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*Aplikace matematiky*, Vol. 21 (1976), No. 6, 431–433

Persistent URL: http://dml.cz/dmlcz/103667

# A PARADOX IN THE THEORY OF LINEAR ELASTICITY

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(Received November 26, 1975)

Let $\Omega = \{x \in E_3;\ \|x\| < 1\}$. Let $\mathscr{D}(\Omega)$ be the class of real functions, each of which is infinitely differentiable and has its support in $\Omega$. Let $W^{1,2}(\Omega)$, $W_0^{1,2}(\Omega)$ be the usual Sobolev spaces. Let us define $C_{ijkl}$, $i, j, k, l = 1, 2, 3$ (the tensor of the elastic coefficients) in $\Omega$ as

$$C_{ijkl}(x) = \tfrac{1}{2}(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) + \delta_{ij}\delta_{kl} + \\ + \frac{3}{\|x\|^2}(\delta_{ij}x_k x_l + \delta_{kl}x_i x_j) + \frac{9}{\|x\|^4} x_i x_j x_k x_l\,, \quad \|x\| \neq 0\,,$$

where $\delta_{ij}$ is the Kronecker symbol delta. Let us denote the strain tensor by $e_{kl} = \frac{1}{2}(\partial u_k/\partial x_l + \partial u_l/\partial x_k)$ (where $u$ is the displacement vector), $k, l = 1, 2, 3$.

Let $u_0 \in [W^{1,2}(\Omega)]^3$. We say that the vector function $u \in [W^{1,2}(\Omega)]^3$ is a generalized solution of the second problem of the mathematical theory of elasticity in $\Omega$ with the boundary condition $u = u_0$ on $\partial\Omega$, if the following conditions are fulfilled:

$$\text{(i)} \qquad \int_\Omega C_{ijkl} \frac{\partial v_i}{\partial x_j} e_{kl}\, \mathrm{d}x = 0 \quad \text{for every} \quad v \in [W_0^{1,2}(\Omega)]^3$$

(we neglect body forces),

$$\text{(ii)} \qquad u - u_0 \in [W_0^{1,2}(\Omega)]^3\,.$$

Put

$$\alpha = \frac{3(1 - \sqrt{17})}{2\sqrt{17}}\,.$$

**Theorem.** *The displacement vector* $u(x) = x\|x\|^\alpha = (x_1\|x\|^\alpha,\ x_2\|x\|^\alpha,\ x_3\|x\|^\alpha)$ *is the generalized solution of the second problem of the mathematical theory of elasticity in* $\Omega$ *with the boundary condition* $u(x) = x$ *on* $\partial\Omega$.

Proof. We shall prove the relation (i). The other one is obvious. If $\|x\| \neq 0$, then

$$\text{(1)} \qquad \frac{\partial}{\partial x_j}(C_{ijkl}e_{kl}) = 0\,, \quad i = 1, 2, 3\,.$$

Let $\varphi$ be an arbitrary function from $[\mathscr{D}(\Omega)]^3$. Let $\psi \in \mathscr{D}(\Omega)$ be such a function that $\psi(x) = 1$ when $\|x\| < \frac{1}{2}$. We write

$$\psi_\varepsilon(x) = \psi(x/\varepsilon)\,, \quad \varphi_\varepsilon(x) = \varphi(x)\,(1 - \psi_\varepsilon(x))\,, \quad \varepsilon \in (0, 1)\,.$$

Then

$$\int_\Omega C_{ijkl} \frac{\partial \varphi_i}{\partial x_j} e_{kl}\, \mathrm{d}x = \int_\Omega C_{ijkl} \frac{\partial \varphi_{\varepsilon i}}{\partial x_j} e_{kl}\, \mathrm{d}x + \int_{\|x\| < \varepsilon} C_{ijkl} \frac{\partial(\varphi_i \psi_\varepsilon)}{\partial x_j} e_{kl}\, \mathrm{d}x\,.$$

The first integral on the right hand side is, according to Green's theorem and to (1), equal to zero. Because $C_{ijkl}$ and $\partial(\varphi_i \psi_\varepsilon)/\partial x_j$ are bounded and $|e_{kl}| \leqq \|x\|^\alpha$, it is

$$\left| \int_\Omega C_{ijkl} \frac{\partial \varphi_i}{\partial x_j} e_{kl}\, \mathrm{d}x \right| \leqq C\varepsilon^{\alpha + 3}\,, \quad C > 0\,.$$

Because $(\alpha + 3) > 0$, the relation (i) holds for every function $v \in [\mathscr{D}(\Omega)]^3$. The set $[\mathscr{D}(\Omega)]^3$ is dense in $[W_0^{1,2}(\Omega)]^3$, hence (i) holds.

The uniqeness of the solution follows from the relation

$$C_{ijkl}(x)\, \xi_{ij}\xi_{kl} \geqq \xi_{ij}\xi_{ij} \quad \text{for every} \quad \xi \in E_6\,,\ \xi_{ij} = \xi_{ji}\,, \quad \|x\| \neq 0\,.$$

From the physical point of view we may compare this deformation to an explosion. When the radius of the sphere $\Omega$ increases by an arbitrary $\varepsilon > 0$, then the points from a neighbourhood of the origin "cross the boundary of $\Omega$" (i.e., for the boundary condition $u_0(x) = \varepsilon x$ it is $\|x + u(x)\| > 1 + \varepsilon$ in a neighbourhood of the origin). The displacement vector and the stress tensor are unbounded.

The tensor $C_{ijkl}$ is constant on the radial lines (except for the origin) and invariant with respect to the rotation about the origin. The behaviour of the derived material is paradoxical. Let us have a constant tensor $\bar{C}_{ijkl} = C_{ijkl}(\frac{1}{2}, 0, 0)$. Consider the cube $\langle 0, 1 \rangle^3$ of derived homogeneous material. In the case of a constant hydrostatic pressure the body extends in the direction of the axis $x_1$. In the case of a pure tension in the direction of the axis $x_1$ the body contracts.

Nonetheless, all the assumptions of the mathematical theory of the linear elasticity are satisfied (i.e., the coefficients $C_{ijkl}$ are measurable, bounded, the form $C_{ijkl}, \xi_{ij}\xi_{kl}$ is elliptic).

*References*


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[3] *L. F. Nye:* Physical properties of crystals, Oxford 1957.

## Souhrn

# PARADOX V TEORII LINEÁRNÍ PRUŽNOSTI

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Uvažujme systém parciálních diferenciálních rovnic lineární pružnosti. Ukážeme, že řešení tohoto systému s omezenou okrajovou podmínkou není (obecně) omezené (tj. nejsou omezené složky vektoru posunutí). Tento příklad je modifikací příkladu z článku E. De Giorgiho [1].



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