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# A PATH INTEGRAL APPROACH TO THE KONTSEVICH QUANTIZATION FORMULA 

ALBERTO S. CATTANEO AND GIOVANNI FELDER


#### Abstract

We give a quantum field theory interpretation of Kontsevich's deformation quantization formula for Poisson manifolds. We show that it is given by the perturbative expansion of the path integral of a simple topological bosonic open string theory. Its Batalin-Vilkovisky quantization yields a superconformal field theory. The associativity of the star product, and more generally the formality conjecture can then be understood by field theory methods. As an application, we compute the center of the deformed algebra in terms of the center of the Poisson algebra.


## 1. Introduction

In a recent paper [K], M. Kontsevich gave a general formula for the deformation quantization BFFLS of the algebra of functions on a Poisson manifold. The deformed product (the "star product") is given in terms of an expansion reminiscent of the Feynman perturbation expansion of a two dimensional field theory on a disc with boundary. We review Kontsevich's formula in Sect. 2.

The purpose of this paper is to describe this quantum field theory explicitly. It turns out that it is a simple bosonic topological quantum field theory on a disc $D$ with a field $X: D \rightarrow M$ taking values in the Poisson manifold $M$ and a one-form $\eta$ on $D$ taking values in the pull-back $X^{*}\left(T^{*} M\right)$ of the cotangent bundle. The formula for the star product is

$$
f \star g(x)=\int_{X(\infty)=x} f(X(1)) g(X(0)) e^{\frac{i}{\hbar} S[X, \eta]} d X d \eta
$$

where $0,1, \infty$ are three distinct points on the boundary of $D$. The integral is normalized in such a way that in the case of the trivial Poisson structure the star product reduces to the ordinary product. The action $S$ is described in Sect. 3 and was originally studied for manifolds without boundary in [i] and SchStr]. In particular the canonical quantization on the cylinder was considered.

In the symplectic case the above formula essentially reduces to the original Feynman path integral formula for quantum mechanics, as pointed out to us by H. Ooguri.

The quantization of the theory is somewhat subtle, due to the presence of a gauge symmetry which only closes on shell, as already noticed in [i]]. In other words, the action $S$ is a function of the fields annihilated by a distribution of vector fields which is only integrable on the set of critical points of $S$. As a consequence, the BRST quantization
fails and one has to resort to the Batalin-Vilkovisky method (see for example BV, W1, 51, AKSZ]).

This method yields a gauge fixed action, which turns out to have a superconformal invariance. Its perturbative expansion around constant classical solutions reproduces Kontsevich's formula.

As an application, we show in Sect. Re b $^{\text {b }}$ quantum field theory methods that there exists a star product equivalent to Kontsevich's whose center consists of the power series in $\hbar$ whose coefficients are in the center of the Poisson algebra. A rigorous proof of this statement will appear elsewhere [CFT].

More generally, we may consider a path integral associated to an arbitrary polyvector field, a formal sum of skew-symmetric contravariant tensor fields of arbitrary rank, the star product being the special case of bivector fields. Correlation functions of boundary fields yield then a map $U$ from polyvector fields to polydifferential operators. Formal properties of this map can be deduced from BV and factorization methods of quantum field theory. This leads to identities, also found by Kontsevich, which may be thought of as the open string analog of the WDVV equations W2, DDV. They may be formulated by saying that $U$ is an $L_{\infty}$ morphism [SchlSt, LS]. They imply the associativity of the star product and, in the general setting of arbitrary polyvector fields, the formality conjecture [K]. These constructions are explained in Sect. 5. 5 .

Although the non-rigorous quantum field theory arguments of this paper are of course no substitute for the proofs in [K] , this approach offers an explanation for why Kontsevich's construction works, and puts it in the context of Feynman's original picture of quantization $[\mathrm{F}]$. Moreover, our approach indicates the way for more general constructions. In particular, one can consider the perturbative expansion around a non-trivial classical solution, one can insert a Hamiltonian and one can consider this quantum field theory on a complex curve of higher genus. We plan to study these variants in the future.

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## 2. The Kontsevich formula

In [区], M. Kontsevich wrote a beautiful explicit solution to the problem of deformation quantization of the algebra of functions on a Poisson manifold $M$. The problem is to find a deformation of the product on the algebra of smooth functions on a Poisson manifold, which to first order in Planck's constant is given by the Poisson bracket.

If $M$ is an open set in $\mathbb{R}^{d}$ with a Poisson structure $\{f, g\}(x)=\sum_{i, j=1}^{d} \alpha^{i j}(x) \partial_{i} f(x) \partial_{j} g(x)$ given by a skew-symmetric bivector field $\alpha$, obeying the Jacobi identity

$$
\begin{equation*}
\alpha^{i l} \partial_{l} \alpha^{j k}+\alpha^{j l} \partial_{l} \alpha^{k i}+\alpha^{k l} \partial_{l} \alpha^{i j}=0, \tag{1}
\end{equation*}
$$

the problem is to find an associative product $\star$ on $C^{\infty}(M)[[\hbar]]$, such that for $f, g \in$ $C^{\infty}(M)$,

$$
f \star g(x)=f(x) g(x)+\frac{i \hbar}{2}\{f, g\}(x)+O\left(\hbar^{2}\right) .
$$

Kontsevich's solution' to this problem may be described as follows. The coefficient of $(i \hbar / 2)^{n}$ in $f \star g$ is given by a sum of terms labeled by diagrams of order $n$. A diagram $\Gamma$ of order $n$ is a graph consisting of $n$ vertices numbered from 1 to $n$ and two vertices labeled by letters $L$ and $R$, for Left and Right. From each of the numbered vertices there emerge two ordered oriented edges that end at numbered vertices or at vertices labeled by letters, so that no edge starts and ends at the same vertex. The two edges emerging from vertex $i$ are called $e_{i}^{1}, e_{i}^{2}$. They are of the form $e_{i}^{a}=\left(i, v_{a}(i)\right)$ for some maps $v_{a}:\{1, \ldots, n\} \rightarrow\{1, \ldots, n, L, R\}$. In fact, a diagram $\Gamma$ can be thought of as an ordered pair $\left(v_{1}, v_{2}\right)$ of maps $\{1, \ldots, n\} \rightarrow\{1, \ldots, n, L, R\}$, such that $v_{a}(i)$ is never equal to $i$.

To each diagram $\Gamma$ of order $n$ there corresponds a bidifferential operator $D_{\Gamma}$ whose coefficients are differential polynomials, homogeneous of degree $n$ in the components $\alpha^{i j}$ of the Poisson structure. The edges indicate how the partial derivatives are acting. For instance the bidifferential operator $(f, g) \mapsto \alpha^{i j}(x) \partial_{i} f(x) \partial_{j} g(x)$ corresponds to the diagram with vertices $1, L, R$ and edges $e_{1}^{1}=(1, L), e_{1}^{2}=(1, R)$. The bidifferential operator $D_{\Gamma}(f \otimes g)=\alpha^{i j} \partial_{i} \alpha^{k l} \partial_{j} \partial_{l} f \partial_{k} g$ corresponds to the diagram $\Gamma$ with vertices $1,2, L, R$ and edges $e_{1}^{1}=(1,2), e_{1}^{2}=(1, L), e_{2}^{1}=(2, R), e_{2}^{2}=(2, L)$.

Kontsevich's formula is then

$$
f \star g=f g+\sum_{n=1}^{\infty}\left(\frac{i \hbar}{2}\right)^{n} \sum_{\Gamma \text { of order } n} w_{\Gamma} D_{\Gamma}(f \otimes g)
$$

The weight $w_{\Gamma}$ is the integral of a differential form over the configuration space $C_{n}(H)=$ $\left\{u \in H^{n}, u_{i} \neq u_{j}(i \neq j)\right\}$ of $n$ ordered points on the upper half plane $H$. It is defined as follows: for any two distinct points $z, w$ in the upper half plane with the Poincaré metric $d s^{2}=\left(d x^{2}+d y^{2}\right) / y^{2}$, let $\phi(z, w)$ be the angle between the (vertical) geodesic connecting $z$ to $i \infty$ and the geodesic connecting $z$ to $w$, measured in counterclockwise direction. Let $d \phi(z, w)=d z \frac{\partial}{\partial z} \phi(z, w)+d w \frac{\partial}{\partial w} \phi(z, w)$ denote the differential of this angle. Then the weight is

$$
w_{\Gamma}=\frac{1}{(2 \pi)^{2 n} n!} \int_{C_{n}(H)} \wedge_{i=1}^{n} d \phi\left(u_{i}, u_{v_{1}(i)}\right) \wedge d \phi\left(u_{i}, u_{v_{2}(i)}\right)
$$

where we set $u_{L}=0$ and $u_{R}=1$. The orientation is induced from the product of the standard orientation of the upper half plane.

[^0]For example, we have two graphs of order one, differing in the ordering of edges. Let us compute the weight of these diagrams. Let $\Gamma$ be the diagram with $e_{1}^{1}=(1, L), e_{1}^{2}=(1, R)$. To compute the integral over $u=u^{1}+i u^{2} \in H$ we introduce new variables $\phi_{0}=\phi(u, 0)$, $\phi_{1}=\phi(u, 1)$. As $\arg (u)$ varies between 0 and $\pi$, the angle $\phi_{0}$ varies from 0 to $2 \pi$. As we vary $u$ on the half-line of constant $\phi_{0}$, the angle $\phi_{1}$ varies between $\phi_{0}$ (at infinity) and $2 \pi$ (at $u=0$ ). Thus this change of variables is a diffeomorphism from the upper half plane to the domain $0<\phi_{0}<\phi_{1}<2 \pi$ in $\mathbb{R}^{2}$. The above description also shows that this diffeomorphism is orientation preserving. Thus $w_{\Gamma}=(2 \pi)^{-2} \int_{0<\phi_{0}<\phi_{1}<2 \pi} d \phi_{0} \wedge$ $d \phi_{1}=1 / 2\left(\phi_{j}=\phi(u, j)\right.$ and $d \phi_{0} \wedge d \phi_{1}$ is positively oriented). The other diagram has $e_{1}^{1}=(1, R), e_{1}^{2}=(1, L)$ and gives the same contribution with the opposite sign. Therefore the coefficient of $(i \hbar / 2)$ is

$$
\frac{1}{2} \alpha^{i j} \partial_{i} f \partial_{j} g-\frac{1}{2} \alpha^{i j} \partial_{j} f \partial_{i} g=\alpha^{i j} \partial_{i} f \partial_{j} g,
$$

by the skew-symmetry of $\alpha$.
Let us conclude this section with some remarks about involutions. The opposite product $f \star_{\text {op }} g$ is related to the product by a change of sign of $\hbar$. Indeed, $D_{\Gamma}(g \otimes f)=$ $D_{\bar{\Gamma}}(f \otimes g)$ where $\bar{\Gamma}$ is obtained from $\Gamma$ by exchanging $R$ and $L$, and $w_{\bar{\Gamma}}=(-1)^{n} w_{\Gamma}$ if $\Gamma$ is of order $n$, since $w_{\bar{\Gamma}}$ is the integral of the pull-back of the differential form defining $w_{\Gamma}$ by the reflection about the axis $\operatorname{Re}(z)=\frac{1}{2}$ which reverses the orientation of $H$. Since the weights $w_{\Gamma}$ are real, this implies that complex conjugation, extended to $C^{\infty}(M)[[\hbar]]$ by setting $\bar{\hbar}=\hbar$, is an antilinear antiautomorphism for the star product.

## 3. A sigma model

3.1. The classical action and its symmetries. We start by introducing a sigma model action. The perturbative expansion of correlation functions of boundary fields on the disc will then be related to the star product.

The model has two real bosonic fields $X, \eta . X$ is a map from the disc $D=\{u \in$ $\left.\mathbb{R}^{2},|u| \leq 1\right\}$ to $M$ and $\eta$ is a differential 1-form on $D$ taking values in the pull-back by $X$ of the cotangent bundle of $M$, i.e. a section of $X^{*}\left(T^{*} M\right) \otimes T^{*} D$. In local coordinates, $X$ is given by $d$ functions $X^{i}(u)$ and $\eta$ by $d$ differential 1-forms $\eta_{i}(u)=\eta_{i, \mu}(u) d u^{\mu}$.

The action reads

$$
S[X, \eta]=\int_{D} \eta_{i}(u) \wedge d X^{i}(u)+\frac{1}{2} \alpha^{i j}(X(u)) \eta_{i}(u) \wedge \eta_{j}(u)
$$

The boundary condition for $\eta$ is that for $u \in \partial D, \eta_{i}(u)$ vanishes on vectors tangent to $\partial D$.

We then claim that the star product is given by the semiclassical expansion of the path integral ${ }^{5}$

$$
f \star g(x)=\int_{X(\infty)=x} f(X(1)) g(X(0)) e^{\frac{i}{\hbar} S[X, \eta]} d X d \eta .
$$

Here $0,1, \infty$ are any three cyclically ordered points on the unit circle (which we secretely view as the completed real line by stereographic projection). Cyclically ordered means that if we start from 0 and move on the circle counterclockwise we first meet 1 and then $\infty$. The path integral is over all $X: D \rightarrow M, \eta \in \Gamma\left(D, X^{*}\left(T^{*} M\right) \otimes T^{*} D\right)$ subject to the boundary conditions $X(\infty)=x, \eta(u)(\xi)=0$ if $u \in \partial D$ and $\xi$ is tangent to $\partial D$. Its semiclassical expansion is to be understood as an expansion around the classical solution $X(u)=x, \eta(u)=0$.

To evaluate this path integral we have as usual to take gauge fixing and renormalization into account.

This action is invariant under the following infinitesimal gauge transformations with infinitesimal parameter $\beta_{i}$, which is a section of $X^{*}\left(T^{*} M\right)$ and vanishes on the boundary of $D$ :

$$
\begin{aligned}
\delta_{\beta} X^{i} & =\alpha^{i j}(X) \beta_{j} \\
\delta_{\beta} \eta_{i} & =-d \beta_{i}-\partial_{i} \alpha^{j k}(X) \eta_{j} \beta_{k} .
\end{aligned}
$$

This symmetry is an extension of more familiar gauge symmetries encountered in special cases. On one extreme we have $\alpha=0$ and the action is invariant under translations of $\eta$ by exact one-forms on $D$. On the other extreme we have the symplectic case where $\alpha^{i j}$ is an invertible matrix so that integrating formally over $\eta$ we get the action $\int_{D} X^{*} \omega$ which is invariant under arbitrary translations $X^{i} \mapsto X^{i}+\xi^{i}$, with $\xi^{i}(u)=0$ on the boundary of $D$. Another special case is the case when $M$ is a vector space and $\alpha$ is a linear function on $M$. In this case $M$ is the dual space to a Lie algebra $\mathfrak{g}$ with KirillovKostant Poisson structure. The Lie bracket of two linear functions $f, g \in \mathfrak{g}=M^{*}$ is just the Poisson bracket and is again a linear function on $M$. Then the classical action is best viewed as a function of a field $X$ taking values in $\mathfrak{g}^{*}$ and a connection $d+\eta$ on a trivial principal bundle on $D$. After an integration by parts, the action becomes the "BF action" S2, BT $S=\int_{D}\langle X, F(\eta)\rangle$ where $F(\eta)$ is the curvature of $d+\eta$. In this case the gauge transformation is the usual gauge transformation (with gauge parameter $-\beta$ ) of a connection and a field $X$ in the coadjoint representation.

[^1]In the general case, the commutator of two gauge transformations is a gauge transformation only on shell, i.e., modulo the equations of motion:

$$
\begin{aligned}
{\left[\delta_{\beta}, \delta_{\beta^{\prime}}\right] X^{i} } & =\delta_{\left\{\beta, \beta^{\prime}\right\}} X^{i} \\
{\left[\delta_{\beta}, \delta_{\beta^{\prime}}\right] \eta_{i} } & =\delta_{\left\{\beta, \beta^{\prime}\right\}} \eta_{i}-\partial_{i} \partial_{k} \alpha^{r s} \beta_{r} \beta_{s}^{\prime}\left(d X^{k}+\alpha^{k j}(X) \eta_{j}\right)
\end{aligned}
$$

Here $\left\{\beta, \beta^{\prime}\right\}_{i}=-\partial_{i} \alpha^{j k}(X) \beta_{j} \beta_{k}^{\prime}$ and $d X^{k}+\alpha^{k j} \eta_{j}=0$ is an Euler-Lagrange equation for the action $S$. In this calculation the Jacobi identity (1]) plays an essential role.

Thus the gauge transformations form a Lie algebra only when acting on critical points (classical solutions) of $S$.

In the BRST formalism one then promotes the infinitesimal gauge parameter $\beta_{i}$ to an anticommuting ghost field (vanishing on the boundary of the disc) and introduces the BRST operator $\delta_{0}$, an odd derivation on the functions of $X, \eta, \beta$ such that

$$
\begin{aligned}
\delta_{0} X^{i} & =\alpha^{i j}(X) \beta_{j} \\
\delta_{0} \eta_{i} & =-d \beta_{i}-\partial_{i} \alpha^{k l}(X) \eta_{k} \beta_{l} \\
\delta_{0} \beta_{i} & =\frac{1}{2} \partial_{i} \alpha^{j k}(X) \beta_{j} \beta_{k} .
\end{aligned}
$$

Then $\delta_{0}$ is a differential on shell, i.e., it squares to zero modulo the equations of motion. More precisely we have $\delta_{0}^{2} X^{i}=\delta_{0}^{2} \beta_{i}=0$ and $\delta_{0}^{2} \eta_{i}=-\frac{1}{2} \partial_{i} \partial_{k} \alpha^{r s} \beta_{r} \beta_{s}\left(d X^{k}+\alpha^{k j}(X) \eta_{j}\right)$. We assign a gradation, the ghost number, to our fields: $\operatorname{gh}\left(X^{i}\right)=\operatorname{gh}\left(\eta_{i}\right)=0, \operatorname{gh}\left(\beta_{i}\right)=1$. The BRST operator has then ghost number one. Additionally we have the gradation of the fields as differential forms on the disc, which will be denoted by deg: $\operatorname{deg}\left(X^{i}\right)=$ $\operatorname{deg}\left(\beta_{i}\right)=0, \operatorname{deg}\left(\eta_{i}\right)=1$.

In the case $M=\mathfrak{g}^{*}$ of linear Poisson structures, the second derivatives of $\alpha$ vanish, and the BRST operator squares to zero.
3.2. The Batalin-Vilkovisky action. If the BRST operator squares to zero only modulo the equations of motion, the usual BRST procedure to evaluate the path integral does not quite work, since it essentially requires a well-defined cohomology to construct physical observables. The generalization of the BRST procedure that works in this case is the Batalin-Vilkovisky method. The recipe is as follows. One first adds antifields $X^{+}, \eta^{+}, \beta^{+}$with complementary ghost number and degree as differential forms on $D$. The assignments of degree (from left to right) and ghost number (from top to bottom) are given by

|  | 0 | 1 | 2 |
| ---: | :---: | :---: | :---: |
| -2 |  |  | $\beta^{+i}$ |
| -1 |  | $\eta^{+i}$ | $X_{i}^{+}$ |
| 0 | $X^{i}$ | $\eta_{i}$ |  |
| 1 | $\beta_{i}$ |  |  |

One then looks for a Batalin-Vilkovisky action $S_{\mathrm{BV}}\left[\phi, \phi^{+}\right]$of ghost number zero depending on fields $\phi^{1}, \phi^{2}, \ldots$ (here $\left.X^{i}, \eta_{i}, \beta_{i}\right)$ and antifields $\phi_{1}^{+}, \phi_{2}^{+}, \ldots$, with $\operatorname{gh}\left(\phi_{\alpha}^{+}\right)=$
$-1-\operatorname{gh}\left(\phi^{\alpha}\right)$ and $\operatorname{deg}\left(\phi_{\alpha}^{+}\right)=2-\operatorname{deg}\left(\phi^{\alpha}\right)$ subject to two requirements. The first requirement is that $S_{\mathrm{BV}}[\phi, 0]$ reduces to the classical action $S[\phi]$ when the antifields are set to zero and the second requirement is that $S_{\mathrm{BV}}$ obeys the quantum master equation

$$
\left(S_{\mathrm{BV}}, S_{\mathrm{BV}}\right)-2 i \hbar \triangle S_{\mathrm{BV}}=0
$$

The BV Laplacian $\triangle$ and the BV antibracket are defined as follows.
Let us introduce temporarily a Riemannian metric on $D$, and denote by $\langle,\rangle_{u}$ the induced scalar product on the exterior algebra of the cotangent space at $u$. The volume form $\sqrt{g} d u^{1} d u^{2}$ will be denoted by $d v(u)$. The Hodge star $*$ then obeys $\langle\alpha, \beta\rangle_{u} d v(u)=$ $\alpha \wedge * \beta$. The expression for the Laplacian is better expressed in terms of the Hodge dual antifields

$$
\phi_{\alpha}^{*}=* \phi_{\alpha}^{+} .
$$

The Laplacian of a function of fields and antifields is

$$
\triangle A=\sum_{\alpha}(-1)^{\operatorname{gh}(\alpha)} \frac{\vec{\delta}^{2} A}{\delta \phi^{\alpha}(u) \delta \phi_{\alpha}^{*}(u)}
$$

The functional derivatives of a function of fields and antifields, collectively denoted by $\psi^{\alpha}$, are the distributions (de Rham currents) defined by

$$
\left.\frac{d}{d t} A(\psi+t \rho)\right|_{t=0}=\int_{D}\left\langle\rho^{\alpha}(u), \frac{\vec{\delta} A}{\delta \psi^{\alpha}(u)}\right\rangle_{u} d v(u)=\int_{D}\left\langle\frac{A \overleftarrow{\delta}}{\delta \psi^{\alpha}(u)}, \rho^{\alpha}\right\rangle_{u} d v(u)
$$

for any test forms $\rho^{\alpha}$ of the same degree and ghost number as $\psi^{\alpha}$.
Note that the Laplacian is the restriction of a distribution on $D^{2}$ to the diagonal, and is thus a singular object in this infinite dimensional context. It should be understood as the limit of a suitably regularized expression.

The Laplacian obeys

$$
\begin{equation*}
\triangle(A B)=\triangle(A) B+(-1)^{\operatorname{gh}(A)}(A, B)+(-1)^{\operatorname{gh}(A)} A \triangle(B) \tag{2}
\end{equation*}
$$

where the Batalin-Vilkovisky antibracket is

$$
(A, B)=\sum_{\alpha} \int_{D}\left(\left\langle\frac{A \overleftarrow{\delta}}{\delta \phi^{\alpha}(u)}, \frac{\vec{\delta} B}{\delta \phi_{\alpha}^{*}(u)}\right\rangle-\left\langle\frac{A \overleftarrow{\delta}}{\delta \phi_{\alpha}^{*}(u)}, \frac{\vec{\delta} B}{\delta \phi^{\alpha}(u)}\right\rangle\right) d v(u)
$$

This antibracket is better defined than the Laplacian, in the sense that if $A$ and $B$ are local functionals of the fields and antifields, such as the action $S$, then the functional derivatives are regular distributions and $(A, B)$ is again a local functional. Moreover, it is independent of the choice of Riemannian metric: it can be expressed without reference to the metric at the cost of introducing signs:

$$
(A, B)=\sum_{\alpha} \int_{D}\left(\frac{A \overleftarrow{\partial}}{\partial \phi^{\alpha}} \wedge \frac{\vec{\partial} B}{\partial \phi_{\alpha}^{+}}-(-1)^{\operatorname{deg} \phi_{\alpha}} \frac{A \overleftarrow{\partial}}{\partial \phi_{\alpha}^{+}} \wedge \frac{\vec{\partial} B}{\partial \phi^{\alpha}}\right)
$$

Here the derivatives of a function $A$ of fields and antifields $\psi_{\alpha}$ are the distributions defined by

$$
\left.\frac{d}{d t} A(\psi+t \rho)\right|_{t=0}=\int_{D} \rho^{\alpha} \wedge \frac{\vec{\partial} A}{\partial \psi^{\alpha}}=\int_{D} \frac{A \overleftarrow{\partial}}{\partial \psi^{\alpha}} \wedge \rho^{\alpha}
$$

for any test forms $\rho^{\alpha}$ of the same degree and ghost number as $\psi^{\alpha}$. The antibracket obeys the graded commutativity relation

$$
(A, B)=-(-1)^{(\operatorname{gh}(A)-1)(\operatorname{gh}(B)-1)}(B, A)
$$

and the Leibnitz rule

$$
\begin{equation*}
(A, B C)=(A, B) C+(-1)^{(\operatorname{gh}(A)-1) \operatorname{gh}(B)} B(A, C) \tag{3}
\end{equation*}
$$

In the general case of field theories with non-trivial renormalization the BV action depends on $\hbar$ through counterterms and the full quantum master equation is solved by a recursive procedure order by order in $\hbar$. Here, as we shall see, the renormalization is rather trivial and the Batalin-Vilkovisky action satisfies separately the equation $\triangle S_{\mathrm{BV}}=0$ and the classical master equation

$$
\left(S_{\mathrm{BV}}, S_{\mathrm{BV}}\right)=0
$$

The classical master equation implies that the BV version of the BRST operator $\delta$ defined by $\delta A=\left(S_{\mathrm{BV}}, A\right)$ is a differential. It obeys the Leibnitz rule $\delta(A B)=\delta A B+$ $(-1)^{\operatorname{gh}(A)} A \delta B$ and it acts on fields and antifields by the rule

$$
\delta \phi^{\alpha}=(-1)^{\operatorname{gh}\left(\phi^{\alpha}\right)} \frac{\vec{\partial} S_{\mathrm{BV}}}{\partial \phi_{\alpha}^{+}}, \quad \delta \phi_{\alpha}^{+}=(-1)^{\mathrm{gh}\left(\phi^{\alpha}\right)+\operatorname{deg}\left(\phi^{\alpha}\right)} \frac{\vec{\partial} S_{\mathrm{BV}}}{\partial \phi^{\alpha}} .
$$

One semi-systematic way to find the BV action, which is under suitable hypotheses unique up to the BV version of canonical transformations, is to start with the obvious action $S_{\mathrm{BV}}^{0}=S+\int_{D} X_{i}^{+} \delta_{0} X^{i}+\eta^{+i} \wedge \delta_{0} \eta_{i}-\beta^{+i} \delta_{0} \beta_{i}$, which has BRST operator $\delta=\delta_{0}$ and then add suitable terms, so that the new BRST operator obeys $\delta^{2}=0$. Since $\delta_{0} \eta^{+i}=\partial S_{\mathrm{BV}}^{0} / \partial \eta_{i}$ contains a term proportional to the equations of motion (plus terms involving antifields) which we need to cancel from $\delta_{0}^{2} \eta_{i}$, it is natural to add a term quadratic in $\eta^{+}$to achieve our goal. It turns out that

$$
\begin{aligned}
S_{\mathrm{BV}}= & S_{\mathrm{BV}}^{0}-\frac{1}{4} \int_{D} \eta^{+i} \wedge \eta^{+j} \partial_{i} \partial_{j} \alpha^{k l}(X) \beta_{k} \beta_{l} \\
= & \int_{D} \eta_{i} \wedge d X^{i}+\frac{1}{2} \alpha^{i j}(X) \eta_{i} \wedge \eta_{j}+X_{i}^{+} \alpha^{i j}(X) \beta_{j}-\eta^{+i} \wedge\left(d \beta_{i}+\partial_{i} \alpha^{k l}(X) \eta_{k} \beta_{l}\right) \\
& -\frac{1}{2} \beta^{+i} \partial_{i} \alpha^{j k}(X) \beta_{j} \beta_{k}-\frac{1}{4} \eta^{+i} \wedge \eta^{+j} \partial_{i} \partial_{j} \alpha^{k l}(X) \beta_{k} \beta_{l}
\end{aligned}
$$

does the job. Moreover $S_{\mathrm{BV}}$ is BRST closed (i.e., it obeys $\delta S_{\mathrm{BV}}=0$ ), which is equivalent to the classical master equation. This is more conveniently shown in the superfield formalism of the next subsection.

We claim that, if the regularization is appropriate, $\triangle S_{\mathrm{BV}}=0$. Indeed the only terms contributing to the Laplacian of the BV action contain both a field and its antifield:

$$
\begin{aligned}
\triangle S_{\mathrm{BV}} & =\triangle \int_{D} X_{i}^{+} \alpha^{i j}(X) \beta_{j}-\eta^{+i} \wedge \partial_{i} \alpha^{k l}(X) \eta_{k} \beta_{l}-\frac{1}{2} \beta^{+i} \partial_{i} \alpha^{j k}(X) \beta_{j} \beta_{k} \\
& =(1-2+1) C \int_{D} \partial_{i} \alpha^{i j}(X) \beta_{j} d v \\
& =0 .
\end{aligned}
$$

Here $C$ is an infinite constant. The factor takes into account the contribution of the first term (1), of the second term ( -2 since the one-form $\eta_{i}$ has two components) and the third term (1). In an appropriate regularization scheme, this cancellation is supposed to be valid before removing the regularization, in spite of the fact that $C$ tends to infinity.

Let us conclude this subsection by discussing the boundary conditions of the various fields. The rule is that Hodge dual antifields must have the same boundary conditions as the fields. The boundary conditions for the fields are that, for $u \in \partial D, \beta_{i}(u)=0$ and $\eta_{i}(u)$ vanishes on vectors tangent to the boundary. Thus $\beta^{+i}(u)=0$ and $\eta_{i}^{+}(u)$ vanishes on vectors normal to the boundary.
3.3. Superfield formalism. It turns out that the calculations simplify if we combine our fields and antifields into superfields. These are functions of the even coordinates $u^{1}, u^{2}$ on $D$ and odd (anticommuting) coordinates $\theta^{1}, \theta^{2}$. Thus a superfield $\phi$ has the form $\phi(u, \theta)=\phi^{(0)}(u)+\theta^{\mu} \phi_{\mu}^{(1)}(u)+\theta^{\mu} \theta^{\nu} \frac{1}{2} \phi_{\mu \nu}^{(2)}$. Its components fields are a scalar function $\phi^{(0)}$, a one-form $\phi^{(1)}=\phi_{\mu}^{(1)} d u^{\mu}$ and a two form $\phi^{(2)}=\frac{1}{2} \phi_{\mu \nu}^{(2)} d u^{\mu} \wedge d u^{\nu}$. The fields of total degree (degree+ghost number) zero combine into even superfields $\tilde{X}^{i}$, the "supercoordinates".

$$
\tilde{X}^{i}=X^{i}+\theta^{\mu} \eta_{\mu}^{+i}-\frac{1}{2} \theta^{\mu} \theta^{\nu} \beta_{\mu \nu}^{+i}
$$

and the fields of total degree one combine into odd superfields $\tilde{\eta}_{i}$, the "super-one-forms":

$$
\tilde{\eta}_{i}=\beta_{i}+\theta^{\mu} \eta_{i, \mu}+\frac{1}{2} \theta^{\mu} \theta^{\nu} X_{i, \mu \nu}^{+} .
$$

Let $D=\theta^{\mu} \partial / \partial u^{\mu}$. This operator acts on component fields as the de Rham differential. Then the BRST operator $\delta$ acts as an odd derivation on functions of the superfields $\tilde{X}^{i}$, $\tilde{\eta}_{i}$ by the rule

$$
\begin{aligned}
\delta \tilde{X}^{i} & =D \tilde{X}^{i}+\alpha^{i j}(\tilde{X}) \tilde{\eta}_{j} \\
\delta \tilde{\eta}_{i} & =D \tilde{\eta}_{i}+\frac{1}{2} \partial_{i} \alpha^{j k}(\tilde{X}) \tilde{\eta}_{j} \tilde{\eta}_{k}
\end{aligned}
$$

It is easy to check that the Jacobi identity implies $\delta^{2}=0$. The action of $\delta$ on components field can then easily be evaluated by comparing coefficients and taking into account the
sign rule $\delta \phi=\delta \phi^{(0)}-\theta^{\mu} \delta \phi_{\mu}^{(1)}+\frac{1}{2} \theta^{\mu} \theta^{\nu} \delta \phi_{\mu \nu}^{(2)}$. One gets

$$
\begin{aligned}
\delta X^{i}= & \alpha^{i j}(X) \beta_{j}, \\
\delta \eta^{+i}= & -d X^{i}-\alpha^{i j}(X) \eta_{j}-\partial_{k} \alpha^{i j}(X) \eta^{+k} \beta_{j}, \\
\delta \beta^{+i}= & -d \eta^{+i}-\alpha^{i j}(X) X_{j}^{+}+\frac{1}{2} \partial_{k} \partial_{l} \alpha^{i j}(X) \eta^{+k} \wedge \eta^{+l} \beta_{j} \\
& +\partial_{k} \alpha^{i j}(X) \eta^{+k} \wedge \eta_{j}+\partial_{k} \alpha^{i j}(X) \beta^{+k} \beta_{j} .
\end{aligned}
$$

and

$$
\begin{aligned}
\delta \beta_{i}= & \frac{1}{2} \partial_{i} \alpha^{k l}(X) \beta_{k} \beta_{l}, \\
\delta \eta_{i}= & -d \beta_{i}-\partial_{i} \alpha^{k l}(X) \eta_{k} \beta_{l}-\frac{1}{2} \partial_{i} \partial_{j} \alpha^{k l}(X) \eta^{+j} \beta_{k} \beta_{l}, \\
\delta X_{i}^{+}= & d \eta_{i}+\partial_{i} \alpha^{k l}(X) X_{k}^{+} \beta_{l}-\partial_{i} \partial_{j} \alpha^{k l}(X) \eta^{+j} \wedge \eta_{k} \beta_{l}+\frac{1}{2} \partial_{i} \alpha^{k l}(X) \eta_{k} \wedge \eta_{l} \\
& -\frac{1}{4} \partial_{i} \partial_{j} \partial_{p} \alpha^{k l}(X) \eta^{+j} \wedge \eta^{+p} \beta_{k} \beta_{l}-\frac{1}{2} \partial_{i} \partial_{j} \alpha^{k l}(X) \beta^{+j} \beta_{k} \beta_{l} .
\end{aligned}
$$

This BRST operator coincides with the one obtained by the Batalin-Vilkovisky procedure. The Batalin-Vilkovisky action is the integral

$$
S_{B V}=\int_{D} L^{(2)} .
$$

of the two-form part $L^{(2)}=\int d^{2} \theta L$ of

$$
L=\tilde{\eta}_{i} D \tilde{X}^{i}+\frac{1}{2} \alpha^{i j}(\tilde{X}) \tilde{\eta}_{i} \tilde{\eta}_{j}
$$

It is BRST closed, i.e., it obeys the master equation. In fact one has

$$
\delta L=D\left(\tilde{\eta}_{i} D \tilde{X}^{i}\right)
$$

so that $\delta L^{(2)}$ is the differential of a one form which vanishes along the boundary.
3.4. The gauge fixed action. We compute the path integral in the Lorentz-type gauge $d * \eta_{i}=0$. The Hodge $*$ operator (alias the almost complex structure) depends on the conformal structure and the orientation of $D \subset \mathbb{R}^{2}$ : in terms of the standard coordinates, $* d u^{1}=d u^{2}, * d u^{2}=-d u^{1}$.

Let us shortly recall the main idea of the Batalin-Vilkovisky formalism in the general setting of 3.2. For any function $\Psi$, the "gauge fixing fermion", of the fields of ghost number -1 , one considers the integral $\int_{L} \mathcal{O} e^{\frac{i}{\hbar} S_{\mathrm{BV}}}$ for an observable $\mathcal{O}$, i.e., a function of fields and antifields which is closed with respect to the quantum BRST operator $\Omega=-i \hbar \triangle+\delta$ :

$$
\Omega \mathcal{O}=-i \hbar \triangle \mathcal{O}+\left(S_{\mathrm{BV}}, \mathcal{O}\right)=0
$$

The integral is taken over the "Lagrangian" submanifold $L$ defined by the equations $\phi_{\alpha}^{+}=\vec{\partial}_{\phi^{\alpha}} \Psi$. Using formally the master equation and the fact that $\mathcal{O}$ is BRST closed,
one then see that these integrals are invariant under variations of $\Psi$ and thus "equal" to the original (ill-defined) path integral with action $S[\phi]=S_{\mathrm{BV}}[\phi, 0]$, which is what one gets if $\Psi=0$.

The problem is then to find a function $\Psi$ which makes the integral well-defined, at least as a perturbative series. One way to do this is to add new fields, called antighosts and Lagrange multipliers together with their antifields, and choose $\Psi$ as the scalar product of the antighost and the gauge fixing condition. The action for these new fields is the simplest and is added to the Batalin-Vilkovisky action $S_{\mathrm{BV}}$.

Let us do this in the case at hand. We first introduce new anticommuting scalar fields (antighosts) $\gamma^{i}$ of ghost number -1 on $D$, and scalar Lagrange multiplier fields $\lambda^{i}$ of ghost number zero, together with their antifields $\gamma_{i}^{+}, \lambda_{i}^{+}$. The boundary condition for $\lambda^{i}$ is Dirichlet: $\lambda^{i}(u)=0, u \in \partial D$, and $\gamma^{i}$ is constant on the boundary. The action for these fields and antifields is $-\int_{D} \lambda^{i} \gamma_{i}^{+}$and is just added to the BV action. The BRST operator acts then as

$$
\delta \lambda=\delta \gamma^{+}=0, \quad \delta \lambda^{+}=-\gamma^{+}, \quad \delta \gamma=\lambda
$$

Clearly the new action also obeys the master equation. The gauge fixing condition $d * \eta=0$ is encoded in the gauge fixing fermion $\Psi=-\int_{D} d \gamma^{i} * \eta_{i}$. On the Lagrangian submanifold we then have $X^{+}=\beta^{+}=\lambda^{+}=0, \gamma_{i}^{+}=d * \eta_{i}$ plus a boundary term whose form will not matter, and $\eta^{+i}=* d \gamma^{i}$. The boundary condition for $\gamma^{i}$ was chosen so as to fulfill the boundary condition for $\eta^{+}$(vanishing on normal vectors). The gauge fixed action is then

$$
\begin{aligned}
S_{\mathrm{gf}}= & \int_{D} \eta_{i} \wedge d X^{i}+\frac{1}{2} \alpha^{i j}(X) \eta_{i} \wedge \eta_{j}-* d \gamma^{i} \wedge\left(d \beta_{i}+\partial_{i} \alpha^{k l}(X) \eta_{k} \beta_{l}\right) \\
& -\frac{1}{4} * d \gamma^{i} \wedge * d \gamma^{j} \partial_{i} \partial_{j} \alpha^{k l}(X) \beta_{k} \beta_{l}-\lambda^{i} d * \eta_{i} .
\end{aligned}
$$

3.5. Superconformal invariance of the gauge fixed action. The original action is invariant under arbitrary diffeomorphisms of the disc. As the gauge fixing condition depends on a choice of conformal structure, the gauge fixed action is only invariant under conformal diffeomorphisms. In fact this invariance is part of a (twisted) superconformal invariance, as we now show. For each vector field $\epsilon(u)=\epsilon^{\mu}(u) \frac{\partial}{\partial u^{\mu}}$ on $D$, tangent to the boundary on $\partial D$, we introduce an odd derivation $\bar{\delta}_{\epsilon}$, depending linearly on $\epsilon$, on functions of our fields:

$$
\begin{array}{lll}
\bar{\delta}_{\epsilon} X^{i}=i(\epsilon) * d \gamma^{i}, & \bar{\delta}_{\epsilon} \lambda^{i}=-i(\epsilon) d \gamma^{i}, \\
\bar{\delta}_{\epsilon} \beta_{i}=i(\epsilon) \eta_{i}, & \bar{\delta}_{\epsilon} \eta_{i}=0, & \bar{\delta}_{\epsilon} \gamma^{i}=0 .
\end{array}
$$

Here $i(\epsilon)$ is the interior multiplication of a differential form on $D$ with a vector field $\epsilon$. A straightforward calculation shows that these derivations, together with the BRST operator obey the twisted supersymmetry algebra relations

$$
[\delta, \delta]_{+}=\left[\bar{\delta}_{\epsilon}, \bar{\delta}_{\epsilon}^{\prime}\right]_{+}=0, \quad\left[\delta, \bar{\delta}_{\epsilon}\right]_{+}=-\mathcal{L}_{\epsilon}
$$

modulo the equations of motion for $S_{\mathrm{gf}}$, with Lie derivative $\mathcal{L}_{\epsilon}=i(\epsilon) \circ d+d \circ i(\epsilon)$.
The gauge fixed action obeys $\delta S_{\mathrm{gf}}=0$ and

$$
\bar{\delta}_{\epsilon} S_{\mathrm{gf}}=\int_{D} \eta_{i} \wedge\left(\mathcal{L}_{\epsilon} * d \gamma^{i}-* \mathcal{L}_{\epsilon} d \gamma^{i}\right)
$$

The latter expression vanishes if $\epsilon$ is conformal, i.e., if $\mathcal{L}_{\epsilon}$ commutes with $*$. Conformal vector fields on the disc form a three dimensional Lie algebra, isomorphic to $\mathfrak{s u}(1,1)$.
3.6. BRST cohomology classes. Observables can be obtained from differential forms on $M$. To a differential $p$-form $\omega=\omega_{i_{1}, \ldots, i_{p}}(x) d x^{i_{1}} \wedge \cdots \wedge d x^{i_{p}}$ and a point $u$ on the boundary of the disc $D$, one associates the observable

$$
\hat{\omega}(u)=\omega_{i_{1}, \ldots, i_{p}}(X(u)) \gamma^{i_{1}}(u) \cdots \gamma^{i_{p}}(u) .
$$

In general, functions of the components of $\tilde{X}(u)$ with $u$ on the boundary are observables, since with our boundary conditions we have $\delta \tilde{X}=0$ on the boundary and the Laplacian of a function depending only a field but not on its antifield or on an antifield but not its field, vanishes. More general observables are considered in Sect. H. $^{\text {. }}$

The gauge fixed action still has a (finite dimensional) residual infinitesimal symmetry $X^{i} \mapsto X^{i}+a^{i}, \gamma^{i} \mapsto \gamma^{i}+g^{i}$ where $a_{i}, g_{i}$ are constant functions on the disc. This translates into zero modes in the integration over the fermions $\gamma^{i}$, and it follows that the only observables that have non-zero integral have ghost number $-\operatorname{dim}(M)$.
3.7. Feynman rules. The Feynman perturbation expansion in powers of $\hbar$ around the classical solution $X(u)=x, \eta(u)=0$ can be now computed. Thus we write $X(u)=$ $x+\xi(u)$ with a fluctuation field $\xi(u)$ with $\xi(\infty)=0$. The Feynman propagators can then be deduced from the "kinetic" part

$$
\begin{aligned}
S_{\mathrm{gf}}^{0} & =\int_{D} \eta_{i} \wedge d \xi^{i}-* d \gamma^{i} \wedge d \beta_{i}-\lambda^{i} d * \eta_{i} \\
& =\int_{D} \eta_{i} \wedge\left(d \xi^{i}+* d \lambda^{i}\right)+\beta_{i} d * d \gamma^{i} .
\end{aligned}
$$

of the gauge fixed action. The other terms of $S_{\text {gf }}$ are considered as perturbations. Thus we have to invert the operators $d \oplus * d: \Omega^{0}(D) \oplus \Omega_{0}^{0}(D) \rightarrow \Omega^{1}(D)$ and $d * d: \Omega^{0}(D) \rightarrow$ $\Omega^{2}(D)$. Here $\Omega^{p}(D)$ is the space of smooth $p$-forms on $D$ and $\Omega_{0}^{0}(D)$ denotes the space of functions with Dirichlet boundary conditions $\lambda^{i}(u)=0, u \in \partial D$. Both operators are surjective but have a one-dimensional kernel consisting of constant functions. Inverses (modulo these kernels) are integral operators: to describe them it is useful to map conformally the disc onto the upper half plane $H_{+}$and use the standard complex coordinate of $H_{+}$. The integration kernel of $(d * d)^{-1}$ is the Green function $\frac{1}{2 \pi} \psi(z, w)$, with

$$
\psi(z, w)=\ln \left|\frac{z-w}{z-\bar{w}}\right| .
$$

The integration kernel of $(d \oplus * d)^{-1}$ is the Green function $G(w, z)=\frac{1}{2 \pi}\left(* d_{z} \psi(z, w) \oplus\right.$ $d_{z} \phi(z, w)$ ), where $d_{z}=d z \frac{\partial}{\partial z}+d \bar{z} \frac{\partial}{\partial \bar{z}}$ is the differential with respect to $z$ and

$$
\phi(z, w)=\frac{1}{2 i} \ln \frac{(z-w)(z-\bar{w})}{(\bar{z}-\bar{w})(\bar{z}-w)}
$$

We have $d_{w} * d_{w} \psi(z, w)=d_{w} * d_{w} \phi(z, w)=2 \pi \delta_{z}(w)$ where $\delta_{z}(w)$ is the Dirac distribution two-form, and the boundary conditions for $w \in \partial H_{+}$are Dirichlet for $\psi$ and Neumann for $\phi$. The propagators are then

$$
\begin{aligned}
\left\langle\gamma^{k}(w) \beta_{j}(z)\right\rangle= & \frac{i \hbar}{2 \pi} \delta_{j}^{k} \psi(z, w), \quad\left\langle\xi^{k}(w) \eta_{j}(z)\right\rangle=\frac{i \hbar}{2 \pi} \delta_{j}^{k} d_{z} \phi(z, w), \\
& \left\langle\lambda^{k}(w) \eta_{j}(z)\right\rangle=\frac{i \hbar}{2 \pi} \delta_{j}^{k} * d_{z} \psi(z, w)
\end{aligned}
$$

Note that $* d_{w} \psi(z, w)=d_{w} \phi(z, w)$ so that $\left\langle * d \gamma^{k}(w) \beta_{j}(z)\right\rangle=\delta_{j}^{k} \frac{i \hbar}{2 \pi} d_{w} \phi(z, w)$. It follows that the propagators combine into a superpropagator

$$
\left\langle\xi^{k}(w) \eta_{j}(z)\right\rangle+\left\langle * d \gamma^{k}(w) \beta_{j}(z)\right\rangle=\frac{i \hbar}{2 \pi} \delta_{j}^{k} d \phi(z, w),
$$

where $d=d_{z}+d_{w}$. In terms of superfields $\tilde{\eta}_{j}(z, \theta)=\beta_{j}(z)+\theta^{\mu} \eta_{j, \mu}(w), \tilde{\xi}^{k}(w, \zeta)=$ $\xi^{k}(w)+\zeta^{\mu} \eta_{\mu}^{+j}(w)$, with $\eta^{+j}=* d \gamma^{j}$, the superpropagator is

$$
\left\langle\tilde{\xi}^{k}(w, \zeta) \tilde{\eta}_{j}(z, \theta)\right\rangle=\frac{i \hbar}{2 \pi} \delta_{j}^{k} D \phi(z, w)
$$

where $D=\theta^{\mu} \frac{\partial}{\partial z^{\mu}}+\zeta^{\mu} \frac{\partial}{\partial w^{\mu}}$.
The perturbation expansion is then obtained by writing $S_{\mathrm{gf}}=S_{\mathrm{gf}}^{0}+S_{\mathrm{gf}}^{1}$ and expanding:

$$
\int e^{\frac{i}{\hbar} S_{\mathrm{gf}}} \mathcal{O}=\sum_{n=0}^{\infty} \frac{i^{n}}{\hbar^{n} n!} \int e^{\frac{i}{\hbar} S_{\mathrm{gf}}^{0}}\left(S_{\mathrm{gf}}^{1}\right)^{n} \mathcal{O}
$$

This expression is calculated using the Wick theorem for Gaussian integrals

$$
\begin{aligned}
& \int e^{\frac{i}{\hbar} S_{\mathrm{gf}}^{0} \tilde{\xi}^{k_{1}}\left(w_{1}, \zeta_{1}\right) \cdots \tilde{\xi}^{k_{N}}\left(w_{N}, \zeta_{N}\right) \tilde{\eta}_{j_{1}}\left(z_{1}, \theta_{1}\right) \cdots \tilde{\eta}_{j_{N}}\left(z_{N}, \theta_{N}\right) \delta_{x}(X(\infty))} \\
& =\sum_{\sigma \in S_{N}}\left\langle\tilde{\xi}^{k_{\sigma(1)}}\left(w_{\sigma(1)}, \zeta_{\sigma(1)}\right) \tilde{\eta}_{j_{1}}\left(z_{1}, \theta_{1}\right)\right\rangle \cdots\left\langle\tilde{\xi}_{\sigma(N)}^{k_{\sigma(N)}}\left(w_{\sigma(N)}, \zeta_{\sigma(N)}\right) \tilde{\eta}_{j_{N}}\left(z_{N}, \theta_{N}\right)\right\rangle .
\end{aligned}
$$

The normalization of the integral is such that $\int \exp \left(\frac{i}{\hbar} S_{\mathrm{gf}}^{0}\right) \delta_{x}(X(\infty))=1$, so that for $\alpha=$ 0 the star product coincides with the ordinary product. Here $\delta_{x}(X(t))=\prod_{i=1}^{d} \delta\left(X^{i}(t)-\right.$ $\left.x^{i}\right) \gamma^{i}(t)$ fixes the value of the zero modes (constant functions) of $X$ and the $\gamma^{\prime}$ 's are needed since the integral is otherwise zero, owing to the presence of zero modes in the integration over $\gamma$. More generally we could insert instead of the delta distribution a factor $\rho(X(\infty)) \gamma^{1}(\infty) \cdots \gamma^{d}(\infty)$, for some top differential form $\omega=\rho(x) d x^{1} \cdots d x^{d}$ on $M$, resulting in a factor $\int_{M} \omega$ in the right-hand side. The Feynman perturbation
expansion is then obtained by expanding the interaction term $S_{\mathrm{gf}}^{1}$ and the observable in powers of $\tilde{\xi} \tilde{\eta}$. This gives the vertices

$$
\begin{equation*}
S_{\mathrm{gf}}^{1}=\frac{1}{2} \int_{D} \int d^{2} \theta \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{j_{1}} \cdots \partial_{j_{k}} \alpha^{i j}(x) \tilde{\xi}^{\tilde{j}_{1}} \cdots \tilde{\xi}^{\tilde{j}_{k}} \tilde{\eta}_{i} \tilde{\eta}_{j} . \tag{4}
\end{equation*}
$$

Here the Berezin integral selects the two-form part of the superfield. We consider the observable of the correct ghost number

$$
\begin{equation*}
\mathcal{O}=f(\tilde{X}(1)) g(\tilde{X}(0)) \delta_{x}(X(\infty)) \tag{5}
\end{equation*}
$$

where $f, g \in C^{\infty}(M)$. Then expanding $f$ and $g$ in powers of $\tilde{\xi}$ we get an expansion in Feynman diagramst. The terms with $n$ vertices (4) are then labeled by the Kontsevich diagrams $\Gamma$ of order $n$, but possibly with tadpoles, i.e., lines that start and end at the same vertex. The term labeled by a diagram $\Gamma$ with lines $\left(j, v_{1}(j)\right),\left(j, v_{2}(j)\right), j=1, \ldots, n$ is $D_{\Gamma}(g \otimes f)$ (see Sect. 2) times

$$
\frac{1}{n!}\left(\frac{i}{\hbar}\right)^{n} \frac{1}{2^{n}}\left(\frac{i \hbar}{2 \pi}\right)^{2 n} \int \wedge_{j=1}^{n} d \phi\left(u_{j}, u_{v_{1}(j)}\right) \wedge d \phi\left(u_{j}, u_{v_{2}(j)}\right)=(-1)^{n} w_{\Gamma}
$$

The factor $1 / \prod k_{j}$ !, where $k_{j}$ is the number of lines pointing to $j$, is compensated by the fact that there are as many terms in the Wick theorem which give the same contribution because $k_{j}$ arguments of $\tilde{\xi}$ are equal to each other.

As explained at the end of Sect. 2, we have $(-1)^{n} w_{\Gamma} D_{\Gamma}(g \otimes f)=w_{\bar{\Gamma}} D_{\bar{\Gamma}}(f \otimes g)$, where $\bar{\Gamma}$ is $\Gamma$ with $R$ and $L$ interchanged. Thus the product obtained here coincides with Kontsevich's, except that it also involves tadpoles diagrams. These have to be considered separately, and require (finite) renormalization, which we proceed to discuss.
3.8. Renormalization. In the perturbation expansion described above, all integrals are absolutely convergent except for those containing tadpole diagrams, which are diagrams with one edge connecting a vertex to itself. The corresponding amplitude contains an ill-defined factor $d \phi(z, z)$, the superpropagator taken at coinciding points.

To make sense of this expression we introduce a point-splitting regularization and define $d \phi(z, z)$ as the limit

$$
d \phi(z, z)=\kappa(z ; \zeta)=\lim _{\epsilon \rightarrow 0} d \phi(z, z+\epsilon \zeta(z))
$$

Here $\zeta(z)$ is a vector field on $D$ which does not vanish in the interior of $D$. This limit exists but depends on the regularizing vector field $\zeta(z)$. Indeed, if we write $\zeta(z)=$ $r(z) e^{i \vartheta(z)}$ in polar coordinates, then

$$
\kappa(z ; \zeta)=d \vartheta(z)
$$

[^2]Thus the Feynman integrals have a finite renormalization ambiguity. One way to fix it is to add a counterterm

$$
\begin{equation*}
S_{\text {c.t. }}=\frac{i \hbar}{2 \pi} \int_{D} \int d^{2} \theta \partial_{i} \alpha^{i j}(\tilde{X}) \tilde{\eta}_{j} \tilde{\kappa}, \quad \tilde{\kappa}=\theta^{\mu} \kappa_{\mu} \tag{6}
\end{equation*}
$$

(or more simply choose the slightly singular $\vartheta=$ constant) which removes the tadpoles diagrams, and one gets precisely the Kontsevich formula. One easily checks that the action with the addition of the counterterm still obeys the classical master equation and, by the same argument as at the end of 3.2, also the quantum master equation.

## 4. Central functions

Using a non-rigourous quantum field theory argument based on BRST cohomology, we can prove the following claim:

There is a star product, equivalent to Kontsevich's, so that every function that is central in the Poisson algebra is also central for the star product.
Two star products $\star, \star^{\prime}$ corresponding to the same Poisson bracket are called equivalent if there is a series $R=R_{0}+\hbar R_{1}+\hbar^{2} R_{2}+\cdots$ with $R_{i}$ differential operators and $R_{0}=\mathrm{Id}$, such that $f \star^{\prime} g=R^{-1}(R f \star R g)$. The argument will also give us a formula for $R$, see (8).

Observe first that the BRST variation of a function on $M$ is given by

$$
\delta f(\tilde{X})=\partial_{i} f(\tilde{X})\left(D \tilde{X}^{i}+\alpha^{i j}(\tilde{X}) \tilde{\eta}_{j}\right)=D f(\tilde{X})+\tilde{\eta}_{j} \alpha^{i j}(\tilde{X}) \partial_{i} f(\tilde{X})
$$

If $f$ is central in the Poisson algebra-that is, if $\alpha^{i j}(X) \partial_{j} f(X)=0$ - then the second term on the right hand side vanishes. $\square$ Writing $f(\tilde{X})$ in components, $f(\tilde{X})=f(X)+$ $\theta^{\mu} \eta_{\mu}^{+i} \partial_{i} f(X)+\cdots$, we get the descent equations

$$
\begin{aligned}
\delta f(X) & =0 \\
\delta\left(\eta^{+i} \partial_{i} f(X)\right) & =-d f(X) .
\end{aligned}
$$

The first equation means that $f(X)$ is an observable. Therefore, the expectation value

$$
\begin{equation*}
h(u ; f, g)(x)=\int f(X(u)) g(X(0)) \delta_{x}(X(\infty)) e^{\frac{i}{\hbar} S} \tag{7}
\end{equation*}
$$

is well defined for any $u$ in the upper half plane. Observe that we put no additional hypotheses on $g$, so that - as in the previous sections- $g(X(v))$ is an observable only if $v$ is on the real axis.

[^3]The second descent equation may then be used to prove that $h$ is independent of $u$. In fact, denoting by $d$ the exterior derivative on the upper half plane, we get

$$
\begin{aligned}
d h(u ; f, g)(x)= & \int d f(X(u)) g(X(0)) \delta_{x}(X(\infty)) e^{\frac{i}{\hbar} S}= \\
& =-\int \delta\left[\eta^{+i}(u) \partial_{i} f(X(u))\right] g(X(0)) \delta_{x}(X(\infty)) e^{\frac{i}{\hbar} S}= \\
& =-\int \delta\left[\eta^{+i}(u) \partial_{i} f(X(u)) g(X(0)) \delta_{x}(X(\infty))\right] e^{\frac{i}{\hbar} S}=0 .
\end{aligned}
$$

Observe that to obtain the last equality one must also check that

$$
\triangle\left[\eta^{+i}(u) \partial_{i} f(X(u)) g(X(0)) \delta_{x}(X(\infty))\right]=0
$$

As a consequence, we get eventually

$$
R f \star g(x)=\lim _{\epsilon \downarrow 0} h(1+i \epsilon ; f, g)(x)=\lim _{\epsilon \downarrow 0} h(-1+i \epsilon ; f, g)(x)=g \star R f(x),
$$

Here for $v$ on the real axis, $R f(X(v))=\lim _{\epsilon \downarrow 0} f(X(v+i \epsilon))$ is the limit of the observable $f(X(u))$ defined on the upper half plane as $u$ tends to the real axis. We claim that $R f$ is given by the one-point function

$$
\begin{equation*}
R f(x)=\int f(X(u)) \delta_{x}(X(\infty)) e^{\frac{i}{\hbar} S}=f(x)+O\left(\hbar^{2}\right) \tag{8}
\end{equation*}
$$

for any point $u$ not on the boundary. This is based on the following factorization argument: if in the integral $h(u ; f, g)$ the point $u$ approaches the boundary, it is as if we considered the integral on two discs connected by a small bridge, with $u$ in the middle of one disc and the insertion point for the observable $g$ on the boundary of the other. In the limit one obtains a path integral for a disc with two points (and the point at infinity) on the boundary. One point is the insertion point for $g$ and at the other the result of the path integral on the disc with one point in the interior is inserted. See Fig. 1 for the case when $u$ approaches -1 . This argument can be made precise looking at the perturbation expansion [CFT] with the result that $R f=f+\hbar^{2} R_{2} f+\cdots$, with $R_{j}$ differential operators.

Thus $f$ is central for the star product $g \star^{\prime} h=R^{-1}(R g \star R h)$, proving the claim at the beginning of the section.

Using this result we may strengthen our claim:
The center of $C^{\infty}(M)[[\hbar]]$ with the star product $\star^{\prime}$ is $Z[[\hbar]]$ where $Z=\{f \in$ $\left.C^{\infty}(M) \mid\{f, \cdot\}=0\right\}$ is the center of the Poisson algebra $C^{\infty}(M)$.
The proof goes as follows. We need to show that if $f=f_{0}+\hbar f_{1}+\cdots$ is central for $\star^{\prime}$ then all coefficients $f_{i}$ are in $Z$. If $\{f, g\}=0$ for arbitrary $g \in C^{\infty}(M)$, then in particular the coefficient of $\hbar$ vanishes, so $\left\{f_{0}, g\right\}=0$ and $f_{0} \in Z$. But this implies, by what we showed above, that $f_{0}$ is central for the star product. Thus $\left(f-f_{0}\right) / \hbar=f_{1}+\hbar f_{2}+\cdots$ is central. By proceeding in this way we see that $f_{1}, f_{2}, \ldots$ are all in $Z$.


Figure 1. The expectation value ( ( $)$ in the limit as $u$ approaches the boundary reduces to a path integral on this surface

## 5. $L_{\infty}$ MORPHISM AND FORMALITY

5.1. The general path integral as a map from polyvector fields to polydifferential operators. The path integral we considered so far is a special case of the following general construction. A polyvector field of degree $p$ is a section of $\wedge^{p+1} T M$, i.e., a skew-symmetric contravariant tensor field of rank $p+1$,

$$
\frac{1}{(p+1)!} \alpha^{j_{0}, \ldots, j_{p}}(x) \frac{\partial}{\partial x^{j_{0}}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_{p}}} .
$$

A polyvector field is a sum $\alpha=\sum_{p=0}^{d-1} \alpha^{(p)}$ of polyvector fields of all nonnegative degrees. The space of polyvector fields is denoted by $T_{\text {poly }}(M)$.

For a multi-index $I=\left(i_{1}, \ldots, i_{d}\right) \in \mathbb{Z}_{\geq 0}^{d}$, let $\partial_{I}=\prod_{k}\left(\partial / \partial x^{k}\right)^{i_{k}}$. A polydifferential operator of degree $m$ is an operator $V: \bar{C}^{\infty}(M)^{\otimes m+1} \rightarrow C^{\infty}(M)$, of the form $V\left(f_{0} \otimes\right.$ $\left.\cdots \otimes f_{m}\right)(x)=\sum V_{I_{0}, \ldots, I_{m}}(x) \partial_{I_{0}} f_{0} \cdots \partial_{I_{m}} f_{m}$, with a finite sum over sets of multi-indices $I_{j}$. A polydifferential operator is a formal sum of polydifferential operators of arbitrary nonnegative degrees. The space of polydifferential operator is denoted by $D_{\text {poly }}(M)$.

To a polyvector field $\alpha$ we may associate a function of fields and antifields:

$$
S=S_{0}+S_{\alpha}
$$

with

$$
S_{0}=\int_{D} \int d^{2} \theta \tilde{\eta}_{j} D \tilde{X}^{j}-\int_{D} \lambda^{i} \gamma_{i}^{+}
$$

as above, and

$$
S_{\alpha}=\sum_{p=0}^{d-1} \int_{D} \int d^{2} \theta \frac{1}{(p+1)!} \alpha^{j_{0}, \ldots, j_{p}}(\tilde{X}(u, \theta)) \tilde{\eta}_{j_{0}}(u, \theta) \cdots \tilde{\eta}_{j_{p}}(u, \theta)
$$

We may then consider correlation functions of boundary fields associated to the functions $f_{0}, \ldots, f_{m}$ on $M$.

$$
\begin{aligned}
U(\alpha)\left(f_{0} \otimes \cdots \otimes f_{m}\right)(x) & =\int e^{\frac{i}{\hbar}\left(S_{0}+S_{\alpha}\right)} \mathcal{O}_{x}\left(f_{0}, \ldots, f_{m}\right), \\
\mathcal{O}_{x}\left(f_{0}, \ldots, f_{m}\right) & =\int_{B_{m}}\left[f_{0}\left(\tilde{X}\left(t_{0}, \theta_{0}\right)\right) \cdots f_{m}\left(\tilde{X}\left(t_{m}, \theta_{m}\right)\right)\right]^{(m-1)} \delta_{x}(X(\infty)) .
\end{aligned}
$$

The path integral is, as before, the integral over the Lagrangian submanifold in the space of fields and antifields determined by our gauge condition $d * \eta=0$. The integral over the $t_{i}$ is the integral over the $m-1$ form part (the coefficient of $\theta_{1} \cdots \theta_{m-1}$ ) of the integrand over the simplex $1=t_{0}>t_{1}>\cdots>t_{m-1}>t_{m}=0$, with the orientation given by the volume form $d t_{1} \wedge \cdots \wedge d t_{m-1}$. It may be viewed as an integral over the moduli space $B_{m}$ of $m+1$ cyclically ordered points on the circle modulo conformal transformations. More explicitly

$$
\mathcal{O}_{x}\left(f_{0}, \ldots, f_{m}\right)=\int_{1>t_{1}>\cdots>t_{m-1}>0} f_{0}(X(1)) \prod_{k=1}^{m-1} \partial_{i_{k}} f\left(X\left(t_{k}\right)\right) \eta^{+i_{k}}\left(t_{k}\right) f_{m}(X(0)) \delta_{x}(X(\infty))
$$

Expanding the path integral in powers of $\hbar$ as in the previous section, we get a map $U$ that associates, to each polyvector field $\alpha$, a formal power series whose coefficients are polydifferential operators.

The perturbative expansion has the form $U(\alpha)=\sum_{n=0}^{\infty} U_{n}(\alpha, \ldots, \alpha ; \hbar)$. Here $U_{n}$ is a multilinear function of $n$ arguments in $T_{\text {poly }}(M)$ with values $D_{\text {poly }}(M)$. The formula for $U_{n}$ is

$$
U_{n}\left(\alpha_{1}, \ldots, \alpha_{n} ; \hbar\right)\left(f_{0} \otimes \cdots \otimes f_{m}\right)(x)=\int e^{\frac{i}{\hbar} S_{0}} \frac{i}{\hbar} S_{\alpha_{1}} \cdots \frac{i}{\hbar} S_{\alpha_{n}} \mathcal{O}_{x}\left(f_{0}, \ldots, f_{m}\right)
$$

Suppose now that, for $i=1, \ldots, n, \alpha_{i}$ is homogeneous of degree $p_{i}$. Then $S_{\alpha_{i}}$ is the integral of the two-form component of $\frac{1}{\left(p_{i}+1\right)!} \alpha_{i}^{j_{0} \ldots j_{p_{i}}}(\tilde{X}) \tilde{\eta}_{j_{0}} \ldots \tilde{\eta}_{j_{p_{i}}}$, and has thus ghost number $p_{i}-1$. This has two consequences: first, since the integral over the $t_{i}$ picks the $m-1$ form component of $\prod f_{i}\left(\tilde{X}\left(t_{i}\right)\right)$, which has ghost number $1-m$, we have the ghost number condition $1-m+\sum_{i=1}^{n}\left(p_{i}-1\right)=0$ or

$$
\begin{equation*}
m=1-n+\sum_{i=1}^{n} p_{i} \tag{9}
\end{equation*}
$$

for the path integral to be non-zero. This means that $U_{n}$ is a map of degree $1-n$ from $T_{\text {poly }}(M)^{\otimes n}$ to $D_{\text {poly }}(M)$. Using this formula we may compute the dependence on $\hbar$ of this integral: the path integral has an overall $\hbar$ to the power $-n+\sum\left(p_{i}+1\right)=n+m-1$ (each vertex has $1 / \hbar$ and each propagator has an $\hbar$ ), and we have

$$
U_{n}\left(\alpha_{1}, \ldots, \alpha_{n} ; \hbar\right)\left(\otimes_{0}^{m} f_{i}\right)=(i \hbar)^{n+m-1} U_{n}\left(\alpha_{1}, \ldots, \alpha_{n}\right)\left(\otimes_{0}^{m} f_{i}\right),
$$

with $U_{n}\left(\alpha_{1}, \ldots, \alpha_{n}\right)=U_{n}\left(\alpha_{1}, \ldots, \alpha_{n} ; \hbar=1 / i\right)$ independent of $\hbar$.

The second consequence is that

$$
\begin{equation*}
U_{n}\left(\ldots, \alpha_{i}, \ldots, \alpha_{j}, \ldots\right)=(-1)^{\left(p_{i}-1\right)\left(p_{j}-1\right)} U_{n}\left(\ldots, \alpha_{j}, \ldots, \alpha_{i}, \ldots\right) \tag{10}
\end{equation*}
$$

i.e., $U_{n}$ is symmetric in a graded sense.
5.2. Special cases. Let us consider in detail some special cases. For $n=0, U_{n}$ is a polydifferential operator of degree $m=1$, and

$$
U_{0}\left(f_{0} \otimes f_{1}\right)=\int e^{\frac{i}{\hbar} S_{0}} f_{0}(\tilde{X}(1)) f_{1}(\tilde{X}(0)) \delta_{x}(X(\infty))=f_{0}(x) f_{1}(x)
$$

is the undeformed product on $C^{\infty}(M)$.
If $n=1$ and $\alpha$ is a polyvector field of degree $p$ then $U_{1}(\alpha)$ is of degree $p$. Let

$$
\alpha=\frac{1}{(p+1)!} \alpha^{j_{0}, \ldots, j_{p}}(x) \frac{\partial}{\partial x^{j_{0}}} \wedge \cdots \wedge \frac{\partial}{\partial x^{j_{p}}},
$$

with $\alpha^{j_{0}, \ldots, j_{p}}$ antisymmetric. The Wick theorem yields in this case

$$
U_{1}(\alpha ; \hbar)\left(f_{0} \otimes \cdots \otimes f_{p}\right)(x)=\frac{i}{\hbar}\left(\frac{i \hbar}{2 \pi}\right)^{p+1} I_{p} \alpha^{j_{0}, \ldots, j_{p}} \partial_{j_{0}} f_{0}(x) \cdots \partial_{j_{p}} f_{p}(x)
$$

Here $I_{p}$ is the integral

$$
I_{p}=\int d \phi(u, 1) \wedge d \phi\left(u, t_{1}\right) \wedge \cdots \wedge d \phi(u, 0)
$$

over $u=u^{1}+i u^{2} \in H$ and $1>t_{1}>\cdots>t_{p-1}>0$, with orientation given by the form $d u^{1} \wedge d u^{2} \wedge d t_{1} \ldots d t_{p-1}$. To compute this integral we proceed as in Sect. 2 and introduce new variables $\phi_{0}=\phi(u, 1), \phi_{j}=\phi\left(u, t_{j}\right)(j=1, \ldots, m-1)$ and $\phi_{p}=\phi(u, 0)$. In the new variables the integration is over the region $2 \pi>\phi_{0}>\cdots>\phi_{p}>0$. We claim that the Jacobian of the change of variables is $(-1)^{p}$. This follows from the fact that $d \phi(u, 0) \wedge d \phi(u, 1)=J d u^{1} \wedge d u^{2}$ with $J>0$ and that $\partial \phi(u, t) / \partial t>0$. Hence,

$$
\begin{aligned}
\int d \phi(u, 1) \wedge d \phi\left(u, t_{1}\right) \wedge \cdots \wedge d \phi(u, 0) & =(-1)^{p} \int_{2 \pi>\phi_{0}>\cdots>\phi_{p}>0} d \phi_{0} \cdots d \phi_{p} \\
& =\frac{(-1)^{p}(2 \pi)^{p+1}}{(p+1)!}
\end{aligned}
$$

and we obtain

$$
U_{1}(\alpha)\left(f_{0} \otimes \cdots \otimes f_{p}\right)(x)=\frac{(-1)^{p+1}}{(p+1)!} \alpha^{j_{0}, \ldots, j_{p}}(x) \partial_{j_{0}} f_{0}(x) \cdots \partial_{j_{p}} f_{p}(x)
$$

5.3. $U$ is an $L_{\infty}$ morphism. The formal properties of the map $U$ can be deduced using the main trick of the BV formalism, which is to use the fact that the integral of the Laplacian of anything is zero. In our situation we have, with $S_{0}=\int_{D} \int d^{2} \theta \tilde{\eta}_{i} D \tilde{X}^{i}-$ $\int_{D} \lambda^{i} \gamma_{i}^{+}$, and $\alpha_{j}(j=1, \ldots, n)$ homogeneous polyvector fields of degree $p_{j}$,

$$
\int \triangle\left(e^{\frac{i}{\hbar} S_{0}} \prod_{i=1}^{n} S_{\alpha_{i}} \mathcal{O}_{x}\left(f_{0}, \ldots, f_{m}\right)\right)=0
$$

To evaluate the left-hand side we use (2) and $\triangle S_{0}=\triangle S_{\alpha}=0$. Also, we use the fact that $\left(S_{0}, S_{\alpha}\right)$ is proportional to $\int_{D} \int d^{2} \theta D\left(\alpha^{j_{0} \ldots j_{p}}(\tilde{X}) \tilde{\eta}_{j_{0}} \cdots \tilde{\eta}_{j_{p}}\right)$ which vanishes because of the boundary conditions for $\tilde{\eta}_{j}$. Thus we get

$$
\begin{aligned}
0= & (-1)^{m-1} \int e^{\frac{i}{\hbar} S_{0}} \prod_{i=1}^{n} S_{\alpha_{i}} \frac{i}{\hbar}\left(S_{0}, \mathcal{O}_{x}\left(f_{0}, \ldots, f_{m}\right)\right) \\
& +\int e^{\frac{i}{\hbar} S_{0}} \sum_{1 \leq j<k \leq n} \epsilon_{j k}\left(S_{\alpha_{j}}, S_{\alpha_{k}}\right) \prod_{i \neq j, k} S_{\alpha_{i}} \mathcal{O}_{x}\left(f_{0}, \ldots, f_{m}\right) .
\end{aligned}
$$

The sign is $\epsilon_{j k}=(-1)^{\left(g_{1}+\cdots+g_{j}\right) g_{j}+\left(g_{1}+\cdots+g_{j-1}+g_{j+1}+\cdots+g_{k-1}\right) g_{k}}$, where $g_{j}=p_{j}-1$ is the ghost number of $S_{\alpha_{j}}$. Now the BV bracket $\left(S_{\alpha_{j}}, S_{\alpha_{k}}\right)$ is again of the form $S_{\alpha}$ :

$$
\left(S_{\alpha_{j}}, S_{\alpha_{k}}\right)=-S_{\left[\alpha_{j}, \alpha_{k}\right]} .
$$

The Schouten-Nijenhuis bracket [, ] is a graded super Lie algebra structure on $T_{\text {poly }} M$. On vector fields it is defined to be the usual Lie bracket, and it is extended to polyvector fields by the Leibnitz rule $\left[\alpha_{1}, \alpha_{2} \wedge \alpha_{3}\right]=\left[\alpha_{1}, \alpha_{2}\right] \wedge \alpha_{3}+(-1)^{p_{1}\left(p_{2}-1\right)} \alpha_{2} \wedge\left[\alpha_{1}, \alpha_{3}\right]$. The Jacobi identity for a bivector field $\alpha$ is $[\alpha, \alpha]=0$.

Moreover, $\left(S_{0}, f(\tilde{X}(t), \theta)\right)=D f(\tilde{X}(t), \theta)$, which in components reads

$$
\left(S_{0}, f(X(t))\right)=0, \quad\left(S_{0}, \partial_{i} f(X(t)) \eta^{+i}(t)\right)=-d f(X(t))
$$

Using this identity and the Leibnitz rule (3) we see that the integral over $B_{m}$ reduces to an integral over the boundary (of a suitable compactification). The compactification may be understood by thinking of $B_{m}$ as the moduli space of discs with $m+1$ marked points on the boundary modulo the action of $S U(1,1)$. The boundary of $\bar{B}_{m}$ consists of discs degenerated into pairs of discs with a point in common. Its various connected components are obtained by distributing the points on the two discs in all possible ways compatible with the cyclic ordering, see Fig. 2. By the usual factorization arguments of quantum field theory, we obtain the following formulae. Let $S_{\ell, n-\ell}$ be the subset of the group $S_{n}$ of permutations of $n$ letters consisting of permutations such that $\sigma(1)<\cdots<$ $\sigma(\ell)$ and $\sigma(\ell+1)<\cdots<\sigma(n)$. For $\sigma \in S_{\ell, n-\ell}$ let us introduce the sign

$$
\epsilon(\sigma)=(-1)^{\sum_{r=1}^{\ell} g_{\sigma(r)}\left(\sum_{s=1}^{\sigma(r)-1} g_{s}-\sum_{s=1}^{r-1} g_{\sigma(s)}\right)} .
$$



Figure 2. A component of the boundary of $B_{m}$. The point $\infty$ is $t_{m+1}$

It is the sign one gets if one puts a product of $n$ elements of degree $g_{1}, \ldots, g_{n}$ of a graded commutative algebra in the order given by $\sigma$. Then we have

$$
\begin{gathered}
\sum_{\ell=0}^{n} \sum_{k=1}^{m-1} \sum_{i=0}^{m-k} \sum_{\sigma \in S_{\ell, n-\ell}} \epsilon(\sigma)(-1)^{k(i+1)}(-1)^{m} U_{\ell}\left(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(\ell)}\right)\left(f_{0} \otimes \cdots \otimes f_{i-1}\right. \\
\left.\quad \otimes U_{n-\ell}\left(\alpha_{\sigma(\ell+1)}, \ldots, \alpha_{\sigma(n)}\right)\left(f_{i} \otimes \cdots \otimes f_{i+k}\right) \otimes f_{i+k+1} \otimes \cdots \otimes f_{m}\right) \\
=\sum_{i<j} \epsilon_{i j} U_{n-1}\left(\left[\alpha_{i}, \alpha_{j}\right], \alpha_{1}, \ldots, \widehat{\alpha}_{i}, \ldots, \widehat{\alpha}_{j}, \ldots, \alpha_{n}\right)\left(f_{0} \otimes \cdots \otimes f_{m}\right)
\end{gathered}
$$

The sign $-(-1)^{k(i+1)}$ comes from the orientation of the faces of the boundary of $\bar{B}_{m}$. A sequence of maps $U_{n}$ with this property and the symmetry (10) is called an $L_{\infty}$ morphism Schist, LS, 区. A consequence of it in this case is the formality conjecture: $U_{1}$, which is (up to sign) the obvious map sending a polyvector field to itself viewed as polydifferential operator, induces an isomorphism of graded Lie algebras from the graded Lie algebra of polyvector fields to the cohomology of the polydifferential operators, viewed as a complex of the Hochschild cochains of $C^{\infty}(M)$, see [区].

A special case of this identity is the associativity of the star product: in this case $\alpha$ is a bivector field (a polyvector field of degree 1) obeying the Jacobi identity $[\alpha, \alpha]=0$. Then we have $U(\alpha)=\sum_{n=0}^{\infty} \hbar^{n} U_{n}(\alpha, \ldots, \alpha)$ and by the ghost number condition (9), every $U_{n}$ is a bidifferential operator. The $L_{\infty}$ identity reduces to the associativity of $U(\alpha)$ :

$$
\sum_{k=0}^{n} U_{k}\left(U_{n-k}\left(f_{0} \otimes f_{1}\right) \otimes f_{2}\right)-\sum_{k=0}^{n} U_{k}\left(f_{0} \otimes U_{n-k}\left(f_{1} \otimes f_{2}\right)\right)=0
$$

where we have suppressed the dependence on $\alpha$ in the notation.
5.4. Tadpoles. The perturbative expansion of $U(\alpha)$ contains tadpoles which can be removed as in 3.8 either by choosing a constant angle $\vartheta$, or by replacing $S_{\alpha}$ by

$$
S_{\alpha}^{\prime}=S_{\alpha}-\frac{i \hbar}{2 \pi} \frac{1}{p!} \int_{D} \int d^{2} \theta \tilde{\kappa} \partial_{k} \alpha^{k, j_{1}, \ldots, j_{p}} \tilde{\eta}_{j_{1}} \cdots \tilde{\eta}_{j_{p}}
$$

The arguments of the previous subsections remain valid with the addition of these counterterms, since $\triangle S_{\alpha}^{\prime}=\left(S_{0}, S_{\alpha}^{\prime}\right)=0$ and $\left(S_{\alpha}^{\prime}, S_{\beta}^{\prime}\right)=-S_{[\alpha, \beta]}^{\prime}$.

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[^0]:    ${ }^{1}$ In $K \pi$ is what is here $i \hbar / 2$. We adopt the notation of the physics literature and work accordingly over the complex numbers. With Kontsevich's conventions one may formulate the problem over the real numbers, which in terms of the physics conventions would mean to have an imaginary Planck constant.
    ${ }^{2}$ We use throughout the paper the Einstein summation convention, meaning that sums over repeated indices are understood

[^1]:    ${ }^{3}$ In the symplectic case, where $\alpha$ comes from a symplectic form $\omega$, one can integrate formally over $\eta$ and this formula may, in the spirit of Feynman [F], be written as

    $$
    f \star g(x)=\int_{\gamma( \pm \infty)=x} f(\gamma(1)) g(\gamma(0)) e^{\frac{i}{\hbar} \int_{\gamma} d^{-1} \omega} d \gamma
    $$

    The integral over trajectories $\gamma: \mathbb{R} \rightarrow M$ is to be understood as an expansion around the classical solution $\gamma(t)=x$, which is a constant function of time since the Hamiltonian vanishes.

[^2]:    ${ }^{4}$ Actually, the arguments of $f$ and $g$ of our original integral are $X(1), X(0)$, rather than the superfields. However, the additional terms in $\tilde{X}(1), \tilde{X}(0)$ do not contribute to the integral since their are of negative ghost number

[^3]:    ${ }^{5}$ In presence of the counterterm (6) the BRST operator is modified and we get an additional term $(i \hbar / 2 \pi) \tilde{\kappa} \partial_{j}\left(\alpha^{j i} \partial_{i} f\right)(\tilde{X})$ in the formula for $\delta f$. This term also vanishes if $f$ is central in the Poisson algebra.

