A one-dimensional diffusion hits points fast

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Abstract

A one-dimensional, continuous, regular, and strong Markov process X with state space E hits any point $z \in E$ fast with positive probability. To wit, if $\boldsymbol{\tau}_z = \inf\{t \geq 0 : X_t = z\}$, then $\mathsf{P}_\xi(\boldsymbol{\tau}_z < \varepsilon) > 0$ for all $\xi \in E$ and $\varepsilon > 0$.

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1 Introduction

Consider a measurable function $\sigma : \mathbb{R} \mapsto \mathbb{R} \setminus \{0\}$ such that $1/\sigma^2$ is locally integrable. Then [4] guarantees the existence of a filtered probability space $(\Omega, \mathscr{F}, \mathfrak{F}, \mathsf{P})$, equipped with a Brownian motion $W = (W_t)_{t \geq 0}$, and the existence of a stochastic process $Z = (Z_t)_{t \geq 0}$ such that

$$Z_t = \int_0^t \sigma(Z_s)\mathrm{d}W_s, \qquad t \geq 0$$

holds. Moreover, Z is strong Markov and continuous. Let now $z \in \mathbb{R}$, $\varepsilon > 0$, and $\boldsymbol{\tau}_z^Z$ denote the first hitting time of z by Z. Then we know that $\mathsf{P}(\boldsymbol{\tau}_z^Z < \infty) > 0$. [11] and [8] ask whether also $\mathsf{P}(\boldsymbol{\tau}_z^Z < \varepsilon) > 0$ holds for all $\varepsilon > 0$. Only a partial answer is provided: If $1/\sigma^4$ is locally integrable (everywhere, apart from countably many points), then the answer is affirmative.

This note answers the question affirmatively in a general setup. To this end, we fix an open, half-open, or closed interval E of $\mathbb{R}$, and denote its interior by $\mathring{E}$ and its closure in $[-\infty, \infty]$ by $\overline{E}$. We then consider a one-dimensional Markov process $X = (X_t)_{t \geq 0}$ with state space E on the filtered space $(\Omega, \mathscr{F}, \mathfrak{F})$, along with a family of probability measures $(\mathsf{P}_\xi)_{\xi \in E}$. We denote the death-time of X by ζ. We assume that X is strong Markov, regular, continuous on $[0, \zeta)$, and $\lim_{t \nearrow \zeta} X_t$ exists and satisfies $\lim_{t \nearrow \zeta} X_t \notin E$ on $\{\zeta < \infty\}$. We set $X_{\zeta+s} = \lim_{t \nearrow \zeta} X_t \in \overline{E}$ for all $s \geq 0$ on $\{\zeta < \infty\}$. If $Y = (Y_t)_{t \geq 0}$ is a stochastic process and $\boldsymbol{\rho}$ a stopping time, then $Y^{\boldsymbol{\rho}} = (Y_t^{\boldsymbol{\rho}})_{t \geq 0} = (Y_{\boldsymbol{\rho} \wedge t})_{t \geq 0}$. Furthermore, if Y is a semimartingale, we let $[Y] = ([Y]_t)_{t \geq 0}$ denote the quadratic variation process of Y.

We now define the stopping times

$$\boldsymbol{\tau}_z = \inf\{t \geq 0 : X_t = z\}, \qquad z \in \overline{E}.$$

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Since X is regular, we have $\mathsf{P}_\xi(\tau_z < \infty) > 0$ for all $\xi \in \mathring{E}$ and $z \in E$. Throughout the note we shall fix a starting point $\xi \in E$ and a target point $z \in \overline{E} \setminus \{\xi\}$ such that $\mathsf{P}_\xi(\tau_z < \infty) > 0$. Then ξ cannot be an absorbing boundary point for X, and we also have $\mathsf{P}_\xi(\tau_y < \infty) > 0$ for all $y \in E$. We are now able to state the main result of this note.

Theorem 1.1. *For all $\varepsilon > 0$, we have $\mathsf{P}_\xi(\tau_z < \varepsilon) > 0$.*

The theorem is proved in the next section. We directly obtain two corollaries.

Corollary 1.2. *The support of τ_z is $[0, \infty]$ under P_ξ; that is, for any open subset $\mathcal{U} \subset [0, \infty]$ we have $\mathsf{P}_\xi(\tau_z \in \mathcal{U}) > 0$.*

Proof. Let us start by fixing $t \geq 0$ and arguing that $\mathsf{P}_\xi(t < \tau_z < \infty) > 0$. Without loss of generality, we may assume that $\xi < z$ and that $\xi \in \mathring{E}$. We now fix a point $y \in \mathring{E}$ with $y < \xi$ and note that there exists some $\delta > 0$ such that the stopping time $\tau_{y,\xi} = \inf\{t \geq \tau_y : X_t = \xi\}$ satisfies

$$\mathsf{P}_\xi\left(\{\delta < \tau_{y,\xi} < 1/\delta\} \cap \{\tau_y < \tau_z\}\right) > 0.$$

Applying the strong Markov property then several times and using the fact $X_{\tau_{y,\xi}} = \xi$ on the event $\{\tau_{y,\xi} < \infty\}$ yields the claim $\mathsf{P}_\xi(t < \tau_z < \infty) > 0$.

In order to argue the statement we need to prove that the survival function $[0, \infty) \ni t \mapsto \mathsf{P}_\xi(\tau_z > t)$ is strictly decreasing. Note that the strong Markov property of X yields

$$\mathsf{P}_\xi(\tau_z > t + \varepsilon) = \mathsf{E}_\xi[\mathbf{1}_{\{\tau_z > t\}} \mathsf{P}_{X(t)}(\tau_z > \varepsilon)] < \mathsf{P}_\xi(\tau_z > t)$$

for all $t \geq 0$ and $\varepsilon > 0$, where the inequality uses $\mathsf{P}_\xi(\tau_z > t) > 0$ and $\mathsf{P}_{X(t)}(\tau_z > \varepsilon) < 1$ on $\{\tau_z > t\}$ with positive probability under P_ξ, thanks to Theorem 1.1. This theorem may be applied since $\mathsf{P}_\xi(t < \tau_z < \infty) > 0$ yields that $\mathsf{P}_{X(t)}(\tau_z < \infty) > 0$ on $\{\tau_z > t\}$ with positive probability under P_ξ. $\qquad\square$

Corollary 1.3. *For each $t > 0$, the support of X_t is $\overline{E}$ under P_ξ; that is, for any open subset $\mathcal{U} \subset \overline{E}$ we have $\mathsf{P}_\xi(X_t \in \mathcal{U}) > 0$.*

Proof. The statement follows directly from Corollary 1.2 and the continuity of X. To add some details, let us fix a point y in some open subset $\mathcal{U} \subset \overline{E}$. The continuity of X now yields the existence of a constant $\delta > 0$ such that $\mathsf{P}_y(\inf_{s \leq \delta} X_s \in \mathcal{U}; \sup_{s \leq \delta} X_s \in \mathcal{U}) > 0$. Next, let us fix $t > 0$ and observe that

$$\mathsf{P}_\xi(X_t \in \mathcal{U}) \geq \mathsf{P}_\xi\left(\tau_y \in (t - \delta, t)\right) \mathsf{P}_y\left(\inf_{s \leq \delta} X_s \in \mathcal{U}; \sup_{s \leq \delta} X_s \in \mathcal{U}\right) > 0$$

by Corollary 1.2. $\qquad\square$

Remark 1.4. We now provide some warnings concerning Thereom 1.1.

- The continuity of X is clearly important in Theorem 1.1. For instance, the compensated Poisson process with state space $E = \mathbb{R}$ is strong Markov and regular, but the assertion of Theorem 1.1 does not hold for it.

- If X is Brownian motion then Theorem 1.1 clearly holds. If X is only a local martingale, the Dambis-Dumbins-Schwarz theorem yields the representation $X = B_{[X]}$ for some Brownian motion B and Lemma 2.3 below yields that $[X]$ is strictly increasing. However, B and $[X]$ are usually not independent. In particular, $[X]$ might slow down as X approaches a point. Thus, an argument for Theorem 1.1 that is based purely on a change of time is incomplete. $\qquad\square$

After we had completed this note, Umut Cetin and Pat Fitzsimmons pointed out to us that Theorem 1.1 could also be derived as follows. First, the theorem could be proved by

studying the (positivity of the) transition density of X directly, as for example presented in Section 4.11 of [7]. Alternatively, Krein's spectral theory of strings yields precise estimates for the transition density of X for short time horizons; see Appendix II of [10]. These estimates then yield Theorem 1.1 as a corollary. The arguments of this note, however, are less analytic and more direct.

2 Proof of Theorem 1.1

Before proving Theorem 1.1, we provide some auxiliary results.

Lemma 2.1. *Let* $v : [0, \infty) \to [0, \infty)$ *denote a nonnegative function with* $v(0) = 0$ *that satisfies* $v(t + s) - v(t) \leq s$ *for all* $s, t \geq 0$. *Then the first variation of* $v|_{[0,t]}$ *is bounded by* $2t$, *for each* $t \geq 0$.

Proof. Note that v can increase by at most t on the interval $[0, t]$. This, in conjunction with the nonnegativity of v, then yields that v can drop by at most t as well, and hence the bound of $2t$. $\qquad\square$

Recall that we have fixed a strong Markov process X with state space E and a starting point $\xi \in E$ for which the following results are formulated.

Proposition 2.2. *Let* $v : \overline{E} \to \mathbb{R}$ *be a measurable function and consider the case that* $\xi \in \mathring{E}$ *and that the strong Markov process* X *is a continuous* P_ξ*–local martingale. Then the function* $[0, \infty) \ni t \mapsto v(X_t)$ *is of finite first variation on compact subintervals of* $[0, \infty)$, P_ξ*–almost surely, if and only if* v *is constant on* $\mathring{E}$.

Since Proposition 2.2 is the core step of this note's argument we provide three different proofs.

Preparation for the proofs of Proposition 2.2. Since X is a P_ξ–local martingale, and hence gets absorbed when hitting a boundary point, v being constant on $\mathring{E}$ implies that $v(X.)$ is of finite first variation; thus it suffices to argue the reverse direction. Hence, from now on, we will assume that $v(X.)$ is of finite first variation on compact subintervals of $[0, \infty)$. Note that $v(X.)$ is of finite first variation variation on $\{\zeta < \infty\}$. If $\mathsf{P}_\xi(\zeta = \infty) > 0$ let $(a_n)_{n \in \mathbb{N}}$ be a strictly decreasing sequence and $(b_n)_{n \in \mathbb{N}}$ a strictly increasing sequence such that $\mathring{E} = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ and $\xi \in (a_1, b_1)$. Moreover, let

$$\zeta_n = \inf\{t \geq 0 : X_t \notin (a_n, b_n)\}, \qquad n \in \mathbb{N}.$$

Then we have $\zeta_n < \infty$ and $v(X^{\zeta_n})$ is of finite first variation for each $n \in \mathbb{N}$. Moreover, note that v is constant on $\mathring{E}$ if and only if v is constant on (a_n, b_n) for each $n \in \mathbb{N}$. Thus, we shall assume, without loss of generality, that $v(X.)$ is of finite first variation.

Next, observe that the Dambis-Dumbins-Schwarz theorem yields the existence of a Brownian motion $B = (B_t)_{t \geq 0}$ with $B_0 = \xi$, possibly on an extension of the probability space, such that $X = B_{[X]}$; see, for instance, Theorem V.1.7 in [12]. Using the fact that $[X]$ is continuous and defining $\rho = [X]_\infty$, the process $v(B^\rho)$ is also of finite first variation.

The first proof relies on an application of the Itô-Meyer-Tanaka formula.

Proof I of Proposition 2.2. Proceeding as in Section 5 in [3] we observe that v is a so called semimartingale function for a Brownian motion killed when hitting the boundary of E and thus, v is locally the difference of two convex functions. More precisely, with $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ as above, $v|_{[a_n, b_n]}$ is the difference of two convex functions. It then suffices to prove that $D^- v|_{[a_n, b_n]} = 0$, where $D^- v|_{[a_n, b_n]}$ denotes its left derivative, for each $n \in \mathbb{N}$. To this end, let

$$\rho_n = \inf\{t \geq 0 : B_t \notin (a_n, b_n)\}, \qquad n \in \mathbb{N}.$$

Then the Itô-Meyer-Tanaka formula yields

$$v(B_{\cdot}^{\rho_n}) = v(\xi) + \int_0^{\cdot \wedge \rho_n} D^- v|_{[a_n,b_n]}(B_t)\mathrm{d}B_t + A_{\cdot}^{\rho_n}, \qquad n \in \mathbb{N},$$

where $A = (A_t)_{t \geq 0}$ is a process of finite first variation. Since $v(B_{\cdot}^{\rho_n})$ is of finite first variation we obtain $\int_0^{\cdot \wedge \rho_n} (D^- v|_{[a_n,b_n]}(B_t))^2 \mathrm{d}t = 0$, and thus $D^- v|_{[a_n,b_n]} = 0$ for each $n \in \mathbb{N}$, as desired. □

We remark that [1] provides a similar proof. The next proof has been suggested by Vilmos Prokaj, to whom we are very grateful. The proof requires the additional assumption that v is of finite first variation and uses local time of Brownian motion.

Proof II of Proposition 2.2. Let $N(x, y)$ denote the number of upcrossings of $[x, y]$ made by B^ρ for all $x, y \in \mathbb{R}$ with $x < y$. Moreover, let $L_\rho(x)$ denote the local time of B^ρ at $x \in \overline{E}$, fix $\varepsilon > 0$, and pick some sufficiently small $\delta > 0$, possibly depending on $\omega \in \Omega$, such that

$$|\delta N(x, x + \delta) - L_\rho(x)| \leq \varepsilon$$

for all $x \in \overline{E}$. Such a δ exists almost surely, thanks to the uniform convergence of Theorem 2 in [2]. Next, define the sequence $(\sigma_k)_{k \in \mathbb{N}_0}$ of stopping times inductively by $\sigma_0 = 0$ and

$$\sigma_{k+1} = \rho \wedge \inf\{t > \sigma_k : |B_t - B_{\sigma_k}| = \delta\}.$$

Suppose that the first variation Ξ of the function $v(B_{\cdot}^\rho)$ is finite almost surely, directly implying that v is continuous on $\mathring{E}$. Then we have

$$\delta \Xi \geq \delta \sum_{k \in \mathbb{N}} |v(B_{\sigma_{k+1} \wedge \rho}) - v(B_{\sigma_k \wedge \rho})| \geq \sum_{i \in \mathbb{Z}, (i\delta, i\delta+\delta) \subset E} |v(i\delta + \delta) - v(i\delta)| \delta N(i\delta, i\delta + \delta)$$

$$\geq \sum_{i \in \mathbb{Z}, (i\delta, i\delta+\delta) \subset E} |v(i\delta + \delta) - v(i\delta)| (L_\rho(i\delta) - \varepsilon).$$

Letting now δ tend to zero and using the continuity of L_ρ, argued in Theorem VI.1.7 in [12], note that

$$0 = \lim_{\delta \downarrow 0} \delta \Xi \geq \int_{\mathring{E}} L_\rho(x)|\mathrm{d}v(x)| - \varepsilon \mathrm{TV}(v),$$

where $\mathrm{TV}(v)$ denotes the variation of v, which is finite by assumption. Next, letting ε tend to zero, taking expectations, and using Tonelli yields

$$\int_{\mathring{E}} \mathsf{E}_\xi[L_\rho(x)]|\mathrm{d}v(x)| = 0.$$

Since each expectation is strictly positive, we obtain that the function v is constant on $\mathring{E}$. □

The third proof follows a pathwise argument and relies less on the one-dimensional character of X. The proof requires the additional assumption that $v(\xi) = 0$, v is nonnegative, and there exists a P_ξ–nullset N such that for all $s, t \geq 0$ and $\omega \in \Omega \setminus N$ we have the upper-Lipschitz condition

$$v(X_{t+s}(\omega)) - v(X_t(\omega)) \leq s. \tag{2.1}$$

Proof III of Proposition 2.2. Again, clearly v is continuous on $\mathring{E}$. Fix now some $\omega \in \Omega$ such that the function $f : [0, \zeta(\omega)) \to \mathbb{R}$, $t \mapsto v(X_t(\omega))$ is of finite first variation, (2.1) holds, and $X(\omega)$ has no point of monotonicity (see Theorem 2.9.13 in [9]). Then f is

continuous and Theorem 3.23(b) in [6] yields that f has a derivative f' almost everywhere. Levy's decomposition theorem, Hahn's decomposition theorem, and Proposition 3.30 in [6] yield the existence of two nonnegative measures μ_- and μ_+, both singular with respect to each other and to Lebesgue measure, such that

$$\mathrm{d}f = f'\mathrm{d}t - \mathrm{d}\mu_- + \mathrm{d}\mu_+.$$

Suppose now that $f'(t) > 0$ for some $t > 0$. Then we must have $f(t + h) - f(t) > 0$ for all sufficiently small $h \in \mathbb{R}$, but then t is a point of monotonicity of $X(\omega)$. This contradicts the choice of ω. Thus $f' \leq 0$ and we get in the same way that $f' = 0$. Therefore, $\mathrm{d}f = -\mathrm{d}\mu_- + \mathrm{d}\mu_+$ Since, on intervals, we have $\mathrm{d}f \leq \mathrm{d}t$ thanks to the upper-Lipschitz condition we get $\mu_+ \leq m + \mu_-$, where m denotes the Lebesgue measure. Thanks to a monotone class argument we also get $\mu_+(D) \leq m(D) + \mu_-(D)$ for all $D \in \mathcal{B}$, the Borel sigma algebra of $[0, \infty)$. Thus, μ_+ is both absolutely continuous and singular with respect to $m + \mu_-$, and we get $\mu_+ = 0$. Finally, since $f \geq 0$ and $f(0) = 0$, we have $\mu_- = 0$, and so f is constant. □

Proposition 2.2 could also be argued as a simple consequence of Theorem 1 in [5].

Lemma 2.3. *Consider the case that the strong Markov process X is a continuous P_ξ–local martingale. Then the quadratic variation process $[X]$ is P_ξ–almost surely strictly increasing on $[0, \widetilde{\zeta})$, where $\widetilde{\zeta} = \inf\{t \geq 0 : X_t \notin \mathring{E}\}$.*

Proof. Proposition III.3.13 and the discussion proceeding it in [12] yield that X cannot be constant on an interval before hitting the boundary of E. Proposition IV.1.13 in [12] then yields the statement. □

Before stating the next lemma we introduce some notation. We observe that E is of the form $E = (a, b)$, $E = [a, b)$, $E = (a, b]$, or $E = [a, b]$ for some $a, b \in [-\infty, \infty]$ with $a < b$. For each $x \in \overline{E}$ we now define the deterministic function $u_x : \overline{E} \mapsto [0, 1]$ by

$$u_x(y) = 1 \wedge \inf\{t \geq 0 : \mathsf{P}_x(\tau_y \leq t) > 0\}, \quad y \in \mathring{E}; \quad u_x(a) = \lim_{y \searrow a} u_x(y); \quad u_x(b) = \lim_{y \nearrow b} u_x(y). \tag{2.2}$$

Note that u_x is nonincreasing before x and nondecreasing after x; thus, in particular, the limits in (2.2) always exist, for each $x \in \overline{E}$. Moreover, u_x is nonnegative, of finite first variation, and satisfies $u_x(x) = 0$, for each $x \in \mathring{E}$. Observe that an equivalent formulation of Theorem 1.1 is the statement that u_ξ is constant, at least, if $\xi \in \mathring{E}$.

Lemma 2.4. *Consider the case that $\xi \in \mathring{E}$ and that the strong Markov process X is a continuous P_ξ–local martingale. The function u_ξ, given in (2.2), satisfies the following two claims.*

(i) *u_ξ is continuous;*

(ii) *there exists a P_ξ–nullset N such that for all $s, t \geq 0$ and $\omega \in \Omega \setminus N$ we have*

$$u_\xi(X_{t+s}(\omega)) - u_\xi(X_t(\omega)) \leq s.$$

Proof. To start, for all $x, w \in \overline{E}$, we have the triangle inequality

$$u_x(\cdot) \leq u_x(w) + u_w(\cdot). \tag{2.3}$$

Indeed, this is clear if either one of the two summands equals one. To see the distributional property of (2.3) otherwise, fix $x, w, y \in \overline{E}$ and assume for the moment that the underlying probability space is the canonical one; see Section I.3 in [12]. Then, for each

path ω we have the inequality $\boldsymbol{\tau}_y(\omega) \leq \boldsymbol{\tau}_w(\omega) + \boldsymbol{\tau}_y(\theta_{\boldsymbol{\tau}_w(\omega)}(\omega))$, where θ denotes the shift operator; that is $\theta_t(\omega)(\cdot) = \omega(t + \cdot)$ for all $t \geq 0$, see also the discussion on page 104 in [12]. Fix now $\varepsilon > 0$ and $t_1 = u_x(w) + \varepsilon/2$ and $t_2 = u_w(y) + \varepsilon/2$. Then we have

$$\mathsf{P}_x(\boldsymbol{\tau}_y \leq t_1 + t_2) \geq \mathsf{P}_x(\boldsymbol{\tau}_w + \boldsymbol{\tau}_y(\theta_{\boldsymbol{\tau}_w}) \leq t_1 + t_2) \geq \mathsf{P}_x(\boldsymbol{\tau}_w \leq t_1; \boldsymbol{\tau}_y(\theta_{\boldsymbol{\tau}_w}) \leq t_2)$$
$$= \mathsf{P}_x(\boldsymbol{\tau}_w \leq t_1)\mathsf{P}_w(\boldsymbol{\tau}_y \leq t_2) > 0,$$

where the equality follows the strong Markov property of X and the last inequality follows from the definition of t_1 and t_2. This yields directly that $u_x(y) \leq t_1 + t_2 = u_x(w) + u_w(y) + \varepsilon$. Letting ε tend to zero then gives (2.3).

Claim (i): First, for any $w \in \mathring{E}$, the continuity of u_w at w follows from the fact that X is not constant on any interval (see the proof of Lemma 2.3), in conjunction with the strong Markov property. Let us now study the continuity of u_ξ at some $y \in \mathring{E}$. Without loss of generality, we may assume that $y > \xi$. The right-continuity then follows from (2.3) and the continuity of u_y at y. For the left-continuity of u_ξ at y, Section 3.3 in [7] or Lemma 4.1, in particular (4.5), in [8] also hold for the case of the regular, strong Markov process X, thanks to Lemma 2.3. Thus, for each $\varepsilon > 0$ there exists $w \in (\xi, y)$ such that $\mathsf{P}_w(\boldsymbol{\tau}_y \leq \varepsilon) > 0$. The left-continuity of u_ξ at y then follows by another application of (2.3).

Claim (ii): Assume first that there exists some $t \geq 0$ such that $\mathsf{P}_w(u_w(X_t) > t) > 0$ for some $w \in \mathring{E}$. This then implies that there exists some $y \in \mathring{E}$ such that $u_w(y) > t$ and $\mathsf{P}_w(X_t > y) > 0$ if $y > \xi$ and $\mathsf{P}_w(X_t < y) > 0$ if $y < \xi$, respectively. This, in conjunction with the continuity of X, however, contradicts the definition of u_w in (2.2). We therefore have

$$\mathsf{P}_w\left(u_w(X_t) \leq t\right) = 1 \qquad \text{for all } t \geq 0 \text{ and } w \in \mathring{E}. \tag{2.4}$$

Fix now $q_1, q_2 \in \mathbb{Q}$. Conditioning and the strong Markov property of X then yield that $\mathsf{P}_\xi(u_\xi(X_{q_1+q_2}) - u_\xi(X_{q_1}) \leq q_2) = 1$ if

$$\mathsf{P}_w\left(u_\xi(X_{q_2}) - u_\xi(w) \leq q_2\right)\big|_{w=X_{q_1}} = 1 \text{ holds } \mathsf{P}_\xi\text{–almost surely.} \tag{2.5}$$

We now note that (2.3) and (2.4) imply (2.5). The claim then follows from the continuity of u_ξ and X. $\qquad\square$

We are now ready to prove this note's main result.

Proof of Theorem 1.1. Let us first consider the case $\xi \in \mathring{E}$. Then, in order to show the statement we may assume that X is stopped when exiting $\mathring{E}$. Moreover, thanks to Propositions VII.3.2, VII.3.4, and VII.3.5 in [12] we may assume, without loss of generality, that X is in natural scale and thus a P_ξ–local martingale. Next, we recall the function u_ξ, given in (2.2). Now Lemma 2.1, in conjunction with Lemma 2.4(ii), yields that the function $[0, \infty) \ni t \mapsto u_\xi(X_t)$ has finite first variation on compact subintervals of $[0, \infty)$, P_ξ–almost surely. Proposition 2.2 now implies that u_ξ is constant. This yields that $u_\xi(z) = u_\xi(\xi) = 0$, and thus, the assertion of the theorem follows if $\xi \in \mathring{E}$.

Next, if $\xi \notin \mathring{E}$ then Proposition III.2.19 in [12] and the assumptions that $\mathsf{P}_\xi(\boldsymbol{\tau}_z < \infty) > 0$ and $\xi \neq z$ yield the existence of a stopping time $\boldsymbol{\rho}$ taking values in $[0, \varepsilon/2]$ such that $\mathsf{P}_\xi(X_{\boldsymbol{\rho}} \in \mathring{E}) > 0$. Another application of the strong Markov property, together with fact that we already argued the case $\xi \in \mathring{E}$ now concludes the proof. $\qquad\square$

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