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A Perturbation Property of the TLS-LP Method

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***Abstract*—We show that the TLS-LP method and the SVD-Prony method are equivalent to the first-order perturbation approximation. In practice, it means that the two methods yield the same estimation variances when the signal-to-noise ratio (SNR) is above a threshold.**

I. Introduction

The TLS-LP method and the SVD-Prony method have been presented recently in [1] and [2], [3], respectively. In [1], the TLS-LP method is claimed to be more robust to noise than the SVD-Prony method, based on intuitive interpretations of "LS" and "TLS" and on simulation results.

In this correspondence, we submit that the TLS-LP method and the SVD-Prony method yield the identical estimates (of frequencies and damping factors) to the first-order approximation. This result explains a simulation observation, presented in [5], that the improved Pisarenko method (equivalent to the TLS-LP method in [1] and the Mini-Norm method in [4]) and the Tufts-Kumaresan (TK) method (i.e., SVD-Prony method) have very close estimation accuracy when the signal-to-noise ratio (SNR) is above a threshold. Note that the estimation variances are linearly proportional to the noise variance when SNR is larger than the threshold [2], [3], [5], [6].

A detailed perturbation analysis of the TK method was presented in [6]. The result shown in this correspondence implies that the perturbation analysis in [6] also applies to the TLS-LP method in [1].

II. An Equivalent Formulation

Adopting the notation used in [1], we define

$$A = \begin{bmatrix} x_0 & x_1 & \cdots & x_{L-1} \\ x_1 & x_2 & \cdots & x_L \\ & & \cdots & \\ x_{N-L-1} & x_{N-L} & \cdots & x_{N-2} \end{bmatrix} \tag{2.1}$$

$$b = \begin{bmatrix} x_L \\ x_{L+1} \\ \cdot \\ x_{N-1} \end{bmatrix} \tag{2.2}$$

where x_k, $k = 0, 1, \cdots, N-1$, is the sequence of superimposed exponentials perturbed by noise. L satisfies $M \le L \le N - M$. We also denote the L-degree polynomial coefficients by

$$c = \begin{bmatrix} c_L \\ c_{L-1} \\ \cdot \\ c_1 \end{bmatrix}. \tag{2.3}$$

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The signal poles are estimated from the roots of the polynomial $P(z) = 1 - \Sigma_{k=1,L} c_k z^{-k}$. The only difference between the TLS-LP method and the SVD-Prony method is the way of computing c from A and b.

According to the TLS-LP method, c is such that $\|c\|$ is minimized subject to

$$[A \ b]_T \begin{bmatrix} c \\ -1 \end{bmatrix} = 0 \tag{2.4}$$

where $\|\cdot\|$ denotes 2-norm, $[\cdot]_T$ denotes rank-M SVD truncation (i.e., except for the M largest singular values of the corresponding matrix, all smaller singular values are set to be zero).

According to the SVD-Prony method, c is such that $\|c\|$ is minimized subject to

$$[A]_T c = b. \tag{2.5}$$

III. First-Order Perturbations

Since the estimated pole z_j ($j = 1, 2, \cdots, M$) should satisfy $P(z_j) = 0$, it follows (also see [6, eq. (20)]) that the perturbation δz_j in estimated z_j is related to the first-order term of the perturbation δc in estimated c in the following way:

$$\begin{aligned} \delta z_j &= (z_j^H \delta c) \Big/ \left(\sum_{k=1,L} k c_k z_j^{-(k+1)} \right) \\ &= (z_j^H \delta c)/D \end{aligned} \tag{3.1}$$

where D is the denominator free of noise, and

$$z_j^H = [z_j^{-L}, \cdots, z_j^{-1}]. \tag{3.2}$$

In (3.1), only δz_j and δc are perturbed by noise but all other quantities are noise free. Equation (3.1) applies to both the TLS-LP and the SVD-Prony. Hence, all we need to show is that the numerator in (3.1), $(z_j^H \delta c)$, is the same for both the TLS-LP and the SVD-Prony in first-order terms of noise in the data sequence x_k.

For the TLS-LP, we let

$$[A \ b]_T = [A' \ b']. \tag{3.3}$$

Then (first-order) differentiating (2.4) yields

$$\delta A' c + A' \delta c - \delta b' = 0 \tag{3.4}$$

which leads to

$$\delta c = A'^{+} \delta [A \ b]_T \begin{bmatrix} c \\ -1 \end{bmatrix} + \varepsilon \tag{3.5}$$

where the superscript "+" denotes pseudoinverse, $\delta[\cdot]_T$ denotes the perturbation in the truncated matrix $[\cdot]_T$ due to noise, and all other quantities on the left side are noise free. ε can be any vector orthogonal to the column space of noiseless $A'^H = A^H$. But note that in noise free case, $z_j^H \varepsilon = 0$.

For the SVD-Prony, one can easily verify, by following the above approach, that

$$\delta c = -A_T^{+} \delta [A_T \ b] \begin{bmatrix} c \\ -1 \end{bmatrix} + \varepsilon \tag{3.6}$$

where all quantities, except those preceded by δ, are noise free.

Note that in (3.5) and (3.6), $A'^{+} = A_T^{+} = A^{+}$ (because they are noise free quantities in the two equations).

From (3.5) and (3.6), we have that for the TLS-LP

$$z_j^H \delta c = -z_j^H A^{+} \delta [A \ b]_T \begin{bmatrix} c \\ -1 \end{bmatrix} \tag{3.7}$$

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and for the SVD–Prony

$$z_j^H \delta c = -z_j^H A^+ \delta[A_l \ b] \begin{bmatrix} c \\ -1 \end{bmatrix}. \quad (3.8)$$

Now we provide the lemma:

Lemma: If $Z = Y + \delta Z$, where Y has rank M and δZ is a small perturbation, and Z_T is the rank-M SVD truncation of Z, then $u_0^H \delta Z_T = u_0^H \delta Z$ to the first-order approximation where μ_0 is any vector in the column space of Y.

This lemma implies that the SVD truncation does not affect the first-order perturbations (under a constraint).

Proof: Using SVD, we write

$$Y = \sum_{i=1}^{M} s_i u_i v_i^H \quad (3.9a)$$

$$Z = \sum_{i=1}^{>M} \sigma_i \alpha_i \beta_i^H \quad (3.9b)$$

$$Z_T = \sum_{i=1}^{M} \sigma_i \alpha_i \beta_i^H \quad (3.10)$$

where the singular values are in descending order. Clearly, if $\delta Z = 0$, (3.9a), (3.9b), and (3.10) are identical. Following the approach in [7], we let the perturbation matrix δZ be kX where k is a small number. Then σ_i, α_i and β_i have Taylor expansions with respect to k at $k = 0$. Substituting those expansions into (3.9a) and dropping second and higher order terms of k, we obtain the first-order approximation equation:

$$\delta Z = \sum_{i=1}^{>M} (\delta\sigma_i u_i v_i^H + s_i \delta\alpha_i v_i^H + s_i u_i \delta\beta_i^H) \quad (3.11)$$

where all quantities, except those preceded by δ, are unperturbed (i.e., noiseless). Since u_0 is in the column space of Y, u_0 is orthogonal to u_i for $i > M$. Also note that in (3.11), $s_i = 0$ for $i > M$ since Y has rank M. Hence, multiplying (3.11) by u_0^H yields

$$u_0^H \delta Z = u_0^H \sum_{i=1}^{M} (\delta\sigma_i u_i v_i^H + s_i \delta\alpha_i v_i^H + s_i u_i \delta\beta_i^H)$$
$$= u_0^H \delta Z_T. \quad (3.12)$$

The second equation in (3.12) follows from the same derivation for (3.11). The lemma is proved.

Referring to (3.7) and (3.8), we see that

$$R[(z_j^H A^+)^H]$$
$$\text{belong-to } R[A^{+H}]$$
$$\text{equal-to } R[A]$$
$$\text{belong-to } R[A \ b] \quad (3.13)$$

where $R[\cdot]$ denotes "column space" (range) of $[\cdot]$. Applying (3.13) and the lemma to (3.7) and (3.8) leads to that for both the TLS–LP and the SVD–Prony

$$z_j^H \delta c = -z_j^H A^+ \delta[A \ b] \begin{bmatrix} c \\ -1 \end{bmatrix} \quad (3.14)$$

where all quantities, except for those preceded by δ, are noise free.

Finally, we note that if A and b are replaced by forward-and-backward versions for the case of undamped sinusoids, the above presentation is also valid.

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Comments on "Complete Discrete 2-D Gabor Transforms by Neural Networks for Image Analysis and Compression"

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In his paper,[1] Daugman uses a three-layer neural network to transform two-dimensional (2-D) discrete signals into another type of representation, namely the 2-D "Gabor" representation. He then used this new representation to analyze images in different ways and mainly to do compression on them.

This correspondence addresses a statement that is made on page 1171, last paragraph: "it would be completely impractical to solve this huge system of simultaneous equations by algebraic methods such as matrix manipulation, since the complexity of such methods grows factorially with the number of simultaneous equations."

We would like to point out that the method of Gauss elimination solves the problem in a low order polynomial time; specifically, $O(N^3)$ arithmetic operations are needed where N is the number of linear equations and the number of unknowns.

Major algorithms include LU decomposition, requiring $O(N^3/3)$ operations; the Householder QR decomposition, requiring $O(2N^3/3)$ operations; and the Givens QR decomposition, requiring $O(4N^3)$ operations, as discussed in detail in many numerical analysis books [1], [2].

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