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A parabolic approach to the Calabi–Yau problem in HKT geometry

Lucio Bedulli[1] · **Giovanni Gentili**[2] · **Luigi Vezzoni**[2]

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Abstract
We consider the natural generalization of the parabolic Monge–Ampère equation to HKT geometry. We prove that in the compact case the equation has always a short-time solution and when the hypercomplex structure is locally flat and admits a compatible hyperkähler metric, then the equation has a long-time solution whose normalization converges to a solution of the quaternionic Monge–Ampère equation first introduced in Alekser and Verbitsky (Isr J Math 176:109–138, 2010). The result gives an alternative proof of a theorem of Alesker (Adv Math 241:192–219, 2013).

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1 Introduction

Let M be a $4n$-dimensional smooth manifold. A *hypercomplex* structure on M is a triple (I, J, K) of complex structures satisfying the quaternionic identities

$$IJ = -JI = K.$$

A Riemannian metric g on (M, I, J, K) is called *hyperhermitian* if it is compatible with each of I, J, K. Every hyperhermitian metric induces the form

$$\Omega = \omega_J + i\omega_K \in \Lambda^{2,0}_I,$$

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✉ Luigi Vezzoni
luigi.vezzoni@unito.it

Lucio Bedulli
lucio.bedulli@univaq.it

Giovanni Gentili
giovanni.gentili@unito.it

1 Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica, Università dell'Aquila, via Vetoio, 67100 L'Aquila, Italy

2 Dipartimento di Matematica G. Peano, Università di Torino, Via Carlo Alberto 10, 10123 Turin, Italy

where $\omega_J = g(J\cdot,\cdot)$ and $\omega_K = g(K\cdot,\cdot)$ are the fundamental forms of (g, J) and (g, K) respectively. The form Ω is nondegenerate, i.e. $\Omega^n \neq 0$ everywhere, and determines g by the relation

$$\Omega(X, Y) = 2g(JX, Y) \tag{1}$$

for every X, Y vector fields of type $(1, 0)$ with respect to I.

A hyperhermitian manifold (M, I, J, K, g) is called HKT (hyperkähler with torsion) if

$$\partial\Omega = 0,$$

where ∂ is with respect to I. HKT manifolds were first introduced by Howe and Papadopoulos in [15], but the characterization in terms of the form Ω is due to Grantcharov and Poon [13] who also showed the existence of HKT structures on some homogeneous hypercomplex manifolds. HKT structures have been studied intensively in the last years and many analogies to Kähler manifolds have been discovered (see e.g. [1–5, 9–13, 16, 21, 24, 26–28] and the references therein). In particular a pluripotential theory has been developed in HKT geometry, according to which every HKT form Ω can be locally written as

$$\Omega = \partial\partial_J v$$

for some real smooth function v (see [3, 5]), where $\partial_J = J^{-1}\bar{\partial}J$ and the action of J on a k-form α is defined as

$$J\alpha(X_1, \dots, X_k) = \alpha(JX_1, \dots, JX_k).$$

Furthermore, in analogy with the complex case, the space of *quaternionic plurisubharmonic functions* has been introduced

$$\mathcal{H}_\Omega = \{\varphi \in C^\infty(M) : \Omega_\varphi := \Omega + \partial\partial_J\varphi > 0\},$$

where "$\Omega_\varphi > 0$" means that Ω_φ induces a hyperhermitian metric via (1).

In [4] Alesker and Verbitsky introduced the following quaternionic version of the Calabi problem in analogy with the famous complex Calabi problem solved by Yau in [29]. Let (M, I, J, K, g) be a compact HKT manifold and let $F \in C^\infty(M)$ be a smooth real valued function. The *quaternionic Monge–Ampère equation* is

$$(\Omega + \partial\partial_J\varphi)^n = b\,\mathrm{e}^F\,\Omega^n, \tag{2}$$

where $F \in C^\infty(M)$ is the datum, while $(\varphi, b) \in \mathcal{H}_\Omega \times \mathbb{R}_+$ is the unknown. Equation (2) is most naturally settled if the canonical bundle of (M, I) is holomorphically trivial. In this case there exists a holomorphic volume form Θ on (M, I) which satisfies the *q-real* condition $J\Theta = \bar{\Theta}$ and b is determined by F:

$$b = \frac{\int_M \Omega^n \wedge \bar{\Theta}}{\int_M \mathrm{e}^F\,\Omega^n \wedge \bar{\Theta}}.$$

So far there are only partial results about the solvability of (2). In [4] it is proved the uniqueness and an a priori C^0 estimate for solutions to (2) when the canonical bundle of (M, I) is holomorphically trivial; in [2, 21] the C^0 estimate is established on any compact HKT manifold, without further assumptions and in [11] the problem is studied on some 8-dimensional nilmanifolds. In [1] it is proved that Eq. (2) can always be solved on HKT manifolds with *locally flat* hypercomplex structure which admit a compatible hyperkähler metric. Here we recall that a hypercomplex structure is called *locally flat* if it is locally isomorphic to $\mathbb{H}^n$.

Manifolds of this kind were firstly considered by Sommese in [20] and simply called *quaternionic manifolds*. Recently in [9] the quaternionic Monge–Ampère equation has been solved on hyperkähler manifolds without the assumption of local flatness.

In the present paper we approach Eq. (2) via the following geometric flow

$$\varphi_t = \log \frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} - F, \quad \varphi(x,0) = 0, \tag{3}$$

where the solution φ is supposed to satisfy $\varphi(\cdot, t) \in \mathcal{H}_\Omega$ for every t and the subscript t denotes the derivative of φ with respect to the variable t. The same dynamic approach was pursued on Kähler manifolds [6], on Hermitian manifolds [14, 23] and on almost Hermitian manifolds [8].

Our main result is the following theorem which provides an alternative proof of Alesker's Theorem [1].

Theorem 1 *Let (M, I, J, K, g) be a compact HKT manifold with (I, J, K) locally flat and assume that there exists a hyperkähler metric $\hat{g}$ on (M, I, J, K). Then there exists a long-time solution $\varphi \in C^\infty(M \times \mathbb{R}_+)$ to the parabolic quaternionic Monge–Ampère equation (3) such that*

$$\tilde{\varphi} = \varphi - \int_M \varphi\, \Omega^n \wedge \bar{\Omega}^n_{\hat{g}}$$

converges in C^∞-topology to a smooth function $\tilde{\varphi}_\infty \in C^\infty(M)$. Moreover if

$$b := \frac{\int_M \Omega^n \wedge \bar{\Omega}^n_{\hat{g}}}{\int_M \mathrm{e}^F \Omega^n \wedge \bar{\Omega}^n_{\hat{g}}},$$

then $(\tilde{\varphi}_\infty, b)$ solves the quaternionic Monge–Ampère equation (2).

Recently we have been made aware of the paper [30] where the parabolic quaternionic Monge–Ampère equation is studied and its long-time behaviour is described with techniques different from ours.

Now we describe the layout of the proof. Since (3) is strongly parabolic, it admits a unique maximal solution $\varphi \in C^\infty(M \times [0, T))$.

- Step 1. From the equation we directly deduce a uniform C^0 bound on φ_t (Lemma 5).
- Step 2. The C^0 estimate for solutions of the quaternionic Calabi–Yau equation (2) then implies a uniform bound on $\operatorname{osc}\varphi$ (Lemma 6).
- Step 3. We use the existence of the hyperkähler metric and the local flatness of the hypercomplex structure in order to establish a uniform upper bound on $\Delta_{\hat{g}}\varphi$ (Lemma 7).
- Step 4. A general result in [7] implies a uniform Hölder estimate on the second derivatives of φ, thus a classical bootstrapping argument using Schauder estimates implies $T = \infty$ and a uniform bound on $|\nabla^k \varphi|$ for $k \geq 1$ (Lemmas 8 and 9).
- Step 5. We prove the convergence of $\tilde{\varphi}$ using an argument due to Phong–Sturm [18] based on an adapted Mabuchi-type functional (Lemma 10).

We point out that the local flatness of the hypercomplex structure plays a role in steps 3 and 4, while the existence of a background hyperkähler metric is only used in step 3.

Remark Flow (3) can be regarded as a geometric flow in Hermitian Geometry. Here we assume that the canonical bundle of (M, I) is trivial and we fix a q-real complex volume form Θ on (M, I). As shown in [4] one has

$$(\Omega + \partial\partial_J\varphi)^n \wedge \bar{\Theta} = i^n(\omega - i\partial\bar{\partial}\varphi)^n \wedge \Phi, \quad \Omega^n \wedge \bar{\Theta} = i^n\omega^n \wedge \Phi$$

where ω is the fundamental form of (g, I) and Φ is a real (n, n)-form which is positive in a weak sense. By setting $u = -\varphi$ we can then rewrite (3) as

$$u_t = -\log \frac{(\omega + i\partial\bar{\partial}u)^n \wedge \Phi}{\omega^n \wedge \Phi} + F, \quad u(0) = 0. \tag{4}$$

Equation (4) reminds the parabolic k-Hessian flow

$$u_t = \log \frac{(\chi + i\partial\bar{\partial}u)^k \wedge \alpha^{n-k}}{\alpha^n} + F, \quad u(0) = 0 \tag{5}$$

studied by Phong and Tô on a complex n-dimensional Hermitian manifold (M, α) in [19], where $1 \leq k \leq n$ and χ is real k-positive $(1, 1)$-form. According to [19] (5) has always a long-time solution whose normalization converges in C^∞-topology to a solution of the k-Hessian equation. Equation (4) differs from the parabolic n-Hessian flow since the role of α^n is replaced by the form Φ which is positive in a weak sense and the theorem of Phong and Tô cannot be directly applied.

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2 Preliminaries

Let (M, I, J, K, g) be a compact HKT manifold with HKT form Ω. Let ∂ be the ∂-operator with respect to I and $\partial_J := J^{-1}\bar{\partial}J \colon \Lambda_I^{r,0} \to \Lambda_I^{r+1,0}$. Then

$$\partial\partial_J = -\partial_J\partial,$$

see [26]. Moreover we assume that the canonical bundle of (M, I) is holomorphically trivial and we let Θ be a q-real holomophic volume form on (M, I). Note that, since Ω is easily seen to be q-real, $\Omega^n \wedge \bar{\Theta}$ is a real volume form; indeed, J acts trivially on top forms and thus

$$\overline{\Omega^n \wedge \bar{\Theta}} = J\Omega^n \wedge J\bar{\Theta} = \Omega^n \wedge \bar{\Theta}.$$

The HKT metric induces the *quaternionic Laplacian* operator

$$\Delta_g\varphi := \frac{\partial\partial_J\varphi \wedge \Omega^{n-1}}{\Omega^n}$$

for $\varphi \in C^\infty(M)$. It is well-known that Δ_g is elliptic and it is straightforward to show that for $\eta, \psi \in C^\infty(M)$ we have

$$\int_M (\Delta_g\eta)\psi\, \Omega^n \wedge \bar{\Theta} = \int_M \eta(\Delta_g\psi)\, \Omega^n \wedge \bar{\Theta}.$$

Moreover the following formula will be useful: for every $\alpha, \beta \in \Lambda_I^{1,0}$

$$\frac{\alpha \wedge J\bar{\beta} \wedge \Omega^{n-1}}{\Omega^n} = -\frac{1}{2n} g(\alpha, \bar{\beta}). \tag{6}$$

The basic example of hyperhermitian manifold is given by an open set A of $\mathbb{R}^{4n}$ with the standard hyperhermitian structure

$$I_0 = \begin{pmatrix} 0 & -\mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbb{1}_n \\ 0 & 0 & \mathbb{1}_n & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 0 & -\mathbb{1}_n & 0 \\ 0 & 0 & 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 & 0 & 0 \\ 0 & -\mathbb{1}_n & 0 & 0 \end{pmatrix}, \quad K_0 = \begin{pmatrix} 0 & 0 & 0 & -\mathbb{1}_n \\ 0 & 0 & -\mathbb{1}_n & 0 \\ 0 & \mathbb{1}_n & 0 & 0 \\ \mathbb{1}_n & 0 & 0 & 0 \end{pmatrix},$$

where $\mathbb{1}_n$ is the $n \times n$ identity matrix. In this case for an $\mathbb{H}$-valued function $u\colon \mathbb{R}^{4n} \to \mathbb{H}$ the following derivatives are defined

$$\partial_{q^r} u := \partial_{x_0^r} u e_0 - \sum_{i=1}^{3} \partial_{x_i^r} u e_i, \quad \partial_{\bar{q}^r} u := \sum_{i=0}^{3} e_i \partial_{x_i^r} u,$$

where to shorten the notation we denote the quaternions $1, i, j, k$ with e_0, e_1, e_2, e_3 and the coordinates on $\mathbb{R}^{4n}$ are taken as $(x_0^1, \dots, x_0^n, x_1^1, \dots, x_1^n, x_2^1, \dots, x_2^n, x_3^1, \dots, x_3^n)$ in order to identify $\mathbb{R}^{4n}$ with $\mathbb{H}^n$. We denote by

$$\mathrm{Hyp}(n, \mathbb{H}) = \{U \in \mathbb{H}^{n,n} : \bar{U} = {}^tU\}$$

the space of hyperhermitian matrices. Any $U \in \mathrm{Hyp}(n, \mathbb{H})$ has real eigenvalues and we can consider the subset $\mathrm{Hyp}^+(n, \mathbb{H})$ of positive-definite hyperhermitian matrices.

Any hyperhermitian Riemannian metric g on (A, I_0, J_0, K_0) defines a smooth map $G\colon A \to \mathrm{Hyp}^+(n, \mathbb{H})$,

$$G_{rs} := g(\partial_{q^r}, \partial_{q^s}),$$

where g is extended $\mathbb{H}$-semilinearly in its components, i.e.

$$g(X, Y\lambda) = g(X, Y)\lambda, \quad g(X\lambda, Y) = \bar{\lambda} g(X, Y)$$

for every $\lambda \in \mathbb{H}$, $X, Y \in \Gamma(TA)$.

In some occasions we will make use of the following real representation of quaternionic matrices $\iota\colon \mathbb{H}^{n,n} \to \{U \in \mathbb{R}^{4n,4n} : I_0 U I_0 = J_0 U J_0 = K_0 U K_0 = -U\}$,

$$\iota(A + iB + jC + kD) := \begin{pmatrix} A & B & C & D \\ -B & A & -D & C \\ -C & D & A & -B \\ -D & -C & B & A \end{pmatrix}.$$

The map ι is an isomorphism of real algebras and

$$\iota(\mathrm{Hyp}(n, \mathbb{H})) = \mathrm{Hyp}(n, \mathbb{R}),$$

where

$$\mathrm{Hyp}(n, \mathbb{R}) = \{U \in \mathrm{Sym}(4n, \mathbb{R}) : I_0 U I_0 = J_0 U J_0 = K_0 U K_0 = -U\}.$$

For any smooth function $u\colon A \to \mathbb{R}$ it is defined the *quaternionic Hessian matrix*

$$(\mathrm{Hess}_{\mathbb{H}} u)_{rs} := u_{\bar{r}s},$$

where we set $u_{\bar{r}s} = \partial_{\bar{q}^r} \partial_{q^s} u$. $\mathrm{Hess}_{\mathbb{H}} u$ is a hyperhermitian quaternionic matrix, in particular the entries $(\mathrm{Hess}_{\mathbb{H}} u)_{rr}$ are real.

The following lemma will be useful. We refer to [3, Proposition 4.1] for a proof (see also [22]).

Lemma 2 *Let g be a HKT metric on (A, I_0, J_0, K_0). Then the matrix associated to g is*

$$G = \kappa \mathrm{Hess}_{\mathbb{H}} u,$$

where $u \in C^\infty(A, \mathbb{R})$ is such that $\Omega = \partial\partial_J u$ is the HKT form associated to g and $\kappa > 0$ is a universal constant.

Next we recall the formulation of the *quaternionic Monge–Ampère equation* (2) on open sets of $\mathbb{H}^n$; for a hyperhermitian $U \in \mathbb{H}^{n,n}$ we will denote by $\det U$ its *Moore determinant* (see [17]).

Lemma 3 *Let g be a HKT metric on (A, I_0, J_0, K_0), $\varphi\colon A \to \mathbb{R}$ a smooth function and Ω the HKT form of g, then*

$$(\Omega + \partial\partial_J\varphi)^n = \frac{\det(G + \kappa\,\mathrm{Hess}_{\mathbb{H}}\varphi)}{\det G}\,\Omega^n, \quad n\frac{\partial\partial_J\varphi \wedge \Omega^{n-1}}{\Omega^n} = \kappa\,\mathrm{Re}\left(\mathrm{tr}(G^{-1}\mathrm{Hess}_{\mathbb{H}}\varphi)\right).$$

Proof The first formula is [3, Corollary 4.6] and the second is simply obtained by linearizing the first one at the origin and using $\det\iota(U) = (\det U)^4$ for any hyperhermitian matrix U (see [3, Theorem 2.4]),

$$\left.\frac{d}{ds}\right|_{s=0} \log\frac{(\Omega + \partial\partial_J(s\varphi))^n}{\Omega^n} = \left.\frac{d}{ds}\right|_{s=0} \log\frac{\det(G + \kappa\mathrm{Hess}_{\mathbb{H}}(s\varphi))}{\det G}$$

which gives

$$\begin{aligned} n\frac{\partial\partial_J\varphi \wedge \Omega^{n-1}}{\Omega^n} &= \left.\frac{d}{ds}\right|_{s=0} \log\frac{\det\iota\,(G + \kappa\mathrm{Hess}_{\mathbb{H}}(s\varphi))^{1/4}}{\det\iota(G)^{1/4}} \\ &= \frac{1}{4}\left.\frac{d}{ds}\right|_{s=0} \log\frac{\det\left(\iota(G + \kappa\mathrm{Hess}_{\mathbb{H}}(s\varphi))\right)}{\det\iota(G)} \\ &= \frac{\kappa}{4}\mathrm{tr}\left(\iota(G)^{-1}\iota(\mathrm{Hess}_{\mathbb{H}}\varphi)\right) = \kappa\mathrm{Re}\left(\mathrm{tr}\left(G^{-1}\mathrm{Hess}_{\mathbb{H}}\varphi\right)\right) \end{aligned}$$

as claimed. □

Finally, we provide a lemma which will be helpful in the proof of the main theorem.

Lemma 4 *Let $U\colon A \to \mathrm{Hyp}^+(n, \mathbb{H})$ be a smooth map and assume that there exists $p \in A$ such that $U(p)$ is diagonal. Let $\hat{g}$ be a hyperhermitian metric on A such that the induced matrix $\hat{G}$ is the identity. Then*

$$\Delta_{\hat{g}} \log(\det U) = -\frac{\kappa}{n}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\frac{1}{U_{ss}}\frac{1}{U_{tt}}|U_{st,x_i^r}|^2 + \sum_{s=1}^{n}\frac{1}{U_{ss}}\Delta_{\hat{g}}U_{ss}$$

at p, where the subindex "x_i^r" denotes the derivative with respect to the corresponding real coordinate.

Proof Since $\det \iota(U) = (\det U)^4$ we directly compute

$$
\begin{aligned}
\partial_{\bar q^r}\partial_{q^r}\log(\det U) &= \sum_{i=0}^{3}\partial^2_{x_i^r}\log(\det U) = \frac{1}{4}\sum_{i=0}^{3}\partial^2_{x_i^r}\log(\det \iota(U))\\
&= \frac{1}{4}\sum_{i=0}^{3}\partial_{x_i^r}\operatorname{tr}\left(\iota(U)^{-1}\iota(U)_{,x_i^r}\right)\\
&= \frac{1}{4}\sum_{i=0}^{3}\operatorname{tr}\left(-\iota(U)^{-1}\iota(U)_{,x_i^r}\iota(U)^{-1}\iota(U)_{,x_i^r} + \iota(U)^{-1}\iota(U)_{,x_i^r x_i^r}\right)\\
&= \frac{1}{4}\sum_{i=0}^{3}\operatorname{tr}\left(\iota\left(-U^{-1}U_{,x_i^r}U^{-1}U_{,x_i^r} + U^{-1}U_{,x_i^r x_i^r}\right)\right)\\
&= \sum_{i=0}^{3}\operatorname{Re}\left(\operatorname{tr}\left(-U^{-1}U_{,x_i^r}U^{-1}U_{,x_i^r} + U^{-1}U_{,x_i^r x_i^r}\right)\right)
\end{aligned}
$$

and at the point p where U takes a diagonal form

$$
\begin{aligned}
\Delta_{\hat g}\log(\det U) &= \frac{\kappa}{n}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\operatorname{Re}\left(-U^{ss}U_{st,x_i^r}U^{tt}U_{ts,x_i^r} + U^{ss}U_{ss,x_i^r x_i^r}\right)\\
&= -\frac{\kappa}{n}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\frac{1}{U_{ss}}\frac{1}{U_{tt}}|U_{st,x_i^r}|^2 + \sum_{s=1}^{n}\frac{1}{U_{ss}}\Delta_{\hat g}U_{ss}
\end{aligned}
$$

and the claim follows. □

3 Proof of the result

Let (M, I, J, K, g) be a HKT manifold with HKT form Ω. Every $\varphi \in \mathcal{H}_\Omega$ induces a HKT metric g_φ and a quaternionic Laplacian $\Delta_\varphi := \Delta_{g_\varphi}$. Consider the operator

$$
P\colon \mathcal{H}_\Omega \to C^\infty(M), \quad P(\varphi) = \log\frac{(\Omega + \partial\partial_J\varphi)^n}{\Omega^n} - F.
$$

The first variation of P is

$$
P_{*|\varphi}(\psi) = n\frac{\partial\partial_J\psi \wedge (\Omega + \partial\partial_J\varphi)^{n-1}}{(\Omega + \partial\partial_J\varphi)^n} = n\Delta_\varphi\psi.
$$

Since Δ_φ is a strongly elliptic operator, Eq. (3) is always well-posed and it admits a unique maximal solution $\varphi \in C^\infty(M \times [0, T))$. Assume further that the canonical bundle of (M, I) is holomorphically trivial and let $\Theta \in \Lambda_I^{2n,0}$ be a q-real holomorphic volume form. We then denote

$$
\tilde\varphi = \varphi - \int_M \varphi\,\Omega^n \wedge \bar\Theta.
$$

We start by proving C^0 bounds for the time derivatives φ_t and $\tilde\varphi_t$ and then use these to prove the C^0 estimate for $\tilde\varphi$. In what follows we denote by C all the uniform constants (which may be different from line to line).

Lemma 5 *There exists a uniform constant C such that*

$$|\varphi_t(x,t)| \leq C, \quad |\tilde{\varphi}_t(x,t)| \leq C$$

for every $(x,t) \in M \times [0,T)$.

Proof Since

$$\frac{\partial}{\partial t} \log \frac{(\Omega + \partial\partial_J \varphi)^n}{\Omega^n} = n \frac{\partial\partial_J \varphi_t \wedge \Omega_\varphi^{n-1}}{\Omega_\varphi^n} = n\Delta_\varphi \varphi_t,$$

we have

$$\varphi_{tt} = n\Delta_\varphi \varphi_t$$

and the parabolic maximum principle implies the a priori C^0 estimate for φ_t. The estimate on $\tilde{\varphi}_t$ immediately follows. □

Lemma 6 *We have*

$$\max_M \varphi - \min_M \varphi \leq C$$

and

$$|\tilde{\varphi}| \leq C,$$

for a uniform constant C.

Proof Since $|\varphi_t|$ is bounded and

$$(\Omega + \partial\partial_J \varphi)^n = \mathrm{e}^{F+\varphi_t}\, \Omega^n,$$

for every fixed t, $\varphi(\cdot,t)$ solves the quaternionic Monge–Ampère equation (2) with datum $F+\varphi_t$. In view of the C^0 estimate for solutions to the quaternionic Monge–Ampère equation [2, 4, 21], φ satisfies the bound

$$\max_M \varphi - \min_M \varphi \leq C, \tag{7}$$

where C depends only on (M,I,J,K,g) and an upper bound of $\max|F+\varphi_t|$. Therefore Lemma 5 implies that the constant C in (7) may be chosen so that it only depends on (M,I,J,K,g) and an upper bound of $\max|F|$. Now, let $(x,t) \in M \times [0,T)$, since $\tilde{\varphi}$ is normalized, there exist (y,t) such that $\tilde{\varphi}(y,t) = 0$, and thus we obtain $|\tilde{\varphi}(x,t)| = |\tilde{\varphi}(x,t) - \tilde{\varphi}(y,t)| = |\varphi(x,t) - \varphi(y,t)| \leq C$ and the claim follows. □

Lemma 7 *Assume that (I,J,K) is locally flat and that there exists a hyperkähler metric $\hat{g}$ on (M,I,J,K). Then*

$$\Delta_{\hat{g}} \varphi \leq C,$$

for a uniform constant C.

Proof Let

$$Q = 2\sqrt{\frac{1}{n}\mathrm{tr}_{\hat{g}}\, g_\varphi} - \varphi.$$

Fix $T' < T$ and let (x_0,t_0) be a point where Q achieves its maximum in $M \times [0,T']$. We may assume without loss of generality that $t_0 > 0$. Since (I,J,K) is locally flat, then in a neighborhood of x_0 we can locally identify M with an open set A of $\mathbb{H}^n$. Let G and $\hat{G}$ be the hyperhermitian matrices in A induced by g and $\hat{g}$ respectively. We may further assume that $G = \mathrm{Hess}_{\mathbb{H}} v$ in A, that $\hat{G}$ is the identity in A and that $U = \mathrm{Hess}_{\mathbb{H}}(v + \kappa\varphi)$ is diagonal at x_0. Let $u = v + \kappa\varphi$. Then in A we have

$$Q = 2\sqrt{\Delta_{\hat{g}} u} - \varphi.$$

Computing at (x_0, t_0), we have

$$\Delta_\varphi Q = \frac{\kappa}{n}\frac{1}{\sqrt{\Delta_{\hat{g}} u}}\sum_{r=1}^{n}\frac{1}{u_{r\bar{r}}}\left(-\frac{1}{2}\frac{1}{\Delta_{\hat{g}} u}|\Delta_{\hat{g}}\, u_r|^2 + \Delta_{\hat{g}} u_{r\bar{r}}\right) - \Delta_\varphi\varphi$$

and, applying Lemmas 3 and 4 we infer

$$\begin{aligned}\partial_t Q &= \frac{1}{\sqrt{\Delta_{\hat{g}} u}}\Delta_{\hat{g}}\varphi_t - \varphi_t = \frac{1}{\sqrt{\Delta_{\hat{g}} u}}\Delta_{\hat{g}}(\log\det(U) - \log\det(G) - F) - \varphi_t\\ &= \frac{1}{\sqrt{\Delta_{\hat{g}} u}}\left(-\frac{\kappa}{n}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\frac{1}{u_{s\bar{s}}}\frac{1}{u_{t\bar{t}}}|u_{s\bar{t},x_i^r}|^2 + \sum_{r=1}^{n}\frac{1}{u_{r\bar{r}}}\Delta_{\hat{g}} u_{r\bar{r}} - \Delta_{\hat{g}}\log\det(G) - \Delta_{\hat{g}} F\right) - \varphi_t\end{aligned}$$

which implies

$$\begin{aligned}&\partial_t Q - \frac{n}{\kappa}\Delta_\varphi Q\\ &\quad = \frac{1}{\sqrt{\Delta_{\hat{g}} u}}\left(\frac{1}{2\Delta_{\hat{g}} u}\sum_{r=1}^{n}\frac{1}{u_{r\bar{r}}}|\Delta_{\hat{g}}\, u_r|^2\right.\\ &\qquad \left.-\frac{\kappa}{n}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\frac{1}{u_{s\bar{s}}}\frac{1}{u_{t\bar{t}}}|u_{s\bar{t},x_i^r}|^2 - \Delta_{\hat{g}}(F + \log\det(G))\right) + \frac{n}{\kappa}\Delta_\varphi\varphi - \varphi_t.\end{aligned}$$

Using the Cauchy–Schwarz inequality and [1, Proposition 3.1] we obtain

$$\begin{aligned}\sum_{r=1}^{n}\frac{1}{u_{r\bar{r}}}|\Delta_{\hat{g}} u_r|^2 &= \sum_{r=1}^{n}\sum_{i=0}^{3}\frac{1}{u_{r\bar{r}}}(\Delta_{\hat{g}} u_{x_i^r})^2 = \frac{\kappa^2}{n^2}\sum_{r=1}^{n}\sum_{i=0}^{3}\frac{1}{u_{r\bar{r}}}\left(\sum_{s=1}^{n}\frac{\sqrt{u_{s\bar{s}}}}{\sqrt{u_{s\bar{s}}}}u_{s\bar{s},x_i^r}\right)^2\\ &\le \frac{\kappa^2}{n^2}\sum_{r,s,t=1}^{n} u_{t\bar{t}}\sum_{i=0}^{3}\frac{1}{u_{r\bar{r}}}\frac{1}{u_{s\bar{s}}}(u_{s\bar{s},x_i^r})^2 = \frac{\kappa}{n}\Delta_{\hat{g}} u\sum_{r,s=1}^{n}\sum_{i=0}^{3}\frac{1}{u_{r\bar{r}}}\frac{1}{u_{s\bar{s}}}(u_{s\bar{s},x_i^r})^2\\ &\le 2\frac{\kappa}{n}\Delta_{\hat{g}} u\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\frac{1}{u_{s\bar{s}}}\frac{1}{u_{t\bar{t}}}|u_{s\bar{t},x_i^r}|^2\end{aligned}$$

i.e.

$$\frac{1}{2\Delta_{\hat{g}} u}\sum_{r=1}^{n}\frac{1}{u_{r\bar{r}}}|\Delta_{\hat{g}} u_r|^2 \le \frac{\kappa}{n}\sum_{r,s,t=1}^{n}\sum_{i=0}^{3}\frac{1}{u_{s\bar{s}}}\frac{1}{u_{t\bar{t}}}|u_{s\bar{t},x_i^r}|^2,$$

from which it follows

$$
\begin{aligned}
0 &\le \partial_t Q - \frac{n}{\kappa}\Delta_\varphi Q \le \frac{n}{\kappa}\Delta_\varphi \varphi - \frac{\Delta_{\hat g}(F+\log\det(G))}{\sqrt{\Delta_{\hat g} u}} - \varphi_t \\
&\le \frac{n}{\kappa} - \frac{n}{\kappa^2}\Delta_\varphi v - \frac{\Delta_{\hat g}(F+\log\det(G))}{\sqrt{\Delta_{\hat g} u}} - \varphi_t
\end{aligned}
$$

at (x_0, t_0), where we have used that it is a maximum point as well as the relation

$$\Delta_\varphi \varphi = 1 - \frac{1}{\kappa}\Delta_\varphi v.$$

Hence

$$\frac{n}{\kappa^2}\Delta_\varphi v \le \frac{n}{\kappa} - \frac{\Delta_{\hat g}(F+\log\det(G))}{\sqrt{\Delta_{\hat g} u}} - \varphi_t$$

at (x_0, t_0). Since $|\varphi_t|$ is uniformly bounded we obtain

$$\Delta_\varphi v(x_0, t_0) \le C + \frac{C}{\sqrt{\sum_{r=1}^n u_{r\bar r}(x_0, t_0)}} \tag{8}$$

for a uniform constant C. In terms of u and G Eq. (3) writes as

$$\frac{1}{\kappa}u_t = \log\det(U) - \log\det(G) - F$$

and then

$$\frac{1}{\kappa}u_t(x_0, t_0) = \log\prod_{r=1}^n u_{r\bar r}(x_0, t_0) - \log\det(G(x_0)) - F(x_0).$$

Lemma 5 implies that $|u_t|$ is uniformly bounded and we deduce that

$$\frac{1}{C} \le \prod_{r=1}^n u_{r\bar r}(x_0, t_0) \le C.$$

Thus in particular by the geometric-arithmetic mean inequality we have $\sum_{r=1}^n u_{r\bar r}(x_0, t_0) \ge C$. Since

$$\Delta_\varphi v(x_0, t_0) = \frac{\kappa}{n}\sum_{r=1}^n \frac{1}{u_{r\bar r}(x_0, t_0)} v_{r\bar r}(x_0),$$

by (8) we finally deduce

$$\sum_{r=1}^n \frac{1}{u_{r\bar r}(x_0, t_0)} \le C.$$

Therefore

$$\Delta_{\hat g} u(x_0, t_0) = \frac{\kappa}{n}\sum_{r=1}^n u_{r\bar r}(x_0, t_0) \le \frac{\kappa}{n}\frac{1}{(n-1)!}\left(\sum_{r=1}^n \frac{1}{u_{r\bar r}(x_0, t_0)}\right)^{n-1}\prod_{r=1}^n u_{r\bar r}(x_0, t_0) \le C.$$

It follows

$$2\sqrt{\Delta_{\hat g} u(x,t)} \le C + \varphi(x,t) - \varphi(x_0, t_0) \le C + \operatorname{osc}\varphi \qquad \text{in } M \times [0, T'],$$

from which, using Lemma 6, we get

$$\Delta_{\hat{g}} u \leq C$$

for a uniform C and the claim is proved. □

Lemma 8 *Assume that* (I, J, K) *is locally flat and that there exists a hyperhermitian metric* $\hat{g}$ *on* (M, I, J, K) *such that*

$$\Delta_{\hat{g}} \varphi \leq C$$

for a uniform constant C*. Then for* $0 < \alpha < 1$ *we have*

$$\|\nabla^2 \varphi\|_{C^\alpha} \leq C$$

for a uniform constant C*.*

Proof We prove the result by applying [7, Theorem 5.1]. First we state some algebraic preliminaries (which are analogous to the complex case [25, Section 2]). Note that, in the notation introduced in Sect. 2, the real representation $\iota\colon \mathbb{H}^{n,n} \to \{H \in \mathbb{R}^{4n,4n} : I_0 H I_0 = J_0 H J_0 = K_0 H K_0 = -H\}$ of quaternionic matrices is monotonic in the sense that when H_1, H_2 are hyperhermitian one has

$$H_1 \leq H_2 \Rightarrow \iota(H_1) \leq \iota(H_2),$$

where $H_1 \leq H_2$ means that all the eigenvalues of $H_2 - H_1$ are non-negative.

Let $\mathrm{p}\colon \mathbb{R}^{4n,4n} \to \{H \in \mathbb{R}^{4n,4n} : I_0 H I_0 = J_0 H J_0 = K_0 H K_0 = -H\}$ be the projection defined as

$$\mathrm{p}(N) := \frac{1}{4}(N - I_0 N I_0 - J_0 N J_0 - K_0 N K_0).$$

Then for any real valued smooth function f and any hyperhermitian matrix H we have

$$\iota(\mathrm{Hess}_{\mathbb{H}} f) = 4\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} f), \quad \det(\iota(H)) = (\det H)^4.$$

Thus, once local quaternionic coordinates are fixed, working as in the proof of Lemma 7, we can rewrite Eq. (3) as

$$\frac{1}{\kappa} u_t = \frac{1}{4} \log\det\left(4\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u)\right) - \log\det(G) - F,$$

where $u = \kappa v + \kappa\varphi$ and v is a HKT potential of Ω. We rewrite the last equation as

$$\frac{1}{\kappa} u_t = \Phi(\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u)) - \log\det(G) - F \tag{9}$$

where for $N \in \mathrm{Sym}(4n, \mathbb{R})$ such that $\det N > 0$ we set

$$\Phi(N) = \frac{1}{4} \log\det\left(4N\right).$$

Fix positive constants $C_1 < C_2$ and let

$$\mathcal{E} := \{N \in \mathrm{Sym}(4n, \mathbb{R}) : C_1 \mathbb{1}_{4n} \leq N \leq C_2 \mathbb{1}_{4n}\}.$$

Then $\mathcal{E}$ is a compact convex subset of $\mathrm{Sym}(4n, \mathbb{R})$. We observe that Φ and p satisfy the assumptions in [7, Theorem 5.1]. Indeed

- Φ is uniformly elliptic in $\mathcal{E}$;
- Φ is concave in $\mathcal{E}$;

- p is linear;
- if $N \geq 0$, then $\mathrm{p}(N) \geq 0$ and $C^{-1}\|N\| \leq \|\mathrm{p}(N)\| \leq C\|N\|$ for C uniform.

Therefore, if we show that $\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u) \in \mathcal{E}$ for a suitable choice of C_1 and C_2 Eq. (9) belongs to the class of equations considered in [7, Theorem 5.1].

Without loss of generality we can fix $x_0 \in M$ and assume that $\hat{G}$ is the identity at x_0. Our assumption $\Delta_{\hat{g}}\varphi \leq C$ implies

$$\sum_{r=1}^{n} u_{r\bar{r}} \leq C \tag{10}$$

at x_0 for a uniform $C > 0$ and thus

$$\mathrm{Hess}_{\mathbb{H}} u \leq C\mathbb{1}_n.$$

On the other hand, Eq. (3) writes as

$$\frac{1}{\kappa}u_t = \log\det(\mathrm{Hess}_{\mathbb{H}} u) - \log\det(G) - F.$$

Thus by Lemma 5

$$\prod_{i=1}^{n} \lambda_i = \det(\mathrm{Hess}_{\mathbb{H}} u) \geq \det(G)\mathrm{e}^{-\frac{1}{\kappa}|u_t|+F} \geq C,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathrm{Hess}_{\mathbb{H}} u$ and $C > 0$ is a uniform constant. From (10) we also infer $\sum_{i=1}^{n} \lambda_i \leq C$ at x_0 which then implies a uniform lower bound for each λ_i at the point x_0, but such bound does not depend on x_0.

Therefore

$$C_1\mathbb{1}_n \leq \mathrm{Hess}_{\mathbb{H}} u \leq C_2\mathbb{1}_n.$$

By applying ι we get

$$C_1\mathbb{1}_{4n} \leq 4\mathrm{p}(\mathrm{Hess}_{\mathbb{R}} u) \leq C_2\mathbb{1}_{4n}.$$

Then we can work as in the proof of [8, Lemma 6.1].

We assume that the domain of u is $B \times [0, T)$ with B diffeomorphic to the unit ball in $\mathbb{R}^{4n}$. If $T < 1$, then Lemma 5 implies

$$|u| \leq CT + C \leq C$$

for a uniform C and [7, Theorem 5.1] implies the result. If $T \geq 1$ we define, for any $a \in (0, T-1)$

$$\hat{u}(x,t) := u(x, t+a) - \inf_{B\times[a,a+1)} u(x,t)$$

for all $t \in [0, 1)$. We immediately deduce

$$\frac{1}{\kappa}\hat{u}_t = \log\det(\mathrm{Hess}_{\mathbb{H}} \hat{u}) - \log\det(G) - F, \qquad \sup_{B\times[0,1)} |\hat{u}(x,t)| \leq C.$$

Invoking again [7, Theorem 5.1], chosen $\varepsilon \in (0, \frac{1}{2})$ and $\alpha \in (0, 1)$ we have

$$\|\nabla^2 u\|_{C^\alpha(B\times[a+\varepsilon,a+1))} = \|\nabla^2 \hat{u}\|_{C^\alpha(B\times[\varepsilon,1))} \leq C$$

where the constant C depends on ε and α. As a was chosen arbitrarily in $(0, T-1)$ we have

$$\|\nabla^2 u\|_{C^\alpha(B\times[\varepsilon,T))} \le C,$$

and the lemma follows. □

Lemma 9 *Assume that there exists* $0<\alpha<1$ *such that*

$$\|\nabla^2\varphi\|_{C^\alpha} \le C$$

for a uniform constant C*. Then* $T=\infty$ *and for every* $k\ge 1$

$$\|\nabla^k\varphi\|_{C^0} \le C$$

for a uniform constant C*.*

Proof Our assumptions imply that the spatial derivatives of φ satisfy a uniformly parabolic equation and uniform bounds on $\|\nabla^k\varphi\|_{C^0}$ with $k\ge 1$ follow by Schauder theory and a standard bootstrapping argument.

Now we shall prove the long-time existence. Assume by contradiction that the maximal time interval $[0,T)$ of existence of φ is bounded. Then the achieved estimates and short-time existence would allow us to extend φ past T, which is a contradiction, thus $T=\infty$. □

Lemma 10 *Assume* $T=\infty$ *and that* $\|\nabla^k\varphi\|_{C^0}$ *is uniformly bounded for every* $k\ge 1$*. Then*

$$\tilde\varphi := \varphi - \int_M \varphi\Omega^n\wedge\bar\Theta$$

converges in C^∞*-topology to a smooth function* $\tilde\varphi_\infty$*. Moreover if*

$$b := \frac{\int_M \Omega^n\wedge\bar\Theta}{\int_M \mathrm{e}^F\Omega^n\wedge\bar\Theta},$$

then $(\tilde\varphi_\infty, b)$ *solves the quaternionic Monge–Ampère equation* (2)*.*

Proof Let

$$f(t) := \int_M \varphi_t\,\Omega^n_\varphi\wedge\bar\Theta = \int_M \log\frac{\Omega^n_\varphi}{\Omega^n}\,\Omega^n_\varphi\wedge\bar\Theta - \int_M F\,\Omega^n_\varphi\wedge\bar\Theta.$$

Using (6) we have

$$\begin{aligned}
f' &= n\int_M\left(\Delta_\varphi\varphi_t + \log\frac{\Omega^n_\varphi}{\Omega^n}\Delta_\varphi\varphi_t - F\Delta_\varphi\varphi_t\right)\Omega^n_\varphi\wedge\bar\Theta\\
&= n\int_M \varphi_t\Delta_\varphi\varphi_t\,\Omega^n_\varphi\wedge\bar\Theta = n\int_M \varphi_t\partial\partial_J\varphi_t\wedge\Omega^{n-1}_\varphi\wedge\bar\Theta = -n\int_M \partial\varphi_t\wedge\partial_J\varphi_t\wedge\Omega^{n-1}_\varphi\wedge\bar\Theta\\
&= -\frac{1}{2}\int_M |\partial\varphi_t|^2_{g_\varphi}\Omega^n_\varphi\wedge\bar\Theta.
\end{aligned}$$

Differentiating again we obtain

$$f'' = -\frac{1}{2}\int_M \tfrac{\partial}{\partial t}|\partial\varphi_t|^2_{g_\varphi}\Omega^n_\varphi\wedge\bar\Theta - \frac{n}{2}\int_M |\partial\varphi_t|^2_{g_\varphi}\Delta_\varphi\varphi_t\,\Omega^n_\varphi\wedge\bar\Theta.$$

Now

$$\frac{\partial}{\partial t}|\partial\varphi_t|^2_{g_\varphi} = -g_\varphi\left(\tfrac{\partial}{\partial t}g_\varphi, \partial\varphi_t\otimes\bar\partial\varphi_t\right) + 2\mathrm{Re}\,g_\varphi(\partial\varphi_{tt}, \bar\partial\varphi_t). \tag{11}$$

For the first term of (11) Cauchy–Schwarz inequality gives

$$-g_\varphi\left(\tfrac{\partial}{\partial t}g_\varphi, \partial\varphi_t\otimes\bar\partial\varphi_t\right)\leq |\tfrac{\partial}{\partial t}g_\varphi|_{g_\varphi}|\partial\varphi_t|^2_{g_\varphi}\leq C|\partial\varphi_t|^2_{g_\varphi}$$

because Ω_φ and g_φ are related by (1) and Ω_φ and $\frac{\partial}{\partial t}\Omega_\varphi$ are uniformly bounded in C^k-norm for every k. For the second term of (11) using (6) again we have

$$\begin{aligned}-n\mathrm{Re}\int_M g_\varphi(\partial\Delta_\varphi\varphi_t, \bar\partial\varphi_t)\Omega^n_\varphi\wedge\bar\Theta &= -2n^2\mathrm{Re}\int_M \partial\Delta_\varphi\varphi_t\wedge\partial_J\varphi_t\wedge\Omega_\varphi^{n-1}\wedge\bar\Theta\\ &= 2n^2\mathrm{Re}\int_M\Delta_\varphi\varphi_t\,\partial\partial_J\varphi_t\wedge\Omega^{n-1}_\varphi\wedge\bar\Theta = 2n^2\int_M(\Delta_\varphi\varphi_t)^2\,\Omega^n_\varphi\wedge\bar\Theta\end{aligned}$$

therefore

$$f''\geq -C\int_M|\partial\varphi_t|^2_{g_\varphi}\,\Omega^n_\varphi\wedge\bar\Theta.$$

Thus we have a non increasing smooth function $f:[0,+\infty)\to\mathbb{R}$ which is bounded from below and such that $f''(t)\geq Cf'(t)$ for some positive constant C. This implies that $\lim_{t\to+\infty}f'(t)=0$, i.e.

$$\lim_{t\to\infty}\int_M|\partial\varphi_t|^2_{g_\varphi}\Omega^n_\varphi\wedge\bar\Theta=0. \tag{12}$$

Now, $\tilde\varphi$ has a uniform C^∞ bound and Ascoli-Arzelà theorem implies that there exists a sequence $\{t_k\}\subseteq\mathbb{R}$, $t_k\to\infty$ such that $\tilde\varphi(\cdot,t_k)$ converges to some $\tilde\varphi_\infty$ in C^∞-topology. Since

$$\tilde\varphi_t=\log\frac{\Omega^n_{\tilde\varphi}}{\Omega^n}-F-\int_M\left(\log\frac{\Omega^n_{\tilde\varphi}}{\Omega^n}-F\right)\Omega^n\wedge\bar\Theta,$$

by (12) we get

$$0=\lim_{t\to\infty}\int_M|\partial\tilde\varphi_t|^2_{g_{\tilde\varphi}}\Omega^n_{\tilde\varphi}\wedge\bar\Theta=\int_M\left|\partial\left(\log\frac{\Omega^n_{\tilde\varphi_\infty}}{\Omega^n}-F\right)\right|^2_{g_{\tilde\varphi}}\Omega^n_{\tilde\varphi_\infty}\wedge\bar\Theta.$$

It follows that

$$\log\frac{\Omega^n_{\tilde\varphi_\infty}}{\Omega^n}-F=C$$

for some constant C, so that

$$\Omega^n_{\tilde\varphi_\infty}=\mathrm{e}^{F+C}\Omega^n.$$

This means that $(\tilde\varphi_\infty,\mathrm{e}^C)$ solves the quaternionic Calabi–Yau equation. Finally, we prove that $\lim_{t\to\infty}\tilde\varphi=\tilde\varphi_\infty$. Assume by contradiction that there exists $\varepsilon>0$ and a sequence $t_k\to\infty$ such that

$$\|\tilde\varphi(\cdot,t_k)-\tilde\varphi_\infty\|_{C^\infty}\geq\varepsilon$$

for every t_k. We may assume that $\tilde\varphi(\cdot,t_k)$ converges in C^∞-topology to $\tilde\varphi'_\infty$. Hence

$$\Omega^n_{\tilde\varphi'_\infty}=\mathrm{e}^{F+C'}\Omega^n.$$

Since $\tilde{\varphi}_\infty$ and $\tilde{\varphi}'_\infty$ solve the same quaternionic Calabi–Yau equation, from uniqueness follows $\tilde{\varphi}_\infty = \tilde{\varphi}'_\infty$ and the lemma is proved. □

Proof of Theorem 1 We put together Lemmas 5–10 proved in this section. Lemmas 5, 6, 7 imply that if φ solves (3), its quaternionic Laplacian $\Delta_{\hat{g}}\varphi$ with respect to the background hyperkähler metric $\hat{g}$ has a uniform upper bound. Hence Lemmas 8 and 9 can be applied and (3) has a long-time solution φ such that $\|\nabla^k\varphi\|_{C^0}$ is bounded for every $k \geq 1$. Therefore, taking $\Theta = \Omega^n_{\hat{g}}$, Lemma 10 implies the last part of the statement. □

4 Further developments

On a hypercomplex manifold (M, I, J, K) with a HKT form Ω a $(1,0)$-form ϑ is defined by the relation

$$\bar{\partial}\Omega^n = \bar{\vartheta} \wedge \Omega^n.$$

If the canonical bundle is holomorphically trivial then we can take $h \in C^\infty(M)$ such that $\partial J\bar{\vartheta} = \partial\partial_J h$. Now the proof of Lemma 10 suggests to consider on a $\mathrm{SL}(n,\mathbb{H})$-manifold with holomorphic q-real volume form Θ the operator $\mathcal{M}$ acting on HKT forms in the $\partial\partial_J$-class of a fixed HKT form Ω_0 as

$$\mathcal{M}(\Omega_\varphi) := \int_M \log\frac{\Omega^n_\varphi}{\Omega^n_0}\,\Omega^n_\varphi \wedge \bar{\Theta} - \int_M h\,\Omega^n_\varphi \wedge \bar{\Theta}.$$

This is related to the following geometric flow of HKT forms

$$\partial_t\Omega = -\partial J\bar{\vartheta}, \quad \Omega(0) = \Omega_0. \tag{13}$$

Indeed working as in the proof of Lemma 10 one can observe that $\mathcal{M}$ is decreasing along flow (13), thus $\mathcal{M}$ plays a role similar to that of the Mabuchi functional in Calabi–Yau geometry. It is easy to prove that the gradient flow of $\mathcal{M}$ can be expressed in terms of the quaternionic potential φ as

$$\varphi_t = \frac{J\bar{\partial}_J\bar{\vartheta}_\varphi \wedge \Omega^{n-1}_\varphi}{\Omega^n_\varphi}, \quad \varphi(0) = 0 \tag{14}$$

and the fixed points of $\mathcal{M}$ are the HKT forms in the $\partial\partial_J$-class of Ω_0 which are balanced, i.e. which induce a balanced metric.

From this perspective we believe that the operator $\mathcal{M}$ and the flow (14) could give new insights in the search of canonical HKT metrics and this will be the subject of a future work.

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