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A PARTITION THEOREM FOR PERFECT SETS

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ABSTRACT. Let P be a perfect subset of the real line, and let the n-element subsets of P be partitioned into finitely many classes, each open (or just Borel) in the natural topology on the collection of such subsets. Then P has a perfect subset whose n-element subsets lie in at most $(n-1)!$ of the classes.

Let C be the set of infinite sequences of zeros and ones, topologized as the product of countably many discrete two-point spaces, and ordered lexicographically. For $X \subseteq C$, let $[X]^n$ be the set of n-element subsets of X. When we describe a finite subset of C by listing its elements, we always assume that they are listed in increasing order. Thus, $[C]^n$ is identified with a subset of the product space C^n, from which it inherits a topology. A subset of C is *perfect* if it is nonempty and closed and has no isolated points.

The purpose of this paper is to prove the following partition theorem, which was conjectured by F. Galvin who proved it [3] for $n \leqslant 3$.

THEOREM. *Let P be a perfect subset of C and let $[P]^n$ be partitioned into a finite number of open (in $[P]^n$) pieces. Then there is a perfect set $Q \subseteq P$ such that $[Q]^n$ intersects at most $(n-1)!$ of the pieces.*

Before setting up the machinery for the proof of this theorem, we point out some of its consequences. First, the therem remains true if C is replaced by the real line $\mathbf{R}$ with its usual topology and order. To see this, it suffices to observe that every perfect subset of $\mathbf{R}$ has a subset (a generalized Cantor set) homeomorphic to C via an order-preserving map and that any one-to-one continuous image in $\mathbf{R}$ of a perfect subset of C is perfect in $\mathbf{R}$. Second, the hypothesis that the pieces of the partition are open can be greatly relaxed. Mycielski [6], [7] has shown that any meager set or any set of measure zero in $[\mathbf{R}]^n$ is disjoint from $[P]^n$ for some perfect $P \subseteq \mathbf{R}$. For the meager case, he obtains the same result with $\mathbf{R}$ replaced by any complete metric space X without isolated points. It follows that, if $[\mathbf{R}]^n$ (or $[X]^n$) is partitioned into finitely many pieces that have the Baire property, then their intersections with $[P]^n$ are open in $[P]^n$ for some perfect P. Similarly, if the pieces are Lebesgue measurable, they become G_δ sets when restricted to $[P']^n$ for suitable perfect P'; since G_δ sets have the Baire property, we can apply the preceding sentence, with P' as X, to get a perfect $P \subseteq P'$ such that the pieces intersected with $[P]^n$ are open in $[P]^n$. Thus, our theorem, as extended by the first remark above, implies the following partition theorem.

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COROLLARY. *If* $[\mathbf{R}]^n$ *is partitioned into finitely many pieces that all have the Baire property or are all measurable, then there is a perfect set* $Q \subseteq \mathbf{R}$ *such that* $[Q]^n$ *meets at most* $(n-1)!$ *of the pieces.*

Some hypothesis about the pieces is necessary, however, for Galvin and Shelah **[4]** have shown that there is a partition of $[C]^2$ into infinitely many pieces such that, for any $Q \subseteq C$ of the cardinality of the continuum (in particular, for any perfect Q), $[Q]^2$ intersects all the pieces.

At the suggestion of the referee, we mention that the Corollary immediately implies Filipczak's result **[2**, Theorem 1] that any continuous (or only Borel measurable) real-valued function defined on a perfect set of reals is monotonic on a perfect subset. Although this application uses only the (trivial) case $n = 2$, it seems reasonable to expect similar applications for the higher cases of our theorem. Another application of the case $n = 2$ occurs in the descriptive set theory of selectors **[1]**.

The proof of the theorem uses the correspondence between perfect subsets of C and perfect trees. A *tree* is a set T of finite sequences of zeros and ones, called the *nodes* of T, such that, for each node s of T, all initial segments and at least one proper extension of s are also in T. A tree T is partially ordered by the inclusion (or initial segment) relation $\subseteq$, and words like "above", "below", and "comparable" always refer to this order. The nth *level* of T is the set of nodes whose length, as sequences, is n. A *path* through T, i.e., a linearly ordered subset containing a point from every level, can be identified with an element of C. The set of all paths through T will be written (T), since the customary notation $[T]$ conflicts with the notation $[X]^n$; (T) is a closed subset of C and every closed subset of C is (T) for a unique tree T. Clearly, $(T_1) \subseteq (T_2)$ if and only if $T_1 \subseteq T_2$.

A *fork* of a tree T is a node s both of whose one-term extensions $s\langle 0\rangle$ and $s\langle 1\rangle$ are in T. A tree is *perfect* if it is nonempty and every node is below a fork; this is equivalent to (T) being a perfect subset of C. Note that, in a perfect tree, every node is below forks in arbitrarily high levels.

The theorem asserts that a perfect set P has a perfect subset Q with certain properties. Since we shall work with the tree representation of perfect sets, we shall be constructing perfect subtrees of given perfect trees. The following terminology will be useful in such constructions. Let a tree T be given. "Remove node s from T" means that T is to be replaced by its largest subtree not containing s. Thus, what is actually removed is not just s but all the nodes above s and all nodes below s but above the highest fork below s. "Kill the fork s of T" means to remove (in the sense just explained) one of the two immediate successors of s. The choice of which successor to remove is arbitrary except when a specific node t above s is to be "retained"; then the immediate successor of s that is not below t is to be removed. Note that, in the resulting tree, s is a node but not a fork. Note also that, if T is perfect, it remains perfect when a fork is killed. "Fix all nodes up to level l" means nothing as far as changing T is concerned but indicates that nodes at level l or lower will remain intact during subsequent operations on T. Finally, if s is a node of T, then $T(s)$ is the subtree of T consisting of all nodes comparable with s; it is perfect if T is.

It will be convenient to work with trees T that are *skew* in the sense that no level contains two distinct forks. Although the following lemma is very easy, we include its proof for future reference and as an example of the terminology introduced above.

LEMMA. *Every perfect tree has a perfect skew subtree.*

PROOF. Enumerate the nodes of the given perfect tree T in a sequence $s_0, s_1, s_2, \ldots,$ and replace T by perfect subtrees $T_0 = T \supseteq T_1 \supseteq T_2 \supseteq \ldots$ as follows. Suppose that, after n steps, we have defined T_n and fixed all nodes up to some level l_n. If $s_n \notin T_n$, do nothing, i.e., set $T_{n+1} = T_n$ and $l_{n+1} = l_n$. Otherwise, find a fork t of T_n, above s_n, at level $l > l_n$. Kill all forks except t in levels l_n through l inclusive, retaining t, and let T_{n+1} be the tree so obtained. Then fix all nodes up to level $l_{n+1} = l + 1$. Since t is a fork in T_k for every k, it is clear that $T' = \bigcap_n T_n$ is a perfect tree. And, since no level $< l_n$ has two forks in T_n, T' is skew. $\square$

In view of this lemma, our theorem is equivalent to the following.

THEOREM. *If T is a perfect skew tree, then, for any partition of $[(T)]^n$ into finitely many open pieces, there is a perfect subtree T' of T such that $[(T')]^n$ meets at most $(n-1)!$ of the pieces.*

For distinct $\alpha, \beta \in (T)$, let $\Delta(\alpha, \beta)$ be the highest common node of the paths α and β through T. By the *pattern* of an n-element set $\{\alpha_0, \ldots, \alpha_{n-1}\} \subseteq (T)$, we mean the linear ordering ρ of $\{1, \ldots, n-1\}$ given by

$$i\rho j \leftrightarrow \Delta(\alpha_{i-1}, \alpha_i) \text{ is in a lower level than } \Delta(\alpha_{j-1}, \alpha_j).$$

(Recall that the α_i are listed in lexicographic order and that T is skew.) Thus, ρ tells us in what order the paths α_i split apart as we proceed up the tree. Since there are $(n-1)!$ linear orderings of $\{1, \ldots, n-1\}$ and all of them are obviously realized as patterns within any perfect subset of (T), we see that the $(n-1)!$ in the conclusion of the theorem is optimal. We also see that, to prove the theorem, it suffices to find a perfect subtree T' such that the partition class of an n-element subset of (T') depends only on its pattern. In fact, it suffices to show that, for any finite open partition of $[(T)]^n$ and any fixed pattern ρ, there is a perfect subtree T' such that all n-element subsets of (T') with the given pattern ρ lie in the same partition class, for we can then repeat the construction for each ρ in turn.

To obtain such a T', it is convenient to work in a more general setting as follows. Let $\mathbf{T} = \langle T_0, \ldots, T_{r-1}\rangle$ be a skew r-tuple of perfect trees; this means not only that each T_i is skew but also that no two distinct T_i's have forks at the same level. Let $\mathbf{n} = \langle n_0, \ldots, n_{r-1}\rangle$ be an r-tuple of positive integers with sum n. By an $\mathbf{n}$*-set in* $\mathbf{T}$ we mean an r-tuple $\boldsymbol{\sigma}$ whose ith entry σ_i is an n_i-element subset of (T_i). The *pattern* ρ of such an $\mathbf{n}$-set $\boldsymbol{\sigma}$ is the linear ordering of the pairs (i, j), with $0 \leqslant i < r$ and $0 < j < n_i$, given by the levels of the nodes $\Delta(\alpha_{i,j-1}, \alpha_{i,j})$, where $\alpha_{i,j}$ is the jth element of σ_i (in lexicographic order). We shall prove the following result, whose special case $r = 1$ is all we need, according to the preceding paragraph, to establish our main theorem. Let the collection $\Pi_{i<r}[(T_i)]^{n_i}$ of $\mathbf{n}$-sets in $\mathbf{T}$ have the product topology.

POLARIZED THEOREM. *Let r, $\mathbf{T}$, $\mathbf{n}$, n be as before and let ρ be any pattern of $\mathbf{n}$-sets. Let the collection of all $\mathbf{n}$-sets in $\mathbf{T}$ be partitioned into finitely many open pieces. Then there exist perfect trees $T_i' \subseteq T_i$ such that all $\mathbf{n}$-sets in $\mathbf{T}'$ with pattern ρ lie in the same piece of the partition.*

The proof of this theorem occupies the remainder of this paper. We proceed by downward induction on r, with n fixed. Since each n_i is required to be positive, the highest possible value of r is n, and if $r = n$ each $n_i = 1$. An $\mathbf{n}$-set from $\mathbf{T}$ is then just an n-tuple $\boldsymbol{\alpha}$ of paths α_i through T_i. Fix such an $\mathbf{n}$-set $\boldsymbol{\alpha}$. Since the partition is open, all $\mathbf{n}$-sets sufficiently close to $\boldsymbol{\alpha}$ lie in the same partition class. Thus, for sufficiently long finite initial segments s_i of α_i, the trees $T_i' = T_i(s_i)$ satisfy the conclusion of the theorem. (When $r = n$, there is only one possible pattern, so patterns play no role in this case.)

We turn our attention to the nontrivial case $r < n$ and assume the polarized theorem for $r + 1$ (and the same n). By reindexing the trees, we may assume that the first element in ordering ρ is $(0, q)$ for some q. This means that, if an $\mathbf{n}$-set $\boldsymbol{\sigma}$ (with $\sigma_i = \{\alpha_{i,j} | j < n_i\}$ as before) has pattern ρ, then the lowest-level branching within any σ_i occurs in σ_0, where $\alpha_{0,0}, \ldots, \alpha_{0,q-1}$ separate from $\alpha_{0,q}, \ldots, \alpha_{0,n_0-1}$ at a fork which we call the *first fork* of $\boldsymbol{\sigma}$, $f(\boldsymbol{\sigma})$. In each of the remaining trees T_i $(1 \leqslant i < r)$, all n_i of the paths $\alpha_{i,j}$ pass through the same node s_i at the level of $f(\boldsymbol{\sigma})$, as otherwise the pattern could not begin with $(0, q)$. We call the r-tuple $\langle f(\boldsymbol{\sigma}), s_1, \ldots, s_{r-1}\rangle$ the *signature* of the $\mathbf{n}$-set $\boldsymbol{\sigma}$. Our immediate goal is to find perfect subtrees $T_i^* \subseteq T_i$ in which all $\mathbf{n}$-sets having pattern ρ and having the same signature lie in the same partition class. (Later, we shall reduce these trees further to eliminate the dependence on the signature.)

Each tree T_i^* will be obtained as the intersection of an inductively defined decreasing sequence of perfect subtrees of T_i. To describe the induction without excessive notation, we use S_i as a variable representing, at each point in the proof, the subtree of T_i obtained at that point. To begin the induction, set $S_i = T_i$ and fix all nodes up to level 0 (in all these trees).

Suppose that, at a later stage, we have obtained perfect subtrees S_i and we have fixed all nodes up to level l. Choose a level l' so high that in each S_i $(i \neq 0)$ every node at level l has at least two successors at level l'; this can be done because S_i is perfect. Kill all forks of S_0 between levels l and l' inclusive, and then fix all remaining nodes up to level l' in all the trees. (The choice of l' and this fixing of nodes, being repeated at every stage of our induction, clearly ensure that T_i^* will be perfect for $i \neq 0$.) Next, choose a fork f of (the new) S_0 above level l', say at level l''. Kill all forks of S_0 between levels l' and $l'' - 1$ inclusive, retaining f, and fix all remaining nodes up to level $l'' + 1$ in all the trees. This guarantees that f will be a fork of T_0^* and indeed the only fork between levels l and l'' inclusive (as T_0 was skew).

We now seek to ensure that all $\mathbf{n}$-sets with pattern ρ and with f as the first fork have their partition classes determined by their signatures. Enumerate the possible signatures $\langle f, s_1, \ldots, s_n\rangle$ that begin with f; there are of course only finitely many of them since each s_i must be at the same level l'' as f and our trees have finite levels. Consider each such signature in turn. An $\mathbf{n}$-set $\boldsymbol{\sigma}$ in $\mathbf{S}$ with pattern ρ and signature $\langle f, s_1, \ldots, s_{r-1}\rangle$ yields, by splitting the first component, an $\mathbf{n}^*$-set $\boldsymbol{\sigma}^* = \langle\{\alpha_{0,0}, \ldots, \alpha_{0,q-1}\}, \{\alpha_{0,q}, \ldots, \alpha_{0,n_0-1}\}, \sigma_1, \ldots, \sigma_{r-1}\rangle$ in $\mathbf{S}^* = \langle S_0(f\langle 0\rangle), S_0(f\langle 1\rangle), S_1(s_1), \ldots, S_{r-1}(s_{r-1})\rangle$, where $\mathbf{n}^*$ is the $(r+1)$-tuple $\langle q, n_0 - q, n_1, \ldots, n_{r-1}\rangle$. The pattern ρ^* of $\boldsymbol{\sigma}^*$ is uniquely determined by ρ, and $\boldsymbol{\sigma}^*$ determines $\boldsymbol{\sigma}$. Partition the $\mathbf{n}^*$-sets in $\mathbf{S}^*$ with pattern ρ^* by putting two such $\boldsymbol{\sigma}^*$'s in the same piece if and only if the corresponding $\boldsymbol{\sigma}$'s are in the same piece of the original partition. Since $\boldsymbol{\sigma}$ is a continuous function of $\boldsymbol{\sigma}^*$, this is an open partition of $\mathbf{n}^*$-sets, and we can apply the induction hypothesis to find perfect subtrees of the S_i^*'s such that all their $\mathbf{n}^*$-sets with pattern ρ^* lie in the same partition class. Prune the trees S_i correspondingly, so that the new $S_0(f\langle 0\rangle)$, $S_0(f\langle 1\rangle)$, $S_1(s_1), \ldots, S_{r-1}(s_{r-1})$ are the subtrees of the S_i^*'s given by the induction hypothesis. Note that this pruning takes place entirely above level $l'' + 1$, because (a) the subtrees given by the induction hypothesis are perfect, hence nonempty, so level l'' is preserved (and also level $l'' + 1$ in S_0), and (b) $\mathbf{T}$ is skew, so no s_i is a fork, so level $l'' + 1$ is preserved in S_i $(i > 0)$. So our previous fixing of nodes is not violated. The pruning guarantees that all $\mathbf{n}$-sets in (the new) $\mathbf{S}$ with pattern ρ and signature $\langle f, s_1, \ldots, s_{r-1}\rangle$ lie in the same partition class. Repeating the process finitely often, we obtain this homogeneity for each signature that begins with f. This completes one step of the induction.

Repeating the preceding two paragraphs infinitely often, we obtain decreasing sequences of perfect subtrees $S_i \subseteq T_i$ whose intersections T_i^* have the following properties. T_i^* is perfect for $i \neq 0$, as we remarked in the course of the construction. The forks in T_0^* are precisely the f's considered at the various induction steps. Therefore, all $\mathbf{n}$-sets in $\mathbf{T}^*$ with pattern ρ have their partition classes uniquely determined by their signatures. Finally, by choosing the fork f carefully at each step, as in the proof of the lemma, we can easily arrange for T_0^* to be perfect. Thus, $\mathbf{T}^*$ has all the desired properties. We now seek to eliminate the dependence of the partition class on the signature, by reducing the trees still further.

Since the partition class of an $\mathbf{n}$-set in $\mathbf{T}^*$ with pattern ρ is determined by its signature, we have an induced partition of signatures of $\mathbf{T}^*$, i.e., of the r-tuples $\mathbf{s} = \langle s_0, \ldots, s_{r-1}\rangle$ such that each $s_i \in T_i^*$, all the s_i are at the same level, and s_0 is a fork of T_0^*. We shall find perfect subtrees $T_i' \subseteq T_i^*$ such that all signatures of $\mathbf{T}'$ are in the same piece of this induced partition. Consequently, all $\mathbf{n}$-sets in $\mathbf{T}'$ with pattern ρ will be in the same piece of the original partition, so the proof will be complete.

We shall assume that the partition of the signatures is into only two pieces; the general case follows by a trivial induction on the number of pieces. The construction of $\mathbf{T}'$ from $\mathbf{T}^*$ will be similar to the construction of $\mathbf{T}^*$ from $\mathbf{T}$, in that we obtain $\mathbf{T}'$ as the intersection of an inductively defined decreasing sequence of perfect subtrees of $\mathbf{T}^*$. As before, we use $\mathbf{S}$ to stand for the subtrees of $\mathbf{T}^*$ currently under consideration. Two preliminary remarks are needed before we begin the construction of $\mathbf{T}'$. First, all fixing of nodes in the construction of $\mathbf{T}^*$ is now rescinded. Second, a signature of $\mathbf{T}^*$ whose components are still present in $\mathbf{T}'$ need not be a signature of $\mathbf{T}'$; it will be one only if the component in T_0' is still a fork.

To each r-tuple $\mathbf{s}$ with $s_i \in T_i^*$, we associate a signature $\hat{\mathbf{s}}$ of $\mathbf{T}^*$ as follows. Let $\hat{s}_0$ be the highest fork $\subseteq s_0$ in T_0^*. (It will not matter how we define $\hat{s}$ if s_0 is below the lowest fork of T_0^*.) For $i \neq 0$, let $\hat{s}_i$ be the predecessor or the lexicographically first successor of s_i at the level of $\hat{s}_0$. Clearly, $\hat{\mathbf{s}} = \langle \hat{s}_0, \ldots, \hat{s}_{r-1} \rangle$ is a signature of $\mathbf{T}^*$. At any future stage of our construction, if the subtrees $\mathbf{S}$ under consideration contain the components of $\mathbf{s}$ and $\hat{\mathbf{s}}$ and if $\hat{s}_0$ is a fork in S_0, then the description of how $\hat{\mathbf{s}}$ is obtained from $\mathbf{s}$ remains correct when $\mathbf{T}^*$ is replaced by $\mathbf{S}$.

Extend the partition of the signatures to a partition of all the r-tuples $\mathbf{s}$ with $s_i \in T_i^*$ by putting each $\mathbf{s}$ into the partition class of $\hat{\mathbf{s}}$. We apply the Halpern-Läuchli partition theorem [5] to this partition of $T_0^* \times \cdots \times T_{r-1}^* = \prod \mathbf{T}^*$. It asserts that there is a natural number h such that one of the partition classes, say c, contains (h, k)-matrices for all natural numbers k. This means that, for each k, there is a sequence $\mathbf{x}$ of nodes x_i at level h in T_i^* and there is a sequence $\mathbf{A}$ of subsets $A_i \subseteq T_i^*$ such that

(1) every successor of x_i at level $h + k$ in T_i^* is below a node in A_i, and

(2) all r-tuples $\mathbf{s} \in \prod \mathbf{A}$ lie in the partition class c.

Fix such an h. There are only finitely many tuples $\mathbf{x}$ as above, so the same $\mathbf{x}$ works for arbitrarily large k, hence for all k. Fix such an $\mathbf{x}$. The process of pruning the trees $\mathbf{T}^*$ begins by setting $S_i = T_i^*(x_i)$. To simplify notation, we assume that $h = 0$; the general case involves adding h to every level mentioned in the sequel.

Suppose, at some stage of the pruning, we have perfect trees $\mathbf{S}$ and we have fixed all nodes up to some level l. Suppose further that the previous stages have involved the removal only of nodes at or below level l along with their successors. Thus, nodes of S_i above level l have the same successors in S_i that they had in T_i^*. This ensures that, when we refer to $\hat{\mathbf{s}}$ below, it will not matter whether the $\hat{}$-operation is computed in $\mathbf{S}$ or in $\mathbf{T}^*$. It also ensures that, if $k > l$, the conclusion of the Halpern-Läuchli theorem holds for $\mathbf{S}$ with the same h and $\mathbf{x}$; we need only intersect the original A_i with the current S_i. The next stage of the pruning is as follows.

As in the construction of $\mathbf{T}^*$, begin by choosing l' so large that, in each S_i ($i \neq 0$), every node at level l has at least two successors at level l'. Kill all forks of S_0 between levels l and l' and fix all nodes up to level l'. As before, this ensures that T_i' will be perfect for $i \neq 0$. Next, choose a fork f in S_0 above level l', say at level $l'' \geqslant l'$. By Halpern-Läuchli, find $A_i \subseteq S_i$ such that

(1) A_i dominates all nodes of level l'' in S_i, and

(2) $\prod \mathbf{A} \subseteq c$.

By (1), find $s_0 \in A_0$ above f. Then $\hat{s}_0$ is above or equal to f, so its level l^* is $\geqslant l'$. Kill all forks of S_0 from level l' to level $l^* - 1$ inclusive retaining s_0 and then fix all nodes of S_0 up to level $l^* + 1$. As before, this ensures that $\hat{s}_0$ is the unique fork of S_0 between levels l and l^* inclusive. For any $i \neq 0$ and any $s_i \in A_i$, let $\hat{s}_i$ be, as before, the predecessor or lexicographically first successor of s_i at level l^*. Then $B_i = \{\hat{s}_i | s_i \in A_i\}$ dominates level l' of S_i because $l^* > l'$ and (1) holds. Furthermore, by (2) and the definition of the partition for r-tuples other than signatures, $\{\hat{s}_0\} \times \Pi_{i \neq 0} B_i$ is included in class c of the original partition of signatures. For $i \neq 0$, remove all nodes of S_i that are not comparable with any node in B_i. Since B_i dominates level l' of S_i, nodes at level l' or lower are unaffected by this removal, so our earlier fixing of these nodes is not violated. All nodes at level l^* in the new S_i ($i \neq 0$) belong to B_i, so all signatures that start with $\hat{s}_0$ are in class c. Fix all nodes up to level $l^* + 1$. This completes one step of the induction.

Repeating the preceding paragraph infinitely often, we obtain decreasing sequences of perfect subtrees $S_i \subseteq T_i^*$ whose intersections T_i' have the following properties. As in the construction of $\mathbf{T}^*$, T_i' is perfect for $i \neq 0$, and careful choice of f at each stage will make T_0' perfect also. (Although the chosen f at any stage need not be a fork of T_0', the $\hat{s}_0$ above it is one.) The only forks in T_0' are the $\hat{s}_0$ considered at the various steps of the induction, so all signatures of $\mathbf{T}'$ are in class c. This completes the proof of the polarized theorem and therefore also the original theorem.

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