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A PARTICLE MECHANICS DESCRIPTION OF ANTISYMMETRIC TENSOR FIELDS

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ABSTRACT

In a previous work we discussed the action for a massless relativistic point particle with a gauged N-extended worldline supersymmetry that yields, upon quantization, a relativistic wave equation for pure spin $N/2$. Here we present further details, emphasizing the $N = 2$ particle model for which a wave function with $O(2)$ charge q can be interpreted as the field strength of a (q-1)-form gauge potential. We present extensions of this model that yield field equations for massless and massive antisymmetric tensors in arbitrary space-time dimension d. We show how to obtain chirality and generalized self-duality conditions.

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1. Introduction

Relativistic free field equations for massless particles of arbitrary spin can be obtained from the quantization of point particle models. The Klein-Gordon equation comes from the spinless particle, the Klein-Gordon operator p^2 being the generator of reparametrizations of the particle worldline. The Dirac equation is similarly obtained from the spinning particle action which has $N = 1$ worldline supersymmetry [1,2]. The anticommuting classical variables λ^μ become the Dirac matrices on quantization and the generator of supersymmetry transformations becomes the Dirac operator. This model can be extended to describe particles of spin greater than $\frac{1}{2}$ [3, 4, 5]. The extended model has local N-extended worldline supersymmetry and a local $O(N)$ gauge invariance. For a particle moving in Minkowski space-time with coordinates x^μ and momentum p_μ the first-order form of the action is

$$S = \int_0^1 dt \left\{ \dot{x}^\mu p_\mu + \frac{i}{2}\lambda_i^\mu \dot{\lambda}_i^\nu \eta_{\mu\nu} - \frac{1}{2}e\eta^{\mu\nu}p_\mu p_\nu - i\psi_i \lambda_i^\mu p_\mu - \frac{i}{2}f_{ij}\lambda_i^\mu \lambda_j^\nu \eta_{\mu\nu} \right\} \quad , \qquad (1.1)$$

where the $\lambda_i^\mu(t)$ $i = 1, ...N$ are the supersymmetry partners of $x^\mu(t)$, $e(t)$ and $\psi_i(t)$ are "worldline supergravity fields" and $f_{ij}(t)$ is the $SO(N)$ gauge "field". The gauge fields e, ψ_i, and f_{ij} are the Lagrange multipliers for the constraints which, when imposed on the particle's quantum wave function, yield a relativistic wave equation for a pure spin $N/2$ particle (in four dimensions). Because the action also has a d-dimensional conformal invariance these wave equations are in fact the conformal wave equations for arbitrary spin. This has been noted independently by Siegel [6], who has further shown that all conformal wave equations are obtained in this way.

In this paper we shall emphasize the $N = 2$ model. In this case we can add to the action (1.1) a "Chern-Simons" term proportional to

$$\int_0^1 dt\ \varepsilon^{ij} f_{ij} \quad , \qquad (1.2)$$

which is not possible for $N > 2$ because the internal symmetry group is then non-Abelian. Without the Chern-Simons term, the $N = 2$ wave-function is necessarily a $\frac{d}{2}$-form, which is further constrained to be harmonic. This means that for d odd the wave-function must vanish. As we shall show, this can be understood in the context of path-integral quantization as a consequence of a global $SO(2)$ anomaly. With the inclusion of (1.2) with coefficient $(q - \frac{d}{2})$ the wave-function becomes a harmonic q-form. Since q is an integer this means that for d odd a non-vanishing wave-function requires a non-vanishing Chern-Simons coefficient. Since the Chern-Simons term breaks $O(2)$ to $SO(2)$ and conformal invariance to dilatation invariance, the global anomaly for d odd can be viewed as a clash between the rigid $O(2)$ and conformal symmetries and the local SO(2) invariance.

Another property of the $N = 2$ model is that it can be formulated in superspace (whereas the required auxiliary fields are not known for $N > 2$). Particle models for massive antisymmetric tensor fields in d dimensions can be obtained by a particle mechanics analogue of dimensional reduction *à la* Scherk and Schwarz [7] from the $N = 2$ massless model in $(d + 1)$ dimensions. If space-time has a compact isometry group G the point particle action will have a rigid G-invariance. If this symmetry is gauged, the particle's wave function will have to satisfy the additional constraint that it be a G-singlet. The simplest case is when G is the $U(1)$ isometry group associated with translation around an S^1 factor of a $(d + 1)$-dimensional spacetime. Then the requirement that the wave function be a $U(1)$ singlet effects an (ordinary) dimensional reduction, producing a set of d-dimensional massless wave equations. Because $U(1)$ is Abelian a modification of the particle action is possible for which the wave-function acquires a non-zero $U(1)$ charge. The resulting d-dimensional equations are those of a massive antisymmetric tensor.

Siegel and Zwiebach have mentioned how a chirality constraint on a Dirac spinor can be incorporated into a particle model [8] by including extra generators that enlarge the $N = 1$ supersymmetry algebra *. At the quantum level the new algebra becomes that of an

* A slightly different approach was described in [9]

$N = 2$ supersymmetry (but the wavefunction remains a spinor, rather than the bispinor of the $N = 2$ models that we have been discussing above). We show how this can be extended to the $N = 2$ models for antisymmetric tensors in $d = 2$ mod 4 to incorporate a self-duality constraint on a $\frac{d}{2}$–form wave function. The algebra has six Grassman odd generators in general and is easily investigated in the language of differential forms. For $d = 2$ the algebra reduces to an $N = 4$ supersymmetry algebra with an additional gauged $O(1,1)$ invariance.

2. Particle actions for arbitrary spin

Quantization of the particle with action (1.1) leads to the (anti)commutation relations

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \qquad \{\lambda^\mu_i, \lambda^\nu_j\} = \eta^{\mu\nu}\delta_{ij} \quad . \tag{2.1}$$

The λ–relations can be realized by matrices in the space $\otimes^N H_1$ where, supposing d to be even, H_1 is the $2^{\frac{d}{2}}$–dimensional space of Dirac spinors. If γ^μ are the usual γ–matrices for H_1 and

$$\gamma_\star = i^{\frac{d-2}{2}}\gamma^0\gamma^1...\gamma^{d-1} \quad , \tag{2.2}$$

which satisfies $\gamma_\star^2 = 1$ and $\{\gamma_\star, \gamma^\mu\} = 0$, we can represent the operators λ^μ_i by

$$\lambda^\mu_i = \frac{1}{\sqrt{2}}\gamma_\star \otimes ... \otimes \gamma_\star \otimes \Gamma^\mu_{(i)} \otimes \mathbb{I} \otimes ... \otimes \mathbb{I} \quad , \tag{2.3}$$

where $\Gamma^\mu_{(i)}$ can be chosen, for each value of $i = 1, ..., N$, to be either γ^μ or $i\gamma_\star\gamma^\mu$.

With this realization of the anticommutation relations the wave function is a multi-spinor $\Psi_{\alpha_1...\alpha_N}$. The $O(N)$ gauge field f_{ij} imposes the algebraic constraint

$$(\gamma_\mu\Gamma\gamma^\mu)^{\alpha_i\alpha_j}\Psi_{\alpha_1...\alpha_i...\alpha_j...\alpha_N} = 0 \quad , \tag{2.4}$$

where Γ is any Dirac matrix (with entries $\Gamma_\alpha^{\ \beta}$). We are using here the notation that $\Gamma^{\alpha\beta} = C^{\alpha\gamma}\Gamma_\gamma^{\ \beta}$ with C the charge conjugation matrix. Let us introduce the further notation

$\Gamma^{(n)}$ for any of the Γ–matrices $\gamma^{\mu_1\cdots\mu_n}$; then $(\Gamma^{(n)})^{\alpha\beta}$ is symmetric or antisymmetric in spinor indices, depending on N and d. From the identity

$$\gamma_\mu \Gamma^{(n)} \gamma^\mu = (-1)^n (d-2n)\Gamma^{(n)} \tag{2.5}$$

we see that (2.4) yields the constraints

$$(\Gamma^{(n)})^{\alpha_i\alpha_j}\Psi_{\alpha_1\ldots\alpha_i\ldots\alpha_j\ldots\alpha_N} = 0 \qquad n \neq \frac{d}{2} \quad . \tag{2.6}$$

For d odd the representation (2.3) fails because $\gamma_\star$ will commute with γ^μ. By taking γ^μ to be a reducible matrix representation of the Clifford algebra generators we could find a suitable replacement for $\gamma_\star$ but the constraint (2.6) would still hold and this implies that $\Psi = 0$ for d odd. We shall see shortly that for $N = 2$ these difficulties can be overcome, and that allowing for an SO(2) Chern-Simons term enables us to obtain the wave equation for antisymmetric tensors of arbitrary rank in any spacetime dimension.

The supergravity fields e and ψ_i impose the dynamical constraints

$$\begin{aligned} p^2\Psi_{\alpha_1\ldots\alpha_N} &= 0 \\ \not{p}_{\beta_i}^{\ \alpha_i}\Psi_{\alpha_1\ldots\alpha_i\ldots\alpha_N} &= 0 \quad \forall\, i \end{aligned} \tag{2.7}$$

on the wave function. For $d = 4$ these are the massless Bargmann-Wigner relativistic wave equations for a pure spin $\frac{N}{2}$ field strength. In particular, for $N = 2$ the wave function is $F_{\mu\nu}$ (the Maxwell field strength tensor), and the equations (2.7) are equivalent to the Bianchi "identity" $\partial_{[\mu}F_{\nu\rho]} = 0$ and the field equation $\partial^\mu F_{\mu\nu} = 0$.

For $N = 2M$ in four dimensions, the wave function can be expressed in the equivalent form $\Psi^{(2M,2)}_{\mu_1\nu_1,\mu_2\nu_2,\ldots,\mu_M\nu_M}$ with antisymmetry in each pair, and total symmetry under interchange of pairs. In addition it is totally traceless and satisfies

$$\Psi_{[\mu_1\nu_1,\mu_2]\nu_2,\ldots,\mu_M\nu_M} = 0 \quad . \tag{2.8}$$

The differential constraints are then

$$\begin{aligned} &\partial^{\mu}\Psi^{(2M,2)}_{\mu\nu_1,\mu_2\nu_2,\ldots,\mu_M\nu_M} = 0 \\ &\partial_{[\rho}\Psi^{(2M,2)}_{\mu_1\nu_1],\mu_2\nu_2,\ldots,\mu_M\nu_M} = 0 \quad . \end{aligned} \tag{2.9}$$

For $N = 4$, for example, we recover the linearized Einstein equations expressed in terms of the Weyl tensor. More generally, in $d = 2n$ dimensions and $N = 2M$, the basic index "block" has n antisymmetrized indices, and the wavefunctions are symmetrized on M blocks, i.e.

$$\Psi^{(2M,n)}_{\mu_{11}\ldots\mu_{1n},\mu_{21}\ldots\mu_{2n},\ldots,\mu_{M1}\ldots\mu_{Mn}} = \Psi^{(2M,n)}_{([\mu_{11}\ldots\mu_{1n}],[\mu_{21},\ldots,\nu_{2n}],\ldots,[\mu_{M1}\ldots\nu_{Mn}])} \quad , \tag{2.10}$$

where the symmetrization refers to the blocks of n indices. In addition

$$\Psi^{(2M,n)}_{[\mu_{11}\ldots\mu_{1n},\mu_{21}]\ldots\mu_{2n},\ldots,\mu_{M1}\ldots\mu_{Mn}} = 0 \quad . \tag{2.11}$$

Again, the $\Psi^{(2M,n)}$'s are totally traceless and obey equations similar to (2.9) . These forms for the wave-functions can be summarized by the $O(n)$ Young tableaux with n rows and $N/2$ columns, where irreducibility also implies tracelessness. For $N = 2M + 1$ in $d = 2n$ dimensions the wavefunction is a tensor-spinor with tensor indices of the same form as $\Psi^{(2M,n)}$. In this case there is also the γ-traceless condition

$$\gamma^{\mu_{11}}\Psi^{(2M+1,n)}_{\mu_{11}\ldots\mu_{1n},\mu_{21}\ldots\mu_{2n},\ldots,\mu_{M1}\ldots\mu_{Mn}} = 0 \quad . \tag{2.12}$$

We shall now consider the $N = 2$ case in more detail. The action is

$$\begin{aligned} S = \int dt \Big\{ &\dot{x}^{\mu}p_{\mu} + \frac{i}{2}\lambda_i^{\mu}\dot{\lambda}_i^{\nu}\eta_{\mu\nu} - \frac{1}{2}e\eta^{\mu\nu}p_{\mu}p_{\nu} - i\psi_i\lambda_i^{\mu}p_{\mu} + \\ &+ f\left(-\frac{i}{2}\varepsilon^{ij}\lambda_i^{\mu}\lambda_j^{\nu}\eta_{\mu\nu} - (q - \frac{d}{2})\right)\Big\} \quad . \end{aligned} \tag{2.13}$$

This action differs from the specialization of (1.1) to $N = 2$ by the addition of the $(q - \frac{d}{2})$ term. This addition is consistent with supersymmetry because the supersymmetry

variation of f vanishes. It is also consistent with worldline diffeomorphism and $SO(2)$ invariance because for these symmetries δf is a total derivative. For $N > 2$ the $SO(N)$ variation of f_{ij} is <u>not</u> a total derivative, so this modification is no longer possible. We are using units with $\hbar = 1$ here; if we were to restore $\hbar$, then the Chern-Simons term $-f(q-d/2)$ would be multiplied by $\hbar$ and would therefore vanish in the classical limit. Note that, with the exception of the Chern-Simons term, all terms of the action are invariant under $O(2)$, rather than $SO(2)$; the additional transformation is $\lambda_1 \to \lambda_1, \lambda_2 \to -\lambda_2, \psi_1 \to \psi_1, \psi_2 \to -\psi_2, e \to e, f \to -f, x \to x, p \to p$.

It will prove convenient to introduce the variables

$$\xi^\mu = \frac{1}{\sqrt{2}}(\lambda_1^\mu + i\lambda_2^\mu) \qquad \bar{\xi}^\mu = \frac{1}{\sqrt{2}}(\lambda_1^\mu - i\lambda_2^\mu)$$
$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \qquad \bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \quad , \tag{2.14}$$

in terms of which the action reads

$$S = \int dt \Big\{ \dot{x}^\mu p_\mu + i\bar{\xi}^\mu \dot{\xi}^\nu \eta_{\mu\nu} - i\psi\bar{\xi}^\mu p_\mu - i\bar{\psi}\xi^\mu p_\mu - \frac{1}{2} e\, p^2$$
$$+ f\big(\frac{1}{2}[\xi^\mu, \bar{\xi}^\nu]\eta_{\mu\nu} - (q - \frac{d}{2})\big) \Big\} \quad . \tag{2.15}$$

Upon quantization, the ξ,$\bar{\xi}$ variables satisfy the anticommutation relations

$$\{\xi^\mu, \xi^\nu\} = \{\bar{\xi}^\mu, \bar{\xi}^\nu\} = 0$$
$$\{\xi^\mu, \bar{\xi}^\nu\} = \eta^{\mu\nu} \quad . \tag{2.16}$$

Since the $\bar{\xi}^\mu$ are a set of mutually anticommuting operators they can be diagonalized on a basis of eigenstates $|\bar{\alpha}\rangle$ for which the eigenvalues $\bar{\alpha}^\mu$ of $\bar{\xi}^\mu$ are anticommuting. Thus

$$\bar{\xi}^\mu |\bar{\alpha}\rangle = \bar{\alpha}^\mu |\bar{\alpha}\rangle \quad . \tag{2.17}$$

We are using here the notation of [10] to which we refer for further details of properties of the fermion coherent states $|\bar{\alpha}\rangle$. The wave function of the $N = 2$ particle is

$$(\langle x| \otimes \langle\alpha|)|\Psi\rangle = \Psi(x, \alpha) \quad , \tag{2.18}$$

which can be expanded as a power series in α^μ. Thus

$$\begin{aligned}\Psi =& F(x) + \alpha^\mu F_\mu(x) + \frac{1}{2}\alpha^\mu\alpha^\nu F_{\mu\nu}(x) + \ldots \\ &+ \frac{1}{p!}\alpha^{\mu_1}\ldots\alpha^{\mu_p} F_{\mu_1\ldots\mu_p}(x) + \ldots + \frac{1}{d!}\alpha^{\mu_1}\ldots\alpha^{\mu_d} F_{\mu_1\ldots\mu_d}(x) \quad ,\end{aligned} \tag{2.19}$$

i.e. Ψ can be viewed as an inhomogeneous differential form on Minkowski space. The constraint imposed by f can now be written in the form

$$(\xi\cdot\bar{\xi} - q)|\Psi\rangle = 0 \quad , \tag{2.20}$$

which states that $|\Psi\rangle$ has $O(2)$ charge q. Equation (2.20) is equivalent to

$$\alpha^\mu \frac{\partial}{\partial\alpha^\mu}\Psi(x,\alpha) = q\Psi(x,\alpha) \quad , \tag{2.21}$$

which is solved by writing $\Psi(x,\alpha)$ as

$$\Psi(x,\alpha) = \frac{1}{q!}\alpha^{\mu_1}\ldots\alpha^{\mu_q} F_{\mu_1\ldots\mu_q}(x) \quad . \tag{2.22}$$

The two independent constraints imposed by ψ_i can now be written as

$$\xi\cdot p|\Psi\rangle = 0 \qquad \bar{\xi}\cdot p|\Psi\rangle = 0 \quad , \tag{2.23}$$

which are equivalent to

$$\partial_{[\mu_1} F_{\mu_2\ldots\mu_{q+1}]} = 0 \qquad \partial^{\mu_1} F_{\mu_1\ldots\mu_q} = 0 \quad , \tag{2.24}$$

respectively. The first equation is solved by $F_{\mu_1\ldots\mu_q} = q\partial_{[\mu_1} A_{\mu_2\ldots\mu_q]}$ and the second is then the usual field equation for the $(q-1)$'th rank antisymmetric tensor gauge potential A. Observe that this analysis holds for d even and d odd.

Because of the anticommutation relations (2.16) the $\frac{d}{2}$ term in the $SO(2)$ constraint is cancelled when the latter is written in the form of eq.(2.20) . In the path-integral approach

to quantization the $\xi,\bar{\xi}$ variables have zero anticommutators. To see how equivalent results are nevertheless obtained in this approach we choose the gauge $e = 1, \psi = \bar{\psi} = 0$ in (2.13) and perform the fermion path-integral. The result is the following effective action for $x^i(t)$:

$$S_{eff} = \int_0^1 dt\{\dot{x}p - \frac{1}{2}p^2 - (q - \frac{d}{2})f - id\ln\ \det[1 + i(\partial_t)^{-1} f] + \text{const.}\} \quad . \tag{2.25}$$

A careful evaluation of the determinant was given in [11]; there is a possible anomaly contribution from the linear term in f; i.e. the "tadpole" in the diagrammatic expansion. A quick, but non rigorous, method of calculating this diagram is to replace the energy sum by an integral, at the same time introducing a regulator mass μ for the resulting infrared divergence. The result is then

$$id[\int_0^1 f(t)dt] \int_{-\infty}^{+\infty} \frac{dE}{2\pi(E + i\mu)} \quad . \tag{2.26}$$

After ultraviolet regulation by a cutoff Λ the energy integral becomes

$$\int_{-\Lambda}^{\Lambda} \frac{dE}{2\pi(E + i\mu)} = -\frac{i}{2}(2k - 1) + O(\frac{\mu}{\Lambda}) \quad , \tag{2.27}$$

where the arbitrary integer k arises from a choice of branch of the logarithm appearing in the evaluation of the integral. Hence the term linear in f in the effective action is

$$(-q + kd) \int_0^1 f(t)dt \quad . \tag{2.28}$$

Under an $SO(2)$ transformation

$$f \to f + ig^{-1}\partial_t g \qquad \xi \to g\xi \quad . \tag{2.29}$$

If g is connected to the identity, the C-S term is invariant provided that $g(0) = g(1)$. This periodicity implies that there are global gauge transformations for which $g = e^{2\pi imt}$, $m \in \mathbf{Z}$. For such transformations

$$\int_0^1 f(t)dt \to \int_0^1 f(t)dt - 2\pi m \quad , \tag{2.30}$$

and hence we obtain the quantization condition

$$q \in \mathbf{Z} \quad . \tag{2.31}$$

The case for which $q = \frac{d}{2}$ (which is consistent only for d even) is special. Only in this case is the $SO(2)$ symmetry of (2.13) a genuine "internal" symmetry for which the Lagrangian, as against the action, is invariant. The $q = \frac{d}{2}$ case is also special in another respect; when $q = \frac{d}{2}$ the action (2.13) has additional invariances. Firstly, the $SO(2)$ invariance is extended to $O(2)$; quantum mechanically this corresponds to the invariance of the equations for the $\frac{d}{2}$-index antisymmetric tensor under the discrete duality transformation $F \rightarrow \star F$(where $\star$ is the Hodge dual). Secondly, for $q = \frac{d}{2}$ the action has a rigid conformal invariance. This is also true for $N > 2$, d even, for which the $(q - \frac{d}{2})$ term is necessarily absent. In terms of the conformal Killing vectors k^μ of Minkowski spacetime, which satisfy $k_{(\mu,\nu)} = \frac{1}{d}\eta_{\mu\nu}k^\rho{}_{,\rho}$, the infinitesimal conformal transformations can be written as

$$\delta x^\mu = k^\mu \qquad \delta p_\mu = -k^\nu{}_{,\mu}p_\nu - \frac{i}{2}\lambda_i^\rho\lambda_i^\sigma k_{\rho,\sigma\mu} \qquad \delta\lambda_{i\mu} = k_{[\mu,\nu]}\lambda_i^\nu$$

$$\delta e = \frac{2}{d}ek^\mu{}_{,\mu} \qquad \delta\psi_i = \frac{1}{d}(\psi_i k^\mu{}_{,\mu} - e\lambda_i^\nu k^\mu{}_{,\mu\nu}) \qquad \delta f_{ij} = i\frac{2}{d}\psi_{[i}\lambda_{j]}^\mu k^\nu{}_{,\nu\mu} \quad . \tag{2.32}$$

In the d=2 case there is of course an infinite number of conformal Killing vectors.

Observe that it is precisely when $q = \frac{d}{2}$ that the equations (2.23) are conformally invariant wave equations. This is an example of a property of a field theory that has a counterpart in a similar property of the underlying point particle action. When $q \neq \frac{d}{2}$ the invariance of the action requires that $k^\mu_{,\mu\nu} = 0$, which is satisfied by the Killing vectors that generate the rigid Poincaré invariance of the particle action, and by the conformal Killing vector that generates dilatations.

The extension of (2.13) to include a background gravitational field was given in [4] (for $q = \frac{d}{2}$) where it was also shown that the background is necessarily flat for $N > 2$. In terms of the variables of (2.14) , the action is

$$S = \int dt\Big\{\dot{x}^\mu p_\mu + i\bar{\xi}^a\dot{\xi}^b\eta_{ab} - i\psi\bar{\xi}^\mu(p_\mu - \frac{i}{2}\omega_{\mu bc}[\xi^b,\bar{\xi}^c])$$

$$-i\bar{\psi}\xi^{\mu}(p_{\mu}-\frac{i}{2}\omega_{\mu bc}[\xi^{b},\bar{\xi}^{c}])+f\big(\frac{1}{2}[\xi^{a},\bar{\xi}^{b}]\eta_{ab}-(q-\frac{d}{2})\big)$$
$$-\frac{1}{2}e\,g^{\mu\nu}(p_{\mu}-\frac{i}{2}\omega_{\mu ab}[\xi^{a},\bar{\xi}^{b}])(p_{\nu}-\frac{i}{2}\omega_{\nu cd}[\xi^{c},\bar{\xi}^{d}])\Big\}\quad , \tag{2.33}$$

where $\lambda^{a}=\lambda^{\mu}e^{a}_{\mu}$ with e^{a}_{μ} the vielbein satisfying $e^{a}_{\mu}e^{b}_{\nu}\eta_{ab}=g_{\mu\nu}$.

We shall now investigate the possibility of an electromagnetic background. We have to add to (1.1) the N–extended supersymmetric generalization of the Lorentz coupling $\int dt\dot{x}^{\mu}A_{\mu}$. This is

$$S_{L}=\int dt\left(\dot{x}^{\mu}A_{\mu}-\frac{i}{2}e\lambda^{\mu}_{i}\lambda^{\nu}_{i}F_{\mu\nu}\right)\quad . \tag{2.34}$$

Under the supersymmetry transformations of (1.1) , which are (in the absence of a gravitational background)

$$\begin{aligned}&\delta x^{\mu}=i\alpha_{i}\lambda^{\mu}_{i}\qquad \delta\lambda^{\mu}_{i}=-\alpha_{i}p^{\mu}\qquad \delta p_{\mu}=0\\&\delta e=2i\psi_{i}\alpha_{i}\qquad \delta\psi_{i}=\dot{\alpha}_{i}-f_{ij}\alpha_{j}\quad \delta f_{ij}=0\quad ,\end{aligned} \tag{2.35}$$

the action S_{L} has the variation

$$\delta S_{L}=i\alpha_{i}\lambda^{\mu}_{j}(\psi_{j}\lambda^{\nu}_{i}-\psi_{i}\lambda^{\nu}_{j})F_{\mu\nu}-\frac{i}{2}e\alpha_{j}\lambda^{\mu}_{i}\lambda^{\nu}_{i}\lambda^{\rho}_{j}\partial_{\rho}F_{\mu\nu}\quad . \tag{2.36}$$

For $N=1$ the first term vanishes identically and the second term by the Bianchi identity $\partial_{[\rho}F_{\mu\nu]}=0$. For $N\geq 2$ the first term vanishes if and only if $F_{\mu\nu}=0$. Thus an electromagnetic coupling is not possible for $N\geq 2$. This was to be expected from the well-known consistency problems with charged spin 1 fields.

Finally, we remark that the particle's wave function is, a priori, complex. The world-line time–reversal invariance of the action implies the existence of an antiunitary operator K that leaves the constraints invariant. For the spin $\frac{1}{2}$ particle it was shown in [10] that

$$K|\Psi\rangle=|\Psi\rangle \tag{2.37}$$

implies that Ψ is Majorana. On general grounds $K^{2}=\pm 1$ and (2.37) is consistent only when $K^{2}=+1$. As expected $K^{2}=-1$ whenever Ψ cannot be Majorana. The doubling

of Ψ in this case can therefore be thought of as an example of Kramer's degeneracy in quantum mechanics. For general N, we can write K as

$$K = (\otimes^N U) K_0 \quad , \tag{2.38}$$

where K_0 is the (basis-dependent) complex conjugation operator and U is a unitary matrix acting on H_1 with the property that $U^* U = \pm 1$. Therefore for N odd $K^2 = \pm 1$ whereas for N even $K^2 = +1$, necessarily. For N even it is therefore always possible to require of the wave function that it be real. In particular, this means that the antisymmetric tensors discussed in this paper can always be chosen to be real.

3. $N = 2$ worldline supergravity in superspace

In this section we present the superspace formulation of the $N = 2$ model, thereby updating the results of [2] and generalizing the $N = 1$ results of [10]. In flat superspace the covariant derivatives are

$$D = i\left(\partial_\theta - \frac{i}{2}\bar{\theta}\partial_t\right) \qquad \bar{D} = i\left(\partial_{\bar{\theta}} - \frac{i}{2}\theta\partial_t\right) \quad , \tag{3.1}$$

which satisfy

$$\{D, \bar{D}\} = i\partial_t \qquad D^2 = \bar{D}^2 = 0 \quad . \tag{3.2}$$

In a general superspace we denote a supervector V in a coordinate basis by $V^M = (v^t, v^\theta, v^{\bar{\theta}})$ and in a tangent space basis by $V^A = (V^0, V^1, V^{\bar{1}})$ with the summation convention

$$V^A U_A = V^0 U_0 + V^1 U_1 - V^{\bar{1}} U_{\bar{1}} \quad . \tag{3.3}$$

The flat superspace supervielbein, $\hat{E}_M^{\ A}$, therefore has the following form in standard coordinates

$$\hat{E}_t^{\ 0} = 1 \qquad \hat{E}_t^{\ 1} = 0 \qquad \hat{E}_t^{\ \bar{1}} = 0$$

$$\hat{E}_\theta{}^0 = \frac{1}{2}\bar{\theta} \qquad \hat{E}_\theta{}^1 = i \qquad \hat{E}_\theta{}^{\bar{1}} = 0$$

$$\hat{E}_{\bar{\theta}}{}^0 = \frac{1}{2}\theta \qquad \hat{E}_{\bar{\theta}}{}^1 = 0 \qquad \hat{E}_{\bar{\theta}}{}^{\bar{1}} = -i \tag{3.4}$$

We take the tangent space group to be $U(1)$, which acts by

$$\delta V^A = V^B L_B{}^A \tag{3.5}$$

with

$$L_1{}^1 = iL \qquad L_{\bar{1}}{}^{\bar{1}} = iL \quad , \tag{3.6}$$

and all others zero. With this structure it is easy to show that the imposition of standard conventional constraints implies that a general superspace of this type is in fact flat;

$$T_{1\bar{1}}{}^0 = -i \quad , \tag{3.7}$$

with all other components of $T_{AB}{}^C$ vanishing. These constraints are invariant under the super Weyl transformations

$$E_M{}^0 \to S E_M{}^0$$

$$E_M{}^1 \to S^{\frac{1}{2}} - iS^{-\frac{1}{2}} E_M{}^0 D_{\bar{1}} S \quad , \tag{3.8}$$

where S is a real scalar superfield.

Since the superspace is flat we could choose a gauge in which the supervielbein takes the standard form (modulo moduli). In order to make contact with the component formalism we instead make the partial gauge choice for which E_M^A has its superconformally flat form, i.e. E_M^A differs from $\hat{E}_M^A$ by a superconformal transformation with parameter V. We then find

$$D_1 = V^{-\frac{1}{2}} D \qquad D_{\bar{1}} = V^{-\frac{1}{2}} \bar{D}$$

$$D_0 = V^{-1}\partial_t + \frac{i}{2} V^{-2}(\bar{D}VD + DV\bar{D}) \quad . \tag{3.9}$$

This gauge choice is preserved under a general superspace coordinate transformation, with parameter ξ^A, provided that

$$D_A\xi^B + \xi^C T_{CA}{}^B + L_A{}^B = H_A{}^B(S = V^{-1}\delta V) \quad , \tag{3.10}$$

where $H_A{}^B(S)$ is the infinitesimal form of the super Weyl transformation, i.e.

$$\delta E_M{}^A = E_M{}^B H_B{}^A(S) \quad , \tag{3.11}$$

as can be derived from (3.8) . One finds

$$\begin{aligned} &\xi^1 = -iD_{\bar{1}}\xi^0 \qquad V^{-1}\delta V = \frac{1}{2}D_0\xi^0 \\ &L = \frac{i}{2}[D_1, D_{\bar{1}}]\xi^0 \quad , \end{aligned} \tag{3.12}$$

which implies that

$$\delta V = \frac{1}{2}[\partial_t + iV^{-2}(\bar{D}VD + DV\bar{D})]\xi^0 \quad . \tag{3.13}$$

The action for a set of d scalar superfields Φ^μ coupled to supergravity is given by

$$S = -2\int dtd^2\theta E D_1\Phi^\mu D_{\bar{1}}\Phi^\nu \eta_{\mu\nu} \tag{3.14}$$

which, in the superconformal gauge, is simply

$$S = -2\int dtd^2\theta\ V^{-1}D\Phi\bar{D}\Phi \quad . \tag{3.15}$$

Integrating over θ and $\bar{\theta}$ and eliminating the auxiliary field, which is the $\theta\bar{\theta}$ component of Φ, we obtain agreement with the component action upon making the identifications

$$\begin{aligned} &e = V| \qquad \Psi = -\frac{1}{\sqrt{2}}V\bar{D}V| \qquad f = \frac{1}{2}[D, \bar{D}]lnV| \\ &x^\mu = \Phi^\mu| \qquad \xi^\mu = \sqrt{2}V^{-\frac{1}{2}}D\Phi^\mu| \quad , \end{aligned} \tag{3.16}$$

where the vertical bar indicates that the superfield is to be evaluated at $\theta = \bar{\theta} = 0$. From the equation for f we see that the superspace integrand of the Chern-Simons form is $-q\ln V$.

It is straightforward to couple the system to a non-trivial gravitational field in this formalism; one simply replaces the flat Minkowski space metric $\eta_{\mu\nu}$ by a general metric $g_{\mu\nu}(\Phi)$ in (3.14) .

4. Massive antisymmetric tensor field equations from particle mechanics

If the background spacetime for the action (2.33) has isometries, generated by the Killing vectors $k^{\mu}_{(r)}$, then the action has corresponding rigid symmetries generated by the Noether charges

$$G_{(r)} = k^{\mu}_{(r)}(x)p_{\mu} \quad . \tag{4.1}$$

These symmetries can be gauged by adding to the action the term

$$S_G = \int d\ell \; l_{(r)}G_{(r)} \quad . \tag{4.2}$$

Upon quantization the wave function must then satisfy the additional constraint

$$k^{\mu}(x)\partial_{\mu}\Psi(x) = 0 \quad . \tag{4.3}$$

This is equivalent to a formulation of field theory in a lower dimensional spacetime. The simplest example occurs for a spacetime of the form $(\text{Minkowski})_d \times S^1$, in which case $G = p_d$ is the charge associated with translations around the circle S^1. The constraint (4.3) becomes

$$\partial_{\mu_d}\Psi(x) = 0 \quad , \tag{4.4}$$

so that Ψ depends only on x^{μ}, $\mu = 0, .., d-1$, and the massless field equations in $(d+1)$ dimensions reduce to massless field equations of d-dimensional Minkowski space. This is

just the usual dimensional reduction, but because $U(1)$ is Abelian it is possible to modify this procedure, as shown by Scherk and Schwarz [6]. In the context of particle mechanics the Scherk–Schwarz mechanism operates by the addition to the action of the term

$$\int dt\ l(G+m) \quad , \tag{4.5}$$

where m is an arbitrary constant. Taking $G = p_d$ we find that the particle wave function must satisfy

$$p_d\Psi = -m\Psi \quad . \tag{4.6}$$

In this case the massless field equations in $(d+1)$ dimensions (which are a consequence of the other constraints) will become massive field equations in d dimensions*. Moreover, since the little group for massless particles in $(d+1)$ dimensions equals that for massive particles in d dimensions, a pure spin $\frac{N}{2}$ massless field will always have just the right number of components for a pure spin $\frac{N}{2}$ massive field in one lower dimension.

Since the action (1.1) is applicable to massless fields with spin $\geq \frac{3}{2}$ only for d even, this mechanism allows us to obtain massive fields of arbitrary spin in odd dimensional spacetimes only. For $N = 2$ there is no such restriction, however, and we can obtain massive antisymmetric tensor field equations for arbitrary d. The appropriate particle action is just (2.13) with the addition of the term (4.5) . Rather than add (4.5) we can simply solve the constraint by substituting $(-m)$ for p_d wherever it appears in the rest of the action. In this way we arrive at the action

$$S = \int dt \Big\{ \dot{x}\cdot p + i\bar{\xi}\cdot\dot{\xi} + i\bar{\zeta}\dot{\zeta} - \frac{1}{2}e(p^2+m^2) - i\bar{\psi}(\xi\cdot p - m\zeta) +$$

$$-i\psi(\bar{\xi}\cdot p - m\bar{\zeta}) + f(\frac{1}{2}[\xi,\bar{\xi}] + \frac{1}{2}[\zeta,\bar{\zeta}] - q + \frac{d}{2}) \Big\} \quad , \tag{4.7}$$

* This mechanism has been used previously in the context of other particle models by Henneaux and Teitelboim [12]

where $\zeta = \frac{1}{\sqrt{2}}(\lambda_1^d + i\lambda_2^d)$. The complex wave function satisfying the $SO(2)$ constraint can be written as

$$\frac{1}{q!}\alpha^{\mu_1}...\alpha^{\mu_q}F_{\mu_1...\mu_q} + \frac{im}{(q-1)!}\beta\alpha^{\mu_1}...\alpha^{\mu_{q-1}}A_{\mu_1...\mu_{q-1}} \quad , \tag{4.8}$$

where F and A are, a priori, also complex, and β is the anticommuting eigenvalue of ζ, $\langle\beta|\zeta = \langle\beta|\beta$. The constraint $Q|\Psi\rangle = 0$ imposed by $\bar{\psi}$ yields

$$F_{\mu_1...\mu_q} = q\partial_{[\mu_1}A_{\mu_2...\mu_q]} \quad , \tag{4.9}$$

which tells us that A is the gauge potential for F. The constraint $\bar{Q}|\Psi\rangle = 0$ then gives the antisymmetric tensor generalization of the Proca equation

$$\partial^\mu F_{\mu\nu_1...\nu_{q-1}} - m^2 A_{\nu_1...\nu_{q-1}} = 0 \quad . \tag{4.10}$$

The factor of i in (4.8) is now seen to be necessary in order that A may ultimately taken to be real. There is presumably some variant on the time reversal identification constraint $K|\Psi\rangle = |\Psi\rangle$ that could be imposed on the wave function $|\Psi\rangle$, ab initio, so as to ensure that A is real, without having to impose it directly on A itself, but we have not investigated this point.

One question that may have occurred to the reader, concerning the general construction given above, is whether the addition of terms to the action of the form (4.2) could spoil symmetries that were present before their addition. It is readily checked for the example above that this does not happen. In the general case it is ensured by the fact that the generators $G_{(r)}$ commute with the other generators. In the next section we will want to add generators that do <u>not</u> commute, but as long as we have a set of generators $\{\phi_l\}$ which close to form an algebra under the Poisson bracket,

$$\{\phi_l, \phi_m\}_{PB} = f_{lm}{}^n\phi_n \quad , \tag{4.11}$$

we can immediately write down a gauge invariant action*. To see this, consider a phase (super)space with coordinates Y^A and let Ω be the supersymplectic form on phase space. Then the Poisson bracket is given by

$$\{f,g\}_{PB} = -f\overleftarrow{\partial}_A \Omega^{AB} \overrightarrow{\partial}_B g \quad , \tag{4.12}$$

with

$$\begin{aligned} \Omega^{AB}\Omega_{BC} &= \delta^A_C \\ \Omega_{AB} &= -(-1)^{ab}\Omega_{BA} \quad . \end{aligned} \tag{4.13}$$

The action

$$S = \int dt\Big[-\frac{1}{2}\dot{Y}^B \Omega_{BA} Y^A - u^l \phi_l\Big] \tag{4.14}$$

has the gauge invariance

$$\delta Y^A = -k^l(\phi_l \overleftarrow{\partial}_B)\Omega^{BA} \qquad \delta u^l = -\dot{k}^l - u^m k^n f_{nm}{}^l \quad . \tag{4.15}$$

It follows that the existence of a gauge invariant action is guaranteed once a set of constraints has been shown to form a closed algebra. What is not guaranteed is that the algebra of the quantum generators with respect to (anti)commutation will be the same as that of the classical generators with respect to the Poisson brackets. We shall encounter examples where the classical and quantum algebras differ, but provided no new generators arise, and the algebra still closes, this causes no problems.

5. Chirality and self–duality

For d even a chirality constraint $\gamma_* \Psi = \pm\Psi$ can be imposed on a Dirac spinor Ψ. Quantization of the $N = 1$ particle model leads to a Dirac equation for a non-chiral

* This construction is well-known. See for example [13]

spinor, but the model can be modified to incorporate a constraint which yields chiral wave-functions. Consider the classical quantity

$$\Lambda_1 = (2i)^{\frac{d}{2}} \lambda^0 \lambda^1 ... \lambda^{d-1} \quad . \tag{5.1}$$

Upon quantization we have

$$\Lambda_1 \rightarrow \hbar^{\frac{d}{2}} i^{\frac{d}{2}} \gamma^0 ... \gamma^{d-1} = i\hbar^{\frac{d}{2}} \gamma_* \quad . \tag{5.2}$$

To obtain a chirality constraint we should therefore add to the action a term of the form

$$\int dt \; g_1(t)(\Lambda_1 + i\hbar^{\frac{d}{2}}) \quad . \tag{5.3}$$

Variation of the Lagrange multiplier g_1 leads to the constraint $P_+ \approx 0$, where

$$P_+ = \Lambda_1 + i\hbar^{\frac{d}{2}} \quad . \tag{5.4}$$

This leads to the quantum mechanical constraint

$$\frac{1}{2}(1 + \gamma_*)\Psi = 0 \quad . \tag{5.5}$$

The Lagrange multiplier term (5.3) is not real, and indeed $(1 \pm \gamma_*)$ is not hermitian with respect to the Dirac scalar product (although for Euclidean signature this term would be real and the corresponding constraint hermitian). However, this lack of reality is in a constraint term and does not present problems for the physical subspace.

The inclusion of the new constraint generates one further constraint, and the non-zero Poisson brackets of the complete set of constraint functions are

$$[Q, Q]_{PB} = 2iH \qquad [P_+, Q]_{PB} = -2i\tilde{Q} \qquad [Q, \tilde{Q}]_{PB} = 0$$

$$[P_+, \tilde{Q}]_{PB} = \begin{cases} -2iQ \; , & \text{d=2} \\ 0 \; , & d \geq 4 \end{cases} \qquad [\tilde{Q}, \tilde{Q}]_{PB} = \begin{cases} 2iH \; , & \text{d=2} \\ 0 \; , & d \geq 4 \end{cases} \quad , \tag{5.6}$$

where $\tilde{Q}$ is a new supersymmetry generator,

$$\tilde{Q} = \frac{2i^{\frac{d-2}{2}}}{(d-1)!}\varepsilon_{\mu_1\ldots\mu_d}\lambda^{\mu_1}\ldots\lambda^{\mu_{d-1}}p^{\mu_d} \quad , \tag{5.7}$$

and Q and H are the original generators of supersymmetry and reparametrizations, respectively. Of course, we should now add to the action a Lagrange multiplier for $\tilde{Q}$.

Upon quantization

$$\tilde{Q} \to \frac{1}{\sqrt{2}}\gamma_\star \gamma \cdot p \quad , \tag{5.8}$$

and the quantum algebra is, for all even d,

$$[P_+, Q] = \tilde{Q} \qquad [P_+, \tilde{Q}] = Q \qquad \{Q, \tilde{Q}\} = 0$$

$$Q^2 = H \qquad \tilde{Q}^2 = -H \quad . \tag{5.9}$$

For $d \geq 4$ there are therefore quantum corrections to the classical algebra. The quantum algebra is a non-compact $N = 2$ extended supersymmetry algebra with an internal $O(1,1)$ symmetry group generated by P_+.

In the $N = 2$ case, the analogue of the chirality constraint gives rise to (anti) self-dual antisymmetric gauge field strength tensors. The $N = 2$ analogue of Λ_2 is, in terms of the complex ξ variables introduced in section 2,

$$\Lambda_2 = \frac{i^n}{(n!)^2}\varepsilon_{\mu_1\ldots\mu_n\nu_1\ldots\nu_n}\xi^{\nu_1}\ldots\xi^{\nu_n}\bar{\xi}^{\mu_n}\ldots\bar{\xi}^{\mu_1} \quad . \tag{5.10}$$

Quantum-mechanically, Λ_2 acting on an inhomogeneous differential form annihilates all but the n-form contribution. On an n-form F we have

$$\Lambda_2 F = i^n \star F \quad , \tag{5.11}$$

where $\star$ is the Hodge dual operator

$$(\star F)_{\mu_1\ldots\mu_n} = \frac{1}{n!}\varepsilon_{\nu_1\ldots\nu_n\mu_1\ldots\mu_n}F^{\nu_1\ldots\nu_n} \quad . \tag{5.12}$$

Thus in order to impose the (anti)self-duality constraint

$$F = \pm i^{n-1} \star F \tag{5.13}$$

we should add to the action the Lagrange multiplier term

$$\int dt g_2(t)(\Lambda_2 \mp i\hbar^{\frac{d}{2}}) \quad , \tag{5.14}$$

where g_2 is a new Lagrange multiplier. Clearly, it is only possible to impose reality and self-duality simultaneously in $d = 2 \bmod 4$ (Minkowski space-time) dimensions. As in the $N = 1$ case the new constraint generates yet further constraints. The quantum algebra of these constraints is most easily analysed in terms of differential forms. We have the correspondences

$$\xi^\mu \leftrightarrow dx^\mu \qquad \bar{\xi}_\mu \leftrightarrow i_\mu$$

$$Q \leftrightarrow d \qquad \bar{Q} \leftrightarrow d^*$$

$$H \leftrightarrow dd^* + d^*d \equiv \Delta \qquad M \leftrightarrow dx^\mu i_\mu - n \quad , \tag{5.15}$$

where d is the exterior derivative operator, i_μ denotes the interior product to be taken with the basis vector $\frac{\partial}{\partial x^\mu}$ and $dx^\mu i_\mu$ on a p-form ω_p gives $p\omega_p$. The operator d^* is the adjoint of d with respect to the scalar product

$$\left(\Psi^{(1)}, \Psi^{(2)}\right) = \sum_{p=0}^{2n} \int \bar{\Psi}_p^{(1)} \wedge \star \Psi_p^{(2)} \quad , \tag{5.16}$$

where the wave functions are inhomogeneous forms,

$$\Psi^{(i)} = \sum_{p=0}^{2n} \Psi_p^{(i)} \qquad \Psi_p^{(i)} \text{ a p form} \quad . \tag{5.17}$$

For $2n$-dimensional Minkowski space

$$d^* = \star d \star \quad . \tag{5.18}$$

We introduce the operators τ_p by

$$\tau_p = \begin{cases} i^{p(p-1)+n-1}\star & \text{on p-forms} \\ 0 & \text{on q-forms } q \neq p \end{cases} \tag{5.19}$$

which have the following properties

$$\tau_{2n-p}\cdot\tau_p = 1 \qquad \tau_p d = -d^*\tau_{p-1} \qquad \tau_p d^* = -d\tau_{p+1} \quad . \tag{5.20}$$

The self-duality constraint is incorporated by extending the $N = 2$ algebra generated by $Q, \bar{Q}, H$ and M to include the operator

$$P = 1 + \tau_n \quad . \tag{5.21}$$

The following additional operators are then generated

$$\begin{aligned} S_1 &= \tau_n d - d\tau_n \qquad \bar{S}_1 = \tau_n d^* - d^*\tau_n \\ S_2 &= I_n d + d I_n \qquad \bar{S}_2 = I_n d^* + d^* I_n \\ K &= \Delta I_n + (dd^*)I_{n+1} + (d^*d)I_{n-1} \quad , \end{aligned} \tag{5.22}$$

where I_p is the projector onto p-forms. The complete algebra therefore has the generators $(H, M, P, K; Q, \bar{Q}, S_1, \bar{S}_1, S_2, \bar{S}_2)$. The non-zero (anti)commutation relations are

$$\{Q, \bar{Q}\} = \Delta \qquad [M, Q] = Q \qquad [M, \bar{Q}] = -\bar{Q}$$

$$[P, Q] = S_1 \qquad [P, S_1] = S_2 \qquad [P, S_2] = S_1$$

$$[P, \bar{Q}] = \bar{S}_1 \qquad [P, \bar{S}_1] = \bar{S}_1$$

$$[M, S_1] = S_1 \qquad [M, S_2] = S_2$$

$$[M, \bar{S}_1] = -S_1 \qquad [M, \bar{S}_2] = -S_2$$

$$\{S_1, \bar{S}_1\} = -K \qquad \{S_2, \bar{S}_2\} = K$$

$$\{\bar{Q}, S_2\} = K \qquad \{Q, \bar{S}_2\} = K \quad . \tag{5.23}$$

Thus there are six (real) odd generators. The subalgebra generated by S_1, S_2, M, P and K is an $N = 4$ supersymmetry algebra with an internal $O(2) \times O(1,1)$ symmetry.

In the $n = 1$ case, K equals H and S_2 coincides with Q. The algebra, which is then spanned by $Q, \bar{Q}, S_1, \bar{S}_1, M, P$ and H, is again an $N = 4$ algebra with an $O(2) \times O(1,1)$ internal symmetry.

For $N \geq 2$ the analogue of Λ_1 and Λ_2 is

$$\Lambda_N \propto \varepsilon_{\mu_1\nu_1\ldots\mu_n\nu_n} \left(\lambda^{\mu_1}_{i_1} \lambda^{\nu_1}_{i_1}\right) \ldots \left(\lambda^{\mu_n}_{i_n} \lambda^{\nu_n}_{i_n}\right) \tag{5.24}$$

For N even the imposition of the generalized self-duality constraint implies that the wave function (in tensorial form) is self-dual on each block of n indices; for N odd the wave function is, in addition, constrained to be chiral. We have not attempted to work out the full algebra of constraints in this case.

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