SIAM J. Matrix Anal. Appl.
Vol. 29, No. 2, pp. 413–433


# A PARTIAL CONDITION NUMBER FOR LINEAR LEAST SQUARES PROBLEMS*

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**Abstract.** We consider here the linear least squares problem $\min_{y\in\mathbb{R}^n} \|Ay-b\|_2$, where $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ is a matrix of full column rank $n$, and we denote $x$ its solution. We assume that both $A$ and $b$ can be perturbed and that these perturbations are measured using the Frobenius or the spectral norm for $A$ and the Euclidean norm for $b$. In this paper, we are concerned with the condition number of a linear function of $x$ ($L^Tx$, where $L \in \mathbb{R}^{n\times k}$) for which we provide a sharp estimate that lies within a factor $\sqrt{3}$ of the true condition number. Provided the triangular $R$ factor of $A$ from $A^TA = R^TR$ is available, this estimate can be computed in $2kn^2$ flops. We also propose a statistical method that estimates the partial condition number by using the exact condition numbers in random orthogonal directions. If $R$ is available, this statistical approach enables us to obtain a condition estimate at a lower computational cost. In the case of the Frobenius norm, we derive a closed formula for the partial condition number that is based on the singular values and the right singular vectors of the matrix $A$.

**Key words.** linear least squares, normwise condition number, statistical condition estimate, parameter estimation

**AMS subject classifications.** 65F20, 65F35, 15A09, 15A12

**DOI.** 10.1137/050643088

## 1. Introduction.

Perturbation theory has been applied to many problems of linear algebra such as linear systems, linear least squares, or eigenvalue problems [1, 4, 11, 18]. In this paper we consider the problem of calculating the quantity $L^Tx$, where $x$ is the solution of the linear least squares problem (LLSP) $\min_{x\in\mathbb{R}^n} \|Ax-b\|_2$, where $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$ is a matrix of full column rank $n$. This estimation is a fundamental problem of parameter estimation in the framework of the Gauss–Markov model [17, p. 137]. More precisely, we focus here on the evaluation of the sensitivity of $L^Tx$ to small perturbations of the matrix $A$ and/or the right-hand side $b$, where $L \in \mathbb{R}^{n\times k}$ and $x$ is the solution of the LLSP.

The interest for this question stems, for instance, from parameter estimation where the parameters of the model can often be divided into two parts: the variables of physical significance and a set of ancillary variables involved in the models. For example, this situation occurs in the determination of positions using the GPS system, where the three-dimensional coordinates are the quantities of interest, but the statistical model involves other parameters such as clock drift and GPS ambiguities [12] that are generally estimated during the solution process. It is then crucial to ensure that the solution components of interest can be computed with satisfactory accuracy. The main goal of this paper is to formalize this problem in terms of a condition number and to describe practical methods to compute or estimate this quantity. Note that as far as the sensitivity of a subset of the solution components is concerned, the matrix $L$ is a projection whose columns consist of vectors of the canonical basis of $\mathbb{R}^n$.

*Received by the editors October 19, 2005; accepted for publication (in revised form) by H. Park September 28, 2006; published electronically April 3, 2007.
http://www.siam.org/journals/simax/29-2/64308.html

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The condition number of a map $g : \mathbb{R}^m \mapsto \mathbb{R}^n$ at $y_0$ measures the sensitivity of $g(y_0)$ to perturbations of $y_0$. If we assume that the data space $\mathbb{R}^m$ and the solution space $\mathbb{R}^n$ are equipped, respectively, with the norms $\|.\|_{\mathcal{D}}$ and $\|.\|_{\mathcal{S}}$, the condition number $K(y_0)$ is defined by

$$K(y_0) = \lim_{\delta \to 0} \sup_{0<\|y_0-y\|_{\mathcal{D}} \leq \delta} \frac{\|g(y_0) - g(y)\|_{\mathcal{S}}}{\|y_0 - y\|_{\mathcal{D}}}, \tag{1.1}$$

whereas the relative condition number is defined by $K^{(rel)}(y_0) = K(y_0)\|y_0\|_{\mathcal{D}}/\|g(y_0)\|_{\mathcal{S}}$. This definition shows that $K(y_0)$ measures an asymptotic sensitivity and that this quantity depends on the chosen norms for the data and solution spaces. If $g$ is a Fréchet-differentiable (F-differentiable) function at $y_0$, then $K(y_0)$ is the norm of the F-derivative $|||g'(y_0)|||$ (see [6]), where $|||.|||$ is the operator norm induced by the choice of the norms on the data and solution spaces.

For the full rank LLSP, we have $g(A, b) = (A^TA)^{-1}A^Tb$. If we consider the product norm $\|(A, b)\|_F = \sqrt{\|A\|_{\mathrm{F}}^2 + \|b\|_2^2}$ for the data space and $\|x\|_2$ for the solution space, then [8] gives an explicit formula for the relative condition number $K^{(rel)}(A, b)$:

$$K^{(rel)}(A, b) = \left\|A^\dagger\right\|_2 \left(\left\|A^\dagger\right\|_2^2 \|r\|_2^2 + \|x\|_2^2 + 1\right)^{\frac{1}{2}} \frac{\|(A, b)\|_F}{\|x\|_2},$$

where $A^\dagger$ denotes the pseudoinverse of $A$, $r = b - Ax$ is the residual vector, and $\|.\|_{\mathrm{F}}$ and $\|.\|_2$ are, respectively, the Frobenius and Euclidean norms. But does the value of $K^{(rel)}(A, b)$ give us useful information about the sensitivity of $L^Tx$? Can it in some cases overestimate the error in components or on the contrary be too optimistic?

Let us consider the following example:

$$A = \begin{pmatrix} 1 & 1 & \epsilon^2 \\ \epsilon & 0 & \epsilon^2 \\ 0 & \epsilon & \epsilon^2 \\ \epsilon^2 & \epsilon^2 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} \epsilon \\ \epsilon \\ \frac{1}{\epsilon} \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} 3\epsilon \\ \epsilon^2 + \epsilon \\ \epsilon^2 + \epsilon \\ 2\epsilon^3 + \frac{2}{\epsilon} \end{pmatrix},$$

where $x$ is the exact solution of the LLSP $\min_{x\in\mathbb{R}^3} \|Ax - b\|_2$. If we take $\epsilon = 10^{-8}$, then we have $x = (10^{-8}, 10^{-8}, 10^8)^T$ and the solution computed in MATLAB using a machine precision $2.22 \cdot 10^{-16}$ is $\tilde{x} = (1.5 \cdot 10^{-8}, 1.5 \cdot 10^{-8}, 10^8)^T$. The LLSP condition number is $K^{(rel)}(A, b) = 2.4 \cdot 10^8$ and the relative errors on the components of $x$ are

$$\frac{|x_1 - \tilde{x}_1|}{|x_1|} = \frac{|x_2 - \tilde{x}_2|}{|x_2|} = 0.5 \quad \text{and} \quad \frac{|x_3 - \tilde{x}_3|}{|x_3|} = 0.$$

Then, if $L = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, we expect a large value for the condition number of $L^Tx$ because there is a 50% relative error on $x_1$ and $x_2$. If now $L = (0, 0, 1)^T$, then we expect that the condition number of $L^Tx$ would be close to 1 because $\tilde{x}_3 = x_3$. For these two values of $L$, the LLSP condition number is far from giving a good idea of the sensitivity of $L^Tx$. Note in this case that the perturbations are due to roundoff errors.

Let us now consider a simple example in the framework of parameter estimation where, in addition to roundoff errors, random errors are involved. Let $b = \{b_i\}_{i=1,\ldots,10}$ be a series of observed values depending on data $s = \{s_i\}$, where $s_i = 10 + i, i = 1, \ldots, 10$. We determine a 3-degree polynomial that approximates $b$ in the least squares sense, and we suppose that the following relationship holds:

$$b = x_1 + x_2\frac{1}{s} + x_3\frac{1}{s^2} + x_4\frac{1}{s^3} \quad \text{with } x_1 = x_2 = x_3 = x_4 = 1.$$

We assume that the perturbation on each $b_i$ is $10^{-8}$ multiplied by a normally distributed random number and denote by $\tilde{b} = \{\tilde{b_i}\}_{i=1,...,10}$ the perturbed quantity. This corresponds to the LLSP $\min_{x\in\mathbb{R}^4} \|Ax - \tilde{b}\|_2$, where $A$ is the Vandermonde matrix defined by $A_{ij} = \frac{1}{s_i^{j-1}}$. Let $\tilde{x}$ and $\tilde{y}$ be the computed solutions corresponding to two perturbed right-hand sides. Then we obtain the following relative errors on each component:

$$\frac{|\tilde{x}_1 - \tilde{y}_1|}{|\tilde{x}_1|} = 2 \cdot 10^{-7}, \frac{|\tilde{x}_2 - \tilde{y}_2|}{|\tilde{x}_2|} = 6 \cdot 10^{-6}, \frac{|\tilde{x}_3 - \tilde{y}_3|}{|\tilde{x}_3|} = 6 \cdot 10^{-5}, \text{ and } \frac{|\tilde{x}_4 - \tilde{y}_4|}{|\tilde{x}_4|} = 10^{-4}.$$

We have $K^{(rel)}(A, b) = 3.1 \cdot 10^5$. Regarding the disparity between the sensitivity of each component, we need a quantity that evaluates more precisely the sensitivity of each solution component of the LLSP.

The idea of analyzing the accuracy of some solution components in linear algebra is by no means new. For linear systems $Ax = b$, $A \in \mathbb{R}^n$ and for LLSP, [3] defines so-called componentwise condition numbers that correspond to amplification factors of the relative errors in solution components due to perturbations of data $A$ or $b$ and explains how to estimate them. In our formalism, these quantities are upper bounds of the condition number of $L^T x$, where $L$ is a column of the identity matrix. We also emphasize that the term "componentwise" refers here to the solution components and must be distinguished from the metric used for matrices and for which [21] provides a condition number for generalized inversion and linear least squares.

For LLSP, [14] provides a statistical estimate for componentwise condition numbers due to either relative or structured perturbations. In the case of linear systems, [2] proposes a statistical approach, based on [13] that enables one to compute the condition number of $L^T x$ in $\mathcal{O}(n^2)$.

Our approach differs from the previous studies in the following aspects:

1. We are interested in the condition of $L^T x$, where $L$ is a general matrix and not only a canonical vector of $\mathbb{R}^n$.
2. We are looking for a condition number based on the F-derivative, and not only for an upper bound of this quantity.

We present in this paper three ways to obtain information on the condition of $L^T x$. The first one uses an explicit formula based on the singular value decomposition (SVD) of $A$. The second is at the same time an upper bound of this condition number and a sharp estimate of it. The third method supplies a statistical estimate. The choice between these three methods will depend on the size of the problem (computational cost) and on the accuracy desired for this quantity.

This paper is organized as follows. In section 2, we define the notion of a partial condition number. Then, when perturbations on $A$ are measured using a Frobenius norm, we give a closed formula for this condition number in the general case where $L \in \mathbb{R}^{n\times k}$ and in the particular case when $L \in \mathbb{R}^n$. In section 3, we establish bounds of the partial condition number in Frobenius as well as in spectral norm, and we show that these bounds can be considered as sharp estimates of it. In section 4 we describe a statistical method that enables us to estimate the partial condition number. In section 5 we present numerical results in order to compare the statistical estimate and the exact condition number on sample matrices $A$ and $L$. In section 6 we give a summary comparing the three ways to compute the condition of $L^T x$ as well as a numerical illustration. Finally some concluding remarks are given in section 7.

Throughout this paper we will use the following notation. We use the Frobenius norm $\|.\|_F$ and the spectral norm $\|.\|_2$ on matrices and the usual Euclidean $\|.\|_2$ on vectors. The matrix $I$ is the identity matrix and $e_i$ is the $i$th canonical vector. We also denote by $\mathrm{Im}(A)$ the space spanned by the columns of $A$ and by $\mathrm{Ker}(A)$ the null space of $A$.

**2. The partial condition number of an LLSP.** Let $L$ be an $n \times k$ matrix, with $k \leq n$. We consider the function

$$\begin{array}{rcll} g : & \mathbb{R}^{m\times n} \times \mathbb{R}^m & \longrightarrow & \mathbb{R}^k, \\ & A, b & \longmapsto & g(A,b) = L^T x(A,b) = L^T(A^TA)^{-1}A^Tb. \end{array} \tag{2.1}$$

Since $A$ has full rank $n$, $g$ is continuously F-differentiable in a neighborhood of $(A,b)$ and we denote by $g'$ its F-derivative. Let $\alpha$ and $\beta$ be two positive real numbers. In the present paper we consider the Euclidean norm for the solution space $\mathbb{R}^k$. For the data space $\mathbb{R}^{m\times n} \times \mathbb{R}^m$, we use the product norms defined by

$$\|(A,b)\|_F = \sqrt{\alpha^2 \|A\|_F^2 + \beta^2 \|b\|_2^2}, \quad \alpha, \beta > 0,$$

and

$$\|(A,b)\|_2 = \sqrt{\alpha^2 \|A\|_2^2 + \beta^2 \|b\|_2^2}, \quad \alpha, \beta > 0.$$

These norms are very flexible since they allow us to monitor the perturbations on $A$ and $b$. For instance, large values of $\alpha$ (resp., $\beta$ ) enable us to obtain condition number problems where mainly $b$ (resp., $A$) are perturbed. A more general weighted Frobenius norm $\|(AT, \beta b)\|_{\mathrm{F}}$, where T is a positive diagonal matrix, is sometimes chosen. This is the case, for instance, in [20], which gives an explicit expression for the condition number of rank deficient linear least squares using this norm.

According to [6], the absolute condition numbers of $g$ at the point $(A,b)$ using the two product norms defined above is given by

$$\kappa_{g,F}(A,b) = \max_{(\Delta A, \Delta b)} \frac{\|g'(A,b).(\Delta A, \Delta b)\|_2}{\|(\Delta A, \Delta b)\|_F}$$

and

$$\kappa_{g,2}(A,b) = \max_{(\Delta A, \Delta b)} \frac{\|g'(A,b).(\Delta A, \Delta b)\|_2}{\|(\Delta A, \Delta b)\|_2}.$$

The corresponding relative condition numbers of $g$ at $(A,b)$ are expressed by

$$\kappa_{g,F}^{(rel)}(A,b) = \frac{\kappa_{g,F}(A,b) \ \|(A,b)\|_F}{\|g(A,b)\|_2}$$

and

$$\kappa_{g,2}^{(rel)}(A,b) = \frac{\kappa_{g,2}(A,b) \ \|(A,b)\|_2}{\|g(A,b)\|_2}.$$

We call the condition numbers related to $L^T x(A,b)$ *partial condition numbers* of the LLSP with respect to the linear operator $L$. The partial condition number defined using the product norm $\|(.,.)\|_F$ is given by the following theorem.

THEOREM 1. *Let $A = U\Sigma V^T$ be the thin singular value decomposition of $A$ defined in* [7] *with $\Sigma = \text{diag}(\sigma_i)$ and $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n > 0$. The absolute condition number of $g(A, b) = L^T x(A, b)$ is given by*

$$\kappa_{g,F}(A, b) = \left\| SV^T L \right\|_2,$$

*where $S \in \mathbb{R}^{n \times n}$ is the diagonal matrix with diagonal elements $S_{ii} = {\sigma_i}^{-1}\sqrt{\frac{{\sigma_i}^{-2}\|r\|_2^2 + \|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}$.*

*Proof.* The demonstration is divided into three parts. In Part 1, we establish an explicit formula of $g'(A, b).(\Delta A, \Delta b)$. In Part 2, we derive an upper bound for $\frac{\|g'(A,b).(\Delta A,\Delta b)\|_2}{\|(\Delta A,\Delta b)\|_F}$. In Part 3, we show that this bound is reached for a particular $(\Delta A, \Delta b)$.

*Part* 1. Let $\Delta A \in \mathbb{R}^{m \times n}$ and $\Delta b \in \mathbb{R}^m$. Using the chain rules of composition of derivatives, we get

$$\begin{aligned} g'(A, b).(\Delta A, \Delta b) = {} & L^T (A^T A)^{-1} \Delta A^T (b - A(A^T A)^{-1} A^T b) \\ & - L^T (A^T A)^{-1} A^T \Delta A (A^T A)^{-1} A^T b + L^T A^\dagger \Delta b, \end{aligned}$$

i.e.,

$$g'(A, b).(\Delta A, \Delta b) = L^T (A^T A)^{-1} \Delta A^T r - L^T A^\dagger \Delta A x + L^T A^\dagger \Delta b. \tag{2.2}$$

We write $\Delta A = \Delta A_1 + \Delta A_2$ by defining $\Delta A_1 = AA^\dagger \Delta A$ (projection of $\Delta A$ onto $\text{Im}(A)$) and $\Delta A_2 = (I - AA^\dagger)\Delta A$ (projection of $\Delta A$ onto $\text{Im}(A)^\perp$). We have $\Delta A_1^T r = 0$ (because $r \in \text{Im}(A)^\perp$) and $A^\dagger \Delta A_2 = 0$. Then we obtain

$$g'(A, b).(\Delta A, \Delta b) = L^T (A^T A)^{-1} \Delta A_2^T r - L^T A^\dagger \Delta A_1 x + L^T A^\dagger \Delta b. \tag{2.3}$$

*Part* 2. We now prove that $\kappa_{g,F}(A, b) \leq \left\| SV^T L \right\|_2$. Let $u_i$ and $v_i$ be the $i$th column of $U$ and $V$, respectively.

From $A^\dagger = V\Sigma^{-1}U^T$, we get $AA^\dagger = UU^T = \sum_{i=1}^n u_i u_i^T$ and since $\sum_{i=1}^n v_i v_i^T = I$, we have $\Delta A_1 = \sum_{i=1}^n u_i u_i^T \Delta A$ and $\Delta A_2 = (I - AA^\dagger)\Delta A \sum_{i=1}^n v_i v_i^T$. Moreover, still using the thin SVD of $A$ and $A^\dagger$, it follows that

$$(A^T A)^{-1} v_i = \frac{v_i}{\sigma_i^2}, \quad A^\dagger u_i = \frac{v_i}{\sigma_i}, \quad \text{and} \quad A^\dagger \Delta b = \sum_{i=1}^n v_i u_i^T \frac{\Delta b}{\sigma_i}. \tag{2.4}$$

Thus (2.3) becomes

$$\begin{aligned} g'(A, b).(\Delta A, \Delta b) &= \sum_{i=1}^n L^T v_i \left[ v_i^T \Delta A^T (I - AA^\dagger) \frac{r}{\sigma_i^2} - u_i^T \Delta A \frac{x}{\sigma_i} + u_i^T \frac{\Delta b}{\sigma_i} \right] \\ &= L^T \sum_{i=1}^n v_i y_i, \end{aligned}$$

where we set $y_i = v_i^T \Delta A^T (I - AA^\dagger) \frac{r}{\sigma_i^2} - u_i^T \Delta A \frac{x}{\sigma_i} + u_i^T \frac{\Delta b}{\sigma_i} \in \mathbb{R}$.

Thus if $Y = (y_1, y_2, \ldots, y_n)^T$, we get $\|g'(A, b).(\Delta A, \Delta b)\|_2 = \left\| L^T V Y \right\|_2$ and then

$$\|g'(A, b).(\Delta A, \Delta b)\|_2 = \left\| L^T V S S^{-1} Y \right\|_2 \leq \left\| SV^T L \right\|_2 \left\| S^{-1} Y \right\|_2.$$

We denote by $w_i = \frac{v_i^T \Delta A^T (I - AA^\dagger) r}{S_{ii}\sigma_i^2} - \frac{u_i^T \Delta A x}{S_{ii}\sigma_i} + \frac{u_i^T \Delta b}{S_{ii}\sigma_i}$ the $i$th component of $S^{-1}Y$. Then we have

$$
\begin{aligned}
|w_i| &\le \alpha \left\| v_i^T \Delta A^T (I - AA^\dagger)^T \right\|_2 \frac{\|r\|_2}{\alpha S_{ii}\sigma_i^2} + \alpha \left\| u_i^T \Delta A \right\|_2 \frac{\|x\|_2}{\alpha S_{ii}\sigma_i} + \beta |u_i^T \Delta b| \frac{1}{\beta S_{ii}\sigma_i} \\
&\le \left( \frac{\|r\|_2^2}{\alpha^2 S_{ii}^2 \sigma_i^4} + \frac{\|x\|_2^2}{\alpha^2 S_{ii}^2 \sigma_i^2} + \frac{1}{\beta^2 S_{ii}^2 \sigma_i^2} \right)^{\frac{1}{2}} \\
&\quad \times (\alpha^2 \left\| (I - AA^\dagger)\Delta A v_i \right\|_2^2 + \alpha^2 \left\| u_i^T \Delta A \right\|_2^2 + \beta^2 |u_i^T \Delta b|^2)^{\frac{1}{2}} \\
&= \frac{S_{ii}}{S_{ii}} (\alpha^2 \left\| (I - AA^\dagger)\Delta A v_i \right\|_2^2 + \alpha^2 \left\| u_i^T \Delta A \right\|_2^2 + \beta^2 |u_i^T \Delta b|^2)^{\frac{1}{2}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left\| S^{-1} Y \right\|_2^2 &\le \sum_{i=1}^n \alpha^2 \left\| (I - AA^\dagger)\Delta A v_i \right\|_2^2 + \alpha^2 \left\| u_i^T \Delta A \right\|_2^2 + \beta^2 |u_i^T \Delta b|^2 \\
&= \alpha^2 \left\| (I - AA^\dagger)\Delta A V \right\|_F^2 + \alpha^2 \left\| U^T \Delta A \right\|_F^2 + \beta^2 \left\| U^T \Delta b \right\|_2^2 \\
&= \alpha^2 \left\| (I - AA^\dagger)\Delta A \right\|_F^2 + \alpha^2 \left\| U^T \Delta A \right\|_F^2 + \beta^2 \left\| U^T \Delta b \right\|_2^2 .
\end{aligned}
$$

Since $\left\| U^T \Delta A \right\|_F = \left\| U U^T \Delta A \right\|_F = \left\| AA^\dagger \Delta A \right\|_F$ and $\left\| U^T \Delta b \right\|_2 = \left\| U U^T \Delta b \right\|_2 \le \|\Delta b\|_2$, we get

$$
\left\| S^{-1} Y \right\|_2^2 \le \alpha^2 \|\Delta A_1\|_F^2 + \alpha^2 \|\Delta A_2\|_F^2 + \beta^2 \|\Delta b\|_2^2 .
$$

From $\|\Delta A\|_F^2 = \|\Delta A_1\|_F^2 + \|\Delta A_2\|_F^2$, we get $\left\| S^{-1} Y \right\|_2^2 \le \|(\Delta A, \Delta b)\|_F^2$ and thus

$$
\|g'(A,b).(\Delta A, \Delta b)\|_2 \le \left\| S V^T L \right\|_2 \|(\Delta A, \Delta b)\|_F .
$$

So we have shown that $\left\| S V^T L \right\|_2$ is an upper bound for $\kappa_{g,F}(A,b)$.

*Part* 3. We now prove that this upper bound can be reached, i.e., that $\left\| S V^T L \right\|_2 = \frac{\left\| g'(A,b).(\Delta A, \Delta b) \right\|_2}{\|(\Delta A, \Delta b)\|_F}$ holds for some $(\Delta A, \Delta b) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$.

Let us consider the particular choice of $(\Delta A, \Delta b)$ defined by

$$
(\Delta A, \Delta b) = (\Delta A_2 + \Delta A_1, \Delta b) = \left( \sum_{i=1}^n \frac{\alpha_i}{\alpha} \frac{r}{\|r\|_2} v_i^T + \sum_{i=1}^n \frac{\beta_i}{\alpha} u_i \frac{x^T}{\|x\|_2}, \sum_{i=1}^n \frac{\gamma_i}{\beta} u_i \right),
$$

where $\alpha_i$, $\beta_i$, $\gamma_i$ are real constants to be chosen in order to achieve the upper bound obtained in Part 2.

Since $\Delta A_1^T r = 0$ and $A^\dagger \Delta A_2 = 0$, it follows from (2.3) and (2.4) that

$$
\begin{aligned}
g'(A,b).(\Delta A, \Delta b) &= L^T (A^T A)^{-1} \sum_{i=1}^n \frac{\alpha_i}{\alpha} \|r\|_2 v_i^T - L^T A^\dagger \sum_{i=1}^n \frac{\beta_i}{\alpha} u_i \|x\|_2 \\
&\quad + L^T A^\dagger \sum_{i=1}^n \frac{\gamma_i}{\beta} u_i \\
&= L^T \sum_{i=1}^n \frac{\alpha_i}{\alpha \sigma_i^2} v_i \|r\|_2 - L^T \sum_{i=1}^n \frac{\beta_i}{\alpha \sigma_i} v_i \|x\|_2 + L^T \sum_{i=1}^n \frac{\gamma_i}{\beta \sigma_i} v_i \\
&= \sum_{i=1}^n L^T v_i \left( \frac{\alpha_i}{\alpha \sigma_i^2} \|r\|_2 - \frac{\beta_i}{\alpha \sigma_i} \|x\|_2 + \frac{\gamma_i}{\beta \sigma_i} \right).
\end{aligned}
$$

Thus by denoting $\xi_i = [L^T v_i \frac{\|r\|_2}{\alpha \sigma_i^2}, -L^T v_i \frac{\|x\|_2}{\alpha \sigma_i}, \frac{L^T v_i}{\beta \sigma_i}] \in \mathbb{R}^{k \times 3}$, $\Gamma = [\xi_1, \ldots, \xi_n] \in \mathbb{R}^{k \times 3n}$, and $X = (\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_n, \beta_n, \gamma_n)^T \in \mathbb{R}^{3n \times 1}$ we get

$$
g'(A,b).(\Delta A, \Delta b) = \Gamma X. \tag{2.5}
$$

Since $\forall i, j$ $\mathrm{trace}((\frac{r}{\|r\|_2} v_i^T)^T (\frac{r}{\|r\|_2} v_i^T)) = \mathrm{trace}((u_i \frac{x^T}{\|x\|_2})^T (u_i \frac{x^T}{\|x\|_2})) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol and $\mathrm{trace}((\frac{r}{\|r\|_2} v_i^T)^T (u_i \frac{x^T}{\|x\|_2})) = 0$, then $\{\frac{r}{\|r\|_2} v_i^T\}_{i=1,\ldots,n}$ and $\{u_i \frac{x^T}{\|x\|_2}\}_{i=1,\ldots,n}$ form an orthonormal set of matrices for the Frobenius norm and we get $\|\Delta A\|_F = \sum_{i=1}^n (\alpha_i^2 + \beta_i^2)$. It follows that

$$
\|(\Delta A, \Delta b)\|_F^2 = \sum_{i=1}^n \alpha_i^2 + \sum_{i=1}^n \beta_i^2 + \sum_{i=1}^n \gamma_i^2 = \|X\|_2^2,
$$

and (2.5) yields

$$
\frac{\|g'(A,b).(\Delta A, \Delta b)\|_2}{\|(\Delta A, \Delta b)\|_F} = \frac{\|\Gamma X\|_2}{\|X\|_2}.
$$

We know that $\|\Gamma\|_2 = \max_X \frac{\|\Gamma X\|_2}{\|X\|_2}$ is reached for some $X = (\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_n, \beta_n, \gamma_n)^T$. Then for the $(\Delta A, \Delta b)$ corresponding to this $X$, we have $\frac{\|g'(A,b).(\Delta A, \Delta b)\|_2}{\|(\Delta A, \Delta b)\|_F} = \|\Gamma\|_2$.

Furthermore we have

$$
\begin{aligned}
\Gamma \Gamma^T &= L^T v_1 \left( \frac{\|r\|_2^2}{\alpha^2 \sigma_1^4} + \frac{\|x\|_2^2}{\alpha^2 \sigma_1^2} + \frac{1}{\beta^2 \sigma_1^2} \right) v_1^T L + \cdots + L^T v_n \left( \frac{\|r\|_2^2}{\alpha^2 \sigma_n^4} + \frac{\|x\|_2^2}{\alpha^2 \sigma_n^2} + \frac{1}{\beta^2 \sigma_n^2} \right) v_n^T L \\
&= L^T v_1 S_{11}^2 v_1^T L + \cdots + L^T v_n S_{nn}^2 v_n^T L \\
&= (L^T V S)(S V^T L).
\end{aligned}
$$

Hence

$$
\|\Gamma\|_2 = \sqrt{\|\Gamma \Gamma^T\|_2} = \|S V^T L\|_2
$$

and $\alpha_1, \beta_1, \gamma_1, \ldots, \alpha_n, \beta_n, \gamma_n$ are such that $\frac{\left\| g'(A,b).(\Delta A, \Delta b) \right\|_2}{\|(\Delta A, \Delta b)\|_F} = \left\| S V^T L \right\|_2$.

Thus $\left\| S V^T L \right\|_2 \leq \kappa_{g,F}(A,b)$, which concludes the proof. □

*Remark* 1. Let $l_j$ be the $j$th column of $L$, $j = 1, \ldots, k$. From

$$S V^T L = \begin{pmatrix} S_{11} {v_1}^T \\ \vdots \\ S_{nn} {v_n}^T \end{pmatrix} (l_1, \ldots, l_k) = \begin{pmatrix} S_{11} {v_1}^T l_1 & \cdots & S_{11} {v_1}^T l_k \\ \vdots & & \vdots \\ S_{nn} {v_n}^T l_1 & \cdots & S_{nn} {v_n}^T l_k \end{pmatrix},$$

it follows that $\left\| S V^T L \right\|_2$ is large when there exist at least one large $S_{ii}$ and an $l_j$ such that ${v_i}^T l_j \neq 0$. In particular, the condition number of $L^T x(A,b)$ is large when $A$ has small singular values and $L$ has components in the corresponding right singular vectors or when $\|r\|_2$ is large.

*Remark* 2. In the general case where $L$ is an $n \times k$ matrix, the computation of $\kappa_{g,F}(A,b)$ via the exact formula given in Theorem 1 requires the computation of the singular values and the right singular vectors of $A$, which might be expensive in practice since it involves $2mn^2$ operations if we use an R-SVD algorithm and if $m \gg n$ (see [7, p. 254]). If the LLSP is solved using a direct method, the $R$ factor of the QR decomposition of $A$ (or equivalently, in exact arithmetic, the Cholesky factor of $A^T A$) might be available. Since the right singular vectors of $A$ are also those of $R$, the condition number can be computed in about $12n^3$ flops (using the Golub–Reinsch SVD [7, p. 254]).

Using $R$ is even more interesting when $L \in \mathbb{R}^n$, since from

$$\left\| L^T A^\dagger \right\|_2 = \left\| R^{-T} L \right\|_2 \text{ and } \left\| L^T (A^T A)^{-1} \right\|_2 = \left\| R^{-1} (R^{-T} L) \right\|_2, \tag{2.6}$$

it follows that the computation of $\kappa_{g,F}(A,b)$ can be done by solving two successive $n \times n$ triangular systems which involve about $2n^2$ flops.

**2.1. Special cases and GSVD.** In this section, we analyze some special cases of practical relevance. Moreover, we relate the formula given in Theorem 1 for

$$\kappa_{g,F}(A,b)$$

to the generalized singular value decomposition (GSVD) (see [1, p. 157], [7, p. 466], and [15, 19]). Using the GSVD of $A$ and $L^T$, there exist $U_A \in \mathbb{R}^{m \times m}, U_L \in \mathbb{R}^{k \times k}$ orthogonal matrices and $Z \in \mathbb{R}^{n \times n}$ invertible such that

$$U_A^T A = \begin{pmatrix} D_A \\ 0 \end{pmatrix} Z \quad \text{and} \quad U_L^T L^T = \begin{pmatrix} D_L & 0 \end{pmatrix} Z$$

with

$$D_A = \operatorname{diag}(\alpha_1, \ldots, \alpha_n), \qquad D_L = \operatorname{diag}(\beta_1, \ldots, \beta_k),$$

$$\alpha_i^2 + \beta_i^2 = 1, \quad i = 1, \ldots, k, \qquad \alpha_i = 1, \quad i = k+1, \ldots, n.$$

The diagonal matrix $S$ can be decomposed in the product of two diagonal matrices

$$S = \Sigma^{-1} D$$

with

$$D_{ii} = \sqrt{\frac{{\sigma_i}^{-2} \|r\|_2^2 + \|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

Then, taking into account the relations

$$\left\|SV^TL\right\|_2 = \left\|L^TVS\right\|_2 = \left\|L^TV\Sigma^{-1}U^TUD\right\|_2 = \left\|L^TA^\dagger UD\right\|_2,$$

$$L^TA^\dagger = U_L \begin{pmatrix} D_L & 0 \end{pmatrix} ZZ^{-1} \begin{pmatrix} D_A^{-1} & 0 \end{pmatrix} U_A^T,$$

we can represent $\kappa_{g,F}(A,b)$ as

$$\kappa_{g,F}(A,b) = \left\|T\widetilde{H}D\right\|_2,$$

where $T \in \mathbb{R}^{k\times k}$ is a diagonal matrix with $T_{ii} = \beta_i/\alpha_i$, $i = 1,\dots,k$, and $\widetilde{H} \in \mathbb{R}^{k\times n}$ is

$$\widetilde{H} = \begin{pmatrix} I & 0 \end{pmatrix} U_A^T U.$$

Note that $\left\|L^TA^\dagger\right\|_2 = \|T\|_2$.

We also point out that the diagonal entries of $T$ are the nonzero generalized eigenvalues of

$$\lambda A^TAz = LL^Tz.$$

There are two interesting special cases where the expression of $\kappa_{g,F}(A,b)$ is simpler.

First, when $r = 0$, i.e., the LLSP problem is consistent, we have

$$D = \sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}\ I$$

and

$$\kappa_{g,F}(A,b) = \left\|T\widetilde{H}\right\|_2 \sqrt{\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

Second, if we allow only perturbations on $b$ and if we use the expression (2.2) of the derivative of $g(A,b)$, we get

$$\kappa_{g,F}(A,b) = \frac{\left\|L^TA^\dagger\right\|_2}{\beta} = \frac{\|T\|_2}{\beta}$$

(see Remark 4 in section 3).

Other relevant cases where the expression for $\kappa_{g,F}(A,b)$ has a special interest are $L = I$ and $L$ is a column vector.

In the special case where $L = I$, the formula given by Theorem 1 becomes

$$\kappa_{g,F}(A,b) = \left\|SV^TL\right\|_2 = \|S\|_2 = \max_i S_{ii} = {\sigma_n}^{-1}\sqrt{\frac{{\sigma_n}^{-2}\|r\|_2^2 + \|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

Since $\left\|A^\dagger\right\|_2 = {\sigma_n}^{-1}$, we obtain that

$$\kappa_{g,F}(A,b) = \left\|A^\dagger\right\|_2 \sqrt{\frac{\|A^\dagger\|_2^2\|r\|_2^2 + \|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

This corresponds to the result known from [8] and also to a generalization of the formula of the condition number in the Frobenius norm given in [6, p. 92] (where only $A$ was perturbed).

Finally, let us study the particular case where $L$ is a column vector, i.e., when $g$ is a scalar derived function.

COROLLARY 1. *In the particular case when L is a vector ($L \in \mathbb{R}^n$), the absolute condition number of $g(A,b) = L^T x(A,b)$ is given by*

$$\kappa_{g,F}(A,b) = \left( \left\| L^T (A^T A)^{-1} \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| L^T A^\dagger \right\|_2^2 \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \right)^{\frac{1}{2}}.$$

*Proof.* By replacing $(A^T A)^{-1} = V \Sigma^{-2} V^T$ and $A^\dagger = V \Sigma^{-1} U^T$ in the expression of $K = (\left\| L^T (A^T A)^{-1} \right\|_2^2 \|r\|_2^2 + \left\| L^T A^\dagger \right\|_2^2 (\|x\|_2^2 + 1))^{\frac{1}{2}}$ we get

$$\begin{aligned} K^2 &= \left\| L^T V \Sigma^{-2} V^T \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| L^T V \Sigma^{-1} U^T \right\|_2^2 \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \\ &= \left\| L^T V \Sigma^{-2} \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| L^T V \Sigma^{-1} \right\|_2^2 \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \\ &= \left\| \Sigma^{-2} V^T L \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| \Sigma^{-1} V^T L \right\|_2^2 \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right). \end{aligned}$$

By writing $z = V^T L$, where $z = (z_1, \ldots, z_n)^T \in \mathbb{R}^n$, we obtain

$$\begin{aligned} K^2 &= \sum_{i=1}^n \frac{z_i^2}{\sigma_i^4} \frac{\|r\|_2^2}{\alpha^2} + \sum_{i=1}^n \frac{z_i^2}{\sigma_i^2} \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \\ &= \sum_{i=1}^n \frac{z_i^2}{\sigma_i^2} \left( \frac{\sigma_i^{-2} \|r\|_2^2 + \|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \\ &= \sum_{i=1}^n S_{ii}^2 z_i^2 \\ &= \left\| S V^T L \right\|_2^2, \end{aligned}$$

and Theorem 1 gives the result. ∎

**3. Sharp estimate of the partial condition number in Frobenius and spectral norms.** In many cases, obtaining a lower and/or an upper bound of $\kappa_{g,F}(A,b)$ is satisfactory when these bounds are tight enough and significantly cheaper to compute than the exact formula. Moreover, many applications use condition numbers expressed in the spectral norm. In the following theorem, we give sharp bounds for the partial condition numbers in the Frobenius and spectral norms.

THEOREM 2. *The absolute condition numbers of $g(A,b) = L^T x(A,b)$ ($L \in \mathbb{R}^{n \times k}$) in the Frobenius and spectral norms can be bounded, respectively, as follows:*

$$\frac{f(A,b)}{\sqrt{3}} \le \kappa_{g,F}(A,b) \le f(A,b),$$

$$\frac{f(A,b)}{\sqrt{3}} \leq \kappa_{g,2}(A,b) \leq \sqrt{2} f(A,b),$$

*where*

$$f(A,b) = \left( \left\| L^T (A^T A)^{-1} \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| L^T A^\dagger \right\|_2^2 \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \right)^{\frac{1}{2}}.$$

*Proof.* *Part* 1. We start by establishing the lower bounds. Let $w_1$ and $w_1'$ (resp., $a_1$ and $a_1'$) be the right (resp., the left) singular vectors corresponding to the largest singular values of $L^T(A^TA)^{-1}$ and $L^TA^\dagger$, respectively. We use a particular perturbation $(\Delta A, \Delta b)$ expressed as

$$(\Delta A, \Delta b) = \left( \frac{r}{\alpha \|r\|_2} w_1^T + \epsilon w_1' \frac{x^T}{\alpha \|x\|_2}, -\epsilon \frac{w_1'}{\beta} \right),$$

where $\epsilon = \pm 1$.

By replacing this value of $(\Delta A, \Delta b)$ in (2.2) we get

$$g'(A,b).(\Delta A, \Delta b) = \frac{\|r\|_2}{\alpha} L^T (A^T A)^{-1} w_1 + \frac{\epsilon}{\alpha \|x\|_2} L^T (A^T A)^{-1} x {w_1'}^T r$$
$$- L^T A^\dagger r \frac{w_1^T x}{\alpha \|r\|_2} - \frac{\epsilon \|x\|_2}{\alpha} L^T A^\dagger w_1' - \frac{\epsilon}{\beta} L^T A^\dagger w_1'.$$

Since $r \in \mathrm{Im}(A)^\perp$ we have $A^\dagger r = 0$. Moreover we have $w_1' \in \mathrm{Ker}(L^T A^\dagger)^\perp$ and thus $w_1' \in \mathrm{Im}(A^{\dagger T} L)$, which can be written $w_1' = A^{\dagger T} L \delta$ for some $\delta \in \mathbb{R}^k$. Then ${w_1'}^T r = \delta^T L^T A^\dagger r = 0$. It follows that

$$g'(A,b).(\Delta A, \Delta b) = \frac{\|r\|_2}{\alpha} L^T (A^T A)^{-1} w_1 - \frac{\epsilon \|x\|_2}{\alpha} L^T A^\dagger w_1' - \frac{\epsilon}{\beta} L^T A^\dagger w_1'.$$

From $L^T (A^T A)^{-1} w_1 = \left\| L^T (A^T A)^{-1} \right\|_2 a_1$ and $L^T A^\dagger w_1' = \left\| L^T A^\dagger \right\|_2 a_1'$, we obtain

$$g'(A,b).(\Delta A, \Delta b) = \left\| L^T (A^T A)^{-1} \right\|_2 \frac{\|r\|_2}{\alpha} a_1 - \epsilon \left( \frac{\|x\|_2}{\alpha} + \frac{1}{\beta} \right) \left\| L^T A^\dagger \right\|_2 a_1'.$$

Since $a_1$ and $a_1'$ are unit vectors, $\|g'(A,b).(\Delta A, \Delta b)\|_2$ can be developed as

$$\|g'(A,b).(\Delta A, \Delta b)\|_2^2 = \left\| L^T (A^T A)^{-1} \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| L^T A^\dagger \right\|_2^2 \left( \frac{\|x\|_2}{\alpha} + \frac{1}{\beta} \right)^2$$
$$- 2\epsilon \left\| L^T (A^T A)^{-1} \right\|_2 \frac{\|r\|_2}{\alpha} \left( \frac{\|x\|_2}{\alpha} + \frac{1}{\beta} \right) \left\| L^T A^\dagger \right\|_2 \cos(a_1, a_1').$$

By choosing $\epsilon = -sign(\cos(a_1, a_1'))$ the third term of the above expression becomes positive. Furthermore we have $(\frac{\|x\|_2}{\alpha} + \frac{1}{\beta})^2 \geq \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}$. Then we obtain

$$\|g'(A,b).(\Delta A, \Delta b)\|_2 \geq \left( \left\| L^T (A^T A)^{-1} \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| L^T A^\dagger \right\|_2^2 \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \right)^{\frac{1}{2}},$$

i.e.,

$$\|g'(A,b).(\Delta A,\Delta b)\|_2 \geq f(A,b).$$

On the other hand, we have

$$\|\Delta A\|_F^2 = \left\|\frac{r}{\alpha\,\|r\|_2} w_1^T\right\|_F^2 + \left\|w_1' \frac{x^T}{\alpha\,\|x\|_2}\right\|_F^2 + 2\;\epsilon\;\text{trace}\left(\left(\frac{r}{\alpha\,\|r\|_2} w_1^T\right)^T \left(w_1' \frac{x^T}{\alpha\,\|x\|_2}\right)\right)$$

and

$$\left\|\frac{w_1'}{\beta}\right\|_2^2 = \frac{1}{\beta^2}$$

with

$$\left\|\frac{r}{\alpha\,\|r\|_2} w_1^T\right\|_F^2 = \left\|w_1' \frac{x^T}{\alpha\,\|x\|_2}\right\|_F^2 = \frac{1}{\alpha^2}, \qquad \text{trace}\left(\left(\frac{r}{\alpha\,\|r\|_2} w_1^T\right)^T \left(w_1' \frac{x^T}{\alpha\,\|x\|_2}\right)\right) = 0.$$

Then $\|(\Delta A,\Delta b)\|_F = \sqrt{3}$ and thus we have $\frac{\|g'(A,b).(\Delta A,\Delta b)\|_2}{\|(\Delta A,\Delta b)\|_F} \geq \frac{f(A,b)}{\sqrt{3}}$ for a particular value of $(\Delta A,\Delta b)$. Furthermore, from $\|(\Delta A,\Delta b)\|_2 \leq \|(\Delta A,\Delta b)\|_F$ we get $\frac{\|g'(A,b).(\Delta A,\Delta b)\|_2}{\|(\Delta A,\Delta b)\|_2} \geq \frac{f(A,b)}{\sqrt{3}}$ (for the same particular value of $(\Delta A,\Delta b)$). Then we obtain $\kappa_{g,F}(A,b) \geq \frac{f(A,b)}{\sqrt{3}}$ and $\kappa_{g,2}(A,b) \geq \frac{f(A,b)}{\sqrt{3}}$.

*Part* 2. Let us now establish the upper bound for $\kappa_{g,F}(A,b)$ and $\kappa_{g,2}(A,b)$.

If $\Delta A_1 = AA^\dagger \Delta A$ and $\Delta A_2 = (I - AA^\dagger)\Delta A$, then it comes from (2.3) that $\forall(\Delta A,\Delta b) \in \mathbb{R}^{m\times n} \times \mathbb{R}^m$

$$\begin{aligned}\|g'(A,b).(\Delta A,\Delta b)\|_2 &\leq \left\|L^T(A^TA)^{-1}\right\|_2 \|\Delta A_2\|_2 \|r\|_2 \\ &\quad + \left\|L^TA^\dagger\right\|_2 \|\Delta A_1\|_2 \|x\|_2 + \left\|L^TA^\dagger\right\|_2 \|\Delta b\|_2 \\ &= YX,\end{aligned}$$

where

$$Y = \left(\frac{\left\|L^T(A^TA)^{-1}\right\|_2 \|r\|_2}{\alpha}, \frac{\left\|L^TA^\dagger\right\|_2 \|x\|_2}{\alpha}, \frac{\left\|L^TA^\dagger\right\|_2}{\beta}\right)$$

and

$$X = (\alpha\,\|\Delta A_2\|_2\,, \alpha\,\|\Delta A_1\|_2\,, \beta\,\|\Delta b\|_2)^T.$$

Hence, from the Cauchy–Schwarz inequality we get

$$\|g'(A,b).(\Delta A,\Delta b)\|_2 \leq \|Y\|_2\,\|X\|_2\,, \tag{3.1}$$

with

$$\|X\|_2^2 = \alpha^2\,\|\Delta A_1\|_2^2 + \alpha^2\,\|\Delta A_2\|_2^2 + \beta^2\,\|\Delta b\|_2^2 \leq \alpha^2\,\|\Delta A_1\|_F^2 + \alpha^2\,\|\Delta A_2\|_F^2 + \beta^2\,\|\Delta b\|_2^2$$

and

$$\|Y\|_2 = f(A,b).$$

Then, since $\|\Delta A\|_F^2 = \|\Delta A_1\|_F^2 + \|\Delta A_2\|_F^2$, we have $\|X\|_2 \leq \|(\Delta A, \Delta b)\|_F$ and (3.1) yields

$$\|g'(A,b).(\Delta A, \Delta b)\|_2 \leq \|(\Delta A, \Delta b)\|_F \, \|Y\|_2 \, ,$$

which implies that

$$\kappa_{g,F}(A,b) \leq f(A,b).$$

An upper bound of $\kappa_{g,2}(A,b)$ can be computed in a similar manner: we get from (2.2) that

$$\begin{aligned}\|g'(A,b).(\Delta A, \Delta b)\|_2 &\leq (\left\|L^T(A^TA)^{-1}\right\|_2 \|r\|_2 + \left\|L^T A^\dagger\right\|_2 \|x\|_2) \, \|\Delta A\|_2 \\ &\quad + \left\|L^T A^\dagger\right\|_2 \|\Delta b\|_2 \\ &= Y'X',\end{aligned}$$

where

$$Y' = \left( \frac{\left\|L^T(A^TA)^{-1}\right\|_2 \|r\|_2 + \left\|L^T A^\dagger\right\|_2 \|x\|_2}{\alpha}, \frac{\left\|L^T A^\dagger\right\|_2}{\beta} \right)$$

and

$$X' = (\alpha \, \|\Delta A\|_2 \, , \beta \, \|\Delta b\|_2)^T.$$

Since $\|X'\|_2 = \|(\Delta A, \Delta b)\|_2$ we have $\kappa_{g,2}(A,b) \leq \|Y'\|_2$. Then using the inequality

$$\left(\left\|L^T(A^TA)^{-1}\right\|_2 \|r\|_2 + \left\|L^T A^\dagger\right\|_2 \|x\|_2\right)^2 \leq 2 \left( \left\|L^T(A^TA)^{-1}\right\|_2^2 \|r\|_2^2 + \left\|L^T A^\dagger\right\|_2^2 \|x\|_2^2 \right)$$

we get $\|Y'\|_2 \leq \sqrt{2} \, \|Y\|_2$ and finally obtain $\kappa_{g,2}(A,b) \leq \sqrt{2} f(A,b)$, which concludes the proof. □

Theorem 2 shows that $f(A,b)$ can be considered as a very sharp estimate of the partial condition number expressed either in Frobenius or spectral norm. Indeed, it lies within a factor $\sqrt{3}$ of $\kappa_{g,F}(A,b)$ or $\kappa_{g,2}(A,b)$.

Another observation is that we have

$$\frac{1}{\sqrt{6}} \leq \frac{\kappa_{g,F}(A,b)}{\kappa_{g,2}(A,b)} \leq \sqrt{3}.$$

Thus even if the Frobenius and spectral norms of a given matrix can be very different (for $X \in \mathbb{R}^{m \times n}$, we have $\|X\|_2 \leq \|X\|_F \leq \sqrt{n} \, \|X\|_2$), the condition numbers expressed in both norms are of the same order. The result is that a good estimate of $\kappa_{g,F}(A,b)$ is also a good estimate of $\kappa_{g,2}(A,b)$.

Moreover (2.6) shows that if the $R$ factor of $A$ is available, $f(A,b)$ can be computed by solving two $n \times n$ triangular systems with $k$ right-hand sides and thus the computational cost is $2kn^2$.

*Remark* 3. We can check in the following example that $\kappa_{g,F}(A,b)$ is not equal to $f(A,b)$. Let us consider

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \; L = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \; \text{and} \;\; b = \begin{pmatrix} 2/\sqrt{2} \\ 1/\sqrt{2} \\ 1 \end{pmatrix}.$$

We have

$$x = (1/\sqrt{2}, 1/\sqrt{2})^T \quad \text{and} \quad \|x\|_2 = \|r\|_2 = 1,$$

and we get

$$\kappa_{g,F}(A,b) = \frac{\sqrt{45}}{4} \; < \; f(A,b) = \frac{\sqrt{13}}{2}.$$

*Remark* 4. Using the definition of the condition number and of the product norms, we can obtain tight estimates for the partial condition number for perturbations of $A$ only (resp., $b$ only) by taking $\alpha > 0$ and $\beta = +\infty$ (resp., $\beta > 0$ and $\alpha = +\infty$) in Theorem 2. In particular, when we perturb only $b$ we have, with the notation of section 2.1,

$$f(A,b) = \frac{\left\|L^T A^\dagger\right\|_2}{\beta} = \frac{\|T\|_2}{\beta} = \kappa_{g,F}(A,b).$$

Moreover, when $r = 0$ we have

$$f(A,b) = \left\|L^T A^\dagger\right\|_2 \left(\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}\right)^{\frac{1}{2}} = \|T\|_2 \left(\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}\right)^{\frac{1}{2}}.$$

*Remark* 5. In the special case where $L = I$, we have

$$f(A,b) = \left(\left\|(A^TA)^{-1}\right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\|A^\dagger\right\|_2^2 \left(\frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}\right)\right)^{\frac{1}{2}}.$$

Since $\left\|(A^TA)^{-1}\right\|_2 = \left\|A^\dagger\right\|_2^2$ we obtain that

$$f(A,b) = \left\|A^\dagger\right\|_2 \sqrt{\frac{\|A^\dagger\|_2^2 \, \|r\|_2^2 + \|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2}}.$$

In that case $\kappa_{g,F}(A,b)$ is exactly equal to $f(A,b)$ due to [8].

Regarding the condition number in the spectral norm, since we have $\|(\Delta A, \Delta b)\|_2 \leq \|(\Delta A, \Delta b)\|_F$ we get $\kappa_{g,2}(A,b) \geq f(A,b)$. This lower bound is similar to that obtained in [6] (where only $A$ is perturbed). As mentioned in [6], an upper bound of $\kappa_{g,2}(A)$ is $\kappa_{g,2}^u(A) = \left\|A^\dagger\right\|_2^2 \|r\|_2 + \left\|A^\dagger\right\|_2 \|x\|_2$. If we take $\alpha = 1$ and $\beta = +\infty$, we notice that $f(A,b) \leq \kappa_{g,2}^u(A) \leq \sqrt{2} f(A,b)$, showing thus that our upper bound and $\kappa_{g,2}^u(A)$ are essentially the same.

*Remark* 6. Generalization to other product norms:

Other product norms may have been used for the data space $\mathbb{R}^{m\times n} \times \mathbb{R}^m$.

If we consider a norm $\nu$ on $\mathbb{R}^2$ such that $c_1\nu(x,y) \leq \sqrt{x^2+y^2} \leq c_2\nu(x,y)$, then we can define a product norm $\|(A,b)\|_{F,\nu} = \nu(\alpha \|\Delta A\|_F, \beta \|\Delta b\|_2)$. For instance, in [9], $\nu$ corresponds to $\|.\|_\infty$. Note that the product norm $\|(.,.)\|_F$ used throughout this paper corresponds to $\nu = \|.\|_2$ and that with the above notation we have $\|(A,b)\|_{F,2} = \|(A,b)\|_F$. Then the following inequality holds:

$$c_1 \|(\Delta A, \Delta b)\|_{F,\nu} \leq \|(\Delta A, \Delta b)\|_F \leq c_2 \|(\Delta A, \Delta b)\|_{F,\nu}.$$

If we denote $\kappa_{g,F,\nu}(A,b) = \max_{(\Delta A,\Delta b)} \frac{\left\| g'(A,b).(\Delta A,\Delta b) \right\|_2}{\|(\Delta A,\Delta b)\|_{F,\nu}}$, we obtain

$$\frac{\kappa_{g,F,\nu}(A,b)}{c_2} \leq \kappa_{g,F}(A,b) \leq \frac{\kappa_{g,F,\nu}(A,b)}{c_1}.$$

Using the bounds for $\kappa_{g,F}$ given in Theorem 2 we can obtain tight bounds for the partial condition number expressed using the product norm based on $\nu$ and when the perturbations on matrices are measured with the Frobenius norm:

$$\frac{c_1}{\sqrt{3}} f(A,b) \leq \kappa_{g,F,\nu}(A,b) \leq c_2 f(A,b).$$

Similarly, if the perturbations on matrices are measured with the spectral norm, we get

$$\frac{c_1}{\sqrt{3}} f(A,b) \leq \kappa_{g,F,\nu}(A,b) \leq c_2\sqrt{2} f(A,b).$$

The bounds obtained for three possible product norms ($\nu = \|.\|_\infty$, $\nu = \|.\|_2$, and $\nu = \|.\|_1$) are given in Table 3.1 when using the Frobenius norm for matrices and in Table 3.2 when using the spectral norm for matrices.

TABLE 3.1
*Bounds for partial condition number (Frobenius norm on matrices).*

| Product norm | $\nu$, $c_1$, $c_2$ | Lower bound (factor of $f(A,b)$) | Upper bound (factor of $f(A,b)$) |
|---|---|---|---|
| $\max\{\alpha \|\Delta A\|_F, \beta \|\Delta b\|_2\}$ | $\|.\|_\infty$, $\frac{1}{\sqrt{2}}$, 1 | $\frac{1}{\sqrt{6}}$ | 1 |
| $\sqrt{\alpha^2 \|\Delta A\|_F^2 + \beta^2 \|\Delta b\|_2^2}$ | $\|.\|_2$, 1, 1 | $\frac{1}{\sqrt{3}}$ | 1 |
| $\alpha \|\Delta A\|_F + \beta \|\Delta b\|_2$ | $\|.\|_1$, 1, $\sqrt{2}$ | $\frac{1}{\sqrt{3}}$ | $\sqrt{2}$ |

TABLE 3.2
*Bounds for partial condition number (spectral norm on matrices).*

| Product norm | $\nu$, $c_1$, $c_2$ | Lower bound (factor of $f(A,b)$) | Upper bound (factor of $f(A,b)$) |
|---|---|---|---|
| $\max\{\alpha \|\Delta A\|_2, \beta \|\Delta b\|_2\}$ | $\|.\|_\infty$, $\frac{1}{\sqrt{2}}$, 1 | $\frac{1}{\sqrt{6}}$ | $\sqrt{2}$ |
| $\sqrt{\alpha^2 \|\Delta A\|_2^2 + \beta^2 \|\Delta b\|_2^2}$ | $\|.\|_2$, 1, 1 | $\frac{1}{\sqrt{3}}$ | $\sqrt{2}$ |
| $\alpha \|\Delta A\|_2 + \beta \|\Delta b\|_2$ | $\|.\|_1$, 1, $\sqrt{2}$ | $\frac{1}{\sqrt{3}}$ | 2 |

**4. Statistical estimation of the partial condition number.** In this section we compute a statistical estimate of the partial condition number. We have seen in section 3 that using the Frobenius or the spectral norm for the matrices gives condition numbers that are of the same order of magnitude. For the sake of simplicity, we compute here a statistical estimate of $\kappa_{g,F}(A,b)$.

Let $(z_1, z_2, \ldots, z_q)$ be an orthonormal basis for a subspace of dimension $q$ ($q \leq k$) that has been randomly and uniformly selected from the space of all $q$-dimensional subspaces of $\mathbb{R}^k$ (this can be done by choosing $q$ random vectors and then orthogonalizing). Let us denote $g_i(A,b) = (Lz_i)^T x(A,b)$.

Since $Lz_i \in \mathbb{R}^n$, the absolute condition number of $g_i$ can be computed via the exact formula given in Corollary 1, i.e.,

$$\kappa_{g_i,F}(A,b) = \left( \left\| (Lz_i)^T (A^TA)^{-1} \right\|_2^2 \frac{\|r\|_2^2}{\alpha^2} + \left\| (Lz_i)^T A^\dagger \right\|_2^2 \left( \frac{\|x\|_2^2}{\alpha^2} + \frac{1}{\beta^2} \right) \right)^{\frac{1}{2}}. \tag{4.1}$$

We define the random variable $\phi(q)$ by

$$\phi(q) = \left( \frac{k}{q} \sum_{i=1}^{q} \kappa_{g_i,F}(A,b)^2 \right)^{\frac{1}{2}}.$$

Let the operator $E(.)$ denote the expected value. The following proposition shows that the root mean square of $\phi(q)$, defined by $R(\phi(q)) = \sqrt{E(\phi(q)^2)}$, can be considered as an estimate for the condition number of $g(A,b) = L^T x(A,b)$.

PROPOSITION 1. *The absolute condition number can be bounded as follows:*

$$\frac{R(\phi(q))}{\sqrt{k}} \leq \kappa_{g,F}(A,b) \leq R(\phi(q)). \tag{4.2}$$

*Proof.* Let $vec$ be the operator that stacks the columns of a matrix into a long vector and let $M$ be the $k \times m(n+1)$ matrix such that $vec(g'(A,b).(\Delta A, \Delta b)) = M \binom{vec(\alpha\Delta A)}{vec(\beta\Delta b)}$. Note that $M$ depends on $A$, $b$, $L$ and not on the $z_i$.

Then we have

$$\kappa_{g,F}(A,b) = \max_{(\Delta A,\Delta b)} \frac{\|g'(A,b).(\Delta A,\Delta b)\|_2}{\|(\Delta A,\Delta b)\|_F} = \max_{(\Delta A,\Delta b)} \frac{\|vec(g'(A,b).(\Delta A,\Delta b))\|_2}{\left\| \begin{pmatrix} vec(\alpha\Delta A) \\ vec(\beta\Delta b) \end{pmatrix} \right\|_2}$$

$$= \max_{z \in \mathbb{R}^{m(n+1)}, z \neq 0} \frac{\|M\ z\|_2}{\|z\|_2} = \|M\|_2 = \left\|M^T\right\|_2.$$

Let $Z = [z_1, z_2, \ldots, z_q]$ be the $k \times q$ random matrix with orthonormal columns $z_i$. From [10] it follows that $\frac{k}{q} \left\|M^T Z\right\|_F^2$ is an unbiased estimator of the Frobenius norm of the $m(n+1) \times k$ matrix $M^T$, i.e., we have $E(\frac{k}{q} \left\|M^T Z\right\|_F^2) = \left\|M^T\right\|_F^2$.

From

$$\left\|M^T Z\right\|_F^2 = \left\|Z^T M\right\|_F^2 = \left\| \begin{pmatrix} z_1^T M \\ \vdots \\ z_q^T M \end{pmatrix} \right\|_F^2$$

we get, since $z_i^T M$ is a row vector,

$$\left\|M^T Z\right\|_F^2 = \sum_{i=1}^{q} \left\|z_i^T M\right\|_2^2.$$

We notice that for every vector $u \in \mathbb{R}^k$, if we consider the function $g_u(A,b) = u^T g(A,b)$, then we have $\left\|u^T M\right\|_F = \|g_u'(A,b)\| = \kappa_{g_u,F}(A,b)$ and therefore

$$\left\|z_i^T M\right\|_F = \kappa_{g_i,F}(A,b).$$

Eventually we obtain

$$\left\|M^T\right\|_F^2 = E\left(\frac{k}{q}\sum_{i=1}^{q}\kappa_{g_i,F}(A,b)^2\right) = E(\phi(q)^2).$$

Moreover, considering that $M^T \in \mathbb{R}^{m(n+1)\times k}$ and using the well-known inequality

$$\frac{\left\|M^T\right\|_F}{\sqrt{k}} \leq \left\|M^T\right\|_2 \leq \left\|M^T\right\|_F,$$

we get the result (4.2). Then we will consider $\phi(q)\frac{\|(A,b)\|_F}{\|L^T\tilde{x}\|_2}$ as an estimator of $\kappa_{g,F}^{(rel)}(A,b)$. ∎

The root mean square of $\phi(q)$ is an upper bound of $\kappa_g(A,b)$, and estimates $\kappa_{g,F}(A,b)$ within a factor $\sqrt{k}$. Proposition 1 involves the computation of the condition number of each $g_i(A,b), i = 1,\ldots,q$. From Remark 2, it follows that the computational cost of each $\kappa_{g_i,F}(A,b)$ is $2n^2$ (if the $R$ factor of the QR decomposition of $A$ is available). Hence, for a given sample of vectors $z_i$, $i = 1,\ldots,q$, computing $\phi(q)$ requires about $2qn^2$ flops.

However, Proposition 1 is mostly of theoretical interest, since it relies on the computation of the root mean square of a random variable, without providing a practical method to obtain it. In the next proposition, the use of the small sample estimate theory developed by Gudmundsson, Kenney, and Laub [10] gives a first answer to this question by showing that the evaluation of $\phi(q)$ using only one sample of $q$ vectors $z_1, z_2, \ldots, z_q$ in the unit sphere may provide an acceptable estimate.

PROPOSITION 2. *Using conjecture* [10, p. 781]*, we have the following result: For any* $\alpha > 10$*,*

$$\Pr\left(\frac{\phi(q)}{\alpha\sqrt{k}} \leq \kappa_{g,F}(A,b) \leq \alpha\phi(q)\right) \geq 1-\alpha^{-q}.$$

*This probability approaches* 1 *very fast as* $q$ *increases. For* $\alpha = 11$ *and* $q = 3$ *the probability for* $\phi(q)$ *to estimate* $\kappa_{g,F}(A,b)$ *within a factor* $11\sqrt{k}$ *is* 99.9%*.*

*Proof.* We define as in the proof of Proposition 1 the matrix $M$ as the matrix related to the *vec* operation representing the linear operator $g'(A,b)$. From [10, eq. (4), p. 781 and eq. (9), p. 783] we get

$$\Pr\left(\frac{\left\|M^T\right\|_F}{\alpha} \leq \phi(q) \leq \alpha\left\|M^T\right\|_F\right) \geq 1-\alpha^{-q}. \tag{4.3}$$

We have seen in the proof of Proposition 1 that $\kappa_{g,F}(A,b) = \left\|M^T\right\|_2$. Then we have

$$\kappa_{g,F}(A,b) \leq \left\|M^T\right\|_F \leq \kappa_{g,F}(A,b)\ \sqrt{k}.$$

It follows that, for the random variable $\phi(q)$, we have

$$\Pr\left(\frac{\kappa_{g,F}(A,b)}{\alpha} \leq \phi(q) \leq \alpha\kappa_{g,F}(A,b)\ \sqrt{k}\right) \geq \Pr\left(\frac{\left\|M^T\right\|_F}{\alpha} \leq \phi(q) \leq \alpha\left\|M^T\right\|_F\right).$$

Then we obtain the result from

$$\Pr\left(\frac{\kappa_{g,F}(A,b)}{\alpha} \leq \phi(q) \leq \alpha\kappa_{g,F}(A,b)\ \sqrt{k}\right) = \Pr\left(\frac{\phi(q)}{\alpha\sqrt{k}} \leq \kappa_{g,F}(A,b) \leq \alpha\phi(q)\right).$$ ∎

We see from this proposition that it may not be necessary to estimate the root mean square of $\phi(q)$ using sophisticated algorithms. Indeed only one sample of $\phi(q)$ obtained for $q = 3$ provides an estimate of $\kappa_{g,F}(A, b)$ within a factor $\alpha\sqrt{k}$.

*Remark* 7. If $k = 1$, then $Z = 1$ and the problem is reduced to computing $\kappa_{g_1}(A, b)$. In this case, $\phi(1)$ is exactly the partial condition number of $L^T x(A, b)$.

*Remark* 8. Concerning the computation of the statistical estimate in the presence of roundoff errors, the numerical reliability of the statistical estimate relies on an accurate computation of the $\kappa_{g_i,F}(A, b)$ for a given $z_i$. Let $A$ be a $17 \times 13$ Vandermonde matrix, $b$ a random vector, and $L \in \mathbb{R}^n$ the right singular vector $v_n$.

Using the *Mathematica* software that computes in exact arithmetic, we obtained $\kappa_{g,F}^{(rel)}(A, b) \approx 5 \cdot 10^8$. If the triangular factor $R$ form $A^T A = R^T R$ is obtained by the QR decomposition of $A$, we get $\kappa_{g,F}^{(rel)}(A, b) \approx 5 \cdot 10^8$. If $R$ is computed via a classical Cholesky factorization, we get $\kappa_{g,F}(A, b)^{(rel)} \approx 10^{10}$.

Corollary 1 and Remark 2 show that the computation of $\kappa_{g,F}(A, b)^{(rel)}$ involves linear systems of the type $A^T Ax = d$, which differs from the usual normal equation for least squares in their right-hand side. Our observation that for this kind of ill-conditioned systems, a QR factorization is more accurate than a Cholesky factorization is in agreement with [5].

**5. Numerical experiments.** All experiments were performed in MATLAB 6.5 using a machine precision of $2.22 \cdot 10^{-16}$.

**5.1. Examples.** For the examples of section 1, we compute the partial condition number using the formula given in Theorem 1.

In the first example we have

$$A = \begin{pmatrix} 1 & 1 & \epsilon^2 \\ \epsilon & 0 & \epsilon^2 \\ 0 & \epsilon & \epsilon^2 \\ \epsilon^2 & \epsilon^2 & 2 \end{pmatrix},$$

and we assume that only $A$ is perturbed. If we consider the values for $L$ that are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $L = (0, 0, 1)^T$, then we obtain partial condition numbers $\kappa_{g,F}^{(rel)}(A)$ that are, respectively, $10^{24}$ and 1.22, as expected since there is 50% relative error on $x_1$ and $x_2$ and there is no error on $x_3$.

In the second example where $A$ is the $10 \times 4$ Vandermonde matrix defined by $A_{ij} = \frac{1}{(10+i)^{j-1}}$ and only $b$ is perturbed, the partial condition numbers $\kappa_{g,F}^{(rel)}(b)$ with respect to each component $x_1, x_2, x_3, x_4$ are, respectively, $4.5 \cdot 10^2, 2 \cdot 10^4, 3 \cdot 10^5, 1.4 \cdot 10^6$, which is consistent with the error variation given in section 1 for each component.

**5.2. Average behavior of the statistical estimate.** We compare here the statistical estimate described in the previous section with the partial condition number obtained via the exact formula given in Theorem 1. We suppose that only $A$ is perturbed and then the partial condition number can be expressed as $\kappa_{g,F}^{(rel)}(A)$. We use the method described in [16] in order to construct test problems $[A, x, r, b] = P(m, n, n_r, l)$ with

$$A = Y \begin{pmatrix} D \\ 0 \end{pmatrix} Z^T \in \mathbb{R}^{m \times n}, \quad Y = I - 2yy^T, \; Z = I - 2zz^T,$$

where $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ are random unit vectors and where $D = n^{-l} \operatorname{diag}(n^l, (n-1)^l, \ldots, 1)$.

$x = (1, 2^2, \ldots, n^2)^T$ is given and $r = Y\binom{0}{c} \in \mathbb{R}^m$ is computed with $c \in \mathbb{R}^{m-n}$ random vector of norm $n_r$. The right-hand side is $b = Y\binom{DZx}{c}$. By construction, the condition number of $A$ and $D$ is $n^l$.

In our experiments, we consider the matrices

$$A = \begin{pmatrix} A_1 & E' \\ E & A_2 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} I \\ 0 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{m_1 \times n_1}$, $A_2 \in \mathbb{R}^{m_2 \times n_2}$, $L \in \mathbb{R}^{n \times n_1}$, $m_1 + m_2 = m$, $n_1 + n_2 = n$, and $E$ and $E'$ contain the same element $e_p$ which defines the coupling between $A_1$ and $A_2$. The matrices $A_1$ and $A_2$ are randomly generated using, respectively, $P(m_1, n_1, n_{r_1}, l_1)$ and $P(m_2, n_2, n_{r_2}, l_2)$.

For each sample matrix, we compute in MATLAB

1. the partial condition number $\kappa_{g,F}^{(rel)}(A)$ using the exact formula given in Theorem 1 and based on the singular value decomposition of $A$;
2. the statistical estimate $\phi(3)$ using three random orthogonal vectors and computing each $\kappa_{g_i,F}(A, b), i = 1, 3$, with the $R$ factor of the QR decomposition of $A$.

These data are then compared by computing the ratio

$$\gamma = \frac{\phi(3)}{\kappa_{g,F}^{(rel)}(A)}.$$

Table 5.1 contains the mean $\overline{\gamma}$ and the standard deviation $s$ of $\gamma$ obtained on 1000 random matrices with $m_1 = 12, n_1 = 10, m_2 = 17, n_2 = 13$ by varying the condition numbers ${n_1}^{l_1}$ and ${n_2}^{l_2}$ of, respectively, $A_1$ and $A_2$ and the coupling coefficient $e_p$. The residual norms are set to $n_{r_1} = n_{r_2} = 1$. In all cases, $\overline{\gamma}$ is close to 1 and $s$ is about 0.3. The statistical estimate $\phi(3)$ lies within a factor 1.22 of $\kappa_{g,F}^{(rel)}(A)$, which is very accurate in condition number estimation. We notice that in two cases $\phi(3)$ is lower than 1. This is possible because Proposition 1 shows that $E(\phi(3)^2)$ is an upper bound of $\kappa_{g,F}(A)^2$ but not necessarily $\phi(3)^2$.

TABLE 5.1
*Ratio between statistical and exact condition numbers of $L^T x$.*

| Condition | | $e_p = 10^{-5}$ | | $e_p = 1$ | | $e_p = 10^5$ | |
|---|---|---|---|---|---|---|---|
| $l_1$ | $l_2$ | $\overline{\gamma}$ | $s$ | $\overline{\gamma}$ | $s$ | $\overline{\gamma}$ | $s$ |
| 1 | 1 | 1.22 | $2.28 \cdot 10^{-1}$ | 1.15 | $2.99 \cdot 10^{-1}$ | 1.07 | $3.60 \cdot 10^{-1}$ |
| 1 | 8 | 1.02 | $3.19 \cdot 10^{-1}$ | 1.22 | $3.05 \cdot 10^{-1}$ | 1.21 | $3.35 \cdot 10^{-1}$ |
| 8 | 1 | $9 \cdot 10^{-1}$ | $3 \cdot 10^{-1}$ | 1.13 | $3 \cdot 10^{-1}$ | 1.06 | $3.45 \cdot 10^{-1}$ |
| 8 | 8 | $9.23 \cdot 10^{-1}$ | $2.89 \cdot 10^{-1}$ | 1.22 | $2.95 \cdot 10^{-1}$ | 1.18 | $3.33 \cdot 10^{-1}$ |

**6. Estimates versus exact formula.** We assume that the $R$ factor of the QR decomposition of $A$ is known. We gather in Table 6.1 the results obtained in this paper in terms of accuracy and flop counts for the estimation of the partial condition number for the LLSP. Table 6.2 gives the estimates and flop counts in the particular situation where

$$m = 1500, \; n = 1000, \; k = 50,$$

$$A_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; L_1 = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix},$$

Table 6.1
*Comparison between exact formula and estimates for $\kappa_{g,F}(A,b)$.*

| $\kappa_{g,F}(A,b)$ | Flops | Accuracy |
|---|---|---|
| Exact formula<br>$n \ll m$ | $12n^3$ | Exact |
| Sharp estimate $f(A,b)$<br>$k \ll n$ | $2kn^2$ | $\frac{f(A,b)}{\sqrt{3}} \leq \kappa_{g,F}(A,b) \leq f(A,b)$ |
| Stat. estimate $\phi(q)$<br>$q \ll k$ | $2qn^2$ | $\frac{\phi(q)}{\alpha\sqrt{k}} \leq \kappa_{g,F}(A,b) \leq \alpha\phi(q)$<br>$Pr \geq 1-\alpha^{-q}$ for $\alpha > 10$ |

Table 6.2
*Flops and accuracy: exact formula versus estimates.*

| $\kappa_{g,F}^{(rel)}(A,b)$ | $f(A,b)\frac{\\|(A,b)\\|_F}{\\|L^T\tilde{x}\\|_2}$ | $\phi(q)\frac{\\|(A,b)\\|_F}{\\|L^T\tilde{x}\\|_2}$ |
|---|---|---|
| $2.09 \cdot 10^2$ | $2.18 \cdot 10^2$ | $11.44 \cdot 10^2$ |
| 12 Gflops | 100 Mflops | 6 Mflops |

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-2} \\ 0 & 0 \end{pmatrix} \text{ and } b = \frac{1}{\sqrt{2}}(2,1,\ldots,1)^T, \ L = \begin{pmatrix} L_1 & 0 \\ 0 & I_{k-2} \\ 0 & 0 \end{pmatrix}.$$

We see here that the statistical estimates may provide information on the condition number using a very small amount of floating point operations compared with the other two methods.

**7. Conclusion.** We have shown the relevance of the partial condition number for test cases from parameter estimation. This partial condition number evaluates the sensitivity of $L^T x$, where $x$ is the solution of an LLSP when $A$ and/or $b$ are perturbed. It can be computed via a closed formula, a sharp estimate, or a statistical estimate. The choice will depend on the size of the LLSP and on the needed accuracy. The closed formula requires $\mathcal{O}(n^3)$ flops and is affordable for small problems only. The sharp estimate and the statistical estimate will be preferred for larger problems especially if $k \ll n$ since their computational cost is in $\mathcal{O}(n^2)$.

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