A PARAMETER ROBUST NUMERICAL METHOD
FOR A TWO DIMENSIONAL
REACTION-DIFFUSION PROBLEM

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ABSTRACT. In this paper a singularly perturbed reaction-diffusion partial differential equation in two space dimensions is examined. By means of an appropriate decomposition, we describe the asymptotic behaviour of the solution of problems of this kind. A central finite difference scheme is constructed for this problem which involves an appropriate Shishkin mesh. We prove that the numerical approximations are almost second order uniformly convergent (in the maximum norm) with respect to the singular perturbation parameter. Some numerical experiments are given that illustrate in practice the theoretical order of convergence established for the numerical method.

1. INTRODUCTION

In this work we consider the two dimensional Dirichlet boundary value reaction-diffusion problem

(1.1) $$Lu = f, (x,y) \in \Omega, \quad u = g, \ (x,y) \in \partial\Omega,$$

where the differential operator is defined by

$$L\omega = -\varepsilon(\omega_{xx} + \omega_{yy}) + b(x,y)\omega.$$

The diffusion parameter satisfies $0 < \varepsilon \leq 1$ and it can be arbitrarily small. The domain $\Omega = (0,1)^2$ is the unit square, the reaction term satisfies $b(x,y) \geq 2\beta > 0$, and we assume that $f, b \in C^{4,\alpha}(\overline{\Omega})$ and $g \in C^{4,\alpha}(\partial\Omega)$. We also assume that there are sufficient compatibility conditions (see [4]) in order that $u \in C^{4,\alpha}(\overline{\Omega})$. This type of problem is characterized by the presence of a regular exponential layer in a neighbourhood of $\partial\Omega$ of width $O(\sqrt{\varepsilon})$. It is well known [3] that uniform meshes are an inappropriate discretization of the domain when these boundary layers are present in the solution.

It is known that the differential operator L satisfies a comparison principle: *For any $y \in C^2(\Omega)$, if $y \geq 0$ on $\partial\Omega$ and $Ly \geq 0$ in Ω, then $y \geq 0$ for all points in the closed domain $\overline{\Omega}$.*

To simplify the notation, we consider the following subsets of the boundary $\partial\Omega$:

$$\Gamma_1 = \{(x,0)|0 \leq x \leq 1\}, \qquad \Gamma_2 = \{(0,y)|0 \leq y \leq 1\},$$
$$\Gamma_3 = \{(x,1)|0 \leq x \leq 1\}, \qquad \Gamma_4 = \{(1,y)|0 \leq y \leq 1\},$$

which are the edges of $\partial\Omega$, and the four corners of the domain are denoted by

$$c_1 = \Gamma_1 \cap \Gamma_2, \qquad c_2 = \Gamma_2 \cap \Gamma_3, \qquad c_3 = \Gamma_3 \cap \Gamma_4, \qquad c_4 = \Gamma_4 \cap \Gamma_1.$$

We adopt the following notation for the boundary conditions:

$$g(x,y) = g_i(x), (x,y) \in \Gamma_i, i = 1,3; \quad g(x,y) = g_i(y), (x,y) \in \Gamma_i, i = 2,4.$$

There is an extensive literature on numerical methods for singularly perturbed reaction diffusion problems (see, for example, [1, 10, 12, 17] and the references therein). Our interest lies in examining parameter-uniform numerical methods [3, 8] for singularly perturbed problems. That is, we are interested in numerical methods for which the following error bound can be theoretically established:

$$\|u - \bar{U}^N\|_\infty \le CN^{-p}, \quad p > 0,$$

where N is the number of mesh elements employed in each coordinate direction, $\bar{U}^N$ is a polynomial interpolant generated by the numerical method, $\|\cdot\|_\infty$ is the global pointwise maximum norm, and C is a constant independent of ε and N. In general the gradients of the solution of (1.1) become unbounded as $\varepsilon \to 0$; however, parameter-uniform numerical methods guarantee that the error in the numerical approximation is controlled solely by the size of N.

Li and Navon [6] studied (1.1) within a finite element framework and established a convergence result in the L_2-norm. Note that the boundary layer function $e^{-x/\sqrt{\varepsilon}}$ is of measure $O(\varepsilon^{0.25})$ in both the L_2-norm $\|\cdot\|_0$ and the standard energy norm $|||f|||^2 = \varepsilon\|f_x\|_0^2 + \varepsilon\|f_y\|_0^2 + \|f\|_0^2$. Hence, the size of the boundary layer function is negligible in these norms. In this paper, we establish a convergence result in the pointwise norm $\|\cdot\|_\infty$, where $\|e^{-x/\sqrt{\varepsilon}}\|_\infty = 1$.

In the original work of Shishkin [13], parameter-uniform numerical methods were established for an extensive class of linear singularly perturbed differential equations in n-dimensions, including convection-diffusion $(-\varepsilon\triangle u + \vec{a} \cdot \nabla u + bu, \ \vec{a} > \vec{0}, \ b \ge 0)$ and reaction-diffusion $(-\varepsilon\triangle u + bu, \ b > 0)$. In [13], the author demonstrated the extent of the class of singularly perturbed problems for which the method (simple finite difference operator combined with an appropriate tensor product of piecewise-uniform meshes) could be applied, by assuming minimal regularity on the data. As in the case of nonsingularly perturbed problems, it is possible to obtain parameter-uniform numerical methods with a higher order of parameter-uniform convergence by restricting the class of problems to problems with smoother data.

In one dimension, standard finite difference operators on an appropriate layer-adapted mesh (Shishkin type, Bakhvalov, or variants [7, 8, 11]) yield, up to a possible logarithmic factor, parameter-uniform first order convergence for convection-diffusion problems $(-\varepsilon u'' + au' + bu, \ a > 0, \ b \ge 0)$. For the one dimensional singularly perturbed reaction-diffusion problem $(-\varepsilon u'' + bu, \ b > 0)$ parameter-uniform second order, again up to a possible logarithmic factor, is possible [9, 15]. In the case of convection-diffusion problems in two space dimensions, first order schemes have been examined by several authors (for example, see [8, 13]). In this paper we present a parameter-uniform second order scheme for the reaction-diffusion problem in two space dimensions.

The analytic properties of the solution to the two dimensional reaction-diffusion problem have been studied by [2] and [4]. In this paper, we take the approach of Shishkin [13] to establish parameter-explicit *a priori* bounds on the derivatives of

the solutions, which are central in deriving our theoretical asymptotic error bound on the numerical approximations.

The paper is structured as follows. In Section 2 we give the bounds of the exact solution and its derivatives showing their asymptotic behaviour with respect to the singular perturbation parameter. Moreover, we give an appropriate decomposition of this exact solution. In Section 3 we analyze the standard central finite difference scheme constructed on a special mesh of Shishkin type, proving its ε-uniform convergence of second order. Finally, in Section 4 we show some numerical results, which illustrate the analytical results previously proved.

Notation. Throughout this paper, C denotes a generic positive constant which is independent of the diffusion parameter ε and the discretization parameter N. We also use the following notation for the partial derivatives:

$$f^{(k,j)} = \frac{\partial^{k+j} f}{\partial x^k \partial y^j}.$$

2. Decomposition and a priori bounds

In this section we examine the asymptotic behaviour of the solution of (1.1) with respect to the singular perturbation parameter ε. This behaviour will be used later in the analysis of the uniform convergence of the finite difference approximations defined in Section 3.

From Han and Kellogg [4] we have that if $f, b \in C^{2,\alpha}(\overline{\Omega})$, $g_s \in C^{4,\alpha}([0,1])$, $s = 1, 2, 3, 4$, and compatibility conditions of second level are satisfied at each of the four corners, that is,

(2.1a) $g_1(0) = g_2(0),$

(2.1b) $-\varepsilon g_1''(0) - \varepsilon g_2''(0) + b(0,0)g_1(0) = f(0,0),$

$\varepsilon g_1^{(4)}(0) - \varepsilon g_2^{(4)}(0) + f^{(2,0)}(0,0) - f^{(0,2)}(0,0)$

$\qquad -b(0,0)g_1''(0) - 2b^{(1,0)}(0,0)g_1'(0) - b^{(2,0)}(0,0)g_1(0)$

(2.1c) $\qquad +b(0,0)g_2''(0) + 2b^{(0,1)}(0,0)g_2'(0) + b^{(0,2)}(0,0)g_2(0) = 0,$

and similarly for the other corners, then $u \in C^{4,\alpha}(\overline{\Omega})$.

Using a stretching argument and classical results from [5, 16], we can establish crude bounds on the derivatives of the solution of the form

(2.2) $\|u^{(k,j)}\| \leq C\varepsilon^{-k/2-j/2}, \quad 0 \leq k+j \leq 4,$

where $\|\cdot\|$ is the maximum norm.

Bounds (2.2) are not sufficient to analyze the uniform convergence of the numerical scheme studied in this paper because they do not explicitly show the presence of boundary layers in $\partial\Omega$. Below we present a decomposition of u and appropriate bounds of its derivatives with respect to ε, which will be used in the error analysis in Section 3. This decomposition was first established by Shishkin [13], where the idea of extending the domain was introduced so that a decomposition of the solution into regular and singular components could be effected without imposing additional artificial compatibility conditions on the data. For the sake of completeness, we present a detailed derivation of these bounds here.

Let $\Omega^* = (-a, 1+a) \times (-a, 1+a)$, $a > 0$, be an extended domain that contains $\overline{\Omega}$ as a subset. Define smooth extensions b^*, f^*, and g_s^* of the functions b, f, and g_s

to $\overline{\Omega}^*$ and $[-a, 1 + a]$, respectively. Note that $f^*|_{\overline{\Omega}} = f$. Let $v^* = v_0^* + \varepsilon v_1^*$, where v_0^* is the solution of the extended reduced problem

$$b^* v_0^* = f^*$$

and v_1^* is the solution of the problem

(2.3) $$L^* v_1^* = \Delta v_0^*, (x, y) \in \Omega^*, \quad v_1^* = 0, \ (x, y) \in \partial\Omega^*.$$

Since $v_0^* \in C^{4,\alpha}(\overline{\Omega}^*)$, then $\triangle v_0^* \in C^{2,\alpha}(\overline{\Omega}^*)$. The extensions of all the functions are taken such that the compatibility conditions at the corners of $\overline{\Omega}^*$ of up to second order (see [4]) are satisfied. Hence $v_1^* \in C^{4,\alpha}(\overline{\Omega}^*)$.

The regular solution v is taken to be the solution of the boundary value problem

(2.4) $$Lv = f, (x, y) \in \Omega, \quad v = v^*, \ (x, y) \in \partial\Omega.$$

Applying the classical results (2.2) to the extended problem (2.3), we deduce that $v^*|_{\overline{\Omega}} = v \in C^{4,\alpha}(\overline{\Omega})$ and

(2.5) $$\|v^{(k,j)}\| \le C(1 + \varepsilon^{1-k/2-j/2}), \quad 0 \le k + j \le 4.$$

The function v is called the regular component of the solution u. Note that by virtue of the extension, it is not necessary to impose compatibility conditions at the corners of $\overline{\Omega}$ so that $v \in C^{4,\alpha}(\overline{\Omega})$.

Associated with the bottom edge Γ_1 of the domain, we have a boundary layer function w_1 that is defined as follows. The domain Ω is extended in the horizontal direction to a domain $\Omega^{**} = (-a, 1 + a) \times (0, 1)$, $a > 0$. The function w_1^* is the solution of

(2.6a) $$L^{**} w_1^* = 0, \quad (x, y) \in \Omega^{**},$$

(2.6b) $$w_1^* = u - v, \quad (x, y) \in \Gamma_1,$$

(2.6c) $$w_1^*(x, 1) = 0, \quad x \in [-a, 1 + a],$$

(2.6d) $$w_1^*(-a, y) = w_1^*(1 + a, y) = 0, \quad y \in [0, 1],$$

and the values of the boundary conditions at the points $(x, 0)$ with $x \in (-a, 0) \cup (1, 1 + a)$ are constructed so that $w_1^* \in C^{4,\alpha}(\overline{\Omega}^{**})$.

So, for $0 \le k \le 4$ the extensions on $(-a, 0) \cup (1, 1 + a)$ are constructed so that

$$(w_1^*)^{(k,0)}(0, 0) = (u - v)^{(k,0)}(0, 0),$$
$$(w_1^*)^{(k,0)}(1, 0) = (u - v)^{(k,0)}(1, 0),$$
$$(w_1^*)^{(k,0)}(-a, 0) = (w_1^*)^{(k,0)}(1 + a, 0) = 0.$$

Moreover, we construct the extensions so that

(2.7) $$(b^*)^{(k,0)}(-a, y) = (b^*)^{(k,0)}(1 + a, y) = 0, \quad 0 \le k \le 1,$$

for all $y \in [0, 1]$.

Using the comparison principle, we deduce that

(2.8) $$|w_1^*(x, y)| \le C \frac{(a + x)(1 + a - x)}{a(1 + a)} e^{-\sqrt{\beta/\varepsilon} y}, \quad (x, y) \in \overline{\Omega}^{**}.$$

The crude bounds (2.2) on the derivatives also apply to w_1^*; that is,

(2.9) $$\|(w_1^*)^{(k,j)}\| \le C\varepsilon^{-k/2-j/2}, \quad 0 \le k + j \le 4.$$

We now sharpen these bounds on the derivatives in the direction orthogonal to the layer. Using (2.8) and the fact that $w_1^*(-a, y) = w_1^*(1 + a, y) = 0$, we get that the derivatives on the sides $x = -a$ and $x = 1 + a$ satisfy the bounds

$$|(w_1^*)^{(1,0)}(-a, y)| \le C e^{-\sqrt{\beta/\varepsilon}y} \quad \text{and} \quad |(w_1^*)^{(1,0)}(1 + a, y)| \le C e^{-\sqrt{\beta/\varepsilon}y}.$$

On the other two sides $|(w_1^*)^{(1,0)}(x, 0)| \le C$ and $(w_1^*)^{(1,0)}(x, 1) = 0$. By differentiating the differential equation (2.6a) w.r.t. x, we get that

$$L^{**}(w_1^*)^{(1,0)} = -(b^*)^{(1,0)}(w_1^*), \quad (x, y) \in \Omega^{**},$$

and using the comparison principle, it follows that

$$(2.10) \qquad\qquad |(w_1^*)^{(1,0)}(x, y)| \le C e^{-\sqrt{\beta/\varepsilon}y}.$$

Note that

$$|(w_1^*)^{(2,0)}(x, 0)| \le C, \; (w_1^*)^{(2,0)}(x, 1) = 0,$$

and from the regularity of w_1^* and the fact that $\varepsilon((w_1^*)^{(2,0)} + (w_1^*)^{(0,2)}) = b^* w_1^*$ in the closed region $\overline{\Omega}^{**}$, we have that

$$(w_1^*)^{(2,0)}(-a, y) = (w_1^*)^{(0,2)}(-a, y) = 0,$$
$$(w_1^*)^{(2,0)}(1 + a, y) = (w_1^*)^{(0,2)}(1 + a, y) = 0.$$

Hence, after differentiating (2.6a) twice w.r.t. x, the maximum principle establishes

$$(2.11) \qquad\qquad |(w_1^*)^{(2,0)}(x, y)| \le C(a + x)(1 + a - x)e^{-\sqrt{\beta/\varepsilon}y}.$$

Using (2.11) and the fact that $(w_1^*)^{(2,0)}(-a, y) = (w_1^*)^{(2,0)}(1 + a, y) = 0$, we get that

$$|(w_1^*)^{(3,0)}(-a, y)| \le C e^{-\sqrt{\beta/\varepsilon}y} \quad \text{and} \quad |(w_1^*)^{(3,0)}(1 + a, y)| \le C e^{-\sqrt{\beta/\varepsilon}y},$$

which yields

$$(2.12) \qquad\qquad |(w_1^*)^{(3,0)}(x, y)| \le \frac{C}{\sqrt{\varepsilon}} \, e^{-\sqrt{\beta/\varepsilon}y}.$$

Differentiating (2.6a) twice w.r.t. y yields the fact that the mixed derivatives satisfy

$$(w_1^*)^{(2,2)}(-a, y) = (w_1^*)^{(2,2)}(1 + a, y) = 0.$$

Using this and differentiating (2.6a) twice w.r.t. x, it follows that

$$(2.13) \qquad\qquad (w_1^*)^{(4,0)}(-a, y) = (w_1^*)^{(4,0)}(1 + a, y) = 0.$$

Hence,

$$(2.14) \qquad\qquad |(w_1^*)^{(4,0)}(x, y)| \le \frac{C}{\varepsilon}.$$

Associated with the bottom edge Γ_1, we define a boundary layer function w_1 to be the solution of the homogeneous problem

$$(2.15a) \qquad\qquad\qquad\qquad L w_1 = 0, \; (x, y) \in \Omega,$$
$$(2.15b) \qquad\quad w_1 = u - v, \; (x, y) \in \Gamma_1, \quad w_1 = 0, \; (x, y) \in \Gamma_3,$$
$$(2.15c) \qquad\qquad w_1(0, y) = w_1^*(0, y), \quad w_1(1, y) = w_1^*(1, y).$$

In an analogous fashion, we can define boundary layer functions w_k, $k = 2, 3, 4$, associated with the three other edges, and the corresponding bounds on the derivatives of these functions will hold.

Associated with the corner c_1, we define a corner layer function z_1 such that

(2.16a) $$Lz_1 = 0, \ (x,y) \in \Omega,$$

(2.16b) $$z_1 = -w_2, \ (x,y) \in \Gamma_1, \quad z_1 = -w_1, \ (x,y) \in \Gamma_2,$$

(2.16c) $$z_1 = 0, \ (x,y) \in \Gamma_3, \quad z_1 = 0, \ (x,y) \in \Gamma_4.$$

Note that $Lw_1 = Lw_2 = 0$ and $w_1, w_2 \in C^{4,\alpha}(\overline{\Omega})$. Thus, the compatibility conditions up to second order (see [4]) hold at the four corners of the domain, which implies that $z_1 \in C^{4,\alpha}(\overline{\Omega})$. The maximum principle and the condition $b \geq 2\beta > 0$, result in the bound

(2.17) $$|z_1(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}x}e^{-\sqrt{\beta/\varepsilon}y}.$$

In an analogous fashion we can define corner layer functions $z_k, \ k = 2,3,4$, associated with the three other corners, and the corresponding bounds hold.

Remark 2.1. From (2.2) we have the bounds

$$\|(w_1^*)^{(0,1)}\|, \|(z_1)^{(0,1)}\|, \|(z_1)^{(1,0)}\| \leq \frac{C}{\sqrt{\varepsilon}}.$$

These bounds can be sharpened using the following argument. Use the barrier function

$$\phi(x,y) = C(e^{-\sqrt{\beta/\varepsilon}y} - e^{-\sqrt{\beta/\varepsilon}(2-y)}),$$

to get that

$$|w_1^*(x,y)| \leq C\frac{1-y}{\sqrt{\varepsilon}} \ e^{-\sqrt{\beta/\varepsilon}y} \quad \text{and} \quad |(w_1^*)^{(0,1)}(x,1)| \leq \frac{C}{\sqrt{\varepsilon}} \ e^{-\sqrt{\beta/\varepsilon}}.$$

Using this and the crude bounds on the derivative of w_1^*, we have that

(2.18) $$|(w_1^*)^{(0,1)}(x,y)| \leq \frac{C}{\sqrt{\varepsilon}} \ e^{-\sqrt{\beta/\varepsilon}y}.$$

Analogous bounds hold for $|(w_3^*)^{(0,1)}(x,y)|, |(w_2^*)^{(1,0)}(x,y)|$, and $|(w_4^*)^{(1,0)}(x,y)|$.

Now we sharpen the bounds on the first derivatives of the function z_1. From the above argument it follows that

$$|w_1^*(x,y)| \leq C(e^{-\sqrt{\beta/\varepsilon}y} - e^{-\sqrt{\beta/\varepsilon}(2-y)}),$$
$$|w_2^*(x,y)| \leq C(e^{-\sqrt{\beta/\varepsilon}x} - e^{-\sqrt{\beta/\varepsilon}(2-x)}).$$

Use the barrier function

$$\psi(x,y) = C(e^{-\sqrt{\beta/\varepsilon}x} - e^{-\sqrt{\beta/\varepsilon}(2-x)})(e^{-\sqrt{\beta/\varepsilon}y} - e^{-\sqrt{\beta/\varepsilon}(2-y)}),$$

with C sufficiently large and the discrete maximum principle to establish that

(2.19) $$|(z_1)^{(1,0)}(x,y)|, |(z_1)^{(0,1)}(x,y)| \leq \frac{C}{\sqrt{\varepsilon}} \ e^{-\sqrt{\beta/\varepsilon}x}e^{-\sqrt{\beta/\varepsilon}y}.$$

Analogous bounds hold for the first derivatives of the other three corner layer functions.

We summarize this section with the following result:

Theorem 2.2. *The solution u of (1.1) may be written as a sum*

$$(2.20) \qquad u = v + \sum_{i=1}^{4} w_i + \sum_{i=1}^{4} z_i,$$

where

$$(2.21) \qquad Lv = f, \quad Lw_i = 0 \quad Lz_i = 0, \quad i = 1, 2, 3, 4.$$

Boundary conditions for v, w_i, z_i, $i = 1, 2, 3, 4$, can be specified so that the following bounds on the derivatives of the components hold:

$$(2.22a) \qquad \|v^{(k,j)}\| \leq C(1 + \varepsilon^{1-k/2-j/2}), \quad 0 \leq k + j \leq 4,$$

$$(2.22b) \qquad |w_1(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}y}, \qquad |w_2(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}x},$$

$$(2.22c) \qquad |w_3(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}(1-y)}, \qquad |w_4(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}(1-x)},$$

$$(2.22d) \qquad \max\{\|w_i^{(k,j)}\|, \|z_i^{(k,j)}\|\} \leq C\varepsilon^{-k/2-j/2}, \quad 0 \leq k + j \leq 4,$$

$$(2.22e) \qquad \begin{aligned} \|w_i^{(k,0)}\| &\leq C(1 + \varepsilon^{1-k/2}), \quad i = 1, 3, \\ \|w_i^{(0,k)}\| &\leq C(1 + \varepsilon^{1-k/2}), \quad i = 2, 4, \ 0 \leq k \leq 4, \end{aligned}$$

$$(2.22f) \qquad |z_1(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}y}e^{-\sqrt{\beta/\varepsilon}x},$$

$$(2.22g) \qquad |z_2(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}(1-y)}e^{-\sqrt{\beta/\varepsilon}x},$$

$$(2.22h) \qquad |z_3(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}(1-y)}e^{-\sqrt{\beta/\varepsilon}(1-x)},$$

$$(2.22i) \qquad |z_4(x,y)| \leq Ce^{-\sqrt{\beta/\varepsilon}y}e^{-\sqrt{\beta/\varepsilon}(1-x)}.$$

3. THE DISCRETE PROBLEM

To discretize problem (1.1), we use the standard central difference operator

$$(3.1) \qquad \begin{aligned} L^N U^N &= -\varepsilon(\delta_x^2 + \delta_y^2)U + bU = f, \quad (x_i, y_j) \in \Omega^N, \\ U &= u, \quad \text{on the boundary } \partial\Omega^N, \end{aligned}$$

where the mesh Ω^N is the tensor product of two one dimensional piecewise-uniform Shishkin meshes; i.e., $\Omega^N = \Omega_x \times \Omega_y$, where Ω_x (similarly for Ω_y) splits the interval $[0,1]$ into three subintervals $[0, \sigma_x]$, $[\sigma_x, 1-\sigma_x]$, and $[1-\sigma_x, 1]$. The mesh distributes $N/4$ points uniformly within each of the subintervals $[0, \sigma_x]$ and $[1 - \sigma_x, 1]$ and the remaining $N/2$ mesh points uniformly in the interior subinterval $[\sigma_x, 1 - \sigma_x]$. To simplify our discussion, we take $\sigma_x = \sigma_y$; these transition points are defined as

$$\sigma = \sigma_x = \sigma_y = \min\left\{\frac{1}{4}, 2\sqrt{\frac{\varepsilon}{\beta}} \ln N\right\}.$$

Below we denote by $h = 4\sigma/N, H = 2(1 - 2\sigma)/N$; $h_{i+1} = x_{i+1} - x_i, k_{i+1} = y_{i+1} - y_i$, $0 \leq i \leq N-1$, and $\bar{h}_i = (h_{i+1} + h_i)/2, \bar{k}_i = (k_{i+1} + k_i)/2$, $1 \leq i \leq N-1$.

It is well known that for the reaction-diffusion problem, central differences are an ε-uniformly stable scheme in the maximum norm; that is, for any mesh function W

$$(3.2) \qquad \|W\| \leq \frac{1}{2\beta}\|L^N W\| + \max_{\partial\Omega^N} |W|.$$

From (3.2) we see that in order to prove uniform convergence, we only need to analyze the local truncation error. Using a standard truncation error argument, we can easily obtain

$$
(3.3) \quad
\begin{aligned}
&|L^N(U-u)(x_i,y_j)| \\
&\leq
\begin{cases}
C\varepsilon(\bar{h}_i\|u^{(3,0)}\| + \bar{k}_j\|u^{(0,3)}\|), & \text{if } x_i = \sigma_x, 1-\sigma_x, \text{ or } y_j = \sigma_y, 1-\sigma_y, \\
C\varepsilon(h_i^2\|u^{(4,0)}\| + k_j^2\|u^{(0,4)}\|), & \text{otherwise.}
\end{cases}
\end{aligned}
$$

Nevertheless, to find appropriate bounds of this error, we need to decompose the discrete solution similarly to the decomposition of the exact solution. The numerical solution can be written in the form

$$(3.4) \qquad U = V + \sum_{k=1}^4 W_k + \sum_{k=1}^4 Z_k,$$

where

$$(3.5) \qquad
\begin{cases}
L^N V = f, \text{ in } \Omega^N, \\
V = v, \text{ in } \partial\Omega^N,
\end{cases}$$

$$(3.6) \qquad
\begin{cases}
L^N W_k = 0, \text{ in } \Omega^N, \quad k = 1,2,3,4, \\
W_k = w_k, \text{ in } \partial\Omega^N,
\end{cases}$$

$$(3.7) \qquad
\begin{cases}
L^N Z_k = 0, \text{ in } \Omega^N, \quad k = 1,2,3,4, \\
Z_k = z_k, \text{ in } \partial\Omega^N.
\end{cases}$$

From (2.5) and (3.3) a straightforward computation gives

$$|L^N(V-v)(x_i,y_j)| \leq
\begin{cases}
C\sqrt{\varepsilon}N^{-1}, & \text{if } x_i = \sigma_x, 1-\sigma_x, \text{ or } y_j = \sigma_y, 1-\sigma_y, \\
CN^{-2}, & \text{otherwise.}
\end{cases}$$

Following [9], we define the barrier function

$$\Psi(x_i,y_j) = C\frac{\sigma_x\sigma_y}{\varepsilon} N^{-2}(\theta(x_i)+\theta(y_j)) + CN^{-2},$$

where $\theta(z)$ is a piecewise-linear polynomial defined by

$$\theta(z) =
\begin{cases}
\frac{z}{\sigma}, & 0 \leq z \leq \sigma, \\
1, & \sigma \leq z \leq 1-\sigma, \\
\frac{1-z}{\sigma}, & 1-\sigma \leq z \leq 1,
\end{cases}
\quad \text{and} \quad
\delta_z^2\theta(z) =
\begin{cases}
-\frac{N}{\sigma}, z = \sigma, \ 1-\sigma, \\
0, \text{ otherwise.}
\end{cases}$$

From the choice of transition points, it follows that

$$0 \leq \Psi(x_i,y_j) \leq C(N^{-1}\ln N)^2,$$

and we also have

$$L^N(\Psi)(x_i,y_j) =
\begin{cases}
C\sigma N^{-1} + (b\Psi)(x_i,y_j), & \text{if } x_i = \sigma_x, 1-\sigma_x, \text{ or } y_j = \sigma_y, 1-\sigma_y, \\
(b\Psi)(x_i,y_j), & \text{otherwise.}
\end{cases}$$

Then the discrete maximum principle gives us that

$$(3.8) \qquad \|V-v\| \leq C(N^{-1}\ln N)^2,$$

which is the appropriate bound for the error associated with the regular component.

To prove ε-uniform bounds of the errors associated with the edge and corner functions, we use an argument based on appropriate barrier functions. As usual for singularly perturbed problems, we consider the barrier functions

$$
B_{w_1;j} = \begin{cases} \prod_{s=1}^{j} \left(1 + k_s \sqrt{\beta/\varepsilon}\,\right)^{-1}, \ j \neq 0, \\ 1, \ j = 0, \end{cases}
$$

$$
B_{w_2;i} = \begin{cases} \prod_{s=1}^{i} \left(1 + h_s \sqrt{\beta/\varepsilon}\,\right)^{-1}, \ i \neq 0, \\ 1, \ i = 0, \end{cases}
$$

$$
B_{w_3;j} = \begin{cases} \prod_{s=j+1}^{N} \left(1 + k_s \sqrt{\beta/\varepsilon}\,\right)^{-1}, \ j \neq N, \\ 1, \ j = N, \end{cases}
$$

$$
B_{w_4;i} = \begin{cases} \prod_{s=i+1}^{N} \left(1 + h_s \sqrt{\beta/\varepsilon}\,\right)^{-1}, \ i \neq N, \\ 1, \ i = N. \end{cases}
$$

These functions are first order Taylor approximations of the exponential functions related to the singular behaviour of the solution of problem (1.1). Note that for all j it holds that

$$
\exp\left(-\sqrt{\beta/\varepsilon}y_j\right) = \prod_{s=1}^{j} \exp\left(-\sqrt{\beta/\varepsilon}k_s\right) \leq B_{w_1;j},
$$

and for $\sigma < 0.25$ and $N/4 \leq j \leq N$ we have

$$
(3.9) \qquad B_{w_1;j} \leq B_{w_1;N/4} = \left(1 + \frac{8\ln N}{N}\right)^{-N/4} \leq CN^{-2},
$$

$$
(3.10) \qquad L^N B_{w_1;j} \geq (b(x_i,y_j) - 2\beta)B_{w_1;j}.
$$

Analogous bounds hold for the other three edge functions.

Proposition 3.1. *If w_k and W_k are the solutions of (2.15) and (3.6), respectively, then, for $k = 1, 2, 3, 4$,*

$$
(3.11) \qquad |w_k(x_i, y_j) - W_k(x_i, y_j)| \leq C(N^{-1}\ln N)^2, \quad (x_i, y_j) \in \Omega^N.
$$

Proof. We assume throughout that $\sigma < 0.25$. The case of $\sigma = 0.25$ is dealt with in this case in a classical fashion by noting that $\varepsilon^{-1} \leq C(\ln N)^2$. We only give the details for the edge layer function w_1. The argument is analogous for the other three boundary layer functions.

From (3.6) and Theorem 2.2 it follows that

$$
(3.12) \quad |W_1(x_i, y_j)| = |w_1(x_i, y_j)| \leq C \exp\left(-\sqrt{\beta/\varepsilon}y_j\right) \leq C B_{w_1;j}, \ (x_i, y_j) \in \partial\Omega^N
$$

on the boundary $\partial\Omega^N$. Also for each internal mesh point $(x_i, y_j) \in \Omega^N$, $0 < i, j < N$, from (3.6), (3.10), and the discrete maximum principle, it follows that

$$
(3.13) \qquad\qquad |W_1(x_i, y_j)| \leq B_{w_1;j}.
$$

Therefore, using Theorem 2.2 and (3.13), we deduce that

$$
|w_1(x_i, y_j) - W_1(x_i, y_j)| \leq |w_1(x_i, y_j)| + |W_1(x_i, y_j)| \leq C B_{w_1;j}.
$$

Hence, for the mesh points that are not close to the edge Γ_1, from (3.9) we have

$$(3.14) \qquad |w_1(x_i, y_j) - W_1(x_i, y_j)| \leq CN^{-2}, \quad 0 \leq i \leq N, \ N/4 \leq j \leq N.$$

To prove similar bounds of the error in the region $\Omega_1^N = \{(x_i, y_j) \mid 0 < i < N, \ 0 < j < N/4\}$, we proceed as follows. Using Taylor expansions, we obtain

$$|L^N[W_1(x_i, y_j) - w_1(x_i, y_j)]| \leq \begin{cases} C\varepsilon(\bar{h}_i \|w_1^{(3,0)}\| + k_j^2 \|w_1^{(0,4)}\|), & i = N/4, 3N/4, \\ C\varepsilon(h_i^2 \|w_1^{(4,0)}\| + k_j^2 \|w_1^{(0,4)}\|), & \text{otherwise.} \end{cases}$$

From Theorem 2.2, it follows that

$$|L^N[W_1(x_i, y_j) - w_1(x_i, y_j)]| \leq \begin{cases} C((N^{-1} \ln N)^2 + N^{-1}\sqrt{\varepsilon}), & i = N/4, \ 3N/4, \\ C(N^{-1} \ln N)^2, & \text{otherwise.} \end{cases}$$

Then, similar to the analysis for the regular component, we use the barrier function

$$\Psi(x_i, y_j) = C \frac{\sigma_x}{\sqrt{\varepsilon}} N^{-2}\theta(x_i) + C(N^{-1} \ln N)^2$$

and the discrete maximum principle, now applied only on $\overline{\Omega}_1^N$, to get

$$(3.15) \qquad |w_1(x_i, y_j) - W_1(x_i, y_j)| \leq C(N^{-1} \ln N)^2, \quad (x_i, y_j) \in \overline{\Omega}_1^N.$$

The result follows from (3.14) and (3.15). $\square$

Proposition 3.2. *If z_k and Z_k are the solutions of* (2.16) *and* (3.7), *respectively, then, for $k = 1, 2, 3, 4$,*

$$(3.16) \qquad |z_k(x_i, y_j) - Z_k(x_i, y_j)| \leq C(N^{-1} \ln N)^2, \quad (x_i, y_j) \in \Omega^N.$$

Proof. Again we only give the proof of (3.16) for the corner layer function z_1 and in the case of $\sigma < 0.25$. In a similar way to the argument given in Proposition 3.1, we obtain

$$|Z_1(x_i, y_j)| \leq C \min\{B_{w_2;i}, B_{w_1;j}\}, \quad \text{if } (x_i, y_j) \in \partial\Omega^N,$$

and

$$|z_1(x_i, y_j) - Z_1(x_i, y_j)| \leq C \min\{B_{w_2;i}, B_{w_1;j}\}, \quad \text{if } (x_i, y_j) \in \Omega^N, 0 < i, j < N.$$

Then, using (3.9), we deduce that

$$|z_1(x_i, y_j) - Z_1(x_i, y_j)| \leq CN^{-2}, \quad (x_i, y_j) \in \Omega^N \backslash \Omega_{1,2}^N,$$

where $\Omega_{1,2}^N = \{(x_i, y_j) \mid 0 < i, j < N/4\}$.

Finally, in $\Omega_{1,2}^N$ the truncation error satisfies

$$|L^N[Z_1(x_i, y_j) - z_1(x_i, y_j)]| \leq C\varepsilon h^2(\|z_1^{(4,0)}\| + \|z_1^{(0,4)}\|) \leq C(N^{-1} \ln N)^2,$$

where we have used Theorem 2.2. Considering the barrier function

$$\Psi(x_i, y_j) = C(N^{-1} \ln N)^2,$$

the discrete maximum principle, used on $\overline{\Omega}_{1,2}^N$, proves the required result. $\square$

Thus, from the bounds (3.8) and Propositions 3.1 and 3.2, we deduce the following result of uniform convergence.

Theorem 3.3. *Let u be the solution of problem* (1.1) *and U the numerical solution of* (3.1) *defined on the piecewise-uniform Shihskin mesh. Then the error at the mesh points satisfies*

$$(3.17) \qquad |(u - U)(x_i, y_j)| \leq C(N^{-1} \ln N)^2, \quad (x_i, y_j) \in \Omega^N.$$

A further property of Shishkin meshes is parameter-uniform interpolation. Let

$$w_I(x, y) = \sum_{i,j=0}^{N} w(x_i, y_j)\phi_i(x)\phi^j(y),$$

where $\phi_i(x)$ is the standard piecewise-linear basis function associated with the interval $[x_{i-1}, x_{i+1}]$. Define a global bilinear approximation to the solution u of (1.1) as

$$\bar{U}^N(x, y) = \sum_{i,j=0}^{N} U^N(x_i, y_j)\phi_i(x)\phi^j(y).$$

Note that from the previous theorem

$$\|u - \bar{U}^N\| \leq \|u - u_I\| + C(N^{-1} \ln N)^2.$$

From the argument in [14], the bounds on the components in the decomposition given in Theorem 2.2, and the sharper bounds on the first derivatives of the layer components given in (2.18), (2.19), we have the global parameter-uniform error bound

$$(3.18) \qquad \|u - \bar{U}^N\| \leq C(N^{-1} \ln N)^2.$$

4. NUMERICAL EXPERIMENTS

In this section we present numerical results obtained by applying the numerical method described in Section 3 to two particular problems of the form (1.1). To estimate the maximum errors, we use a variant of the double mesh principle. The two mesh difference is calculated using

$$D_\varepsilon^N = \left(\max_{0 \leq i,j \leq N} |\tilde{U}_{2i,2j}^{2N} - U_{i,j}^N| \right),$$

where $\{\tilde{U}^{2N}\}$ is the numerical solution on a mesh that contains the mesh points (x_i, y_j) of $\overline{\Omega}^N$ and also the midpoints $x_{i+1/2} = (x_i + x_{i+1})/2, y_{i+1/2} = (y_i + y_{i+1})/2$, $i = 0, 1, \ldots, N - 1$. From these values we define the ε-uniform differences by $D^N = \max_\varepsilon D_\varepsilon^N$, the numerical orders of convergence are calculated by $p_\varepsilon^N = \log(D_\varepsilon^N/D_\varepsilon^{2N})/\log 2$, and the ε-uniform order is given by $p_{\mathrm{uni}}^N = \log(D^N/D^{2N})/\log 2$.

The first test problem is

$$(4.1) \qquad \begin{aligned} &-\varepsilon\Delta u + (1 + x^2 y^2)u = 0, \quad (x, y) \in (0, 1)^2, \\ &u(x, 0) = (1 - x)^2, \quad 0 \leq x \leq 1, \\ &u(0, y) = 1 - y, \quad 0 \leq y \leq 1, \\ &u(x, 1) = u(1, y) = 0, \quad 0 \leq x, y \leq 1. \end{aligned}$$

Figure 1 shows the numerical solution for $\varepsilon = 10^{-6}$ and $N = 64$. From this figure we see that the solution has a corner layer at (0,0) and two boundary layers near the edges $x = 0$ and $y = 0$ of the unit square. Note that the reduced solution is identically zero and that $\|u\| = 1$.
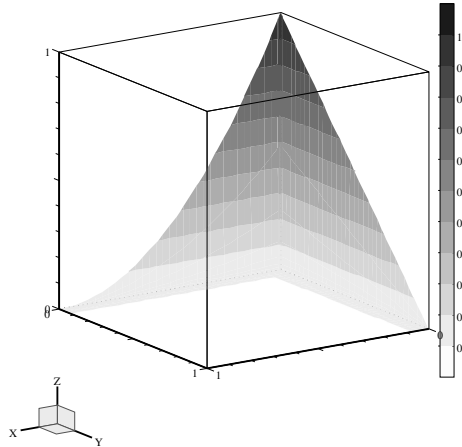
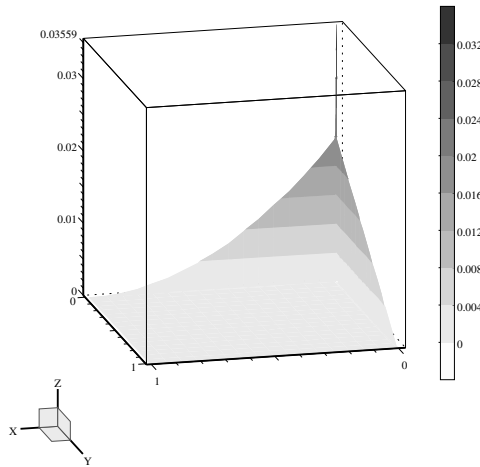FIGURE 1. Numerical solution of problem (4.1) ($\varepsilon = 10^{-6}$, $N = 64$)

FIGURE 2. Approximate errors $\|U^{32} - \tilde{U}^{512}\|$ for problem (4.1) ($\varepsilon = 10^{-6}$)

Table 1 displays the maximum differences D_ε^N and the numerical orders of convergence p_ε^N for $\varepsilon = 1, 2^{-2}, 2^{-4}, \ldots, 2^{-26}$. These results indicate that the uniform order of convergence is in agreement with Theorem 3.3, even though we do not have sufficient compatibility conditions (for example in the corner $(0,0)$ only the condition of order zero holds) for problem (4.1).

Figure 2 shows the approximate pointwise errors for $\varepsilon = 10^{-6}$ calculated by comparing the numerical solution obtained with $N = 32$ with the numerical solution obtained using a finer mesh having $N = 512$, in such way that the transition points of the mesh Ω^{32} and the finer mesh $\tilde{\Omega}^{512}$ coincide. The finer mesh $\tilde{\Omega}^{512}$ is obtained by dividing each subinterval of the coarse mesh Ω^{32} into sixteen subintervals of equal length. We see that the maximum error occurs in the corner layer region.

TABLE 1. Maximum differences D_ε^N and the numerical orders of convergence p_ε^N for problem (4.1)

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$
$\varepsilon = 1$	$3.961E-5$	$9.938E-6$	$2.488E-6$	$6.221E-7$
	1.995	1.998	2.000	
$\varepsilon = 2^{-2}$	$3.740E-5$	$9.579E-6$	$2.420E-6$	$6.075E-7$
	1.965	1.985	1.994	
$\varepsilon = 2^{-4}$	$2.651E-4$	$6.804E-5$	$1.718E-5$	$4.310E-6$
	1.962	1.986	1.995	
$\varepsilon = 2^{-6}$	$1.208E-3$	$3.136E-4$	$7.931E-5$	$1.989E-5$
	1.945	1.984	1.995	
$\varepsilon = 2^{-8}$	$4.791E-3$	$1.308E-3$	$3.360E-4$	$8.460E-5$
	1.873	1.961	1.990	
$\varepsilon = 2^{-10}$	$1.740E-2$	$5.065E-3$	$1.361E-3$	$3.470E-4$
	1.780	1.895	1.972	
$\varepsilon = 2^{-12}$	$2.472E-2$	$1.025E-2$	$3.922E-3$	$1.334E-3$
	1.270	1.386	1.556	
$\varepsilon = 2^{-14}$	$2.514E-2$	$1.035E-2$	$3.964E-3$	$1.345E-3$
	1.281	1.384	1.559	
$\varepsilon = 2^{-16}$	$2.535E-2$	$1.040E-2$	$3.985E-3$	$1.351E-3$
	1.286	1.384	1.561	
$\varepsilon = 2^{-18}$	$2.546E-2$	$1.042E-2$	$3.995E-3$	$1.354E-3$
	1.289	1.383	1.561	
$\varepsilon = 2^{-20}$	$2.551E-2$	$1.043E-2$	$4.000E-3$	$1.355E-3$
	1.290	1.383	1.562	
$\varepsilon = 2^{-22}$	$2.553E-2$	$1.044E-2$	$4.003E-3$	$1.356E-3$
	1.291	1.383	1.562	
$\varepsilon = 2^{-24}$	$2.555E-2$	$1.044E-2$	$4.004E-3$	$1.356E-3$
	1.291	1.383	1.562	
$\varepsilon = 2^{-26}$	$2.555E-2$	$1.044E-2$	$4.005E-3$	$1.356E-3$
	1.291	1.383	1.562	
D^N	$2.555E-2$	$1.044E-2$	$4.005E-3$	$1.356E-3$
p_{uni}^N	1.291	1.383	1.562	

TABLE 2. Values of $C_{p^*}^N$ for problem (4.1)

$N = 32$	$N = 64$	$N = 128$	$N = 256$
3.791	3.790	3.558	2.948

Finally, following [3], we estimate the ε-uniform error constant. From the value

$$p^* = \min_N p_{\text{uni}}^N,$$

we calculate (see Table 2)

$$C_{p^*}^N = \frac{D^N \, N^{p^*}}{1 - 2^{-p^*}}.$$

Then, the ε-uniform error constant is defined by
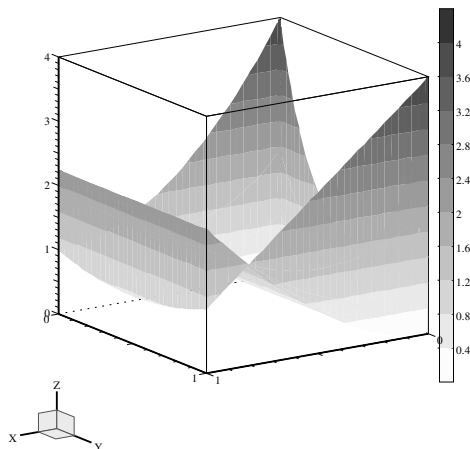
$$C_{p^*}^* = \max_N C_{p^*}^N,$$

FIGURE 3. Numerical solution of problem (4.2) ($\varepsilon = 10^{-6}$, $N = 64$)

which in this case takes the value 3.791. As in [3], we propose the following parameter-uniform error estimate for the numerical approximations U^N of (3.1) to the solution u of problem (1.1)

$$\|u - U^N\| \le 3.79 N^{-1.29}, \quad N \ge 32.$$

The second test problem we consider is

$$-\varepsilon \Delta u + (1+x)^2 u = (x^2 + 2x)^2, \quad (x,y) \in (0,1)^2,$$

(4.2)
$$u(x,0) = 1 + 3(1-x)^2, \quad 0 \le x \le 1,$$
$$u(0,y) = 4(1 - 2y(1-y))^2, \quad 0 \le y \le 1,$$
$$u(x,1) = 1 + 3(1-x), \quad 0 \le x \le 1,$$
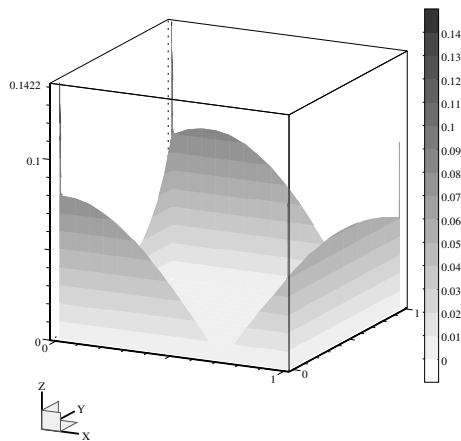$$u(1,y) = 1 - y(1-y), \quad 0 \le y \le 1.$$

FIGURE 4. Approximate errors $\|U^{32} - \tilde{U}^{512}\|$ for problem (4.2) ($\varepsilon = 10^{-6}$)

TABLE 3. Maximum differences D_ε^N and the numerical orders of convergence p_ε^N for problem (4.2)

ε	$N = 32$	$N = 64$	$N = 128$	$N = 256$
$\varepsilon = 1$	$1.445E-3$	$3.618E-4$	$9.049E-5$	$2.263E-5$
	1.998	1.999	2.000	
$\varepsilon = 2^{-2}$	$1.069E-3$	$2.678E-4$	$6.698E-5$	$1.675E-5$
	1.998	1.999	2.000	
$\varepsilon = 2^{-4}$	$1.765E-3$	$4.558E-4$	$1.151E-4$	$2.888E-5$
	1.953	1.985	1.995	
$\varepsilon = 2^{-6}$	$6.291E-3$	$1.708E-3$	$4.376E-4$	$1.102E-4$
	1.881	1.965	1.990	
$\varepsilon = 2^{-8}$	$2.209E-2$	$6.430E-3$	$1.724E-3$	$4.393E-4$
	1.780	1.899	1.973	
$\varepsilon = 2^{-10}$	$6.813E-2$	$2.249E-2$	$6.536E-3$	$1.741E-3$
	1.599	1.783	1.908	
$\varepsilon = 2^{-12}$	$9.762E-2$	$4.070E-2$	$1.731E-2$	$6.363E-3$
	1.262	1.234	1.444	
$\varepsilon = 2^{-14}$	$9.993E-2$	$4.126E-2$	$1.739E-2$	$6.394E-3$
	1.276	1.247	1.443	
$\varepsilon = 2^{-16}$	$1.011E-1$	$4.152E-2$	$1.743E-2$	$6.410E-3$
	1.284	1.253	1.443	
$\varepsilon = 2^{-18}$	$1.017E-1$	$4.165E-2$	$1.745E-2$	$6.418E-3$
	1.287	1.255	1.443	
$\varepsilon = 2^{-20}$	$1.020E-1$	$4.171E-2$	$1.746E-2$	$6.422E-3$
	1.289	1.257	1.443	
$\varepsilon = 2^{-22}$	$1.021E-1$	$4.175E-2$	$1.746E-2$	$6.424E-3$
	1.290	1.257	1.443	
$\varepsilon = 2^{-24}$	$1.022E-1$	$4.176E-2$	$1.747E-2$	$6.425E-3$
	1.291	1.258	1.443	
$\varepsilon = 2^{-26}$	$1.022E-1$	$4.177E-2$	$1.747E-2$	$6.426E-3$
	1.291	1.258	1.443	
D^N	$1.022E-1$	$4.177E-2$	$1.747E-2$	$6.426E-3$
p_{uni}^N	1.291	1.258	1.443	

TABLE 4. Values of $C_{p^*}^N$ for problem (4.2)

$N = 32$	$N = 64$	$N = 128$	$N = 256$
13.744	13.434	13.438	11.822

Now the solution has four boundary and corner layers as is shown in Figure 3. Note that the reduced solution is not identically zero and that $\|u\| = 4$.

Table 3 displays the differences and the order in this case for the same values of ε as in the previous example. Again we only have zero-order compatibility conditions; nevertheless we observe orders of convergence tending toward two.

Figure 4 shows the pointwise errors for $\varepsilon = 10^{-6}$ using the same technique as before. Again we see that the maximum error occurs in the layer regions.

Finally, Table 4 displays the values of $C_{p^*}^N$ and in bold we indicate the ε-uniform error constant for problem (4.2).

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