A Path to the Arrow-Debreu Competitive Market Equilibrium

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Abstract: We present polynomial-time interior-point algorithms for solving the Fisher and Arrow-Debreu competitive market equilibrium problems with linear utilities and n players. The algorithm for solving the Fisher problem is a modified primal-dual path-following algorithm, and the one for solving the Arrow-Debreu problem is a primal-based algorithm. Both of them have the arithmetic computation complexity bound of $O(n^4 \log(1/\epsilon))$ for computing an ϵ equilibrium solution. If the problem data are rational numbers and their bit-length is L, then the bound to generate an exact solution is $O(n^4 L)$ which is in line with the best complexity bound for linear programming of the same dimension and size. This is a significant improvement over the previously best bound $O(n^8 L)$ for solving the two problems. We also present a continuous path leading to the set of the Arrow-Debreu equilibrium, similar to the central path developed for linear programming interior-point methods. This path is derived from the weighted logarithmic utility and barrier functions and the fixed point theorem. The defining equations are bilinear and possess some primal-dual structure for the application of Newton's path-following method.

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1 Introduction

We consider the Arrow-Debreu competitive market equilibrium problem which was first formulated by Leon Walras in 1874 [26]. In this problem every one in a population of n players has an initial endowment of a divisible good and a utility function for consuming all goods—own and others. Every player sells the entire initial endowment and then uses the revenue to buy a bundle of goods such that his or her utility function is maximized. Walras asked whether prices could be set for everyone's good such that this is possible. An answer was given by Arrow and Debreu in 1954 [1] who showed that such equilibrium would exist if the utility functions were concave. Their proof was non-constructive and did not offer any algorithm to find such equilibrium prices.

Fisher was the first to consider algorithm to compute equilibrium prices for a related and different model where players are divided into two catalogs: producer and consumer. Consumers have money to buy good and maximize their individual utility functions; producer sell their goods for money. The equilibrium prices is an assignment of prices to goods so as when every consumer buys an maximal bundle of goods then the market clears, meaning that all the money is spent and all goods are sold. Fisher's model is a special case of Walras' model when money is also considered a commodity so that Arrow and Debreu's result applies.

Eisenberg and Gale [11, 15] gave a convex optimization setting to formulate Fisher's model with linear utility functions. They constructed an concave objective function that is maximized at the equilibrium. Thus, finding an equilibrium became solve a convex optimization problem, and it could be solved by using the Ellipsoid method in polynomial time. Here, polynomial time means that one can compute an ϵ approximate equilibrium in a number of arithmetic operations bounded by polynomial in n and $\log \frac{1}{\epsilon}$. Devanur et al. [9] recently developed a "combinatorial" algorithm for solving Fisher's model with linear utility functions too [1]. Either the ellipsoid method or the combinatorial algorithm has a running time in the order of $O(n^8 \log(1/\epsilon))$. Both approaches, Eisenberg-Gale and Devanur et al.,

[1] There were critical errors in their initial conference paper, but they corrected them in a journal version.

did not apply to the more general Walras model. The ϵ based complexity result seems more appropriate for analyzing these problems because general solutions may be irrational even all input data are rational.

Solving the Arrow-Debreu problem was proved to be more difficult. Eaves [12] showed that the problem with linear utility can be formulated as a linear complementarity problem (e.g. Cottle et al. [7]) so that Lemke's algorithm could compute the equilibrium, if it existed, in a finite time. It was also proved there that the equilibrium solution values were rational as solutions to an n^2-dimension system of linear equations of the original rational inputs. In a later paper [13], Eaves also proved that the problem with Cobb-Douglass utility could be solved in strongly polynomial time of $O(n^3)$. Other effective algorithms to solve the problem include Primak [24], Dirkse and Ferris [10], and Rutherford [25], see the excellent survey by Ferris and Pang [14]. None of these are proved to be polynomial-time algorithms.

More recently, however, Jain [16] has showed that Walras's model can be also formulated as a convex optimization, more precisely, a convex inequality problem, so that the Ellipsoid method again can be used in solving it. Remarkably, he found out a clean set of posinomial inequalities to describe the problem which is necessary and sufficient. This set of inequalities can be logarithmically transferred into a set of convex inequality, which technique was used for geometric programming in early 60's. Similar inequalities were written in the past but with additional inequalities, which were not convex transferable, according to Jain.

The goal of this paper is threefold. First, we develop a polynomial-time interior-point algorithm to solve Fisher's model with linear utility. The complexity bound, $O(n^4 \log \frac{1}{\epsilon})$, of the algorithm is significantly lower than either the Ellipsoid or "combinatorial" algorithm mentioned above. Secondly, we present an interior-point algorithm, which is not primal-dual, for solving the Arrow-Debreu competitive market equilibrium problem with linear utility. The algorithm has an efficient barrier function for every convex inequality where the self-condordant coefficient is at most 2. Thus, the number of arithmetic operations of the algorithm is again bounded by $O(n^4 \log \frac{1}{\epsilon})$, which is substantially lower than the one obtained by the ellipsoid method. If the input data are rational, then an exact solution can be obtained by solving the identified system of linear equations, such as Eaves' modle, when $\epsilon < 2^{-L}$, where L is the bit length of the input data. Thus, the arithmetic operation bound becomes $O(n^4L)$, which is in line with the best complexity bound for linear programming of the same dimension and size.

Finally, we develop a convex optimization setting for Walras' model, and present a continuous path leading to the set of the Arrow-Debreu equilibrium, similar to the central path developed for linear programming interior-point methods (see, e.g., Megiddo [20]). The path is derived from the weighted logarithmic utility and barrier functions and the fixed point theorem. The defining equations are bilinear and possess some primal-dual structure for the application of Newton's method.

2 An interior-point algorithm for solving the Fisher equilibrium

In Fisher's model the players are divided into two catalogs: producer and consumer. Consumer i, $i \in C$, has given money w_i to spend and buy good to maximize their individual utility functions; produce j's, $j \in P$, sell their goods for money. The equilibrium prices is an assignment of prices to goods so as when every consumer buys an maximal bundle of goods then the market clears, meaning that all the money is spent and all goods are sold. Eisenberg and Gale [11] gave a convex optimization setting, where, without losing generality, each producer has one unit good:

$$\begin{array}{ll} \text{maximize} & \sum_{i\in C} w_i \log\left(\sum_{j\in P} u_{ij}x_{ij}\right) \\ \text{subject to} & \sum_{i\in C} x_{ij} = 1, \quad \forall j \in P \\ & x_{ij} \geq 0, \quad \forall i, j. \end{array}$$

Here, player i, $i \in C$, has a linear utility function

$$u(x_{i\cdot}) = u(x_{i1}, ..., x_{in}) = \sum_j u_{ij}x_{ij},$$

where $u_{ij} \geq 0$ is the given utility coefficient of player i for producer j's good and x_{ij} represents the amount of good bought from producer j by consumer i. They proved that the optimal dual vector of the convex problem is the market clearing price.

Jain has the following economic interpretation. Consider a person, he has some utility function. This utility function is his measurement of his happiness in terms of his own measuring scale. Consider two different persons. They may have two different utility functions. They may be using different scales or units for measuring their utilities. But suppose these two persons are in a single family, say husband and wife. If they want to measure the total happiness of the family then what should be a natural way of combining the individual's utility to measure the aggregate utility of the family? Or, the society at large? In the Eisenberg-Gale model the weights used are the amount of money a person possess, which is known and fixed in Fisher's model. This is very natural as shown by the following thought experiment. Suppose we multiply the utilities with *uniform* weights for each player. Let us consider each person in a family. In this family, only the wife goes to the market to buy stuff for the family. So if the market, via open market rules, maximizes the product of utilities of all the players then the weight of the family is only one in the product. But if both husband and wife go to the market then the weight is two. But we know that there is nothing in the rules of the open market which depends on the number of people going to the market. Instead the open market rules depend upon the amount of goods and money brought into the market. So for Eisenberg and Gale, w_i is the amount of money the player has. Hence they showed that the open market maximizes the product of utilities, where this product is taken over all the money in the system so that the market is clear. In fact, based on this principal, Eisenberg-Gale showed the existence of an equilibrium for the Fisher model.

2.1 The weighted analytic center

The Eisenberg-Gale model can be rewritten as

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{i\in C} w_i \log(u_i) \\ \text{subject to} \quad & \textstyle\sum_{i\in C} x_{ij} = 1, \quad \forall j \in P \\ & u_i - \textstyle\sum_{j\in P} u_{ij}x_{ij} = 0, \quad \forall i \in C \\ & u_i,\ x_{ij} \geq 0, \quad \forall i, j. \end{aligned} \tag{1}$$

Consider a more general problem

$$\begin{aligned} \text{maximize} \quad & \textstyle\sum_{j=1}^{n} w_j \log(x_j) \\ \text{subject to} \quad & Ax = b, \\ & x \geq 0, \end{aligned} \tag{2}$$

where given A is an $m \times n$ dimension matrix with full row rank and b is a m dimension vector, and w_j is the weight on each of the n variables. An x who satisfies the constraint is called a primal feasible solution, while the maximal solution to the problem is called the weighted analytic center.

If the feasible set is bounded and has an interior, the optimality conditions of the weighted analytic center are:

$$\begin{aligned} Sx &= w, \\ Ax &= b,\ x \geq 0, \\ -A^T y + s &= 0,\ s \geq 0, \end{aligned} \tag{3}$$

where y and s are called dual variable and slacks, respectively, and S is the diagonal matrix with slack vector s on its diagonals. When $w_j > 0$ and integral for all j, a weight-scaling interior-point algorithm was developed by Atkinson and Vaidya [2] where the arithmetic operation complexity bound is $O(n^3 \log(\frac{\max(w)}{\min(w)}))$ to compute a solution such that

$$\begin{aligned} \|Sx - w\| &\leq O(\min(w)), \\ Ax &= b,\ x \geq 0, \\ -A^T y + s &= 0,\ s \geq 0. \end{aligned}$$

They start with an approximate analytic center where all weights equal $\min(w)$, and then scale them up to w iteratively. It is not clear how their algorithm can be adapted or analyzed when some of w_j are zeros, which is the case of Fisher's model (1).

2.2 A modified primal-dual path-following algorithm

In this subsection, we modify the standard primal-dual path-following algorithm (e.g., Kojima et al. [19], Monteiro and Adler [22] and Mizuno et al. [21]) for solving problems (2) and (1) and analyze their complexity to computing an ϵ-solution for any $\epsilon > 0$:

$$\begin{array}{rcl} \|Sx - w\| & \leq & \epsilon, \\ Ax & = & b,\ x \geq 0, \\ -A^T y + s & = & 0,\ s \geq 0. \end{array} \tag{4}$$

Let $x > 0$ and $s > 0$ be a primal and dual interior-point point pair such that

$$\|Sx - \hat{w}\| \leq \eta\mu, \tag{5}$$

where μ represents the duality gap, η is a positive constant less than 1, and

$$\hat{w}_j = \max\{\mu,\ w_j\}. \tag{6}$$

Such a point pair is called an approximate central-path point pair of the primal and dual feasible set.

Now we solve a prima-dual system of linear equations for d_x, d_y and d_s:

$$\begin{array}{rcl} Sd_x + Xd_s & = & \hat{w}^+ - Xs, \\ Ad_x & = & 0, \\ -A^T d_y + d_s & = & 0, \end{array} \tag{7}$$

where

$$\hat{w}_j^+ = \max\{(1 - \frac{\eta}{\sqrt{n}})\mu, w_j\}. \tag{8}$$

Note that $d_x^T d_s = d_x^T A^T d_y = 0$ here. The arithmetic operations of solving the system is to form the normal matrix ADA^T, where D is a diagonal matrix whose diagonals are strictly positive, and factorize it.

After obtain (d_x, d_y, d_s) let

$$\begin{aligned} x^+ &:= x + d_x, \\ y^+ &:= y + d_y, \\ s^+ &:= s + d_s. \end{aligned} \tag{9}$$

Then, we prove that x^+ and (y^+, s^+) are an interior-point feasible pair, and

$$\|(S^+)x^+ - \hat{w}^+\| \le \eta\mu^+ \tag{10}$$

where

$$\mu^+ = (1 - \frac{\eta}{\sqrt{n}})\mu,$$

so that the computation can repeat.

First, it is helpful to re-express d_x and d_s. Let

$$\begin{aligned} p &:= X^{-.5}S^{.5}d_x, \\ q &:= X^{.5}S^{-.5}d_s, \\ r &:= (XS)^{-.5}(\hat{w}^+ - Xs), \end{aligned} \tag{11}$$

Note that

$$p + q = r \quad \text{and} \quad p^T q = 0$$

so that p and q represent an orthogonal decomposition of r.

Secondly, from (5,6,8), we have

$$x_j s_j \ge \hat{w}_j - \eta\mu \ge (1 - \eta)\mu$$

and

$$\|\hat{w}^+ - Xs\| = \|\hat{w}^+ - \hat{w} + \hat{w} - Xs\| \le \|\hat{w}^+ - \hat{w}\| + \|\hat{w} - Xs\| \le \eta\mu + \eta\mu = 2\eta\mu,$$

which implies that

$$\|r\| \le \|(XS)^{-.5}\| \|\hat{w}^+ - Xs\| \le \frac{2\eta\sqrt{\mu}}{\sqrt{1-\eta}}.$$

Moreover, it is also proved in Mizuno et al. [21] that

$$\|p\|^2 + \|q\|^2 = \|r\|^2 \quad \text{and} \quad \|Pq\| \le \frac{\sqrt{2}}{4}\|r\|^2.$$

Thus,

$$\begin{aligned}
\|(S^+)x^+ - \hat{w}^+\|^2 &= \|(S + D_s)(x + d_x) - \hat{w}^+\|^2 \\
&= \|Sx + Sd_x + Xd_s - \hat{w}^+ + D_s d_x\|^2 \\
&= \|D_s d_x\|^2 \\
&= \|Pq\|^2 \\
&\le \frac{\sqrt{2}}{4}\|r\|^2 \\
&\le \frac{\sqrt{2}\eta^2}{1-\eta}\mu \\
&\le \frac{\sqrt{2}\eta^2}{(1-\eta)^2}\mu^+.
\end{aligned}$$

Thus, if we choose constant η such that

$$\frac{\sqrt{2}\eta^2}{(1-\eta)^2} \le \eta$$

then condition (10) holds. Moreover,

$$\begin{aligned}
\|X^{-1}(x^+ - x)\| &= \|X^{-1}d_x\| \\
&= \|(XS)^{-.5}p\| \\
&\le \|(XS)^{-.5}\| \|p\| \\
&\le \frac{\|p\|}{\sqrt{(1-\eta)\mu}} \\
&\le \frac{\|r\|}{\sqrt{(1-\eta)\mu}} \\
&\le \frac{2\eta}{1-\eta} < 1,
\end{aligned}$$

which implies that $x^+ > 0$. Similarly, we have $s^+ > 0$. That is, (x^+, y^+, s^+) is a feasible interior-point pair.

We can generate an initial point pair $x^0 > 0$ and $s^0 > 0$ such that

$$\|S^0 x^0 - \mu^0 e\| \leq \eta \mu^0$$

where $\mu^0 = \max(w)$ and e is the vector of all ones. Such a point pair is called an approximate analytic center of the primal and dual feasible set. In problem (1), the primal feasible set has an interior and it is bounded, which implies that the dual feasible set has interior. The complexity to generate such an initial point pair is $O(n^3(\log \frac{1}{\epsilon}))$ arithmetic operations. Since the dual feasible set is homogeneous, we can always scale (y, s) so that $\mu^0 = \max(w)$.

Note that μ is decreased at a geometric rate $(1 - \eta/\sqrt{n})$ and it starts at $\max(w)$. Also, if $w_j = 0$ for some j, then

$$s_j x_j \leq \frac{\epsilon}{\sqrt{n}}$$

from

$$|s_j x_j - \mu| \leq \eta \mu$$

as soon as $\mu \leq \frac{\epsilon}{\sqrt{n}(1+\eta)}$. Thus, we have

Theorem 1. *The primal-dual path-following algorithm solves the partial weight analytic center problem (2) in $O(\sqrt{n}\log(n\max(w)/\epsilon))$ iterations and each iteration solves a system of linear equations in $O(nm^2 + m^3)$ arithmetic iterations. If Karmarkar's rank-one update technique is used, the average arithmetic operations can be reduced to $O(n^{1.5}m)$ arithmetic operations.*

If the predictor and corrector algorithm of Mizuno et al. [21] is used, the quadratic convergence result of [27] applies to solving problem (2). We have

Corollary 1. *The primal-dual predictor-corrector algorithm solves the partial weight analytic center problem (2) in $O(\sqrt{n}(\log(n\max(w)C(A,b)) + \log\log(1/\epsilon))$ iterations and each* *iteration solves a system of linear equations in $O(nm^2 + m^3)$ arithmetic iterations. Here, $C(A, b)$ is a positive fixed number depending on data A and b, and if A and b are rational numbers then $C(A, b) \leq 2^{O(L(A,b))}$ where $L(A, b)$ is the bit-length of A and b.*

These result indicate that the complexity of the Fisher equilibrium problem is completely in line with linear programming of same dimension and size.

2.3 Complexity analysis of solving the Fisher equilibrium

In solving Fisher's problem with n producers and n consumers in (1), the number of variables becomes $n^2 + n$ and the number of equalities is $2n$. We can assign the initial x^0 such that

$$x_{ij}^0 = \frac{1}{n}, \quad \forall i, j$$

so that

$$u_i^0 = \frac{1}{n} \sum_{j \in P} u_{ij}, \quad \forall i.$$

Set the dual variable with equality constraint j

$$y_j^0 = 2n\beta$$

and dual variable with equality constraint i be

$$y_i^0 = \frac{\beta}{u_i^0}.$$

Then, we have slack variable s_i^0 and u_i^0

$$s_i^0 u_i^0 = \beta, \quad \forall i$$

and slack variable s_{ij}^0 and x_{ij}^0

$$s_{ij}^0 x_{ij}^0 = (y_j^0 - y_i^0 u_{ij})/n = 2\beta - \frac{u_{ij}\beta}{\sum_{j \in P} u_{ij}}, \quad \forall i, j$$

which is between β and 2β. Using at most $O(\log(n))$ interior-point iterations, we will have an interior-point pair satisfying condition (5) (e.g., see [28]).

Moreover, matrix A of (1) is sparse and each of its columns has at most two nonzeros. Thus, ADA^T can be formed in at most $O(n^2)$ operations, and it can be factorized in $O(n^3)$ arithmetic operations. Thus, we have

Theorem 2. *The modified primal-dual path-following algorithm solves the Fisher equilibrium problem (1) with n producers and n consumers in at most $O(n \log(n \max(w)/\epsilon))$ iterations and each iteration solves a system of linear equations in $O(n^3)$ arithmetic iterations.*

In addition to the feasibility conditions, the optimality condition of the Eisenberg-Gale formulation can be written as

$$\begin{aligned} \sum_{j \in P} u_{ij} x_{ij} &\geq \frac{m_i u_{ij}}{p_j}, \quad \forall i, j \\ x_{ij} \sum_{j \in P} u_{ij} x_{ij} &= x_{ij} \frac{m_i u_{ij}}{p_j}, \quad \forall i, j. \end{aligned}$$

Thus, an optimal solution x_{ij} of the Eisenberg-Gale formulation is the solution of the system equations:

$$\begin{aligned} \sum_{j \in P} u_{ij} x_{ij} &= \frac{m_i u_{ij}}{p_j}, \quad (i, j) \in B^* \\ \sum_{i \in C} x_{ij} &= 1, \quad \forall j \\ x_{ij} &= 0, \quad \forall (i, j) \notin B^* \end{aligned}$$

where B^* is the set of the optimal basic-variables x_{ij} which are positive at the optimal solution.

If we view products $p_j x_{ij}$ as new variales y_{ij}, then the system becomes a system of *linear equations*:

$$\begin{aligned} \sum_{j \in P} u_{ij} y_{ij} &= m_i u_{ij}, \quad (i, j) \in B^* \\ \sum_{i \in C} y_{ij} &= p_j, \quad \forall j \\ y_{ij} &= 0, \quad \forall (i, j) \notin B^* \end{aligned}$$

Hence, $y_{ij} = p_j x_{ij}$ and p_j must be rational numbers and their size is bounded by the bit-length L of all input data u_{ij} and m_i. Thus, the same linear programming interior-point algorithm rounding techniques (e.g., [28]) can be applied to identify B^* in $O(nL)$ the interior-point algorithm itereations, which implies that

Corollary 2. *The modified primal-dual path-following algorithm solves the Fisher equilibrium problem (1) with n producers and n consumers in at most $O(nL)$ iterations and each iteration solves a system of linear equations in $O(n^3)$ arithmetic iterations, where L is the bit-length L of the input data u_{ij} and m_i.*

Our result is a significant improvement from $O(n^8 L)$ arithmetic operations of either the ellipsoid method and the combinatorial algorithm mentioned earlier.

3 An interior-point algorithm for solving the Arrow-Debreu equilibrium

Again, with out loss of generality, assume that each of the n players has exactly one unit of divisible good, and let player i, $i = 1, ..., n$, has a linear utility function

$$u(x_{i\cdot}) = u(x_{i1}, ..., x_{in}) = \sum_j u_{ij} x_{ij},$$

where u_{ij} is the given utility coefficient of player i for player j's good and x_{ij} represents the amount of good bought from player j by player i. The main difference between Fisher's and Walras' models is that, in the latter, each player is both producer and consumer and the budget of player i is *not* given and will be the price assigned to his or her good. Nevertheless, we can still write a (parametric) convex optimization model:

$$\begin{array}{ll} \text{maximize} & \sum_{i=1}^n w_i \log\left(\sum_{j=1}^n u_{ij} x_{ij}\right) \\ \text{subject to} & \sum_{i=1}^n x_{ij} = 1, \quad \forall j \\ & x_{ij} \geq 0, \quad \forall i, j, \end{array}$$

or

$$\begin{array}{lll} \text{maximize} & \sum_{i=1}^n w_i \log(u_i) & (12) \\ \text{subject to} & \sum_{i=1}^n x_{ij} = 1, \quad \forall j & \\ & u_i - \sum_{j=1}^n u_{ij} x_{ij} = 0, \quad \forall i & \\ & u_i,\ x_{ij} \geq 0, \quad \forall i, j, & \end{array}$$

where we wish to select weights w_i's such that the optimal dual prices equal them respectively.

For given w's, the necessary and sufficient optimality conditions of the model are:

$$\begin{array}{rcl} u_i \pi_i & = & w_i,\ \forall i \\ x_{ij}(p_j - u_{ij}\pi_i) & = & 0,\ \forall i, j \\ p_j - u_{ij}\pi_i & \geq & 0,\ \forall i, j \\ \sum_{i=1}^n x_{ij} & = & 1,\ \forall j \\ u_i - \sum_{j=1}^n u_{ij} x_{ij} & = & 0,\ \forall i \\ u_i,\ \pi_i,\ x_{ij} & \geq & 0,\ \forall i, j, \end{array}$$

where p is the dimension-n optimal dual price vector of the first n equality constraints and π is the dimension-n optimal dual price vector of the second n equality constraints in (12). We call the first equation the weighted centering condition, the second the complementarity condition, the third the dual feasibility, and the third and fourth the primal equality.

Next, we will prove that there is indeed a $w \geq 0$ such that $p_i = w_i$ in these conditions, that is, there are (u, x) and (p, π) such that

$$\begin{array}{rcl} u_i \pi_i & = & p_i,\ \forall i \\ x_{ij}(p_j - u_{ij}\pi_i) & = & 0,\ \forall i, j \\ p_j - u_{ij}\pi_i & \geq & 0,\ \forall i, j \\ \sum_{i=1}^n x_{ij} & = & 1,\ \forall j \\ u_i - \sum_{j=1}^n u_{ij} x_{ij} & = & 0,\ \forall i \\ u_i,\ \pi_i,\ x_{ij} & \geq & 0,\ \forall i, j, \end{array} \tag{13}$$

3.1 The self-dual weighted analytic center

Consider a more general problem

$$\begin{array}{rc} \text{maximize} & \sum_{j=1}^{l} w_j \log(x_j) \\ \text{subject to} & Ax = b, \\ & x \geq 0, \end{array} \tag{14}$$

where given A is an $m \times n$ matrix with full row rank,

$$b = \begin{pmatrix} e \\ 0 \end{pmatrix} \in R^m,$$

and e is the $l(\leq m)$-dimension vector of all ones.

We prove the following theorem

Theorem 3. *Assume that the feasible set of (14) is bounded and has a nonempty interior, and the dual feasibility $A^T y \geq 0$ imply $y_1, ..., y_l \geq 0$. Then, there exist $w_1, ..., w_l \geq 0$ such that the optimal dual prices, corresponding to the first l equality constraints of (14), equal $w_1, ..., w_l$, respectively. When w_j's satisfy this property, we call the solution of (14) the self-dual weighted analytic center of the feasible set of (14).*

Proof. For any given $w_1, ..., w_l \geq 0$, and, without loss of generality, let $\sum_{j=1}^{l} w_j = M$ for some positive constant M, the optimality conditions of (14) are

$$\begin{array}{rcl} s_j x_j & = & w_j, \; j = 1, ..., l \\ s_j x_j & = & 0, \; j = l+1, ..., n \\ s - A^T y & = & 0, \\ Ax & = & b, \\ x, \; s & \geq & 0. \end{array} \tag{15}$$

These conditions are necessary and sufficient since the feasible set of (14) is bounded and has a nonempty interior.

Summing up the top n equalities, we have

$$\sum_{j=1}^{n} s_j x_j = \sum_{j=1}^{l} w_j = M.$$

But from the rest conditions

$$\sum_{j=1}^{n} s_j x_j = x^T s = x^T (Ay) = (Ax)^T y = b^T y = \sum_{i=1}^{l} y_i.$$

From the assumption, $y_i \geq 0$ for $i = 1, ..., l$ from $s = A^T y \geq 0$. Thus, $y(w) := (y_1, ..., y_l)$ is a mapping from $w := (w_1, ..., w_l)$ in the simplex $\{w_i \geq 0 : \sum_{i=1}^{l} w_i = M\}$ to itself.

In general, this mapping may not be one-to-one. But if we let

$$y(w) = \lim_{\mu \to 0^+} y(\mu)$$

where $y(\mu)$ is the unique solution to

$$\begin{aligned} s_j x_j &= w_j,\ j = 1, ..., l \\ s_j x_j &= \mu,\ j = l+1, ..., n \\ s - A^T y &= 0, \\ Ax &= b, \\ x,\ s &\geq 0. \end{aligned}$$

Then, the mapping is one-to-one and continuous from the weighted analytic center theory. Therefore, the result follows from the fixed point theorem. ∎

Theorem 3 establishes an alternative proof of the existence of the Arrow-Debreu equilibrium. It also implies that the conditions for the self-dual weighted analytic center of the feasible set of (14) can be written as

$$\begin{aligned} s_j x_j &= y_j,\ j = 1, ..., l \\ s_j x_j &= 0,\ j = l+1, ..., n \\ s - A^T y &= 0, \\ Ax &= b, \\ x,\ s &\geq 0. \end{aligned}$$

Note that the system is homogeneous in (y, s) so that we may add a normalizing constraint

$$\sum_{j=1}^{l} y_j = M$$

to the conditions.

From the above conditions but excluding the second one, we have

$$\sum_{j=l+1}^{n} s_j x_j = s^T x - \sum_{j=1}^{l} s_j x_j = s^T x - \sum_{j=1}^{l} y_j = b^T y - \sum_{j=1}^{l} y_j = 0,$$

that is, the second condition is implied by the rest of them. This fact was first proved by Jain [16] for the Arrow-Debreu equilibrium problem, which is a special case of problem (14).

Corollary 3. *Assume that the feasible set of (14) is bounded and has a nonempty interior, and the dual feasibility $A^T y \geq 0$ imply $y_1, ..., y_l \geq 0$. Then, the self-dual weighted analytic center of the feasible set of (14) satisfies the following necessary and sufficient conditions:*

$$\begin{aligned} s_j x_j &= y_j, \ j = 1, ..., l \\ s - A^T y &= 0, \\ Ax &= b, \\ x, \ s &\geq 0. \end{aligned} \tag{16}$$

3.2 A convex minimization formulation

Jain [16] has also shown that $p_i > 0$ for all i under certain rational conditions on u_{ij} in (13). Thus, by deleting the complementarity condition and substituting u_i and π_i from the equalities, the Arrow-Debreu equilibrium is a point (x_{ij}, p_j) that satisfies

$$\begin{aligned} \textstyle\sum_j u_{ij} x_{ij} &\geq u_{ij} \tfrac{p_i}{p_j}, \ \forall i, j \\ \textstyle\sum_i x_{ij} &= 1, \ \forall j \\ p_i &> 0, \ \forall i \\ x_{ij} &\geq 0, \ \forall i, j. \end{aligned} \tag{17}$$

Then, the problem can be formulated as the following optimization Phase I problem:

$$\begin{aligned} \text{minimize} \quad & \theta \\ \text{subject to} \quad & \textstyle\sum_i x_{ij} = 1 + \theta \quad \forall j \\ & \textstyle\sum_j u_{ij} x_{ij} \geq u_{ij} \frac{p_i}{p_j} \quad \forall i, j : \; u_{ij} \neq 0 \\ & x_{ij} \geq 0, \; p_i > 0 \quad \forall i, j. \end{aligned} \tag{18}$$

Here θ can be viewed as an inflated units of each player's good, i.e., initially every player pretends to have $1 + \theta$ units of good. Then θ is gradually moved down to 0. One can easily see that the problem is strictly feasible with a suitably large θ. Furthermore,

Lemma 1. *For any feasible solution of Problem (18), we must have $\theta \geq 0$.*

Proof. For all i, j, we have

$$x_{ij} p_j \sum_j u_{ij} x_{ij} \geq p_i u_{ij} x_{ij}.$$

Summing over j of the inequalities, we have

$$\left(\sum_j x_{ij} p_j \right) \left(\sum_j u_{ij} x_{ij} \right) \geq p_i \left(\sum_j u_{ij} x_{ij} \right).$$

Thus,

$$\sum_j x_{ij} p_j \geq p_i.$$

Summing over i of the inequalities, we have

$$\sum_i \sum_j x_{ij} p_j \geq \sum_i p_i,$$

or

$$(1 + \theta) \sum_j p_j \geq \sum_i p_i$$

which implies $\theta \geq 0$. ∎

According to Arrow and Debreu [1], we must also have

Lemma 2. *The minimal value of Problem (18) is $\theta = 0$.*

3.3 The logarithmic transformation and efficient barrier functions

Let $y_i = \log p_i,\ \forall i$. Then problem (18) becomes

$$\begin{array}{lc} \text{minimize} & \theta \\ \text{subject to} & \sum_i x_{ij} - \theta = 1 \quad \forall j \\ & \sum_j u_{ij}x_{ij} \geq u_{ij}e^{y_i - y_j} \quad \forall i,j:\ u_{ij} \neq 0 \\ & x_{ij} \geq 0, \quad \forall i,j. \end{array} \tag{19}$$

Note that the new problem is a convex optimization problem since $e^{y_i - y_j}$ is a convex function in y. This type of transformations has been used in Geometric Programming.

The question: is there an efficient barrier function for inequality

$$\sum_j u_{ij}x_{ij} \geq u_{ij}e^{y_i - y_j},\ \ u_{ij} \neq 0.$$

The answer is "yes", and its barrier function is

$$\log\left(\sum_j u_{ij}x_{ij}\right) + \log\left(\log\left(\sum_j u_{ij}x_{ij}\right) - \log u_{ij} - y_i + y_j\right)$$

where its self-condordant parameter is 2, see Proposition 5.3.3 of Nesterov and Nemirovskii [23]. One may also construct the dual, the Legendre transformation, of the barrier function.

Thus, we can formulate the problem as a barrier optimization problem:

$$\begin{array}{ll} \text{minimize} & \theta - \mu \sum_{i,j}\left(\log x_{ij} + \log\left(\sum_j u_{ij}x_{ij}\right) + \log\left(\log\left(\sum_j u_{ij}x_{ij}\right) - \log u_{ij} - y_i + y_j\right)\right) \\ \text{subject to} & \sum_i x_{ij} - \theta = 1 \quad \forall j, \end{array} \tag{20}$$

where the barrier parameter $\mu > 0$. Thus, one can develop an interior-point path-following or potential reduction algorithm to compute an ϵ-optimal solution, i.e., $\theta < \epsilon$. Since the total self-condordant coefficient of the barrier function is $O(n^2)$, and each iteration uses at most $O(n^3)$ arithmetic operations due to the block and sparse structure of the Hessian of the barrier function and the constraint matrix, we have

Theorem 4. *There is an interior-point algorithm to generate a solution to problem (18) with $\theta < \epsilon$ in $O(n \log \frac{1}{\epsilon})$ iterations and each iteration uses $O(n^3)$ arithmetic operations.*

Note that this worst-case complexity bound is significantly lower than the Ellipsoid method used by Jain [16].

3.4 Alternative optimization setting

An alternative Phase I problem is

$$\begin{array}{ll} \text{minimize} & \theta \\ \text{subject to} & \sum_i x_{ij} = 1 \quad \forall j \\ & \theta \cdot \sum_j u_{ij} x_{ij} \geq u_{ij} \frac{p_i}{p_j} \quad \forall i, j : \; u_{ij} \neq 0 \\ & x_{ij} \geq 0, \; p_i > 0 \quad \forall i, j. \end{array} \tag{21}$$

Initially, $\theta > 1$, which is an inflated factor for the utility value. The problem is to drive θ to 1.

Let $y_i = \log p_i, \; \forall i$ and $\kappa = \log \theta$. Then problem (18) becomes

$$\begin{array}{ll} \text{minimize} & \kappa \\ \text{subject to} & \sum_i x_{ij} = 1 \quad \forall j \\ & \sum_j u_{ij} x_{ij} \geq u_{ij} e^{y_i - y_j - \kappa} \quad \forall i, j : \; u_{ij} \neq 0 \\ & x_{ij} \geq 0, \quad \forall i, j. \end{array} \tag{22}$$

Again, the new problem is a convex optimization problem since $e^{y_i - y_j - \kappa}$ is a convex function in y and κ, and the minimal value of the problem is 0.

3.5 Rounding to the exact solution

Eaves [12] showed that the Arrow-Debreu problem with linear utility can be formulated as a complementarity problem, where an optimal solution x_{ij} and price p is the solution of the homogeneous system:

$$\begin{aligned}\sum_j u_{ij} p_j x_{ij} &= u_{ij} p_i, \quad (i,j) \in B^* \\ \sum_i p_j x_{ij} &= p_j, \quad \forall j \\ x_{ij} &= 0, \quad \forall (i,j) \notin B^*\end{aligned}$$

where B^* is the set of the optimal basic-variables x_{ij} which are positive at the optimal solution.

Again, if we view products $p_j x_{ij}$ as new variales y_{ij}, then the system becomes a homogeneous system of *linear equations*. Hence, $y_{ij} = p_j x_{ij}$ and p_j (after normalizing say $\sum_j p_j = n$) must be rational numbers and their size is bounded by the bit-length L of all input data u_{ij}. Thus, the same linear programming interior-point algorithm rounding technique can be applied to identify B^* in $O(nL)$ the interior-point algorithm itereations, which implies that

Corollary 4. *There is an interior-point algorithm to compute a solution of problem (18) with n producers and n consumers in at most $O(nL)$ iterations and each iteration solves a system of linear equations in $O(n^3)$ arithmetic iterations, where L is the bit-length L of the input data u_{ij}.*

Our result is a significant improvement over the ellipsoid method of Jain.

4 A path to the Arrow-Debreu equilibrium

Now, we move our attention to whether there is a direct interior-point algorithm in solving the Arrow-Debreu equilibrium problem, similar to the primal-dual path-following algorithm for linear programming and the Fisher equilibrium. Such an algorithm may have many economical and practical appears.

Consider the convex optimization problem for a fixed scalar $0 \le \mu \le 1$ and a nonnegative weight vector w with $\sum_i w_i = n^2$:

$$\begin{array}{ll} \text{maxmize} & \mu \sum_{i,j} \log(x_{ij}) + \sum_i w_i(1-\mu) \log\left(\sum_j u_{ij} x_{ij}\right) \\ \text{subject to} & \sum_i x_{ij} = 1, \quad \forall j \\ & x_{ij} \geq 0, \quad \forall i, j. \end{array} \tag{23}$$

4.1 Economic interpretations

The objective of (23), when $\mu = 0$, is the same objective function which Eisenberg-Gale used for Fisher's model.

We now present economic interpretations for $\mu > 0$. When $\mu = 1$, then the objective function becomes the logarithmic barrier function and the unique maximizer of (23) is the analytic center of the feasible set, namely, $x_{ij} = \frac{1}{n}$ for all i, j. This is probably an ideal socialist solution if all players are homogeneous.

In our setting, the combined objective function represents a balance between socialism and individualism. Here $w_i(1-\mu)$ is the weight for the log-utility value of player i. If again, w_i represents the amount of money player i possesses, $\sum_i w_i = n^2$ represents the total wealthy of the population, and μ represents player i's tax rate to be collected for social welfare. The leftover amount, $w_i(1-\mu)$, would be the weight used in Eisenberg-Gale to make the market clear. Here, the total collected tax amount is $n^2\mu$ and the tax rate is uniformly applied among the payers.

Mike Todd also pointed out that the objective function is really the convex combination of the two different utility functions, one is un-weighted and the other is weighted, representing two different idealisms.

4.2 The fixed point theorem

Unlike Fisher's problem, we really don't know how much money w each player possesses in Walras' model—it is up to what prices p_i's assigned to them, since they have to sell their goods at these prices for revenues. But prices are the optimal dual variables or Lagrangian multipliers of the n equality constraints in (23). Then, the natural question is, is there a vector w such that the optimal dual prices of (23) equal to w_i's, respectively. We give an affirmative answer in the following theorem.

Theorem 5. *For any scalar $0 \leq \mu \leq 1$, there exists a weight vector $w \geq 0$ and $\sum_i w_i = n^2$ such that the optimal dual price vector of (23) equals w.*

Proof. When $\mu = 1$, i.e., the tax rate equals 1, the (unique) prices would be

$$w_i = p_i = n \quad \text{and} \quad x_{ij} = \frac{1}{n} \ \forall i, j.$$

Consider $0 \leq \mu < 1$. Denote the compact simplex by

$$S(n^2) = \{y \in R^n : \sum_i y_i = n^2, \ y_i \geq 0, \ \forall i\}.$$

From the convex optimization theory, the necessary and sufficient optimality conditions of an x in (23) are

$$\begin{array}{rcl} \frac{\mu}{x_{ij}} + \frac{w_i(1-\mu)u_{ij}}{\sum_j u_{ij}x_{ij}} & \leq & p_j, \ \forall i, j \\ x_{ij}\left(p_j - \frac{\mu}{x_{ij}} - \frac{w_i(1-\mu)u_{ij}}{\sum_j u_{ij}x_{ij}}\right) & = & 0, \ \forall i, j \\ \sum_i x_{ij} & = & 1, \ \forall j \\ x_{ij} & \geq & 0, \ \forall i, j. \end{array} \tag{24}$$

where p_j is the optimal dual price or Lagrangean multiplier for equality constraint j. The first set of constraints is dual feasibility, the second set is called complementarity, and the last two are primal feasibility.

Summing up the complementarity equation over i and noting $\sum_i x_{ij} = 1$, we have

$$p_j = n\mu + \sum_i \frac{w_i(1-\mu)u_{ij}x_{ij}}{\sum_j u_{ij}x_{ij}} \geq n\mu \geq 0, \ \forall j.$$

Summing the above equation over j, we have

$$\sum_j p_j = n^2\mu + \sum_j \sum_i \frac{w_i(1-\mu)u_{ij}x_{ij}}{\sum_j u_{ij}x_{ij}}$$

$$= n^2\mu + \sum_i \frac{w_i(1-\mu)}{\sum_j u_{ij}x_{ij}} \sum_j u_{ij}x_{ij} = n^2\mu + \sum_i w_i(1-\mu) = n^2\mu + n^2(1-\mu) = n^2.$$

That is, $p \in S(n^2)$. For given u_{ij}'s and fixed $\mu \geq 0$, we may think $p \in S(\mu)$ being a mapping of $w \in S(n^2)$, that is, $p(w)$ is a mapping from the simplex to itself. From the Fixed Point Theorem, there exists $w \in S(n^2)$ such that

$$p(w) = w,$$

which completes the proof. ∎

Note that summing up the complementarity equation in (24) over j when $w = p$, we have

$$\sum_j p_j x_{ij} = n\mu + \sum_j \frac{w_i(1-\mu)u_{ij}x_{ij}}{\sum_j u_{ij}x_{ij}} = n\mu + w_i(1-\mu) = n\mu + p_i(1-\mu).$$

That is, the individual payment spent by player i equals his net income (after tax) plus $n\mu$ which can be viewed as a tax amount refunded back to each player uniformly.

4.3 The case $\mu = 0$

When $\mu = 0$, with $p = w = p(w)$, the optimality conditions (24) become (13), which is exactly the necessary and sufficient conditions for p being an Arrow-Debreu equilibrium price vector.

There may be academic advantage of our constructed proof, however. First this proof can be seen as an extension of Eisenberg-Gale proof. Second, this proof reduces the Walras model (by Arrow-Debreu's setting) to Fisher Model. This justifies an approximation algorithm of Jain et al. [18] to compute an approximate equilibrium. Their approximation algorithm reduces the Walras setting to Fisher setting, and it can be simply stated as

1. Starts with arbitrary w_i's.
2. Compute the $p_i(w)$'s.

3. Replace the w_i's with $p_i(w)$'s plus a residual, and repeat the loop until the p_i's computed are almost equal to the w_i's used in the loop. (It is proved that the residual keeps going down linearly in the process.)

Their simple and elegant algorithm converges in a time bounded by $\frac{1}{\epsilon}$.

4.4 The case $\mu > 0$

In this case, we have an interior primal maximal solution, i.e., $x_{ij} > 0$. Again let $p = w = p(w)$ in the optimality conditions of (23). Moreover, let $y_i = \sum_j u_{ij} x_{ij}$ and $q_i = \frac{p_i(1-\mu)}{\sum_j u_{ij} x_{ij}}$. Then we have

$$\begin{array}{rcl} x_{ij}(p_j - u_{ij}q_i) & = & \mu, \ \forall i,j \\ y_i q_i - p_i(1-\mu) & = & 0, \ \forall i \\ \sum_i x_{ij} & = & 1, \ \forall j \\ y_i - \sum_j u_{ij}x_{ij} & = & 0, \ \forall i \\ \sum_i p_i & = & n^2, \\ y_i, \ q_i & \geq & 0, \ \forall i \\ x_{ij}, \ (p_j - u_{ij}q_i) & \geq & 0, \ \forall i,j. \end{array} \tag{25}$$

Since both the primal and dual solutions of the strictly convex optimization are unique interior points for any given $0 < \mu \leq 1$, they can be written as $(x_{ij}(\mu), y_i(\mu), q_i(\mu), p_i(\mu))$. Similar to the central path theory of linear programming (e.g., [20]), we have

Theorem 6. *For any given $\mu > 0$, the system (25) has a unique solution $(x_{ij}(\mu), y_i(\mu), q_i(\mu), p_i(\mu))$, and they form a continuous path for $\mu \in (0, \infty)$. Moreover, when $\mu \rightarrow 0^+$, the solution converges the Arrow-Debreu equilibrium.*

System (25) has linear and bilinear equations, which are similar to the central path equations for linear programming and primal-dual path-following Newton's methods might be applicable.

5 Final Remark

Jain has pointed [17] out that existence proof of Theorem 5 can also be done using a thought experiment without even referring to convex programs. Suppose we have an Arrow-Debreu setting. Suppose every player non-deterministically guesses his potential income w_i's. He takes an advance loan from a bank based on his potential income. He goes to the market with that money. There the market becomes Fisher setting. He sells his goods according to the Fisher equilibrium $p_i(w)$'s. If it turns out that the amount of money he makes is the same as the loan he took from the bank then the non-deterministic guesses agents made were an Arrow-Debreu equilibrium. Note that, Fisher setting guarantees that the summation of $p_i(w)$'s is the same as the summation of w_i's as indicated in our proof. So using the fixed point theorem there is a set of correct guesses. An advantage of this "thought" proof is that it works even for non-linear and concave utilities.

The general self-dual weighted analytic center introduced in this paper seems to have more application in matrix games and other fixed-point problems. We expect more equilibrium problems can be transferred to convex optimization problem where efficient interior-point algorithms may apply.

Other questions remain, such as how to handle general concave utility functions? Some answers have been given by Codenotti, Jain, Varadarajan and Vazirani [4, 5, 6]. Are there direct primal-dual interior-point algorithms for solving the Arrow-Debreu equilibrium? The path developed in this paper may give an answer.

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