# A PAIR OF BIORTHOGONAL POLYNOMIALS FOR THE SZEGÖ-HERMITE WEIGHT FUNCTION

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ABSTRACT. A pair of polynomial sequences  $\{S_n^{\mu}(x;k)\}$  and  $\{T_m^{\mu}(x;k)\}$  where  $S_n^{\mu}(x;k)$  is of degree n in  $x^k$  and  $T_m^{\mu}(x;k)$  is of degree m in x, is constructed. It is shown that this pair is biorthogonal with respect to the Szegö-Hermite weight function  $|x|^{2\mu}\exp(-x^2)$ ,  $(\mu > -1/2)$  over the interval  $(-\infty,\infty)$  in the sense that

$$\int_{-\infty}^{\infty} |x|^{2\mu} \exp(-x^2) S_n^{\mu}(x;k) T_m^{\mu}(x;k) dx = 0, \text{ if } m \neq n$$

$$\neq 0, \text{ if } m = n$$

where  $m, n = 0, 1, 2, \ldots$  and k is an odd positive integer.

Generating functions, mixed recurrence relations for both these sets are obtained. For k=1, both the above sets get reduced to the orthogonal polynomials introduced by professor Szegö.

KEYS WORDS AND PHRASES. Szego-Hermite weight function, Biorthogonal pair, Generating functions, Recurrence relations, etc.

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# 1. INTRODUCTION.

The biorthogonality conditions are useful in the computations involving the penetration of gamma rays through matter as well as in determining the moments of a hypergeometric distribution function. The notion of biorthogonality dates back to Didon [1] and Deruyts [2]. The questions of constructing biorthogonal pairs of polynomials corresponding to the weight functions of classical orthogonal polynomials were taken up by Konhauser [3] for the Laguerre weight function  $\mathbf{x}^{\alpha} = \mathbf{x}^{-\mathbf{x}}$ , by Toscano [4], Chai [5], Carlitz [6] and Madhekar and Thakare [7] for the Jacobi weight function  $(1-\mathbf{x})^{\alpha}$  (1+x) and by Thakare and Madhekar [4] for the Hermite weight function  $\exp(-\mathbf{x}^2)$ . The Szegö-Hermite polynomials  $\mathbf{H}^{\mu}_{\mathbf{n}}(\mathbf{x})$  are orthogonal w.r.t. the Szegö-Hermite weight function  $|\mathbf{x}|^{2\mu}\exp(-\mathbf{x}^2)$ ,  $(\mu > -1/2)$  over the interval  $(-\infty,\infty)$  and these are found

useful in connection with Gauss-Jacobi mechanical quadrature, see Szegö [8]. For  $\mu$  = 0, Szegö-Hermite polynomials are just the classical Hermite polynomials. 2. A BIORTHOGONAL SYSTEM.

We shall construct a pair of biorthogonal polynomials w.r.t. the Szego-Hermite weight function  $|x|^{2\mu} \exp(-x^2)$ ,  $\mu > -1/2$ . Consider the following pair of polynomial sequences.

$$S_{n}^{\mu}(x;k) = 2^{n}\Gamma((kn + k - k\epsilon)/2 + \mu + \epsilon)$$

$$\cdot \sum_{j=0}^{\lfloor 1/2 \rfloor} (-1)^{j} {\lfloor n/2 \rfloor \choose j} x^{nk-2kj} / \Gamma((kn+1+\epsilon)/2 - kj + \mu). \qquad (2.1)$$

$$T_n^{\mu}(x;k) \ = \ (-1)^{\left \lceil n/2 \right \rceil} 2^n \quad \sum_{r=0}^{\left \lceil n/2 \right \rceil} \ x^{n-2r}/(\left \lceil n/2 \right \rceil - r) \,! \quad \sum_{s=0}^{\left \lceil n/2 \right \rceil - r} \ (-1)^s \left( \begin{array}{c} \left \lceil n/2 \right \rceil - r \\ s \end{array} \right)$$

• 
$$((2s+(k+1)\varepsilon + 2\mu+1)/2k)_{\lceil n/2 \rceil}$$
, (2.2)

where the value of  $\epsilon$  is 0 or 1 according to even or odd nature of n. Throughout this paper  $\epsilon$  always has this meaning; and [p] is the greatest integer less than or equal to p.

It is fairly easy to verify after reverting the order of summation for even and odd integers that

$$S_{2n}^{\mu}(x;k) = (-1)^{n} 2^{2n} \Gamma(kn+\mu+k/2) \sum_{j=0}^{n} (-1)^{j} {n \choose j} x^{2kj} / \Gamma(kj+\mu+1/2),$$

$$= (-1)^{n} 2^{2n} n! \left[ \Gamma(kn+\mu+k/2) / \Gamma(kn+\mu+1/2) \right] Z_{n}^{\mu-1/2} (x^{2};k); \qquad (2.3)$$

$$S_{2n+1}^{\mu}(x;k) = (-1)^{n} 2^{2n+1} \Gamma(kn+\mu+1+k/2) \sum_{j=0}^{n} (-1)^{j} (\frac{n}{j}) \frac{x^{2kj+k}}{\Gamma(kj+\mu+1+k/2)}$$
$$= (-1)^{n} 2^{2n+1} n! x^{k} Z_{n}^{\mu+k/2} (x^{2};k); \qquad (2.4)$$

$$T_{2n}^{\mu}(x;k) = (-1)^{n} 2^{2n} \sum_{r=0}^{n} \frac{2^{2r}}{r!} \sum_{s=0}^{r} (-1)^{s} (\frac{r}{s}) ((s+\mu+1/2)/k)_{n},$$

$$= (-1)^{n} 2^{2n} n! \quad Y_{n}^{\mu-1/2}(x^{2};k), \qquad (2.5)$$

$$T_{2n+1}^{\mu}(x;k) = (-1)^{n} 2^{2n+1} \sum_{r=0}^{n} (x^{2r+1}/r!) \sum_{s=0}^{r} (-1)^{s} (\frac{r}{s}) ((s+\mu+1+k/2)/k)_{n},$$

$$= (-1)^{n} 2^{2n+1} n! \times Y_{n}^{\mu+k/2} (x^{2};k). \qquad (2.6)$$

Here  $Z_n^{\alpha}(x;k)$  and  $Y_n^{\alpha}(x;k)$  is a pair of Konhauser [3] biorthogonal polynomials w.r.t. the Laguerre weight function  $x^{\alpha}\exp(-x)$  over  $(0,\infty)$  and are given by

$$Z_{n}^{\alpha}(x;k) = \frac{\Gamma(kn + \alpha + 1)}{n!} \qquad \sum_{j=0}^{n} (-1)^{j} {n \choose j} \frac{x^{kj}}{\Gamma(kj + \alpha + 1)}$$
(2.7)

$$Y_n^{\alpha}(x;k) = \frac{1}{n!} \sum_{r=0}^{n} \frac{x^r}{r!} \sum_{s=0}^{r} (-1)^s (\frac{r}{s}) ((s+\alpha+1)/k_n); \text{ see Carlitz [9]}$$
 (2.8)

where  $\alpha > -1$ , and k is a postive integer, and

$$\int_{0}^{\infty} x^{\alpha} e^{-x} Z_{n}(x;k) Y_{m}^{\alpha}(x;k) dx = \frac{\Gamma(kn+\alpha+1)}{n!} \delta(n,m) \text{ with } \delta(n,m)$$
 (2.9)

the Kronecker's delta. Using [10] one readily obtains the following biorthogonality condition for the sets  $\{S_n^{\mu}(x;k)\}$  and  $\{T_m^{\mu}(x;k)\}$ :

$$\int_{-\infty}^{\infty} |x|^{2\mu} \exp(-x^2) S_n^{\mu}(x;k) T_m^{\mu}(x;k) dx$$

$$= 2^{2n} [n/2]! \Gamma(\mu + \varepsilon + (kn + k - k\varepsilon)/2) \delta(n,m) . \tag{2.10}$$

An independent proof of (2.10) is also possible by using the identity of Carlitz [9, p. 249]: m -r

$$(-j)_{m} = \sum_{r=0}^{m} (kj+c+m-r) \sum_{s=0}^{m-r} (-1)^{s} (m-r) ((s+c+1)/k)_{m}.$$

One has to note, however, that k is involved in  $S_n^{\mu}(x;k)$  and  $T_m^{\mu}(x;k)$  must be an odd positive integer in view of the existence theorem for biorthogonality due to Konhauser [10, p.255].

One readily obtains

$$\Gamma(kn+k+\mu+1/2)$$
  $S_{2n+1}^{\mu}(x;k) = 2x^{k} \Gamma(kn+\mu+1+k/2) S_{2n}^{\mu+(k+1)/2}(x;k)$ , and (2.11)

$$T^{\mu}_{/n+1}(x;k) = 2x \ T^{\mu+(k+1)/2}_{2n}(x;k),$$
 (2.12)

D 
$$S_{2n}^{\mu}(x;k) = 4 \text{ nk } x^{k-1} \frac{\Gamma(kn+\mu+k/2)}{\Gamma(kn+\mu+1/2)} S_{2n-1}^{\mu+(k-1)/2} (x;k)$$
. (2.13)

# SOME PROPERTIES.

Using the relationship (2.3) to (2.6) it is fairly easy to obtain many results for the Szegő-Hermite biorthogonal pair of polynomials from the known results for the Konhauser biorthogonal sets. The results stated below could also be proved directly. Recall the Calvez and Ge'nin [11] generating function in the form (see also Srivastava [12]):

$$\sum_{n=0}^{\infty} \left( \begin{array}{c} m+n \\ n \end{array} \right) Y_{m+n}^{\alpha}(x;k) t^{n} = R^{(1+\alpha+mk)} \exp\{x(1-R)\} Y_{m}^{\alpha}(xR;k), \tag{3.1}$$

where m is any integer  $\geq 0$  and R =  $(1-t)^{-1/k}$ . By handling even and odd cases separately, from (2.5) and (2.6) respectively, one obtains

$$\sum_{n=0}^{\Sigma} T_{2m+n}^{\mu} (x;k)t^{n}/[n/2]!$$
 (3.2)

= 
$$v_U^{(\mu+mk+(1+k)/2)}$$
 [ $v^{-k}$   $T^{\mu}_{2m}(xv;k)$  + t  $T^{\mu}_{2m+1}(xv;k)$ ] where  $v^{\mu}=(1+4t^2)^{-1/2k}$  and

 $V = \exp\{x^2[1-(1+4t^2)^{-1/k}]\}$ . The special case with m=0 is worth noting. Using (3.2) for even case and then applying (2.12) one obtains in a combined form the recurrence relation for the second set

$$T_{n}^{\mu}(x;k) = \sum_{m=0}^{\lfloor n/2 \rfloor} (-1)^{m} 2^{2m} \left( \frac{n/2}{m} \right) \left( \frac{\mu - \lambda}{k} \right)_{m} T_{n-2m}^{\lambda}(x;k), \lambda \neq \mu \text{ and } \lambda, \mu > -1/2. (3.3)$$

Taking  $\mu$  = 0, and n even in (3.3) and using the biorthogonality condition (2.10) we have the integral

$$\int_{-\infty}^{\infty} |x|^{2\lambda} \exp(-x^2) S_{2m}^{\lambda}(x;k) T_{2n}(x;k) dx$$
(3.4)

=  $(-1)^n 4^{m+n} (-n)_m (-\lambda/k)_{n-m} \Gamma(km+\lambda+k/2)$  where with  $\mu$  = 0,  $T_{2n}(x;k)$  is the second biorthogonal set suggested by the Hermite polynomials; see Thakare and Madhekar [4]. The integral (3.4) says that  $T_{2n}(x;k)$  are othogonal to  $|x|^{2\lambda} S_{2m}^{\lambda}(x;k)$  w.r.t. the weight function  $\exp(-x^2)$  when  $n > m+\lambda/k$ .

Consider the generating function first given by Genin and Calvez [13]; (see also Karande and Thakare [14], Prabhakar [15]):

$$\sum_{n=0}^{\infty} (c)_{n} Z_{n}^{\alpha} (x;k) t^{n} / (1+\alpha)_{kn} = (1-t)^{-c} {}_{1} F_{k} \left[ \begin{bmatrix} c & ; \\ & \\ \Delta(k,1+\alpha) & ; \end{bmatrix} tx^{k} / (1-t)k^{k} \right]$$
(3.5)

where |t| < 1 and  $\Delta(m, \delta)$  stands for the sequence of parameters  $\delta/m$ ,  $(\delta+1)/m$ , ...,  $(\delta+m-1)/m$ , (m>1). Using (2.3) one obtains from (3.5), an expression involving even  $S_{2n}^{\mu}(x;k)$  which after putting to use relation (2.11) gives a corresponding relation for odd  $S_{2n+1}^{\mu}(x;k)$ . This resulting expression further with the help of the relation

$$\sum_{n=0}^{\infty} (c)_{n} S_{2n+1}^{\mu}(x;k) t^{2n+1}/n! (\mu+k/2)_{nk}$$
(3.6)

= 
$$t(k+2\mu+k\theta)/(k+2\mu)$$
  $\sum_{n=0}^{\infty}$  (c)  $_{n}S_{2n+1}^{\mu}(x;k)$   $t^{2t}/n!(\mu+1+k/2)_{nk}$ , where  $\theta=t$ ,  $d/dt$ 

yields

$$\sum_{n=0}^{\infty} \frac{(c)_n}{n! (\mu + k/2)_{nk}} S_{2n+1}^{\mu}(x;k) t^{2n+1} = 2tx^k U^{-2k(1+c)} (U^{-2k} - \frac{8ckt^2}{k+2}) \cdot (3.7)$$

• 
$$1^{F_k}\begin{bmatrix} c; & W \\ \Delta(k,\mu+1+k/2); \end{bmatrix} + \frac{16 \text{ ckt}^3 \times 3^k U^{2k}(c+2)}{(k+2\mu)(1+\mu+k/2)_k} & 1^{F_k} \begin{bmatrix} c+1; & W \\ \Delta(k,1+\mu+3k/2; \end{bmatrix}$$

where  $W = 4x^{2k}t^2/(1+4t^2)k^k$ .

In fact, one obtains after combining even case with (3.7) the following generating function for the first biorthogonal set  $\{S_n^{\mu}(x;k)\}$ :

$$\frac{\sum_{n=0}^{\infty} \frac{(c) [n/2]}{[n/2]! (\mu+k/2)_{k[n/2]}} S_{n}^{\mu}(x;k) t^{n} = \frac{(\mu+k/2)}{(\mu+1/2)} U^{2kc} {}_{1}F_{k} \begin{bmatrix} c; \\ \Delta(k,\mu+1/2); \end{bmatrix} W \end{bmatrix} (3.8)$$

$$+ 2tx^{k} U^{2k(1+c)} (U^{-2k} - \frac{8ckt^{2}}{k+2\mu}) {}_{1}F_{k} \begin{bmatrix} c; \\ \Delta(k,1+\mu+k/2); \end{bmatrix} W$$

$$+ \frac{16}{(k+3\mu)} \frac{ckt^{3} x^{3k} U^{2k(c+2)}}{(1+\mu+k/2)_{k}} {}_{1}F_{k} \begin{bmatrix} c+1; \\ \Delta(k,1+\mu+3k/2; \end{bmatrix} W$$

We finally state the differential equation satisfied by the first set  $\{S_n^\mu(x;k)\}$  in the form

$$[x^{2}(xD+2\mu+1+\epsilon)]^{k} \{x^{1-2k} (D-\epsilon k/x) S_{n}^{\mu}(x;k)\}$$
 (3.9)

=  $(2x^2)^k$  {x D  $S_n^\mu(x;k)$  - nk  $S_n^\mu(x;k)$ }, and a differential recurrence relation for the second set

$$k T_{n+2}^{\mu}(x;k) = -2xD T_{n}^{\mu}(x;k) - 2(1+m1+2\mu-2x^{2}) T_{n}^{\mu}(x;k)$$
 (3.10)

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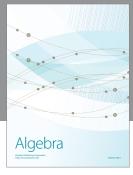
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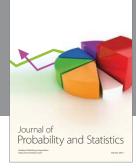
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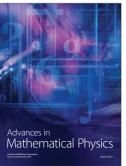






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