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A new perturbation theory for solving the Vlasov equation is derived. The theory is especially designed to cope with time secularities and nonanalyticity in the expansion parameter (the field strength). The method is based on the use of a statistical set of exact particle orbits instead of the unperturbed orbits conventionally used in perturbation solutions of the Vlasov equation. A principal result of the theory is a modification of the particle–wave interaction and a "broadening" of the associated resonant denominator $(\omega - \mathbf{k}\cdot\mathbf{v})^{-1}$. The nature of the time secularities associated with the streaming modes $\exp i\mathbf{k}\cdot\mathbf{v}t$ is discussed. A simple application to velocity-space diffusion and trapping and its effect on wave growth is described.

I. INTRODUCTION

THE Vlasov equation, together with Maxwell's equations, constitute a complete plasma description when particle discreteness (collisional phenomena) can be ignored. For the regime of strong turbulence, we assume this to be the case. Consequently, the theory of plasma turbulence is concerned with obtaining the solution, in some sense, of the nonlinear Vlasov equation.

There seems to be no possibility of obtaining, in any reasonable form, the complete set of such solutions, and indeed they would be no more useful than the detailed set of solutions to the N-body Liouville equation. In fact, it is clear that the microstate solutions to the Vlasov equation can exhibit as detailed a structure as the N-body distribution functions, because the Vlasov distribution function is capable of being distorted or driven into arbitrarily fine grain structures. What is needed are "average" solutions which emphasize the course grain or macroscopic behavior, and do not delve into the details of the microstates.

The development of the quasi-linear and mode coupling theories[1–5] has been successful in creating such a theory for weakly turbulent regimes. The original formulation of this theory was obtained by an iterative solution of the Vlasov equation; however, it is now clear that the solutions of the truncated hierarchy equations (such as the BBGKY equations) constitute an identical theory in the limit of zero plasma parameter. This weakly turbulent theory pictures a turbulent plasma as consisting of small amplitude waves which obey linear wave mechanics, but which interact weakly with each other and with the average distribution function. The theory is not a microstate theory. It does not predict the phases of the waves, and much of the fine grain phase-space structure of the distribution function is "averaged out." However, it is not well understood what the theory's region of validity is or what theory should replace it when it is no longer valid. The difficulty in answering these questions is, in part, mathematical, arising from the perturbation-theoretic nature of the solution.

In deriving a perturbation solution to the Vlasov equation, two fundamental difficulties arise: (a) nonanalyticity in the perturbation parameter[6] ϵ (which is usually proportional to the field strength) and (b) time-secularities in the individual terms of the perturbation solution.[3] To derive a perturbation theory which successfully copes with these problems, it is first of all necessary to lump all nonanalytic dependence on ϵ into the coefficients of the perturbation expansion. Secondly, the expansion must be such that the coefficients are not secular. The desired expansion should be of the form

$$\sum_n A_n(\epsilon, t)\epsilon^n,$$

where $A_n(\epsilon, t)$ is of order unity for all time, and the series converges for all ϵ. Such an expansion is, of course, not unique, but it does have the property that, for small ϵ, accurate solutions can be obtained by computing only the first few coefficients A_n. If this turns out to be a tractable problem, then one has a useful perturbation theory.

[1] W. E. Drummond and D. Pines, Ann. Phys. (N. Y.) **28**, 478 (1964).
[2] A. A. Vedenov, E. P. Velekhov, and R. Z. Sagdeev, Nucl. Fusion Suppl. Pt. 2, 465 (1962).
[3] E. Frieman and P. Rutherford, Ann. Phys. (N. Y.) **28**, 134 (1964).
[4] A. A. Kadomtsev, *Plasma Turbulence* (Academic Press Inc., New York, 1965).
[5] R. E. Aamodt and W. E. Drummond, Phys. Fluids **7**, 1816 (1965).
[6] I. B. Bernstein, J. M. Green, and M. D. Kruskal, Phys. Rev. **108**, 546 (1957).

Existing perturbation theories of the Vlasov equation do not fully cope with either of the difficulties just described. In obtaining solutions, one has no guarantee that *all* secularities have been removed or that the expansion even converges. If it is an asymptotic series, one does not know how accurate it is. Therefore, it is difficult to ascertain the validity of the existing weakly turbulent theory. Furthermore, it seems clear that many phenomena, such as turbulent heating, shock waves, and trapping, cannot be adequately described. In addition, nonlinear calculations of certain instabilities show that the growth rate of the unstable wave is either not suppressed or suppressed at a very large value of the wave amplitude.[4,7] This raises the possibility that some other (heretofore excluded) mechanisms may become important before such large amplitude spectrums can develop.

In order to provide a more rigorous basis from which to construct a general theory of nonlinear plasma behavior, an improved plasma perturbation theory is essential. Towards this end, an exact and convergent perturbation theory is presented in the following pages. The basis of this theory consists in using a statistical ensemble of particle orbits instead of the unperturbed orbits usually employed in plasma perturbation theory. In the conventional perturbation theory, particles with unperturbed orbits of velocity $\mathbf{v}$ moving through the wave $\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)$ give rise to the familiar resonant denominator $1/(\mathbf{k}\cdot\mathbf{v} - \omega)$ of plasma physics. This resonance is, of course, fundamental to the damping and growth of waves in both the linear and the weakly turbulent nonlinear theories, because it describes the resonant (as well as the nonresonant) interaction of wave and particle, and the energy exchange between them. A principal result of the new theory is a modification of this resonance.

The theory can be interpreted in terms of a "test-wave model." In this model, one studies the propagation and the interaction of a few "test" waves coexisting with a set of random phase "background waves." The initial phases of all the background waves are averaged over, so that only the initial phases of the test waves enter. It turns out that the effect of the background waves can be incorporated in the theory by using the (perturbed) trajectories of particles moving in the random phase background waves. This is the origin of the statistical ensemble of trajectories. The effects of the test waves are obtained by a simple perturbation theory using the perturbed trajectories. One then computes the propagation characteristics of single test waves, the interaction between two test waves, three test waves, and systematically builds up a description of *all* the waves in a system. One can prove that this scheme actually converges to the exact solution!

[7] F. C. Hoh, Phys. Fluids 8, 969 (1965).

In Sec. II, we establish the validity of the test-wave model. In Sec. III, we use the model to derive a *general* solution to the Vlasov equation in terms of the solution to the random phase background wave problem. In Sec. IV, this general solution is written in terms of the *ensemble average* solution to the random phase problem. In Secs. VI and VII, the ensemble average solution is investigated. And in Secs. V and VIII applications to plasma turbulence are discussed.

II. THE DEPENDENCE OF THE DISTRIBUTION FUNCTION ON THE PHASES OF THE ELECTRIC FIELD

The method we employ for solving the Vlasov–Maxwell equations consists of two distinct pieces. First, we assume knowledge of the electric field $\mathbf{E}(\mathbf{r}, t)$ (for simplicity we neglect the magnetic field, which may be treated in a manner similar to $\mathbf{E}$) and endeavor to find the resulting particle motion; i.e., given $\mathbf{E}(\mathbf{r}, t)$, we shall find the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ from the Vlasov equation:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} + \frac{q}{m}\mathbf{E}(\mathbf{r}, t)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f(\mathbf{r}, \mathbf{v}, t) = 0. \qquad (2.1)$$

As a second step we must, of course, require that the f so determined does, in fact, produce the assumed $\mathbf{E}$. We must, in other words, impose self-consistency:

$$\frac{\partial}{\partial \mathbf{r}}\cdot\mathbf{E}(\mathbf{r}, t) = 4\pi n q \int d\mathbf{v}\, f(\mathbf{r}, \mathbf{v}, t). \qquad (2.2)$$

Again, for simplicity, we include only the longitudinal field. We assume the plasma to be a cube of volume L^3

The spatial dependence of the electric field may be expanded in a Fourier series:

$$\mathbf{E}(\mathbf{r}, t) = \sum_{\mathbf{k}} \boldsymbol{\varepsilon}_{\mathbf{k}}(\mathbf{r}, t), \qquad \boldsymbol{\varepsilon}_{\mathbf{k}} = \mathbf{E}_{\mathbf{k}}(t) \exp(i\mathbf{k}\cdot\mathbf{r} + i\beta_{\mathbf{k}}), \qquad (2.3)$$

$$f(\mathbf{r}, \mathbf{v}, t) = \sum_{k} f_{\mathbf{k}}(\mathbf{v}, t) e^{i\mathbf{k}\cdot\mathbf{r}}. \qquad (2.4)$$

The phases $\beta_{\mathbf{k}}$ will be referred to as "initial phases," since the phase of $\boldsymbol{\varepsilon}_{\mathbf{k}}(t)$ *relative* to $\boldsymbol{\varepsilon}_{\mathbf{k}}(t_0)$ does not depend on $\beta_{\mathbf{k}}$, but the phase of $\boldsymbol{\varepsilon}_{\mathbf{k}}(t_0)$ does contain $\beta_{\mathbf{k}}$ as an additive constant.

As $L \to \infty$ the amplitude $\boldsymbol{\varepsilon}_{\mathbf{k}}$ need not become a continuous function of $\mathbf{k}$. For this reason, it is more convenient to use a Fourier series than a Fourier integral.

As a first step we are interested in obtaining the time dependence of f, given the initial value of f and the electric field for all t. Obviously f depends on the phase factors $\exp(i\beta_{\mathbf{k}})$ which contain the initial phases of the electric field. Furthermore, f must be a periodic function of each $\beta_{\mathbf{k}}$ with period 2π, and therefore f has a Fourier series expansion in each $\beta_{\mathbf{k}}$:

$$f = \sum_{n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots = -\infty}^{+\infty} \exp(in_{\mathbf{k}}\beta_{\mathbf{k}} + in_{\mathbf{k}'}\beta_{\mathbf{k}'} + \cdots) \cdot F(n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots). \qquad (2.5)$$

The exponent in (2.5) contains the complete set of phases $\beta_{\mathbf{k}}$, and hence the expansion coefficients $F(n_{\mathbf{k}}, \cdots)$ do not depend on any of the initial phases. At $t = t_0$, f cannot depend on the initial phases of the electric field so that

$$F(n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots, t_0) = 0, \text{ unless } n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots = 0.$$

Although the derivation of this equation is trivial, its implications are not. To understand this, we observe that if (2.5) is averaged over some set of initial phases $\beta_{\mathbf{k}}$, the coefficients F are unaffected, and the phase-factor product in front is either unaffected or averages to zero, depending on whether or not it contains any of the phases being averaged over. Thus, the effect on averaging over some of the initial phases is to eliminate some of the terms (i.e., those involving the phases which were averaged) in (2.5) and to leave the rest unchanged.

This property of (2.5) suggests that, if we wish to obtain a particular coefficient, e.g., $F(n_{\mathbf{k}} = 1)$, we may solve a subsidiary, bur simpler, problem instead of the original one. In this subsidiary problem, we consider an ensemble of systems in which $\beta_{\mathbf{k}}$ is the same in each system, but all other initial phases are randomly distributed (from 0 to 2π) among the systems of the ensemble. The Fourier series expansion of ensemble average distribution function obtained in this way will have a term proportional to $\exp i\beta_{\mathbf{k}}$ and whose coefficient is precisely the same $F(n_{\mathbf{k}} = 1)$ that appears in (2.5). It is, therefore, possible to obtain the solution to our original problem by determining each of the F's in (2.5) by solving a subsidiary but, as we shall see, less difficult problem. The fact that the initial phases of the background waves in the subsidiary problem are uncorrelated with each other and with the initial value of f does not prevent the F's, so calculated from being used in (2.5) to describe an actual system in which all the initial phases have some precise relationship with each other and with f.

III. THE TEST-WAVE PROBLEM

We now proceed to obtain the coefficients $F(n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots)$ in (2.5) by solving the subsidiary or test-wave problem previously described. To begin, the total set of waves is divided into two sets. One set is composed of all waves whose wave numbers are contained in the set $(\mathbf{k}, \mathbf{k}', \cdots)$, corresponding to the indices of $F(n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots)$. These are called test waves. The second set are called background waves and contain all the remaining waves. As we have seen, the initial phases of the background waves do not enter into the calculation of $F(n_{\mathbf{k}}, n_{\mathbf{k}'}, \cdots)$ and will be averaged over. In general, we shall consider only a few test waves, but a large number of background waves. The Vlasov equation becomes

$$\frac{\partial}{\partial t} f + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} f + \frac{q}{m} \underbrace{\sum_B \boldsymbol{\varepsilon}_{\mathbf{k}}\cdot\frac{\partial f}{\partial \mathbf{v}}}_{\text{sum over background waves}} + \frac{q}{m} \underbrace{\sum_T \boldsymbol{\varepsilon}_{\mathbf{k}}\cdot\frac{\partial f}{\partial \mathbf{v}}}_{\text{sum over test waves}} = 0. \qquad (3.1)$$

Now, define the propagation operator $U(t, t_0)$ as the solution to

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}}\right)U + \frac{q}{m} \sum_B \boldsymbol{\varepsilon}_{\mathbf{k}}\cdot\frac{\partial U}{\partial \mathbf{v}} = 0, \qquad (3.2)$$

$$U(t_0, t_0) = I. \qquad (3.3)$$

One can write the solution to (3.1) as

$$f(\mathbf{r}, \mathbf{v}, t) = -\frac{q}{m}\int_{t_0}^{t} d\tau \sum_T \mathbf{E}_{\mathbf{k}}(\tau) U(t, \tau) \cdot \exp(i\mathbf{k}\cdot\mathbf{r} + i\beta_{\mathbf{k}})\cdot\frac{\partial}{\partial \mathbf{v}} f(\mathbf{r}, \mathbf{v}, \tau) + U(t, t_0) f(\mathbf{r}, \mathbf{v}, t_0). \qquad (3.4)$$

Let us begin by considering the simple case of a single test wave $\boldsymbol{\varepsilon}_{\mathbf{k}}$. We are specifically interested in the dependence of f on the initial phase $\beta_{\mathbf{k}}$ of the test wave. We denote by $\tilde{F}(n) \exp in\beta_{\mathbf{k}}$ the sum of all terms in (2.5) whose $\beta_{\mathbf{k}}$ dependence is $\exp in\beta_{\mathbf{k}}$. In other words, we define $\tilde{F}(n)$ to be

$$\tilde{F}(n) = \frac{1}{2\pi}\int_0^{2\pi} d\beta_{\mathbf{k}} \exp(-in\beta_{\mathbf{k}}) f, \qquad (3.5)$$

where

$$f = \sum_{n=-\infty}^{+\infty} \tilde{F}(n) \exp(in\beta_{\mathbf{k}}). \qquad (3.6)$$

We shall use the brackets $\langle\ \rangle$ to indicate an average over the phases of the background waves. According to (2.5),

$$\langle \tilde{F}(n) \rangle = F(n_{\mathbf{k}}). \qquad (3.7)$$

Substituting (3.6) in (3.4) gives

$$\tilde{F}(n, t) = -\frac{q}{m}\int_{t_0}^{t} d\tau\, \mathbf{E}_{\mathbf{k}}(\tau)U(t, \tau)e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\tilde{F}(n-1, \tau) - \frac{q}{m}\int_{t_0}^{t} d\tau\, \mathbf{E}_{-\mathbf{k}}(\tau)U(t, \tau)e^{-i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\tilde{F}(n+1, \tau) + U(t, t_0)\tilde{F}(n, t_0) \tag{3.8}$$

or simply

$$\tilde{F}(n) = \tilde{E}\tilde{F}(n-1) + \tilde{E}^*\tilde{F}(n+1) + \tilde{F}(n)^0, \tag{3.9}$$

where $\tilde{F}(n)^0$ is the last term in (3.8) and where $\tilde{E}$ and $\tilde{E}^*$ are the obvious operators. We shall be interested in solving this equation in the limit of vanishingly small $\mathbf{E}_{\mathbf{k}}$. There are a variety of ways to transform a given problem to achieve this limit without loss of generality. However, in the present case, it will suffice to consider a homogeneous turbulent plasma of dimension L^3 and then to let $L \to \infty$, but keep the energy density $\sum_{\text{all } \mathbf{k}} |\mathbf{E}_{\mathbf{k}}|^2$ finite. In this limit $|\mathbf{E}_{\mathbf{k}}| \to 0$, since it is proportional to $L^{-\frac{3}{2}}$, but the energy density $\sum_{\text{all } \mathbf{k}} |\mathbf{E}_{\mathbf{k}}|^2$ remains constant, since

$$\sum_{\text{all } \mathbf{k}} \to \frac{L^3}{\pi^3}\int d\mathbf{k}.$$

We shall say a quantity is of order E^n, i.e., $O(E^n)$, if it is proportional to $|\mathbf{E}_{\mathbf{k}}|^n$ as $|\mathbf{E}_{\mathbf{k}}| \to 0$. Accordingly, in Eq. (3.9), the operators $\tilde{E}$ and $\tilde{E}^*$ are of order E.

The first few equations around $n = 0$ are

$$\tilde{F}(-1) = \tilde{E}\tilde{F}(-2) + \tilde{E}^*\tilde{F}(0), \tag{3.10}$$

$$\tilde{F}(0) = \tilde{E}\tilde{F}(-1) + \tilde{E}^*\tilde{F}(1) + \tilde{F}(0)^0, \tag{3.11}$$

$$\tilde{F}(1) = \tilde{E}\tilde{F}(0) + \tilde{E}^*\tilde{F}(2), \tag{3.12}$$

$$\tilde{F}(2) = \tilde{E}\tilde{F}(1) + \tilde{E}\tilde{F}(3). \tag{3.13}$$

As explained earlier, only the $n = 0$ equation has an initial value ($t = t_0$) term, $\tilde{F}(0)^0$, which is of order zero. By successive substitution, we easily get

$$\tilde{F}(0) = \tilde{F}(0)^0 + O(E^2), \tag{3.14}$$

$$\tilde{F}(1) = \tilde{E}\tilde{F}(0)^0 + O(E^3), \tag{3.15}$$

$$\tilde{F}(2) = \tilde{E}\tilde{E}\tilde{F}(0)^0 + O(E^4), \tag{3.16}$$

$$\vdots$$

$$\tilde{F}(n) = \tilde{E}^n\tilde{F}(0)^0 + O(E^{n+2}). \tag{3.17}$$

As $E \propto L^{-\frac{3}{2}} \to 0$, the second terms in (3.14)–(3.17) become negligible corrections to the first terms and may be neglected. This is the same result that would be obtained for $\tilde{F}(n)$ by solving (3.4), using ordinary iteration and then picking out all terms proportional to exp $in\beta_{\mathbf{k}}$, but eliminating all those terms which contain $\tilde{E}$ and $\tilde{E}^*$ in the same product. These latter terms are always at least L^{-3} times smaller (as we have just seen).

Averaging the initial phases of the background waves in (3.14)–(3.17), produces the following coefficients that appeared in (2.5):

$$F(n_k = 0) = \langle\tilde{F}(0)\rangle = \langle U(t, t_0)\rangle f(t_0), \tag{3.18}$$

$$F(n_{\mathbf{k}} = 1) = \langle\tilde{F}(1)\rangle = -\frac{q}{m}\int_{t_0}^{t} d\tau\, \mathbf{E}_{\mathbf{k}}(\tau) \cdot\left\langle U(t, \tau)e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}U(\tau, t_0)\right\rangle f(t_0), \tag{3.19}$$

$$F(n_{\mathbf{k}} = 2) = \langle\tilde{F}(2)\rangle = \frac{q^2}{m^2}\int_{t_0}^{t} d\tau\int_{t_0}^{\tau} d\tau'\, \mathbf{E}_{\mathbf{k}}(\tau)\mathbf{E}_{\mathbf{k}}(\tau') \cdot\left\langle U(t, \tau)e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}U(\tau, \tau')e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}U(\tau', t_0)\right\rangle f(t_0). \tag{3.20}$$

As used in (3.18)–(3.20), U is the solution of (3.2) with all waves except the test wave $\mathbf{\varepsilon}_{\mathbf{k}}$. However, as $L \to \infty$, we can include all waves in computing U when it appears inside the average brackets, since the addition of a finite number of test waves will produce a vanishingly small change in U as $L \to \infty$.

One can easily generalize the argument to show that the leading contribution to $F(n_{\mathbf{k}}, \cdots, n_{\mathbf{k}'})$ is obtained by iterating Eq. (3.4) with the set of test waves $(\mathbf{\varepsilon}_{\mathbf{k}}, \cdots, \mathbf{\varepsilon}_{\mathbf{k}'})$ and treating all other waves as background waves. All terms containing $\mathbf{E}_{\mathbf{k}''}\mathbf{E}^*_{\mathbf{k}''}$, where $\mathbf{E}_{\mathbf{k}''}$ is a test wave, must be omitted. Each term in the iterated solution is averaged over the initial phases of the background waves. $F(n_{\mathbf{k}}, \cdots, n_{\mathbf{k}'})$ is the sum of the coefficients of all terms whose initial phase dependence is exp $in_{\mathbf{k}}\beta_{\mathbf{k}} + \cdots + in_{\mathbf{k}'}\beta_{\mathbf{k}'}$. The coefficient $F(n_{\mathbf{k}}, \cdots, n_{\mathbf{k}'})$ is of order $E^{n_{\mathbf{k}}+\cdots+n_{\mathbf{k}'}}$, and the omitted corrections are of order $\sum_{\text{test}} |\mathbf{E}_{\mathbf{k}''}|^2$ times smaller or less. If all background waves were test waves, i.e., if one does perturbation theory with unperturbed orbits, it is well known that these correction terms are of order $\sum_{\text{all } \mathbf{k}} |\mathbf{E}_{\mathbf{k}}|^2$ times the "leading" term and cannot be neglected, since $\sum_{\text{all } \mathbf{k}} |\mathbf{E}_{\mathbf{k}}|^2$ remains constant as $|\mathbf{E}_{\mathbf{k}}| \propto L^{-\frac{3}{2}} \to 0$. However, in our case, by putting all the waves (except a few test waves) into the operator U as background waves, the correction to the leading term is of order $\sum_{\text{test}} |\mathbf{E}_{\mathbf{k}}|^2$ which vanishes as L^{-3} as $L \to \infty$.

To obtain $f(\mathbf{r}, \mathbf{v}, t)$ one may, according to (2.5), sum all the coefficients obtained in the manner just described. This produces the following series for f:

$$f(\mathbf{r}, \mathbf{v}, t) = \langle U(t, t_0)\rangle f(t_0) - \frac{q}{m}\int_{t_0}^{t} d\tau \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(\tau) \cdot \exp(i\beta_{\mathbf{k}})\left\langle U(t, \tau)e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}} U(\tau, t_0)\right\rangle f(t_0) + \frac{q^2}{m^2}\int_0^t d\tau \int_0^\tau d\tau' \sum_{\substack{\mathbf{k} \\ \mathbf{k}\neq-\mathbf{k}'}}\sum_{\mathbf{k}'} \mathbf{E}_{\mathbf{k}}(\tau)\mathbf{E}_{\mathbf{k}'}(\tau') \cdot \exp(i\beta_{\mathbf{k}'} + i\beta_{\mathbf{k}}) \cdot\left\langle U(t, \tau)e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}} U(\tau, \tau')e^{i\mathbf{k}'\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}} U(\tau', t_0)\right\rangle \cdot f(t_0) + \cdots . \quad (3.21)$$

The U operator appearing in (3.21) is the solution to (3.2), which, in the limit $L \to \infty$, may be considered to include *all* waves, test as well as background. As pointed out earlier, the addition of a finite set of test waves in the defining equation for U will not change U in the limit $L \to \infty$, since the wave amplitudes vanish as $L^{-\frac{3}{2}}$.

The averages appearing in (3.21) are over the initial phases of all waves, assuming they are uncorrelated.

As an alternate way of viewing the solution (3.21) to the Vlasov equation (3.1), consider the equation

$$\frac{\partial f^{(i)}}{\partial t} + \mathbf{v}\cdot\frac{\partial f^{(i)}}{\partial \mathbf{r}} + \frac{q}{m}\sum_{\text{all }\mathbf{k}} \mathbf{E}_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r} + i\alpha_{\mathbf{k}})\cdot\frac{\partial f^{(i)}}{\partial\mathbf{v}} + \frac{q}{m}\sum_{\text{all }\mathbf{k}} \mathbf{E}_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r} + i\beta_{\mathbf{k}})\cdot\frac{\partial f^{(i-1)}}{\partial\mathbf{v}} = 0. \quad (3.22)$$

The set of waves has been included twice, once with the actual phases $\beta_{\mathbf{k}}$ and once with another set of phases $\alpha_{\mathbf{k}}$. (3.21) can be obtained by iterating this equation as indicated by the superscripts to obtain $f^{(\infty)}$, discarding all terms which contain *both* $\exp i\beta_{\mathbf{k}}$ and $\exp -i\beta_{\mathbf{k}}$ as factors, and finally averaging each of the remaining terms over $\alpha_{\mathbf{k}}$. The first set of waves in (3.22) accounts for phase independent effects and the second set for phase dependent effects.

IV. AN EXPANSION FOR $U(t, t_0)$

If the initial value $f(t_0)$ is deleted from the right-hand side of (3.21), then the right-hand side may be equated to the operator U. This equation is a nonlinear integral equation for U. If this equation is iterated, we obtain an expansion of the ensemble average quantities appearing in the integrands in powers of $\mathbf{E}_{\mathbf{k}}$ and $\langle U\rangle$:

$$U(t, t_0) = \langle U(t, t_0)\rangle - \frac{q}{m}\int_{t_0}^{t} d\tau \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(\tau) \cdot \exp(i\beta_{\mathbf{k}})\left\{\langle U(t, \tau)\rangle e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\langle U(\tau, t_0)\rangle - \frac{q}{m}\int_{\tau}^{t} d\tau' \int_{t_0}^{\tau} d\tau'' \sum_{\mathbf{k}'} \mathbf{E}_{\mathbf{k}'}(\tau')\mathbf{E}_{-\mathbf{k}'}(\tau'') \cdot\langle U(t, \tau')\rangle e^{i\mathbf{k}'\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\langle U(\tau', \tau)\rangle e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\langle U(\tau, \tau'')\rangle e^{-i\mathbf{k}'\cdot\mathbf{r}}\frac{\partial}{\partial\mathbf{v}}\langle U(\tau'', t_0)\rangle + \cdots\right\} + \frac{q^2}{m^2}\int_{t_0}^{t} d\tau \int_{t_0}^{\tau} d\tau' \sum_{\substack{k \\ k\neq-k'}}\sum_{k'} \mathbf{E}_{\mathbf{k}}(\tau)\mathbf{E}_{\mathbf{k}'}(\tau') \cdot\exp(i\beta_{\mathbf{k}} + i\beta_{\mathbf{k}'})\left\{\langle U(t, \tau)\rangle e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\langle U(\tau, \tau')\rangle e^{i\mathbf{k}'\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\langle U(\tau', t_0)\rangle + \cdots\right\} + \cdots . \quad (4.1)$$

It is not difficult to prove that this expansion converges if the operator $\partial/\partial\mathbf{v}\langle U(t, t_0)\rangle$ is bounded in the following sense:

$$\left|\int_{t_0}^{t} dt' \frac{\partial}{\partial\mathbf{v}}\langle U(t', t_0)\rangle h\right| \leq \int_{t_0}^{t} dt' \, g(t', t_0) \, |\max h| < \infty . \quad (4.2)$$

In (4.2) $g(t', t_0)$ is a *function*, not an operator. h is an arbitrary bounded function, and $|\max h|$ is its maximum value as a function of $\mathbf{r}$ and $\mathbf{v}$. For instance, if $\langle U\rangle$ were the diffusion operator

$$[D(t - t_0)]^{-\frac{3}{2}} \int d\mathbf{v}_0 \exp\left[-\frac{|\mathbf{v} - \mathbf{v}_0|^2}{D(t - t_0)}\right],$$

it would clearly be bounded in the sense of (4.2). On the other hand, if $\langle U\rangle$ were the unperturbed orbit operator

$$\int d\mathbf{v}_0 \, \delta(\mathbf{v} - \mathbf{v}_0),$$

then $(\partial/\partial\mathbf{v})\langle U\rangle$ would not be bounded, and the series would not converge. It is for essentially the same reason that the conventional iterative solution of the Vlasov equation using unperturbed orbits does not converge.

We will assume that the spectrum $|\mathbf{E}_\mathbf{k}|^2$ is such that $(\partial/\partial\mathbf{v})\langle U(t, t_0)\rangle$ is bounded. In qualitative terms, this simply means that the solution to (6.1) is a continuous function for $t > t_0$. The details of the convergence of (4.1) and its relation to the spectrum will be deferred to a later paper.

(4.1) provides an explicit expression for the actual "microstate" solution operator $U(t, t_0)$, provided that we know $\langle U(t, t_0)\rangle$. $\langle U\rangle$ is the ensemble average solution to a simpler but related problem in which the field phases are random and uncorrelated. The averaged operator $\langle U\rangle$ is a "macroscopic" quantity and does not have the intricate, phase dependent structure that the microstate operator U has.

V. SELF-CONSISTENCY

Substituting (4.1) into Poisson's equation (2.2), one obtains a nonlinear integral equation for $\mathbf{E}(\mathbf{r}, t)$

$$\sum_\mathbf{k} i\mathbf{k}\cdot\boldsymbol{\mathcal{E}}_\mathbf{k}(t)$$

$$= 4\pi nq \int dv\ [\text{right-hand side of (4.1)}]f(\mathbf{r}, \mathbf{v}, t_0). \quad (5.1)$$

The solution of Eq. (5.1) is evidently very complicated. In the remaining pages, we shall confine ourselves to a few approximate considerations to illustrate some aspects of the statistical trajectory method. Consider the "linear" portion of (5.1), i.e., the first two terms on the right-hand side. The remaining nonlinear terms lead to mode coupling effects which we will not consider.

The Fourier transform of $f(\mathbf{r}, \mathbf{v}, t)$ is defined in (2.4). For the kth mode, the first two terms are $\mathbf{E}_\mathbf{k}(t)e^{i\mathbf{k}\cdot\mathbf{r}}$

$$= \frac{4\pi nq^2 i\mathbf{k}}{mk^2}\int d\mathbf{v}\int_{t_0}^{t} d\tau\ \mathbf{E}_\mathbf{k}(\tau)\langle U(t, \tau)\rangle e^{i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial\langle f_0(\tau)\rangle}{\partial\mathbf{v}}$$

$$- \frac{4\pi nqi\mathbf{k}}{k^2}\int d\mathbf{v}\ \langle U(t, t_0)\rangle f_\mathbf{k}(t_0)e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (5.2)$$

If the $\langle U(t, \tau)\rangle$ operator were the usual "unperturbed" orbit operator that replaces $\mathbf{r}$ with $\mathbf{r} - \mathbf{v}(t - \tau)$, then (5.2) would reduce to the customary linearized equation for $\mathbf{E}_\mathbf{k}(t)$ obtained from the Vlasov equation by the usual perturbation method. In this case, if $\mathbf{E}_\mathbf{k}(t) \propto \exp(-i\omega t)$ ($\omega \approx$ real), one obtains in the usual way that the real part of the time integral in (5.2) is proportional to $\delta(\mathbf{k}\cdot\mathbf{v} - \omega)$, and the wave growth rate is given by

$$\frac{1}{|\mathbf{E}_\mathbf{k}|^2}\frac{\partial}{\partial t}|\mathbf{E}_\mathbf{k}|^2 = \pi\frac{\omega_p^3}{k}\int d\mathbf{v}\ \delta(\mathbf{k}\cdot\mathbf{v} - \omega)\frac{\partial\langle f_0(v)\rangle}{\partial\mathbf{v}}\cdot\hat{\mathbf{k}}. \quad (5.3)$$

The delta function in the integrand is characteristic of the linear picture of particle–wave resonance. As we shall see in the following section, the statistical trajectory leads to a resonance of finite width due to particle trapping. The broadening of the resonance can have a decisive effect on wave growth.

VI. THE DETERMINATION OF $\langle U\rangle$

The equation for $\langle f\rangle$ is obtained by ensemble averaging (3.2). We include all waves in the sum over wave numbers. We obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}\right)\langle f\rangle + \frac{q}{m}\frac{\partial}{\partial\mathbf{v}}\cdot\sum_\mathbf{k}\langle\boldsymbol{\mathcal{E}}_\mathbf{k}f\rangle = 0. \quad (6.1)$$

If we impose the initial condition

$$f(t_0) = \delta(\mathbf{r} - \mathbf{r}_0)\ \delta(\mathbf{v} - \mathbf{v}_0), \quad (6.2)$$

then $\langle f(t)\rangle$ becomes a Green's function, and the operator $\langle U(t, t_0)\rangle$ can be written

$$\langle U(t, t_0)\rangle = \int d\mathbf{x}_0\int d\mathbf{v}_0\ \langle f(t)\rangle. \quad (6.3)$$

Substituting f [as given by (2.5)] into (6.1), it is obvious that only the term proprotional to $\exp(-i\beta_\mathbf{k})$ can contribute to (6.1), because of the phase average. According to (4.1), this term is

$$F(n_\mathbf{k} = -1)\exp(-i\beta_\mathbf{k}) = -\frac{q}{m}\int_{t_0}^{t} d\tau\ \mathbf{E}_{-\mathbf{k}}(\tau)$$

$$\cdot\exp(-i\beta_\mathbf{k})\langle U(t, \tau)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\langle f(\tau)\rangle + O(L^{\frac{1}{2}}E^3). \quad (6.4)$$

Substituting this expression into (6.1) and neglecting terms of order $L^{\frac{1}{2}}E^3$, we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial\mathbf{r}}\right)\langle f\rangle = \frac{\partial}{\partial\mathbf{v}}\cdot\frac{q^2}{m^2}\int_{t_0}^{t} d\tau\ \sum_k \mathbf{E}_\mathbf{k}(t)\mathbf{E}_{-\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{r}}$$

$$\cdot\langle U(t, \tau)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{v}}\langle f(\tau)\rangle. \quad (6.5)$$

We now assume that, in the integrand of (6.5), we can make the replacement

$$\langle U(t, \tau)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}\frac{\partial}{\partial\mathbf{v}}\langle f(\tau)\rangle = [\langle U(t, \tau)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}]\frac{\partial}{\partial\mathbf{v}}\langle f(t)\rangle. \quad (6.6)$$

The replacement of $\partial/\partial\mathbf{v}\langle f(\tau)\rangle$ by $\partial/\partial\mathbf{v}\langle f(t)\rangle$ is based on the fact that the integrand is nonzero in only a small interval around $t = \tau$, and that $\langle f\rangle$ does not change significantly during this interval. This is one of the conventional arguments used to derive the Fokker–Planck equation.[8] The neglect of the operator $\langle U(t, \tau)\rangle$ on $(\partial/\partial\mathbf{v})\langle f\rangle$ is based on approxi-

[8] S. Chandrasekhar, Rev. Mod. Phys. 15, 1 (1943).

mately the same argument, i.e., $\langle U\rangle$ does not change $(\partial/\partial\mathbf{v})\langle f\rangle$ significantly during the time interval that the integrand is nonzero.

One ordinarily bases these approximations on the peaking of the autocorrelation function

$$\textstyle\sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(t)\mathbf{E}_{-\mathbf{k}}(\tau) \text{ at } t = \tau.$$

The width of this peak is the autocorrelation time τ_{AC}. However, when the spectrum of the electric field is narrow, the width of the autocorrelation of the field is not, in which case we shall assume that $e^{i\mathbf{k}\cdot\mathbf{r}}\langle U(t, \tau)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}$ provides the required limitation of the τ integration to a small region around $t = \tau$. The width of this function is called the trapping time τ_{TR} and is defined following Eq. (7.7). Obviously, these approximations are valid only for a sufficiently smooth dependence of $\langle f(\mathbf{r}, \mathbf{v}, t)\rangle$ on $\mathbf{r}$ and $\mathbf{v}$, for otherwise $\langle f(\mathbf{r}, \mathbf{v}, t)\rangle$ will change significantly in a time equal to the width of the resonance.

Although the approximation (6.6) is not very accurate over the entire range to which we shall apply it, it appears to be at least qualitatively correct.

The use of (6.6) converts (6.5) into a diffusion equation:

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{D}(\mathbf{v})\cdot\frac{\partial}{\partial \mathbf{v}}\right]\langle f(\mathbf{r}, \mathbf{v}, t)\rangle = 0, \qquad (6.7)$$

where

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2}\int_0^t d\tau \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(t)\mathbf{E}_{-\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{r}}\langle U(t, \tau)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}. \qquad (6.8)$$

If $\mathbf{E}(\mathbf{r}, t)$ represents a stationary time series, then $\langle U(t, \tau)\rangle = \langle U(t - \tau)\rangle$. In this case the diffusion coefficient can be written

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2}\int_0^{t-t_0} d\tau \cdot \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(T + \tau)\mathbf{E}_{-\mathbf{k}}(T)e^{i\mathbf{k}\cdot\mathbf{r}}\langle U(\tau)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}}. \qquad (6.9)$$

For $t - t_0 \gg \tau_{AC}$ we can replace the upper limit by ∞.

It is revealing to observe that

$$\langle U(t)\rangle e^{-i\mathbf{k}\cdot\mathbf{r}} = \langle U(t)e^{-i\mathbf{k}\cdot\mathbf{r}}\rangle = \langle e^{-i\mathbf{k}\cdot\mathbf{r}_c(\mathbf{r},\mathbf{v},-t)}\rangle, \quad (6.10)$$

where $\mathbf{r}_c(\mathbf{r}, \mathbf{v}, -t)$ is a trajectory at time $-t$ with initial values $\mathbf{r}$, $\mathbf{v}$. Therefore, the integrand of (6.9) contains an average of an exponential of all trajectories in the ensemble. Also, $\langle\exp[-i\mathbf{k}\cdot\mathbf{r}_c(-t)]\rangle$ satisfies (6.7) with the initial value $\exp(-i\mathbf{k}\cdot\mathbf{r})$. Therefore, this equation, together with (6.8) for $D(\mathbf{v})$, comprises a coupled set determining $D(\mathbf{v})$ and $\langle\exp[-i\mathbf{k}\cdot\mathbf{r}_c(-t)]\rangle$:

$$\mathbf{D}(\mathbf{v}) = \frac{q^2}{m^2}\int_0^\infty dt \sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}(T+t)\mathbf{E}_{-\mathbf{k}}(T)e^{i\mathbf{k}\cdot\mathbf{r}}\langle e^{-i\mathbf{k}\cdot\mathbf{r}_c(-t)}\rangle, \qquad (6.11)$$

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{r}} - \frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{D}(\mathbf{v})\cdot\frac{\partial}{\partial \mathbf{v}}\right]\langle e^{-i\mathbf{k}\cdot\mathbf{r}_c(-t)}\rangle = 0. \quad (6.12)$$

It is well known that one of the most persistent secularities in plasma perturbation theory arises from the substitution of the particle streaming modes, $f \propto e^{i\mathbf{k}\cdot\mathbf{v}t}$ into the acceleration term

$$(q/m)\mathbf{E}\cdot\partial f/\partial\mathbf{v}$$

of the Vlasov equation. The physical origin of this secularity is the constantly increasing difference between the trajectory $\mathbf{r} = \mathbf{v}t$ and the actual particle orbit, $\mathbf{r} = \mathbf{r}_c(t)$. The attempt to improve on the orbit by perturbation theory leads to the secularity. However, the use of the average trajectory function $\langle\exp -i\mathbf{k}\cdot\mathbf{r}_c(-t)\rangle$ avoids this problem by summing such secular terms in closed form. As will become apparent, the conventional neglect of such streaming secularities cannot be justified, since the corresponding orbit perturbations are associated with trapping and other important effects.

In the conventional derivation of $\mathbf{D}(\mathbf{v})$, one uses unperturbed orbits which are equivalent to setting $\mathbf{D}(\mathbf{v})$ equal to zero in (6.12). In this case

$$\langle e^{-i\mathbf{k}\cdot\mathbf{r}_c(-t)}\rangle = e^{-i\mathbf{k}\cdot\mathbf{r}+i\mathbf{k}\cdot\mathbf{v}t}. \qquad (6.13)$$

If

$$\mathbf{E}_{\mathbf{k}}(t) = \mathbf{E}_{\mathbf{k}}(0)e^{-i\omega_k t}, \qquad (6.14)$$

then (6.11) reduces to

$$\mathbf{D}(\mathbf{v}) = \pi\frac{q^2}{m^2}\sum_{\mathbf{k}} \mathbf{E}_{\mathbf{k}}\mathbf{E}_{-\mathbf{k}}\,\delta(\mathbf{k}\cdot\mathbf{v} - \omega). \qquad (6.15)$$

(The sum, of course, is to be interpreted as an integral in the limit $L \rightarrow \infty$.) This is the standard Fokker–Planck result for the diffusion coefficient in velocity space.[8] The occurrence of the delta function $\delta(\mathbf{k}\cdot\mathbf{v} - \omega)$ means that the diffusion of particles of velocity $\mathbf{v}$ is caused only by waves whose phase velocity is *exactly* $\mathbf{v}$. As we shall see, this result is valid only in the limit that the wave-energy density vanishes. For a finite energy density, the delta function is replaced with a finite width resonance function. This is the same resonance that occurs in the formula for wave growth (5.3) and the statistical trajectory will cause a similar modification there.

VII. THE WAVE-PARTICLE INTERACTION—APPROXIMATE EXPRESSIONS FOR THE DIFFUSION COEFFICIENT $D(v)$

The solution of (6.12) with an arbitrary $\mathbf{D}(\mathbf{v})$ has not been obtained. However, in view of the approximate nature of (6.11) and (6.12), an approximate solution will be adequate for our purpose. We will consider a one-dimensional problem in which all $\mathbf{E}_k$ and $\mathbf{k}$ lie along the x axis. If D is not a function of the velocity u along the x axis ($u = v_x$), then the solution to (6.12) is simply

$$\langle e^{-i\mathbf{k}\cdot\mathbf{r}_c(-t)}\rangle = e^{-ikx+ikut-\frac{1}{3}k^2Dt^3}. \qquad (7.1)$$

We shall assume this result is also approximately true for velocity dependent D and simply use $D(u)$ in (7.1). Substituting this into (6.11) and using (6.13), we obtain an integral equation for $D(u)$.

$$D(u) = \frac{q^2}{m^2}\int_0^\infty dt \sum_k |E_k|^2 e^{i(ku-\omega)t-\frac{1}{3}k^2D(u)t^3}. \qquad (7.2)$$

If we use the notation

$$R[ku-\omega, D(u)] = \text{Real}\int_0^\infty dt\, e^{i(ku-\omega)t-\frac{1}{3}k^2D(u)t^3}. \qquad (7.3)$$

Then, the expression for the diffusion equation becomes

$$D(u) = \frac{q^2}{m^2}\sum_k |E_k|^2 R[ku-\omega, D(u)]. \qquad (7.4)$$

As a function of ku $-\ \omega$, R has a maxima at the origin, where $R = (\frac{1}{3})!(\frac{1}{3}k^2D)^{-\frac{1}{3}}$. R goes to zero for $|ku-\omega| \gtrsim (\frac{1}{3}k^2D)^{\frac{1}{3}}$ and the area under R is π. In order to deal with (7.4) in a simple way, we shall approximate R by the following function:

$$R(ku-\omega, D) = \begin{matrix} \pi/2kw & |ku-\omega| < kw, \\ 0 & |ku-\omega| > kw, \end{matrix} \qquad (7.5)$$

where $w = (D/3k)^{\frac{1}{3}}$ is k^{-1} times the half width of the resonance function $R(ku-\omega, D)$. This function has the same area and approximately the same height, width, and shape as the actual R function.

It is convenient to write $|E_k|^2$ as a function of phase velocity instead of wave number by setting

$$(L/\pi)\, |E_k|^2 = |E_u|^2, \qquad dk = k(du/u), \qquad (7.6)$$

where $u = \omega/k$. We assume the wave spectrum is centered at k_0 in wave number space and lies between the phase velocities u_1 and u_2. Remembering that $\sum_k \to L/\pi \int dk$, we can use (7.5) and (7.6) to write (7.4) in the form

$$D(u) = \frac{\pi q^2}{2m^2} w^{-1}\int_{u-w}^{u+w} |E_{u'}|^2 \frac{du'}{u'}. \qquad (7.7)$$

The value of $D(u)$ depends critically on the ratio of two fundamental times scales in the problem.

(a) $(\frac{1}{3}k_0D)^{-\frac{1}{3}} = (k_0w)^{-1}$ is the time required for a particle's position to diffuse over a wavelength k_0^{-1}. This is apparent from (7.1). In other words, it is the time for the particle's position to become randomized with respect to wave phases due to its random acceleration. We shall call this the particle "trapping time" τ_{TR}.

(b) $k_0|(u_1-u_2)|^{-1}$ is the time for the relative phases of the waves to randomize due to the difference in phase velocities, $|u_1-u_2|$, of the waves. This is the autocorrelation time τ_{AC} of the electric field.

The evaluation of the diffusion coefficient from (7.7) is simple in the two limiting cases:

1. $\tau_{TR} \gg \tau_{AC}$. In this case $w \ll |u_1-u_2|$ and (7.7) reduces to

$$D(u) = \frac{\pi q^2}{m^2 u}|E_u|^2. \qquad (7.8)$$

This is the conventional result (6.15) for the diffusion coefficient.

2. $\tau_{TR} \ll \tau_{AC}$. In this case $w \gg |u_1-u_2|$ and (7.7) becomes

$$\begin{aligned} D(u) &= \frac{\pi q^2}{2m^2}\left(\frac{D(u)}{3k_0}\right)^{-\frac{1}{3}}\int_{-\infty}^{+\infty}\frac{du'}{u'}|E_{u'}|^2 \\ &= \left[\frac{\pi q^2 3^{\frac{1}{3}}}{2m^2k_0^{\frac{2}{3}}}\int_{-\infty}^{\infty}\frac{du'}{u'}k_0\,|E_{u'}|^2\right]^{\frac{3}{4}} \\ &\qquad \text{if}\quad u_1 - w \lesssim u \lesssim u_2 + w, \\ D(u) &= 0 \quad \text{otherwise.} \end{aligned} \qquad (7.9)$$

The quantity $\int(du'/u')k_0\,|E_u|^2$ is 8π times the electric energy density.

As one can glean from (7.7) or (7.9), the quantity w is the maximum difference between particle velocity and wave-phase velocity for which a particle can still be diffused by the wave. If the value of D given by (7.9) is used, one obtains

$$w = \left(\frac{D}{3k_0}\right)^{\frac{1}{3}} = \left\{\left(\frac{\pi}{6}\right)^{\frac{1}{2}}\frac{q}{m}k_0^{-1}\left[\int_{-\infty}^{+\infty}\frac{du'}{u'}k_0\,|E_{u'}|^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \qquad (7.10)$$

One can easily verify that, aside from numerical factors, this is simply the maximum speed a particle can have in the wave frame and still be trapped by a wave of amplitude

$$\left[\int_{-\infty}^{+\infty}\frac{du'}{u'}k_0\,|E_{u'}|^2\right]^{\frac{1}{2}}$$

(the rms value of the actual electric field) and wavelength k_0^{-1}.

On the average, a particle's velocity will change by an amount $w = (D/3k_0)^{\frac{1}{3}}$ in a time $\tau_{TR} = (k_0^2D/3)^{-\frac{1}{3}}$. Except for a numerical factor, the diffusion constant can be written w^2/τ_{TR}, a rather plausible result.

We have assumed that all waves have real frequencies ω. If there is a small imaginary part, ω_i, then, as is well known,[4,5] R has an additional "long range" term $\omega_i/(ku - \omega)^2$, and there is a corresponding term added to the diffusion coefficient for $|u - \omega/k| > w$,

$$\frac{q^2}{m^2} \sum_k |E_k|^2 \frac{\omega_i}{(ku - \omega)^2}.$$

This term describes the nonresonant or adiabatic interaction between particles and waves. All nonresonant particles oscillate in the electric field of the wave with an amplitude proportional to the field strength. If the field strength increases, the oscillation amplitude increases, and the distribution function is broadened. Actually, this oscillation energy is more properly viewed as a part of the total wave energy.

VIII. SIMPLE APPLICATION TO PLASMA WAVES

We now apply the considerations of the previous two sections to the quasi-linear portion of the integral equation for $E_k(t)$, (5.2). Using (6.6), (6.10), (7.1), and (7.3) in this equation, we obtain the following expression for wave growth in lieu of (5.3):

$$|E_k|^{-2} \frac{\partial}{\partial t} |E_k|^2 = \frac{\omega_p^3}{k} \int du\, R[ku - \omega, D(u)] \frac{\partial f_0(u)}{\partial u}, \tag{8.1}$$

where $\omega = \omega_p$. Thus the statistical trajectory has replaced the delta-function resonance of (5.3) with the finite width resonance $R[ku - \omega, D(u)]$.

Both of these expressions for the rate of wave energy gain (5.3) or (8.1) can also be obtained by computing the rate of kinetic energy loss from $\langle f_0 \rangle$ due to the resonant interaction with each wave. Considering one wave at a time and using (6.7) and (7.4), we obtain

$$\frac{\partial}{\partial t} \langle f_0 \rangle = \frac{\partial}{\partial u} \frac{q^2}{m^2} |E_k|^2 R[ku - \omega, D(u)] \frac{\partial}{\partial u} \langle f_0 \rangle. \tag{8.2}$$

From energy conservation, one knows that half the kinetic energy gained by $\langle f_0 \rangle$ from resonant interaction comes from the electric field of the wave. The other half comes from the kinetic energy of the wave. Therefore

$$2 \frac{\partial}{\partial t} \frac{1}{8\pi} |E_k|^2 = -\int du \frac{1}{2} nmu^2 \frac{\partial}{\partial t} \langle f_0 \rangle. \tag{8.3}$$

The substitution of (8.2) into (8.3) produces the expression (8.1) for wave growth. If we approximate

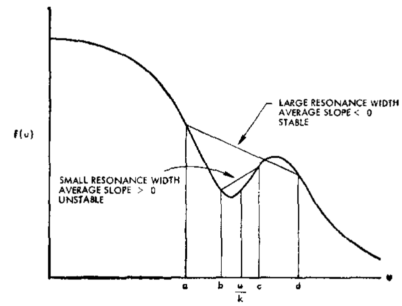

FIG. 1. Wave–particle resonances for a wave of phase speed w/k.

R by (7.5), the expression for the growth rate becomes simply

$$|E_k|^{-2} \frac{\partial}{\partial t} |E_k|^2 = \frac{\pi \omega_p^3}{2k^2} w^{-1} \left[\left\langle f_0\!\left(\frac{\omega}{k} + w\right) \right\rangle - \left\langle f_0\!\left(\frac{\omega}{k} - w\right) \right\rangle \right]. \tag{8.4}$$

Viewed in the light of (8.2), a single wave causes a diffusion in velocity space to occur. The net flow for each value of u is in the direction $-D(u)(\partial/\partial u) \langle f_0(u) \rangle$. But, to find the total kinetic energy change of resonant particles, one must average this expression over the region of velocity space of width $2w$ in which $D(u)$ is nonzero. The finite width is due to the finite width-resonance function R.

The situation is illustrated schematically in Fig. 1 for a wave of phase speed ω/k. For a small resonance width $u_c - u_b$, the average slope is positive so the net particle flow is towards decreasing u. Therefore the particles lose energy and the wave will grow. As the wave grows, the resonance width, $2w$, increases. For the larger resonance width $u_d - u_a$, the average slope is negative, and therefore more particles will diffuse towards increasing u than decreasing u. The energy of the resonant particles will show a net increase and the wave will damp.

In principle, any instability will ultimately stabilize in the manner if $|E_k|^2$ becomes large enough. Of course, some other mechanism may limit the growth before this can happen.

The three equations (6.7), (7.7), and (8.4) comprise a simple set of one-dimensional quasi-linear equations which include trapping:

$$\left[\frac{\partial}{\partial t} - \frac{\partial}{\partial u} D(u) \frac{\partial}{\partial u}\right] \langle f_0(u) \rangle = 0, \tag{8.5}$$

$$D(u) = \frac{q^2 \pi}{2m^2 u} w^{-1} \int_{u-w}^{u+w} |E_{u'}|^2 \, du', \tag{7.7}$$

$$\frac{\partial}{\partial t} |E_u|^2 = |E_u|^2 \frac{\pi \omega_p^3}{2k^2} w^{-1} [\langle f_0(u + w) \rangle - \langle f_0(u - w) \rangle], \tag{8.6}$$

where

$$w = \text{resonance width } (D/3k_0)^{\frac{1}{2}}. \quad (8.7)$$

It is a simple matter to derive the three-dimensional version of these equations. We have purposely kept the analysis simple and somewhat approximate so as not to obscure the underlying physics.

To be sure, the considerations of Secs. VI–VIII are very approximate. However, the purpose is to show in a simple way the origin and the nature of trapping phenomena within the context of Eqs. (5.2) and (6.5). Obviously, it would be desirable to have a more accurate treatment of these two equations, as well as mode coupling effects contained in the nonlinear terms of (5.1).

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Magnetohydrodynamic Channel Flows with Nonequilibrium Ionization

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The behavior of a magnetohydrodynamic channel, with finite segmented electrodes, when a plasma in a nonequilibrium state and with tensor conductivity flows through it with constant (uniform) velocity is presented. The nonequilibrium effect is taken into account by assuming the plasma electrical conductivity to be a linear function of the current density. The problem, which is highly nonlinear, is solved by expanding in powers of the proportionality constant to obtain a recursive set of linear equations. In this paper only the zeroth- and first-order solutions are presented. They are obtained numerically. For the cases studied, no shorting due to leakage between adjacent electrodes exists. Despite this inclusion of the nonequilibrium effect causes a reduction of the Hall (axial) electric field.

I. INTRODUCTION

MANY practical situations exist in which it would be desirable to have a nonequilibrium plasma condition in a magnetohydrodynamic channel flow.[1] In general, however, such a nonequilibrium condition is most easily created in a low-pressure plasma (less than one atmosphere) with a strong magnetic field (high induced electric field $\sim uB$), so that the Hall effect must be allowed for. As a result, a Hall current will flow axially unless the electrodes are segmented and an axial Hall voltage established. When the segments are finite there is an interaction between the extent of segmentation and the nonequilibrium effect due to the current crowding into one corner of the electrode. It will be the objective of the present work to analyze this current distribution when the electrodes are finite and nonequilibrium ionization is a factor.

A number of authors have analyzed the current distribution in a magnetohydrodynamic channel with constant σ and nonuniform geometry. The end currents occurring at the channel entrance or exit were among the first to be considered; the scalar conductivity case being solved first,[2] and the tensor conductivity case being treated later.[3,4] Then, as it was recognized that the Hall effect would require electrode segmentation, this problem was also treated, again for the tensor conductivity case.[5] In all cases, the flow velocity was assumed to be constant as was the scalar electrical conductivity and the Hall parameter $\omega\tau$ (where ω is the electron cyclotron frequency and τ is the mean time between electron–heavy-particle collisions). As a result, the equation to be solved in each case was Laplace's equation and the problem was a linear one, although in some cases the boundary condition was an unusual one.

In order to include nonequilibrium ionization in

[1] G. W. Sutton and A. Sherman, *Engineering Magnetohydrodynamics* (McGraw–Hill Book Company, Inc., New York, 1965).

[2] G. W. Sutton, H. Hurwitz, Jr., and H. Poritsky, Trans. AIEE Commun. Electron. **80**, 687 (1962).

[3] B. Podolsky and A. Sherman, J. Appl. Phys. **33**, 1414 (1962).

[4] G. W. Sutton, J. Appl. Phys. **34**, 396 (1963).

[5] H. Hurwitz, Jr., R. W. Kilb, and G. W. Sutton, J. Appl. Phys. **32**, 205 (1961).