sigma(\lambda_1)}\big(m_2^2\big)^{\sigma(\lambda_2)}\big(m_3^2\big)^{\sigma(\lambda_3)} \tag{40}$$

are monomial symmetric polynomials in m_1^2, m_2^2 and m_3^2 and where

$$\begin{aligned} &c(t,m_1,m_2,m_3) \\ &\quad = \big(-2m_1^2+m_2^2+m_3^2\big)t^3+\big(6m_1^4-3m_2^4 \\ &\qquad -3m_3^4-7m_1^2m_2^2-7m_1^2m_3^2+14m_2^2m_3^2\big)t^2 \\ &\qquad +\big(-6m_1^6+3m_2^6+3m_3^6+11m_1^4m_2^2+11m_1^4m_3^2 \\ &\qquad -8m_1^2m_2^4-8m_1^2m_3^4-3m_2^4m_3^2-3m_2^2m_3^4\big)t \\ &\qquad +\big(2m_1^8-m_2^8-m_3^8-5m_1^6m_2^2-5m_1^6m_3^2 \\ &\qquad +m_1^2m_2^6+m_1^2m_3^6+4m_2^6m_3^2+4m_2^2m_3^6 \\ &\qquad +3m_1^4m_2^4+3m_1^4m_3^4-6m_2^4m_3^4+2m_1^4m_2^2m_3^2 \\ &\qquad -m_1^2m_2^4m_3^2-m_1^2m_2^2m_3^4\big) \end{aligned} \tag{41}$$

(for details, cf. Ref. [3]). Using Eqs. (29), (30) and (31), one obtains

$$\begin{aligned} p_3\big(-p^2\big) &= \int_0^\infty \bigg[\frac{p_0(-p^2)}{p^4}\bigg((px)J_1(px)+\frac{1}{2}(px)^2 \\ &\quad \times \big(J_2(px)-J_0(px)\big)\bigg) \\ &\quad -2\frac{p_1(-p^2)}{p^2}(px)J_1(px)+4p_2\big(-p^2\big)J_0(px)\bigg] \\ &\quad \times K_0(m_1x)K_0(m_2x)K_0(m_3x)x\,dx \\ &= \int_0^\infty \bigg[\bigg(4p_2\big(-p^2\big)-\frac{p_0(-p^2)}{p^4}(px)^2\bigg)J_0(px) \\ &\quad -2\frac{p_1^*(-p^2)}{p^2}(px)J_1(px)\bigg] \\ &\quad \times K_0(m_1x)K_0(m_2x)K_0(m_3x)x\,dx, \end{aligned} \tag{42}$$

where $p_1^*(t)=p_1(t)+p_0(t)/t$. Using different values for p and m_3, in terms of m_1 and m_2 we obtain a 3D-plot with MATHEMATICA which shows again stochastic fluctuations of the order of 10^{-4}. In the course of our numerical checks we were able to identify two typos in the coefficients of $c(t,m_1,m_2,m_3)$ in the preprint version of Ref. [3] which we have corrected.

6 Conclusions

Using configuration space techniques, we were able to check numerically the differential equations for sunrise-type diagrams found in the literature. The precision of our numerical test is still quite moderate, but gives sufficient confidence in the validity of the differential equations derived by other means. For example, the introduction of artificial "typos" in the coefficients of the differential equations are easily discovered. More rigorous tests would require the use of more stable integration routines than those provided by MATHEMATICA. For the future we hope to find independent routes to discover further relations between Bessel moments which may lead to generalizations of the present findings to cases involving three-loop or even higher order sunrise-type diagrams.

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