ESSAYS ON NONLINEAR DYNAMICS IN OPTIMAL GROWTH MODELS

A Ph.D. Thesis

by

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İhsan Doğramacı Bilkent University

Ankara

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To Ethem Sarısülük, Ali İsmail Korkmaz, Medeni Yıldırım, Mehmet Ayvalıtaş, Fadime Ayvalıtaş, Ahmet Atakan, Abdullah Cömert, Hasan Ferit Gedik

et à l'avenir qui déjà dure longtemps

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Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

MUSTAFA KEREM YÜKSEL

In Partial Fulfillment of the Requirements For the Degree
of
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in

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ECONOMICS
İHSAN DOĞRAMACI BİLKENT UNIVERSITY
ANKARA

January 2014

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Hüseyin Çağrı Sağlam
Supervisor

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ABSTRACT

ESSAYS ON NONLINEAR DYNAMICS IN OPTIMAL GROWTH MODELS

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Ph.D., Department of Economics

Supervisor: Assist. Prof. Dr. Hüseyin Çağrı Sağlam

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Economic models with time delay have long been considered in economic theory. It is considered that delay forces the economic system into persistent cycles which can be interpreted as intrinsic crises of the capitalist economy. The effect of delay on economic dynamics is analyzed by Hopf bifurcation according to the recent developments in economics and mathematics. Hopf bifurcation depends on the existence of a pair of pure imaginary eigenvalues of the Jacobian matrix evaluated at the steady state. However, recent studies are inconsistent in a determinate way to decide whether the optimal growth model with investment lags admits persistent cycles or not.

In the second chapter of this thesis, the author tries to sharpen the analysis of one sector optimal growth model with one control and one state variables and time delay.

We firstly give a brief outline of the mathematical history and 'know-how' of delays in economic models, as well as its interpretation, and then, we further the analysis set of the model of Asea and Zak (1999) and try to introduce of a new technique for the exposition of the eigenvalues of the characteristic equation of these type of models in a generalized framework.

In the third chapter we introduce a new technique (see Louisell, 2001) to the study of economic models with delays and incorporate this technique to evaluate the cycle-inducing effects of capital dependent population growth in economic models with time delay. We employ the Solow-Kalecki framework and show that the presence of capital dependent population growth induces cycles. Other than the introduction of a new technique into the area of economics, one particular contribution of this chapter is that the results clearly shows that delay is not sufficient in inducing cycles even in the most simple economic models.

In the forth chapter, we show that Hopf bifurcation may emerge in an overlapping generations resource economy through a feedback mechanism between population and resource availability. In overlapping generations resource economy models, the cycle inducing factor is mainly the nonlinearity of the regeneration of the resources. On the contrary, we assume linear regeneration and yet, endogenize the population growth rate. We show that the interaction between instantenous population growth and regeneration rate triggers persistent cycles in the economy.

In the fifth chapter, we employ a continuous delay structure in the process of recruitment in the population growth in an optimal growth model and hence obtain cyclic solutions. We exploit Erlangian process in the population growth mechanism. As far as we know, the incorporation of Erlangian process in optimal growth models is handled in this chapter for the first time in economic literature. Through this mechanism, not only the population is considered as a function of per capita capital, or in other words, population growth is endogenized, but also the current level of population growth is linked with those of older generations. We find out that the interaction between the effect of older generations' fertility choices and the accumulation of capital induces cyclic behaviour in the economy.

The sixth and the last chapter concludes with future research agenda.

Overall, the thesis considers the effects of delay and endogenized population on the economies of interest (Solow, overlapping generations, optimal growth model) economically and tries to introduce the existing methods and develop new ones to investigate the effects of delay and endogenized population on the eigenvalues of the Jacobians that the drive the economies of interest at their steady states.

Keywords: Hopf Bifurcation, Overlapping Generations Models, Endogenous Population Growth, Nonlinear Dynamics, Bifurcations

ÖZET

OPTİMAL BÜYÜME MODELLERİNDE

DOĞRUSAL OLMAYAN DİNAMİKLER ÜZERİNE MAKALELER

YÜKSEL, Mustafa Kerem

Doktora, Ekonomi Bölümü

Tez Yöneticisi: Yrd. Doç. Dr. Hüseyin Çağrı Sağlam

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Zaman gecikmeli iktisat modelleri, iktisat kuramının uzun süredir gündemindedir. Bu modellerde yer alan zaman gecikmesinin iktisadi sistemi, kapitalist ekonominin içsel krizleri olarak yorumlanabilecek sürekli çevrimlere zorladığı düşünülmektedir. İktisat kuramındaki ve matematikteki gelişmelerle, zaman gecikmesinin iktisadi dinamiklere etkisinin çözümlenmesinde Hopf çatallaşması kullanılmaya başlamıştır. Bu çatallaşma durağan durumda hesaplanan Jacobi matrisinin yalın sanal özdeğer çiftine sahip olmasına bağlıdır. Bununla beraber, yapılan çalışmalar yatırım gecikmeli

optimal büyüme modellerinin çözümlerinde sürekli çevrimlerin bulunup bulunmadığı konusunda kesin bir yargıda bulunamamaktadır.

Bu tezin ikinci bölümünde, bir kontrol ve bir durum değişkeni olan zaman gecikmeli bir sektörlü optimal büyüme modelinin çözümlemesi geliştirilmeye çalışılmıştır. İlk olarak matematiksel bir tarihçe ve zaman gecikmeli iktisadi modellerin teknik bilgisi, ve bunun yanında da yorumu serimlenmiştir. Daha sonra ise, Asea Zak (1999)'da ortaya konan çözümleme kümesi genişletilmiş ve genelleştirilmiş bir çerçevede bu tip modellerin karakteristik denklemlerinin özdeğerlerinin ortaya çıkarılması için yeni bir teknik önerilmiştir.

Tezin üçüncü bölümünde, zaman gecikmeli iktisadi modellerin incelenmesi için yeni bir teknik (bkz. Louisell, 2001) önerilmiş ve bu teknik zaman gecikmeli iktisadi modellerde sermayeye bağımlı nüfûs büyümesinin çevrim-yaratıcı etkilerini irdelemekte kullanılmıştır. Solow-Kalecki çerçevesi kullanılarak sermaye bağımlı nüfûs büyümesinin çevrimleri tetiklediği gösterilmiştir. Bu bölümde, yeni tekniğin iktisat alanına tanıtılmasının dışında, zaman gecikmesinin en basit iktisadi modellerde bile çevrimleri tetiklemeyebileceği ortaya konmuştur.

Tezin dördüncü bölümünde ardışık nesiller kaynak ekonomisinde nüfûs ve kaynak bulunabilirliği arasındaki geribesleme mekanizmasının Hopf çatallaşmasını doğurabileceği gösterilmiştir. Ardışık nesiller kaynak ekonomilerinde çevrimleri yaratan temel etken kaynakların yenilenmesinin doğrusal olmamasıdır. Buna karşın, burada doğrusal yenilenme kullanılmış ama nüfûs artışı hızı içselleştirilmiştir. Anlık nüfûs büyümesi ve yenilenme oranı arasındaki etkileşimin ekonomideki sürekli çevrimleri tetiklediği görülmüştür.

Tezin beşinci bölümünde optimal büyüme modelinde nüfûs artışının modellenmesinde sürekli zaman gecikmesi yapıcı kullanışmış ve çevrimler çözümler olduğu gösterilmiştir. Nüfûs büyüme mekanizmasında Erlang sürecinden faydalanılmıştır. İktisad kuramında optimal büyüme modellerinde Erlang sürecinin kullanılması, bizim bilebildiğimiz kadarıyla ilk defa burada ele alınmıştır. Bu mekanizma sayesinde sadece nüfûs kişi başı sermayenin bir fonksiyonu olarak, yani içselleştirilerek düşünülmekle kalmamış, aynı zamanda bugünkü nüfûs büyümesi geçmiş nesillerin nüfûs büyümesi ile de ilişkilendirilmiştir. Geçmiş nesillerin doğurganlık tercihleri ile sermaye birikimi arasındaki etkileşimin ekonomideki çevrimsel davranışları tetiklediği gösterilmiştir.

Altıncı ve son bölümde gelecekteki araştırma gündemi serimlenmiştir.

Genel olarak bu tezde, zaman gecikmesinin ve içselleştirilmiş nüfûsun (Solowcu, ardışık nesiller, optimal büyüme modelleri gibi model) ekonomilerindeki iktisadi etkisi tartışılmış, zaman gecikmesinin ve içselleştirilmiş nüfûsun ilgili ekonomilerin durağan durumlarını yönlendiren Jacobi matrislerinin özdeğerleri üzerindeki etkilerini inceleyen varolan yöntemler tanıtılmış ve yenileri geliştirilmiştir.

Anahtar Kelimeler: Hopf Çatallaşması, Ardışık Nesiller Modeli, İçsel Nüfûs Artış Hızı, Doğrusal Olmayan Dinamikler, Çatallaşmalar

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CHAPTER 1

INTRODUCTION

Cycles have been on the agenda of researchers in the area of economics for at least two centuries. Cycles (crises) are assumed to be welfare-costly and thus, the stabilization of cycles (or, inevitability of crises) has been a major political and academic topic. Main approaches are explained in detail in the chapters to come. However, in essence, we can assert that there are basically two schools of interpretations: Those who believe that cycles are caused by exogenous shocks (exogenous in the sense that the shock is from a noneconomic variable); or those who believe that cycles are intrinsic to the economic behaviour.

The degree of mathematical sophistication in these models limits us in the sense that it is only through these kind of attempts at the heart of economics that we can understand how sensitive the economic model to different components and assumptions in the model. Unfortunately, we lack the necessary tools to complete a thorough analysis in a general framework, in other words, an analysis that covers all the models with all the possible assumptions. Thus, we have to consider particular models with particular deviations from the existing literature. In that sense, our models may lack the immediate and direct policy implications for policy makers. Yet, if cycles matter practically, we have to make them an object of our study theoretically, as well. Their existence, the causes, their amplitudes, their frequencies (or periods), their qualities (persistent or decaying cycles), their stability, short run dynamics, welfare implications, optimality or suboptimality etc. should be considered in a theoretical framework.

In the thesis, we attempt to summarize the historical discussions about the characteristics of cycles. We contribute to the literature by extending the existing tools and refreshing the approaches in economics to understand and present mechanisms of cycle-inducing investment lags and endogenous population growth and their implications for the macroeconomic dynamics. In other words, we try to establish the limits and possibilities of nonlinear dynamics (i.e., cycles) *vis-à-vis* investment lags and endogenous population growth. The interesting dynamics (limit cycles, i.e., persistent cyclic behaviour) occur when these ingredients cause permanent adjustment failures among the economic variables in the economy.

In order to address the effects of delay and endogenous population growth in macroeconomic models, we try to answer the the cycle puzzle in optimal growth models with time delay in Chapter (2); we show that delay or endogenous population growth alone may not be sufficient for the occurrence of cyclic dynamics in even the most basic economic models or, in other words, cycles depend on the interaction between the lagged capital accumulation and the instantenous population growth in Chapter (3); we incorporate endogenous population growth mechanism in an overlapping generations resource economy and show that cyclic solutions exist even in the absence of unrealistic cycle-inducing assumptions in the existing literature in Chapter (4) and finally in Chapter (5), the endogenous population growth is handled with continuos delay that links the past generations capital dependent fertility choices with the most recent ones'.

1.1 Historical Background

"Once a principle is set in motion, it works by its own impetus through all its consequences, whether the economists like it or not." F. Engels[1]

Just in the beginning of his monumental work *The Age of Revolution 1789-1848* (first publication 1962), Eric J. Hobsbawn was wise to state that "words are witnesses which often speak louder than documents" and in the sentences to follow, he listed some words which had been invented or gained meaning (in terms of their modern usage) within this period, words such as "capitalism", "industry", "working class" etc. and more strikingly "(economic) crises" and "statistics."

Economic crises entered in economic literature as early as Jean-Baptiste Say (1803). By 1830, there were inquiries on early theories of cycles and crises and, certainly there was some awareness of periodicity of times of prosperity and distress[2]. (Besomi, 2008) According to Besomi (2008), one of the first accounts of "waves" were by Thomas Tooke who in his 1823 publication *Thoughts and Details on the High and Low Prices of the Last Thirty Years*, who attributed these crises mainly to exogenous events such as bad seasons etc., and later incorporated some endogenous factors.

[1] Outlines of a Critique of Political Economy, Deutsche-Französische Jahrbücher, 1844. (in Marx-Engels Collected Works, Vol.3, pg. 424)

[2] According to Besomi (2008) Wade (1833) supplied dates for some crises years (p. 150): 1763,1772, 1793, 1811, 1816, 1825–6. Jevons (1878) also gave years of crises: 1763, 1772–3, 1783, 1793, (1804–5?), 1815, 1825 (p. 231).

Wade, J. 1833. *History of the middle and working classes; with a popular exposition of the economical and political principles which have influenced the past and present condition of the industrious orders. Also an Appendix of prices, rates of wages, population, poor-rates, mortality, marriages, crimes, schools, education, occupations, and other statistical information, illustrative of the former and present state of society and of the agricultural, commercial, and manufactoring classes*, London: Effingham Wilson (reprinted: New York: Kelly, 1966). 2nd edition 1834, 3rd edition 1835.

Jevons, W.S. 1878 "Commercial crises and sun-spots", Pt. 1, *Nature*, vol. XIX, 14 November, pp. 33–37. Reprinted in *Investigations in Currency and Finance*, ed. by H. S. Foxwell, London: Macmillan, 1884, pp. 221–35 (Besomi, 2008).

Hyde Clarke (1838) was of interest with the idea that "cycles in nature and society, are subject to an elementary mathematical law" (Besomi, 2008). Although Clarke was not specifically interested in economics, an enormous literature built upon crises and cycles in economics. Citing Besomi (2008); Coquelin[3] (1848) asserted that "commercial perturbations have become in certain countries in some degree periodical"; Lawson[4] (1848) declared these periods would be five to seven years; Jevons (1878) claimed a strict periodicity of 11 years in his survey with reference to "most writers". One should note that early investigators were eager to identify the reasons of cycles to exogenous shocks to the system, such as wars, bad seasons, embargoes, oppressive duties, the dangers and difficulties of transportation, social unrest increasing uncertainty, arbitrary exactions, jobbing and speculations etc. The common point was that these shocks either disrupts the proper working of the system or the proper functioning of the exchange or production mechanisms (Besomi, 2008). These crises were assumed to be corrected in the course of the self-adjusting nature of the economy just after the exogenous determinant is removed.

A second group of analysts were then trying to model these cycles as a part of the natural course of the economy. This group of researchers views cycles as a resultant behaviour intrinsic to economic activity, not disjunct occurrences. This approach forced them to identify the cyclic phenomenon and characterize it. Quoting Besomi (2008), the transition from the exogenous shock models to "proper theories of the cycle was a gradual process that took several decades, and was only completed at the eve of World War I with the theories of Tugan-Baranowsky, Spiethoff, Mitchell, Bouniatian, Aftalion and a few others." Once again, Wade was one of the first who "explicitly spoke of a commercial cycle intrinsic to a mercantile society," and "in-

[3]Coquelin, C. 1848. "Les Crises Commerciales et la Liberté des Banques," *Revue des Deux Mondes XXVI*, 1 November, pp. 445–70. Abridged as Coquelin 1850 (Besomi, 2008).

[4]Lawson, J. A. 1848. *On commercial panics: a paper read before the Dublin Statistical Society*, Dublin (Besomi, 2008).

separable from mercantile pursuits" (see Besomi 2008). Moreover, as the cause of fluctuations, Wade was one of the first to come up with the idea of "the lag between change in price, change in demand and change in production, on which the principal cyclical mechanism implicitly relied, becomes apparent" (Besomi, 2008).

In accordance with Besomi (2008), Persons (1926) also divides theorists into two groups (without giving exact references, but by just mentioning names) according to the their approach to cycles. We can replicate its taxonomy here. The first group consists of economist who emphasize on factors other than economic institutions:

- Periodic agricultural cycles generate economic cycles: W. S. Jevons, H. S. Jevons, H. L. Moore
- Uneven expansion in the output of organic and inorganic materials is the cause of the modern crisis: Werner Sombart
- A specific disturbance, such as an unusual harvest, the discovery of new mineral deposits, the outbreak of war, invention, or other "accidents" may disturb economic equilibrium and set in motion a sequence which, however, will not repeat itself unless another specific disturbance occurs: Thornstein Veblen, Irving Fischer, A. B. Adams
- Variations in the mind of the business community (affected, of course, by specific economic disturbances) are the dominating cause of trade cycles: A. C. Pigou, Ellsworth Huntington, M. B. Hexter.

The second group economists are those who emphasize on factors related to economic institutions:

- Given our economic institutions (particularly capitalistic production and private property) it is their tendency to development which results in business fluctuations: Joseph Schumpeter, Gustav Cassel, E. H Vogel, R. E. May, C. F. Bickerdike.

- The capitalistic or roundabout system of production is the primary cause of business fluctuations: Arthur Spiethoff, D. H. Robertson, Albert Aftalion, T. E. Burton, G. H. Hull, L. H. Frank, T. W. Mitchell, J. M. Clark.

- Excessive accumulation of capital equipment, accompanied by maldistribution of income, is responsible for lapses from prosperity to depression: Mentor Bouniatian, Tugan-Baranowsky, John A. Hobson, M. T. England, W. H. Beveridge, N. Johannsen, E. J. Rich.

- The fluctuation of money profits is the center from which business cycles originate (eclectic theories): W. C. Mitchell, Jean Lescure, T. N. Carver.

- The nature of the flow of money and credit, under our present monetary system, is the element responsible for the interruption of business prosperity: R. G. Hawtrey, Major C. H. Douglas, W. T. Foster and Waddill Catchings, A. H. Hansen, W. C. Schluter, H. B. Hastings, H. Abbati, W. H. Wakinshaw, P. W. Martin, Bilgram and Levy.

Persons (1926) also gives the justification of this classification with reference to essential points of the theories thereafter.

One should also notice that the two groups are also bifurcated in their terminology which is very apt with their theoretical background. Those who understood crises as disconnected events shaped their language accordingly with frequent use of "crises"; yet those who evaluate cycles as a part of the state of the economy exploits the use of the word "cycle". The crises theorists tried to identify to reasoning of each crisis with a particular exogenous shock which lies in the background of all the crisis. W.

S. Jevons (1878), for example, thought that the sunspots with the exact periodicity of 10.45 years are the main cause of crop-failures of which he believed to repeat every 10.44 years and this results with an economic burst. H. S. Jevons considered heat emissions by the sun with the periodicity of 3.5 years to be prior reason of crop cycles and thus the economic cycles. Irving Fischer was the one who put forward most common causes of fluctuations as increase in the quantity of money, shock to business confidence, short crops and invention. Ellsworth Huntington, interestingly, makes a connection between business cycles and mental attitude of the community which depends on health. M. B. Hexter tried to find a link between fluctuations in birth-rate and in death-rate and fluctuations in business enterprise (see Persons, 1926).

On the other hand, those who are tied with the cycles perspective tried to find a causality in the economic system where one state logically precedes the other (Besomi, 2008). Joseph Schumpeter, for example, thought cycles to be "essentially a process of adapting the economic system to the gains or advances of the respective periods of expansion" (Persons, 1926). R. E. May blames increased productivity of labour; Albert Aftalion indicates the existence and the universality of the new industrial technique which has caused the appearance and repetition of economic cycles; L. H. Frank explains cycles with his theory of variations in the rates of production-consumption of consumers' goods; Mentor Bouniatian comes up with two ideas: (1) the idea that the modification of the social utility of wealth, resulting from changes in the relation between the production of goods and the need for them, is a cause of the general advance of prices in a period of prosperity [...] and of decline in a crisis, (2) the idea that the time-using capitalistic process [...] is at the basis of a period of advance. (Persons, 1926)[5]

[5] A more detailed list of theories and explanations can be found in Persons (1926).

As the theories of fluctuations improved from crises to cycles the question "how" takes place of the question "why" (Besomi, 2008). Ragnar Frisch (1933) offers to define the dynamics in a theory within a mathematical setup[6]. Frisch and Holme (1935) tries to identify the roots of a characteristic equation of a specific type of mixed difference and differential equation which occurs in economic dynamics of Michal Kalecki. (Kalecki will be discussed later.)

The crises of capitalist mode of production had also a particular place in marxist economic literature. Besomi (2008) references the "the young Friedrich Engels" who gives an elegant dialectical interpretation in his *Outlines of a Critique to Political Economy* (1844, pp. 433-4). Besomi quotes Engels with the following passage:

"The law of competition is that demand and supply always strive to complement each other, and therefore never do so. The two sides are torn apart again and transformed into flat opposition. Supply always follows close on demand without ever quite covering it. It is either too big or too small, never corresponding to demand; because in this unconscious condition of mankind no one knows how big supply or demand is. If demand is greater than supply the price rises and, as a result, supply is to a certain degree stimulated. As soon as it comes on to the market, prices fall; and if it becomes greater than demand, then the fall in prices is so significant that demand is once again stimulated. So it goes on unendingly—a permanently unhealthy state of affairs—a constant alternation of overstimulation and flagging which precludes all advance—a state of perpetual fluctuation without ever reaching its goal. This law with its constant adjustment, in which whatever is lost here is gained there, is regarded as something excellent by the economist. It is his chief glory—he cannot see enough of it, and considers it in all its possible and impossible applications. Yet it is

[6] Frisch (1933) was a model of persistent fluctuations as a result of the superposition of random exogenous shocks upon a damped system (Besomi, 2006). These type of models will be revised and finally evolve into real business cycle models.

obvious that this law is purely a law of nature and not a law of the mind. It is a law which produces revolution. The economist comes along with his lovely theory of demand and supply, proves to you that 'one can never produce too much', and practice replies with trade crises, which reappear as regularly as the comets, and of which we have now on the average one every five to seven years. For the last eighty years these trade crises have arrived just as regularly as the great plagues did in the past—and they have brought in their train more misery and more immorality than the latter. Of course, these commercial upheavals confirm the law, confirm it exhaustively—but in a manner different from that which the economist would have us believe to be the case. What are we to think of a law which can only assert itself through periodic upheavals?"

Although neither Marx nor Engels put forward a complete theory of this cyclic crises, they assumed that cycles are intrinsically embedded in the nature of capitalist production. Marx calls these "realization crises" which are based on the failure of the realization of the expected profits of the capitalist. Failure is assumed to be rooted in the overproduction of the economy due to insufficient planning, which Marx refers as the "anarchy of the capitalist production". It is Michal Kalecki who tried to find a mathematical reasoning for the marxist approach in a series of papers during 1930s and later. In one of his most influential articles, Kalecki introduces lag structure in the economy to explore the cyclic behaviour, in which he shows rigorously for the first time that business cycles depends endogenously to production (investment) lags. (Kalecki, 1935) (A brief exposition of Kaleckian model is still to be discussed with the literature that builds upon.)

Before discussing in detail the Kaleckian setup and other models, we should track the improvement of mathematical apparatus. Apparently, after a seminar by Kalecki at a meeting in the Econometric Society at Leyden, Frisch and Holme (1935) is first to analyze the roots of difference-differential equations of the form $\dot{y}(t) = ay(t) - cy(t - \theta)$ and characterize the main properties with respect to the roots according to the exogenous (empirical econometric) parameters a and c. It is James and Belz (1938) who contributes to the mathematics of the characterization of the problem further. James and Belz (1938) suggests that "a solution of a difference-differential equation might be developed in terms of an infinite series of characteristic solutions" and investigates "the conditions under which such a development is possible." In addition, this paper gives methods "for determining the coefficients of the development, when it exists" and shows that the solutions of certain forms of integro-differential equations "can be given in the form of an infinite series derived from a consideration of related difference-differential equations." Hayes (1950) partially closes the literature on roots by giving the properties of the roots of transcendental equations of the form $\tau(s) = se^s - a_1 e^s - a_2 = 0$ which is nothing but the resultant characteristic equation of a subset of difference-differential equations with constant coefficients, which frequently occurs in dynamic economic systems with delays. As Zak (1999) points out, the first thorough analysis of a general class of delay differential equations is by Bellman and Cooke (1963) with later fundamental work by Hale (1977).

Kalecki (1935)[7] introduces production lags, a time delay between the investment decisions and delivery of the capital goods, to show the generation of endogenous cycles. He employs a linear delay differential equation of the deviation of investment[8]

[7] A brief exposition of the Kalecki (1935) model and its properties can be found in Zak (1999) and Szydłowski (2002). These texts reproduces Kalecki's results with contemporary techniques which are also employed in this thesis.

[8] Michal Kalecki studied the underlying forces of cycles in economy throughout his life and his bunch of theories vary from linear difference differential equation systems to exogenous factors. As Besomi (2006), in his study about Kalecki's business cycle theories, pointed out Kalecki "either failed to provide a rigorous proof of the stability of the cycle when the model was endogenous or failed to provide an explanation of the cycle relying on the properties of the economic system, resorting instead to exogenous shocks to explain the persistence of fluctuations." Kalecki interpreted cycles as the dynamic expression of the "intrinsic antagonism of capitalism" however he "acknowledged the existence of disturbing factors, from which he abstracted in order to isolate a *pure* cycle." Besomi (2006) also reports that "Kalecki's models describes damped fluctuations around a line of stationary equilibrium and rely for the persistence o fluctuations on exogenous shocks" and moreover, all his

which is denoted by J. The investment equation[9] is of the form $\dot{J}(t) = AJ(t) - BJ(t-\theta)$. Kaleckian models exhibit endogenous cycles by employing simple time lag in a linear delay differential equation. Lags in the model serves two purposes: (1) Lag structure is empirically significant[10] and (2) first order linear ordinary differential equations are known to be unable to give cyclic solutions while linear delay differential equations may exhibit endogenous cycles. Apart from showing that there can exist endogenously driven cycles in the economy rather than crises determined by exogenous shocks, Kalecki develops the mathematical techniques to characterize the stability properties in linear delay differential equations. Obviously, one had to wait for Hayes (1950) for a full understanding of the stability properties in one delay linear differential equations, although Kalecki (1935) presents a thorough stability analysis (Zak, 1999). Kaldor (1940) criticizes Kalecki (1935) by pointing out that the drawback of the model is that "the existence of an undamped cycle can be shown only as a result of a happy coincidence, of a particular constellation of the various time-lags and parameters assumed" and "the amplitude of the cycle depends on the size of the initial shock." Instead, Kaldor (1940) proposes a nonlinear investment decision to obtain cycles of the economy. Inspired by Kaldor (1940), Ichimura (1954) explores the possibility of an economic system with a unique limit cycle; Chang and Smyth (1971) reexamine the model and state the necessary and sufficient conditions of an existence of a limit cycle; Grasman and Wentzel (1994) considers the coexistence of a limit cycle and an equilibrium. The dynamics of Kaldor-Kalecki type of models have been extensively studied on a series of papers by Krawiec and Szydłowski (1999, 2000, 2001, 2005), Szydłowski (2003) and Krawiec, Szydłowski and Toboła (1999).

models "crucially depend for cyclicality upon one or more reaction lags."

[9]The exact linear delay differential equation studied by M. Kalecki (1935, p. 332) is $\dot{J}(t) = \frac{m}{\theta}J(t) - \frac{m+n\theta}{\theta}J(t-\theta)$ where m and n are assumed to be constants.

[10]Kalecki (1935, pp. 337-338) estimates the lag between the curves of beginning and termination of building schemes (dwelling, industrial and public buildings) as 8 months and lags between orders and deliveries in the machinery-making industry as 6 months based on the data supplied by German *Institut fuer Konjunkturforschung*. He assumed "that the average duration of θ is 0.6 years."

Kaldor-Kalecki models have two mechanisms which would lead to cyclic behaviour, one being the nonlinearity of the investment function and the other being the time delay in investment (Krawiec and Szydłowski, 2001). Krawiec and Szydłowski (1999, 2001) proves that it is the time to build assumption rather that the nonlinear (s-shaped) investment function that leads to the generation of cycles.

The main tool in these papers for creating cycles is Hopf bifurcation. "In 1942, Hopf published the ground-breaking work in which he presented the conditions necessary for the appearance of periodic solutions, represented in phase space by a limit cycle." (Szydłowski, 2002). With reference to the contributors of the study of the sufficient conditions under which periodic orbits occur from stationary states are called Poincaré–Andronov–Hopf theorems. As Kind (1999) points out, it is generally easy to prove Hopf bifurcation since it doesn't require any information on the nonlinear parts of the equation system. Moreover, in systems with the dimension higher than two, Hopf bifurcation may be the only tool for the analysis of the cyclical equilibria, since the Poincaré-Bendixson theorem is not applicable. Furthermore, when the conditions of Hopf bifurcation are satisfied, it guarantees both the existence and uniqueness of periodic trajectories (Krawiec and Szydłowski, 1999). However, Hopf theorem gives no information on the number and the stability of closed orbits. On the other hand, nonlinear parts can be used in the calculation of a stability coefficient in order to determine the stability properties of the closed orbits (Kind, 1999). Guckenheimer and Holmes (1983, Thm. 3.4.2, pp. 151-153) both gives the theory and an example in that direction. Feichtinger (1992) is an example of such a calculation in economic literature.

Zak (1999) summarizes Kalecki's contribution and extends his results to a general equilibrium setup, which has been an open research area until then[11]. Zak (1999)

[11] Zak (1999, p. 325ff) also claimed that Kaleckian cycle in Kalecki (1935) was nothing but Hopf cycles.

inserts a production lag into a basic one sector Solowian model and shows that the results also admits Hopf cycles under certain conditions. Later, Krawiec and Szydłowski (2004) reproduces the results and improves the analysis of the same model. Zak (1999) also copies the results of an important contribution to the literature which marked an important "false" attempt to extend the same analysis to the optimal growth models with lags. Asea and Zak (1999) is the first to lay out the main tools and shows that there exists a cyclic behaviour in these type of models. However, this paper contains an error on the first order dynamic equations which erroneously lead to Hopf cycles. The corrected characteristic equation[12] is not easy to analyze to find out whether the roots satisfy Hopf conditions, so studies afterwards turn to numerical analysis to reveal periodic behaviour. Winkler *et al.* (2004), Winkler *et al.* (2005), Collard *et al.* (2006), Collard *et al.* (2008), Brandt-Pollmann *et al.* (2008) are among such studies.

Unlike Solowian systems which result with a characteristic equation of the form $h(\lambda) \stackrel{def}{=} \lambda - Ae^{-\lambda r} = 0$; in optimal growth models, one should deal with more complex characteristic equations. Apart from the nonlinearity of the utility and production functions, optimal growth model is governed by a 2×2 system of equations (one for state and the other for control dynamics), so the degree of the polynomial is greater, if one can mention about degree of quasi-polynomials. Collard *et al.* (2006) numerically shows that the advanced terms in Euler equations governing the dynamic system dampen the fluctuation caused by the lags through a kind of smoothing effect (They call this phenomenon 'time-to-build echo'). Short run dynamics of time-to-build echoes are further studied by Collard *et al.* (2008) in which one can find the associated numerical simulations. Winkler *et al.* (2004) provides numerical solutions

[12] Winkler *et al.* (2003) gives the correct dynamics and characteristic equations for any utility and production function. In Collard *et al.* (2008) one can find the correct dynamics and characteristic equations for a specific concave production function ($f(k) = Ak^{\alpha}$) and in Collard *et al.* (2006) the case of CES utility function ($u(c) = \frac{c^{1-\sigma}-1}{1-\sigma}$) and the same production technology is studied.

of models of time delay optimal growth models for a linear limitational production function, while Winkler *et al.* (2005) gives a numerical analysis of a time-lagged capital accumulation optimal growth model with Leontief type of production functions. Brandt-Pollmann *et al.* (2008) extends the numerical solutions to objective functions with state externalities.

Dockner (1985) is of special interest since it opens a new line of research of Hopf cycles in economy. Dockner (1985) give the root characteristics (local stability properties) of a 4×4 system of dynamic equations in a simple form, where these 4×4 is the resultant dynamics of nonlinear optimal control problems with one control and two state variables. These results have been exploited extensively by Wirl in a series of papers[13], with models of two states, one inducing an externality on the objective function. Note that the etiology of cycles in these models are the externality which is expressed with one of the state variables in objective function, rather than time delays in the evolution of states. The optimality of such cycles has been studied by Dockner and Feichtinger (1991). Optimality of cycles (in a similar two state approach) in more specific setups has also been studied. Wirl (1994) investigates cyclical optimality in a Ramsey model with wealth effects and Wirl (1995) repeats the same for renewable resource stocks can be exemplified. Wirl (1992) simplifies the findings of Dockner (1985) in economic framework of two-dimensional optimal control models and gives an economic interpretation to the necessary conditions for cyclic behaviour. Wirl (1994) repeats and extends Wirl (1992). Wirl (1997, 1999, 2002) further extend the results to optimal control problems with one state and an externality. Since the externality is not included in the Hamiltonian of the optimal control problem, the model has a 3×3 dynamics, yet the findings are in similar direction. Wirl (1999) constructs an environmental model and repeats the analysis. Wirl (2004) analyzes a model of optimal saving with optimal intertemporal renewable resources in terms of

[13]See various papers in references.

thresholds and cycles.

One should also mention the seminal work by Kydland and Prescott (1982). In their paper, Kydland and Prescott (1982) formulates a discrete time theoretical framework and showed that US post-war economy fitted well. This is one of the major studies that supports the idea that the time-to-build assumption contributes to the cyclical behaviour in the economy even when the simplest equilibrium growth model is employed.

1.2 Characteristic Equation of Dynamic Systems and Its Roots

A dynamic system of differential equations induces a characteristic equation of which the placement of the roots of the equations in the complex plane gives clues about the behaviour (stability, indeterminacy etc.) of the system. The characteristic equation determines the behaviour of the system near its *steady state* (i.e. *equilibrium point*). Following Hale and Lunel (1993, p. 17), a linear differential equation of the form $\dot{x}(t) = Ax(t) + Bx(t-r)$ has a nontrivial solution $ce^{\lambda t}$ (c, constant) if and only if $h(\lambda) \stackrel{def}{=} \lambda - A - Be^{-\lambda r} = 0$. Because of the transcendental function of λ, this is not a polynomial but is the type of functional form which is called *quasi-polynomials.* The analysis of quasi-polynomials in economics dates back to Kalecki (1935). In his paper, Kalecki (1935) introduces a *gestation period* to the model and ended up with a quasi-polynomial. Later, Frisch and Holme (1935) and James and Belz (1938) contribute to the literature on the characteristic solutions of mixed difference and differential equations. However, a major breakthrough in the analysis is by Hayes (1950). Hayes give the properties of certain difference-differential equations, mainly the ones of the form $h(\lambda) \overset{def}{=} \lambda e^{\lambda r} - Ae^{\lambda r} - B = 0$[14]. Note that this equation is equivalent in roots with the equation above.

Periodic solutions to dynamic systems are also analyzed extensively in control theory. One way to detect limit cycles is Hopf bifurcation. Hopf bifurcation discards tedious calculations and provides a powerful and easy tool to detect limit cycles. Kind (1999) confirms this by stating "in most cases the proof of a Hopf bifurcation is not difficult because it does not require any information on the nonlinear parts of the equation system. Moreover, in systems whose dimensions are higher than two, Hopf bifurcation theorem may constitute the only tool for the analysis of cyclical equilibria, since the Poincaré–Bendixson theorem is not applicable in these cases". Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter and meanwhile a cycle either emerges from or collapses in to the fixed point (Asea and Zak, 1999). Under the circumstances the system can either have a stable fixed point surrounded by an unstable cycle (called a *subcritical* Hopf bifurcation); or a stable fixed point loses its stability and a stable cycle appears (called a *supercritical* Hopf bifurcation) as the parameter(s) approaches to a critical value (Asea and Zak, 1999). Both cases can be economically significantly meaningful. Supercritical case which implies a stable cycle can be considered as a stylized business cycle or growth cycles and the subcritical case can correspond to the corridor stability (Kind, 1999).

Let us state the Poincaré-Andronov-Hopf Theorem (Hale and Koçak, 1991, Thm. 11.12, p. 344) here, for the sake of completeness:

Theorem 1.1 *(Poincaré-Andronov-Hopf) Let* $\dot{\mathbf{x}} = A(\mu)\mathbf{x} + \mathbf{F}(\mu, \mathbf{x})$ *be a* C^k*, with* $k \geq 3$*, planar vector field depending on a scalar parameter* μ *such that.* $\mathbf{F}(\mu, \mathbf{0}) = \mathbf{0}$ *and* $D_x\mathbf{F}(\mu, \mathbf{0}) = \mathbf{0}$ *for all sufficiently small* $|\mu|$*. Assume that the linear part* $A(\mu)$ *at the origin has the eigenvalues* $\alpha(\mu) \pm i\beta(\mu)$ *with* $\alpha(0) = 0$ *and* $\beta(0) \neq 0$*. Furthermore,*

[14] For a summary of the roots of certain types of quasi-polynomials, see Özbay (2000, pp. 110-113)

suppose that the eigenvalues cross the imaginary axis with nonzero speed, that is, $\frac{d\alpha}{d\mu}(0) \neq 0$*. Then, in any neighborhood* U *of the origin in* $\mathbb{R}^2$ *and any given* $\mu_0 > 0$ *there is a* $\overline{\mu}$ *with* $|\overline{\mu}| < \mu_0$ *such that the differential equation* $\dot{\mathbf{x}} = A(\bar{\mu})\mathbf{x} + \mathbf{F}(\bar{\mu}, \mathbf{x})$ *has a nontrivial periodic orbit in* U*.*

According to the above theorem, one can summarize the sufficient conditions for Hopf Bifurcation as follows:

- (H1) $A(\mu)$[15] has *only* one pair of pure imaginary eigenvalues. (*Pre-Hopf Condition*)[16]

- (H2) Pure imaginary eigenvalues cross the imaginary axis with nonzero speed, i.e., $\frac{d\alpha}{d\mu}(0) \neq 0$. (*Transverse Crossing*)

The pre-Hopf condition is necessary for Hopf Bifurcation. Therefore, if this condition is not met Hopf Bifurcation doesn't exist for the system. This implies that limit cycles do not occur via Hopf Bifurcation, if not via any other way[17].

In the second chapter of this thesis, the author tries to sharpen the analysis of one sector optimal growth model with one control and one state variables and time delay. We firstly give a brief outline of the mathematical history and 'know-how' of delays in economic models, as well as its interpretation, and then, we further the analysis set of the model of Asea and Zak (1999) and try to introduce of a new technique for the

[15] Note that $A(\mu)$ is nothing but the Jacobian matrix that results from *linearization* of the system. If $\bar{\mathbf{x}}$ is the equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, then the linear differential equation $\dot{\mathbf{x}} = D\mathbf{f}(\bar{\mathbf{x}})\mathbf{x} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{x}) & \frac{\partial f_1}{\partial x_2}(\mathbf{x}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{x}) & \frac{\partial f_2}{\partial x_2}(\mathbf{x}) \end{pmatrix}$ is the *linear variational equation* or the *linearization* of the vector field $\mathbf{f}$ at the equilibrium point $\bar{\mathbf{x}}$. (Hale and Koçak, 1991, Defn. 9.4, p. 267)

[16] The name is given by the author of the thesis.

[17] Asea and Zak (1999, p. 1164ff) mentions other ways in which periodic orbits may arise. Heteroclinic orbits are given as an option, yet there are stated to be "rare".

exposition of the eigenvalues of the characteristic equation of these type of models in a generalized framework.

In the third chapter we introduce a new technique (see Louisell, 2001) to the study of economic models with delays and incorporate this technique to evaluate the cycle-inducing effects of capital dependent population growth in economic models with time delay. We employ the Solow-Kalecki framework and show that the presence of capital dependent population growth induces cycles. Other than the introduction of a new technique into the area of economics, one particular contribution of this chapter is that the results clearly shows that delay is not sufficient in inducing cycles even in the most simple economic models.

In the forth chapter, we show that Hopf bifurcation may emerge in an overlapping generations resource economy through a feedback mechanism between population and resource availability. In overlapping generations resource economy models, the cycle inducing factor is mainly the nonlinearity of the regeneration of the resources. On the contrary, we assume linear regeneration and yet, endogenize the population growth rate. We show that the interaction between instantenous population growth and regeneration rate triggers persistent cycles in the economy.

In the fifth chapter, we employ a continuous delay structure in the process of recruitment in the population growth in an optimal growth model and hence obtain cyclic solutions. We exploit Erlangian process in the population growth mechanism. As far as we know, the incorporation of Erlangian process in optimal growth models is handled in this chapter for the first time in economic literature. Through this mechanism, not only the population is considered as a function of per capita capital, or in other words, population growth is endogenized, but also the current level of population growth is linked with those of older generations. We find out that the interaction between the effect of older generations' fertility choices and the accumulation of capital induces cyclic behaviour in the economy.

The sixth and the last chapter concludes with future research agenda.

Overall, the thesis considers the effects of delay and endogenized population on the economies of interest (Solow, overlapping generations, optimal growth model) economically and tries to introduce the existing methods and develop new ones to investigate the effects of delay and endogenized population on the eigenvalues of the Jacobians that the drive the economies of interest at their steady states.

CHAPTER 2

OPTIMAL GROWTH MODELS WITH DELAY: PRELIMINARY RESULTS

The question of the effects of delay in economic models is not exhaustively studied in economic theory. However the history of such analysis can be roughly separated into two phases which is determined by the current state of the economic theory and the elaboration of mathematical tools at hand. In one of his most influential articles, Kalecki introduces lag structure in the economy to explore the cyclic behaviour, which he shows rigorously for the first time that business cycles depend endogenously to production (investment) lags (Kalecki, 1935).

Before discussing in detail the Kaleckian setup and other models, we should track the improvement of mathematical apparatus. Apparently, after a seminar by Kalecki at a meeting in the Econometric Society at Leyden, Frisch and Holme (1935) is the first to analyze the roots of difference-differential equations of the form $\dot{y}(t) = ay(t) - cy(t-\theta)$ and characterizes the main properties with respect to the roots according to the exogenous (empirical econometric) parameters a and c. It is James and Belz (1938) who contributes to the mathematics of the characterization of the problem further. James and Belz (1938) suggests that "a solution of a difference-differential equation might be developed in terms of an infinite series of characteristic solutions" and investigates "the conditions under which such a development is possible." In addition, this paper gives methods "for determining the coefficients of the development, when it exists" and shows that the solutions of certain forms of integro-differential equations "can be given in the form of an infinite series derived from a consideration of related difference-differential equations." Hayes (1950) partially closes the literature on roots by giving the properties of the roots of transcendental equations of the form $\tau(s) = se^s - a_1 e^s - a_2 = 0$ which is nothing but the resultant characteristic equation of a subset of difference-differential equations with constant coefficients, which frequently occurs in dynamic economic systems with delays. As Zak (1999) points out, the first thorough analysis of a general class of delay differential equations is by Bellman and Cooke (1963) with later fundamental work by Hale (1977).

Kalecki (1935)[18] introduces production lags, a time delay between the investment decisions and delivery of the capital goods, to show the generation of endogenous cycles. He employs a linear delay differential equation of the deviation of investment which is denoted by J[19]. The investment equation[20] is $\dot{J}(t) = AJ(t) - BJ(t-\theta)$.

[18] A brief exposition of the Kalecki (1935) model and its properties can be found in Zak (1999) and Szydłowski (2002). These texts reproduces Kalecki's results with contemporary techniques which are also employed in this thesis.

[19] Michal Kalecki studied the underlying forces of cycles in economy throughout his life and his bunch of theories vary from linear difference differential equation systems to exogenous factors. As Besomi (2006), in his study about Kalecki's business cycle theories, pointed out Kalecki "either failed to provide a rigorous proof of the stability of the cycle when the model was endogenous or failed to provide an explanation of the cycle relying on the properties of the economic system, resorting instead to exogenous shocks to explain the persistence of fluctuations." Kalecki interpreted cycles as the dynamic expression of the "intrinsic antagonism of capitalism" however he "acknowledged the existence of disturbing factors, from which he abstracted in order to isolate a *pure* cycle." Besomi (2006) also reports that "Kalecki's models describes damped fluctuations around a line of stationary equilibrium and rely for the persistence o fluctuations on exogenous shocks" and moreover, all his models "crucially depend for cyclicality upon one or more reaction lags."

[20] The exact linear delay differential equation studied by M. Kalecki (1935, p. 332) is $\dot{J}(t) = \frac{m}{\theta}J(t) - \frac{m+n\theta}{\theta}J(t-\theta)$ where m and n are assumed to be constants.

Kaleckian models exhibit endogenous cycles by employing simple time lags in a linear delay differential equation. Lags in the model serve two purposes: (1) Lag structure is empirically significant[21] and (2) first order linear ordinary differential equations are known to be unable to give cyclic solutions while linear delay differential equations may exhibit endogenous cycles. Apart from showing that there can exist endogenously driven cycles in the economy rather than crises determined by exogenous shocks, Kalecki develops the mathematical techniques to characterize the stability properties in linear delay differential equations. Obviously, one had to wait for Hayes (1950) for a full understanding of the stability properties in one delay linear differential equations, although Kalecki (1935) presented a thorough stability analysis (Zak, 1999).

Kaldor (1940) criticizes Kalecki (1935) by pointing out that the drawback of the model is that "the existence of an undamped cycle can be shown only as a result of a happy coincidence, of a particular constellation of the various time-lags and parameters assumed" and "the amplitude of the cycle depends on the size of the initial shock." Instead, Kaldor (1940) proposes a nonlinear investment decision to obtain cycles of the economy. Inspired by Kaldor (1940), Ichimura (1954) explores the possibility of an economic system with a unique limit cycle; Chang and Smyth (1971) reexamine the model and state the necessary and sufficient conditions of an existence of a limit cycle; Grasman and Wentzel (1994) considers the coexistence of a limit cycle and an equilibrium. The dynamics of Kaldor-Kalecki type of models have been extensively studied on a series of papers by Krawiec and Szydłowski (1999, 2000, 2001, 2005) and Krawiec, Szydłowski and Toboła (1999). Kaldor-Kalecki models have two mechanisms which would lead to cyclic behaviour, one being the nonlinearity of the investment function and the other being the time delay in investment (Krawiec

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and Szydłowski, 2001). Krawiec and Szydłowski (1999, 2001) prove that it is the time to build assumption rather that the nonlinear (s-shaped) investment function that leads to the generation of cycles.

The main tool in these papers for detecting cycles is Hopf bifurcation. "In 1942, Hopf published the ground-breaking work in which he presented the conditions necessary for the appearance of periodic solutions, represented in phase space by a limit cycle" (Szydłowski, 2002). With reference to the contributors of the study of the sufficient conditions under which periodic orbits occur from stationary states, these theorems are called Poincaré–Andronov–Hopf theorems[22]. As Kind (1999) points out, it is generally easy to prove Hopf bifurcation since it doesn't require any information on the nonlinear parts of the equation system. Moreover, in systems with the dimension higher than two, Hopf bifurcation may be the only tool for the analysis of the cyclical equilibria, since the Poincaré-Bendixson theorem is not applicable. Furthermore, when the conditions of Hopf bifurcation are satisfied, it guarantees both the existence and the uniqueness of periodic trajectories (Krawiec and Szydłowski, 1999). However, Hopf theorem gives no information on the number and the stability of closed orbits. On the other hand, nonlinear parts can be used in the calculation of a stability coefficient in order to determine the stability properties of the closed orbits (Kind, 1999). Guckenheimer and Holmes (1983, Thm 3.4.2, pp. 151-153) both gives the theory and an example in that direction.

According to the Hopf theorem, one can summarize the sufficient conditions for Hopf Bifurcation as follows:

- (H1) $A(\mu)$, namely, the Jacobian of the nonlinear system, has *only* one pair of pure imaginary eigenvalues. (*Pre-Hopf Condition*)

[22] Poincaré–Andronov–Hopf Theorem is given in Theorem (1.1).

- (H2) Pure imaginary eigenvalues cross the imaginary axis with nonzero speed, i.e., $\frac{d\alpha}{d\mu}(0) \neq 0$. (*Transverse Crossing*)

In other words, the roots (eigenvalues) of the Jacobian should loose stability at the critical level of parameter μ which is called the *Hopf parameter*.

Zak (1999) summarizes Kalecki's contribution and extends his results to a general equilibrium setup, which has been an open research area until then[23]. Zak (1999) inserts a production lag into a basic one sector Solowian model and shows that the results also admits Hopf cycles under certain conditions. Later, Krawiec and Szydłowski (2002, 2003, 2004) reproduce the results and improved the analysis of the same model.

Zak (1999) merges the economic contributions of Kalecki (1935) and Solow (1956) together with that of the mathematical contributions of Hayes (1950) and Hopf (1942). Zak (1999) presents a Solow-Kalecki model in which capital accumulates according to the rule,

$$\dot{k}(t) = sf(k(t-\tau)) - \delta k(t-\tau), \tag{1}$$

so that at time t, the productive capital is $k(t-\tau)$. Now, it is easy to show that delay differential equation in (1) exhibits Hopf cycles around its steady state. Zak (1999) states that "Hopf cycles are precisely the cycles that Kalecki found for his model, although his demonstrations of cycles predate Hopf's work and thus were not so called."

For the presentational purposes, we hereby present the "know-how" of Zak (1999).

[23] Zak (1999, p. 325ff) also claims that Kaleckian cycle in Kalecki (1935) is nothing but an Hopf cycle.

The characteristic equation for the capital accumulation equation in (1) is

$$h(\lambda) \equiv \lambda - Be^{-\tau\lambda} = 0,$$

where $B = sf'(k^*) - \delta$. from linearization and the steady state condition. Let $\lambda = \mu + i\omega$ be the roots to the characteristic equation, then by Euler equation ($e^{i\omega} = \cos\omega + i\sin\omega$), we have

$$\mu - Be^{-\mu\tau}\cos\tau\omega = 0, \tag{2}$$

$$\omega + Be^{-\mu\tau}\sin\tau\omega = 0. \tag{3}$$

Equations (2) and (3) completely characterizes the root distribution of the delay differential equation in (1). Zak (1999) shows that there is a parameter combination that leads to a pure imaginary couple of complex eigenvalues which satisfies the transverse-crossing condition.

To sum up, we may state that the main aim is to model the economic dynamics in **reduced form, without external shocks** so that the model can be used to explain business **cycles**. In that line, Kalecki (1935) shows rigorously that lags produce cycles endogenously and Zak (1999) extends the idea to Solowian economies. Later, Krawiec and Szydłowski (2002, 2003, 2004) further analyze the dynamics and other aspects in a series of papers. This is summarized in the Table (1).

$$\underbrace{\left\{\begin{array}{c}\text{Kalecki (1935)}\\ \Downarrow \\ \text{Hayes (1950)}\end{array}\right\} + \text{Hopf (1942)} + \text{Solow (1956)}}_{\text{Zak (1999)}}$$
$$+$$
$$\text{Krawiec and Szydłowski (2002, 2003, 2004)}$$

Table 1: Solow-Kalecki models

2.1 Delay in Optimal Growth Models

Zak (1999) also copies the results of an important contribution to the literature which marks an important "false" attempt to extend the same analysis to the optimal growth models with lags. The Solow-Kalecki idea has been revived and extended to the Ramsey type optimal growth model by Asea and Zak (1999). In their paper, Asea and Zak (1999) tries to determine the steady state characteristics of the following model:

$$
\begin{aligned}
&\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-rt} u(c(t))dt \\
&\text{subject to} \\
&\qquad \dot{k}(t) = f(k(t-\tau)) + \delta k(t-\tau) - c(t), \\
&\qquad k(t) = \phi(t), t \in [-\tau, 0],
\end{aligned}
\tag{4}
$$

where $r, \tau > 0, \delta \geq 0$ are discount rate, time delay and depreciation, respectively. According to Asea and Zak (1999), the first order conditions are as follows:

$$
\begin{aligned}
\dot{c}(t) &= \frac{u'(c)}{u''(c)} \left[r + \delta - f'(k(t-\tau)) \right], \\
\dot{k}(t) &= f(k(t-\tau)) + \delta k(t-\tau) - c(t),
\end{aligned}
$$

with the characteristic equation,

$$
h(\lambda) \equiv \lambda^2 - \lambda B e^{-\tau\lambda} - C e^{-\tau\lambda} = 0. \tag{5}
$$

Although characteristic equation in (5) is harder to solve than the previous one, it is still solvable and Asea and Zak (1999) shows that the root distribution contains a pair of pure imaginary eigenvalues and that the model exhibits Hopf cycles.

Asea and Zak (1999) is the first to lay out the main tools and shows that there exists a cyclic behaviour in these type of models. However, this paper contains an error in the first order dynamic equations which erroneously lead to Hopf cycles. The corrected first order conditions are

$$\dot{c}\,(t) \;=\; r\frac{u'(c(t))}{u''(c(t))}+\frac{u'(c(t+\tau))}{u''(c(t))}\left[f'(k(t))-\delta\right], \tag{6}$$

$$\dot{k}\,(t) \;=\; f\left(k(t-\tau)\right)+\delta k(t-\tau)-c(t), \tag{7}$$

with the characteristic equation

$$h(\lambda)\equiv\left(r-re^{\lambda\tau}-\lambda\right)\left(re^{r\tau}e^{-\lambda\tau}-\lambda\right)-\frac{u_c}{u_{cc}}e^{-r\tau}f_{kk}=0. \tag{8}$$

Note that the first order conditions constitute a system of delay and advance type of differential equations. As Collard *et al.* (2008) aptly states "unfortunately, as soon as the dynamics of these models are characterized by a **forward looking component**, the lack of numerical methods to solve these problems makes the **quantitative** evaluation of their transitional dynamics difficult."

From this point on, the literature develops on three distinct lines of research. The complexity of the characteristic equation prevents to produce analytical results and thus, some researchers incline towards numerical simulations. Winkler *et al.* (2004), Collard *et al.* (2008) and Brandt-Pollmann *et al.* (2008) are those who try numerical simulations to comprehend the dynamic behaviour of optimal growth models with delay. The main findings are summarized by Winkler *et al.* (2004) who states that "both the frequency and the amplitude of the cycles depend on the length of the investment period," and by Collard *et al.* (2008) who states that "for a large delay the economy converges to the steady state by **oscillations**, but consumption **smoothing** mitigates the induced echo effects through an advanced Euler-type differential." Furthermore, Collard *et al.* (2006) numerically shows that the advanced term in Euler equation governing the dynamic system dampens the fluctuation caused by the lags through a kind of smoothing effect (They call this phenomenon 'time-to-build echo'). Short run dynamics of time-to-build, i.e. echoes, are further studied by Collard *et al.* (2008) in which one can find the associated numerical simulations. Winkler *et al.* (2004) provides numerical solutions of models of time delay optimal growth models for a linear limitational production function, while Winkler *et al.* (2005) gives a numerical analysis of a time-lagged capital accumulation optimal growth model with Leontief type of production functions. Brandt-Pollmann *et al.* (2008) extends the numerical solutions to objective functions with state externalities.

Note that the deficiency of numerical simulations when it comes to Hopf bifurcation is that Hopf bifurcation depends on the precise calibration of the Hopf parameter and without such calibration it may be impossible to hit the limit cycle solution simply by the randomization of parameters. Moreover, the quasi-polynomial associated with the characteristic equation naturally contains infinitely many complex roots which would result in cyclic behaviours. Considering the conditions which exclude completely unstable solutions, like that of transversality condition, it is natural that a random choice of parameters would result in decaying cycles that is, for the most part, in accordance with the results and interpretation of Collard *et al.* (2008).

Another line is AK simplification. Assuming that the production schedule follows an AK production technology simplifies the $\dot{c}$ equation in the first order conditions. The resulting first order conditions are,

$$
\begin{aligned}
\dot{c}\,(t) &= r\frac{u'(c(t))}{u''(c(t))} + \frac{u'(c(t+\tau))}{u''(c(t))}\left[A-\delta\right], \\
\dot{k}\,(t) &= f\left(k(t-\tau)\right) + \delta k(t-\tau) - c(t).
\end{aligned}
$$

with the characteristic equation of

$$h(\lambda) \equiv \left(r - re^{\lambda\tau} - \lambda\right)\left(re^{r\tau}e^{-\lambda\tau} - \lambda\right) = 0.$$

Note that the resulting characteristic equation is easier to handle. Bambi (2008) exploits the simplified characteristic equation and finds Hopf cycles and Winkler (2008) solves $\dot{c}$ equation first and then using the solution solves $\dot{k}$ equation (See Barro and Sala-i Martin, 1995, Ch. 4.1).

Although the 'AK simplification' approach enables some analytical results, the main question of whether there exist limit cycles under concave production remains unanswered. Though there is no clear justification, the third approach is to show the non-existence of such persistent cycles. Benhabib and Rustichini (1991), Caulkins *et al.* (2010) and Hartl and Kort (2010) represent the school of 'lack-in-faith in cycles'. Caulkins *et al.* (2010) states that "here we in some sense **defend** the traditional emphasis on models without delays by showing that an important class of models with delays can be transformed into equivalent optimal control problems without delays," and "the existence of an equivalent problem without delays implies that the optimal solution to the model with delays **cannot involve oscillation**." Thus, Caulkins *et al.* (2010) argues for the "non-oscillatory behaviour under exponential depreciation."

The efforts are summarized in the Table (2).

In this chapter, we try to formulate a new method to further comprehend the root distribution of the characteristic equation of an optimal growth model with concave production function and delayed investment structure.

$$\underbrace{\left\{\begin{array}{c}\text{Kalecki (1935)}\\ \Downarrow \\ \text{Hayes (1950)}\end{array}\right\} \; + \; \text{Hopf (1942)} \; + \; \text{Ramsey (1927)}}_{\text{Asea and Zak (1999)}}$$

Num. Simulation	*AK* Prod'n	Concave Prod'n
Cycles Die Out	Limit Cycles	No Limit Cycles
Winkler *et al.* (2004),	Bambi (2008),	Caulkins *et al.* (2010),
Collard *et al.* (2008),	Winkler (2008),	Hartl *et al.* (2010),
etc.	etc.	etc.

Table 2: Ramsey-Kalecki models

2.2 Roots of the Characteristic Equation: Some Preliminary Results

We have already stated the first order conditions of Ramsey-Kalecki model (4) in equations (6) and (7) with the associated characteristic equation:

$$h(\lambda;\Lambda) \equiv \left(r - re^{\lambda\tau} - \lambda\right)\left(re^{r\tau}e^{-\lambda\tau} - \lambda\right) - \Lambda = 0, \tag{9}$$

where $\Lambda = \frac{u_c}{u_{cc}} e^{-r\tau} f_{kk}\Big|_{c_{ss},k_{ss}} \in \mathbb{R}$. Also note that the steady state conditions are

$$r = e^{-r\tau}(f_k - \delta), \quad \text{and} \quad c = f(k) - \delta k.$$

$h(\lambda;\Lambda)$ is simply a quasi-polynomial, i.e., it can be generalized as

$$H(\lambda) := A(\lambda) + B(\lambda)e^{\lambda\tau} + C(\lambda)e^{-\lambda\tau},$$

where $A(.), B(.)$ and $C(.)$ are real coefficient polynomials of various degrees, themselves.

Lemma 2.1 *Let*

$$H(\lambda) := A(\lambda) + B(\lambda)e^{\lambda\tau} + C(\lambda)e^{-\lambda\tau},$$

be such that $A(.), B(.)$ *and* $C(.)$ *are polynomials various degrees. Then the roots come in complex conjugate pairs.*

Proof. *If*

$$H(\lambda) := A(\lambda) + B(\lambda)e^{\lambda\tau} + C(\lambda)e^{-\lambda\tau} = 0,$$

then we have

$$[H(\lambda)]^* = H(\lambda^*) = A(\lambda^*) + B(\lambda^*)e^{\lambda^*\tau} + C(\lambda^*)e^{-\lambda^*\tau} = 0.$$

■

By Lemma (2.1), we may assert that the root distribution of $h(\lambda; \Lambda) = 0$ in (9) is in complex conjugate pairs.

The solution strategy in Zak (1999) depends on the identification of the roots of a real coefficient quasi polynomial of the form

$$h(\lambda) \equiv \lambda - Be^{-\tau\lambda} = 0.$$

Note that in equation (9), $\Lambda \neq 0$ prevents us to use this solution strategy, since otherwise, we may identify the roots of equation (9) by simply identifying the roots of $r - re^{\lambda\tau} - \lambda = 0$ and $re^{r\tau}e^{-\lambda\tau} - \lambda = 0$ with the already well known techniques. This is only possible when the production technology is AK, since then, $\Lambda = 0$. This is already applied by Bambi (2008).

In this chapter, we propose a new method which increases the complexity of the system by two by adding another (complex) unknown, say $m = \alpha + i\beta \in \mathbb{C}$, but, this reduces the system to two "lower dimensional" complex coefficient quasi-polynomials. Define

$$h'(\lambda; m) \equiv \left(r - re^{\lambda\tau} - \lambda - m\right)\left(re^{r\tau}e^{-\lambda\tau} - \lambda + m\right) = 0$$

such that

$$h(\lambda) = h'(\lambda; m).$$

In other words, suppose there exists a complex number $m = \alpha + i\beta \in \mathbb{C}$ such that

$$m(r - re^{\tau\lambda} - re^{r\tau}e^{-\tau\lambda}) - m^2 = \Lambda.$$

Definition 2.1 *Let $H(\mathbf{x})$ be a vector of quasi-polynomials with unknowns $\mathbf{x} \in \mathbb{C}^n$.*

$$\mathcal{S}[H(\mathbf{x})] = \{\mathbf{x} \in \mathbb{C}^n : H(\mathbf{x}) = 0\}.$$

Define the solution set of the characteristic equation $h(\lambda) = 0$ as

$$\mathcal{S}[h(\lambda; \Lambda)] = \{\lambda \in \mathbb{C} : h(\lambda; \Lambda) = 0\},$$

and the solution set of the characteristic equation $h(\lambda; m) = 0$ as

$$\mathcal{S}[h'(\lambda; m)] = \{(\lambda, m) \in \mathbb{C}^2 : h'(\lambda; m) = 0\}.$$

Note that Lemma (2.1) implies that if $\mathbf{x} \in \mathcal{S}[H(\mathbf{x})]$, then its complex conjugate is also a solution, that is, $\mathbf{x}^* \in \mathcal{S}[H(\mathbf{x})]$.

Theorem 2.2 *If λ is a solution to $h(\lambda; \Lambda) = 0$, i.e., $\lambda \in \mathcal{S}[h(\lambda; \Lambda)]$, then for $m = \alpha + i\beta \in \mathbb{C}$ such that*

$$m(r - re^{\tau\lambda} - re^{r\tau}e^{-\tau\lambda}) - m^2 = \Lambda,$$

we have $(\lambda, m)\in\mathcal{S}[h'(\lambda; m)]$. *Moreover, if* $(\lambda, m)\in\mathcal{S}[h'(\lambda; m)]$ *such that*

$$m(r - re^{\tau\lambda} - re^{r\tau}e^{-\tau\lambda}) - m^2 = \Lambda,$$

then $\lambda\in\mathcal{S}[h(\lambda; \Lambda)]$.

Theorem (2.2) is important in the sense that now the whole root distribution of $h(\lambda; \Lambda) = 0$ can be characterized as the solutions to $h'(\lambda; m) = 0$ given that m satisfies $m(r - re^{\tau\lambda} - re^{r\tau}e^{-\tau\lambda}) - m^2 = \Lambda$. In other words, the roots that govern the dynamics of the economy can be obtained as the solutions $(\lambda; m)$ to the following system of quasi-polynomials

$$h_1(\lambda, m) \ : \ = \left(r - re^{\lambda\tau} - \lambda - m\right)\left(re^{(r-\lambda)\tau} + (r - \lambda) + m\right) = 0, \tag{10}$$

$$h_2(\lambda, m) \ : \ = m(r - re^{\tau\lambda} - re^{(r-\lambda)\tau}) - m^2 - \Lambda = 0. \tag{11}$$

$h_1(\lambda, m)$ is simply $h'(\lambda, m)$ in which the terms are reorganized so that the symmetry appears as daylight. Now define

$$g(\lambda, m) = \left(r - re^{\lambda\tau} - \lambda - m\right).$$

Then, $h_1(\lambda, m)$ can be restated as

$$h_1(\lambda, m) = g(\lambda, m)g(r - \lambda, m).$$

Lemma 2.2 $\mathcal{S}[h'(\lambda; m)] = \mathcal{S}[h'(r - \lambda; m)]$

Proof. *Note that*

$$
\begin{aligned}
h_1(\lambda, m) &= h_1(r-\lambda, m), \\
h_2(\lambda, m) &= h_2(r-\lambda, m).
\end{aligned}
$$

The rest follows. ∎

Thus, the eigenvalues are symmetrical with respect to $\operatorname{Re}\lambda = \frac{r}{2} > 0$.

Definition 2.2 *Denote the solutions* (λ, m) *of* $g(\lambda; m) = 0$ *and* $h_2(\lambda, m) = 0$ *by*

$$\mathcal{S}_1[g(\lambda; m), h_2(\lambda, m)];$$

the solutions (λ, m) *of* $g(r-\lambda; m) = 0$ *and* $h_2(r-\lambda, m) = 0$ *by*

$$\mathcal{S}_2[g(r-\lambda; m), h_2\,(r-\lambda, m)],$$

and the solutions (λ, m) *of* $g(\lambda; m) = 0,\ g(r-\lambda; m) = 0,$ *and* $h_2(\lambda, m) = 0$ *by*

$$\mathcal{S}_{12}[g(\lambda; m), g(r-\lambda; m), h_2(\lambda, m)].$$

Theorem 2.3 *We have*

1. *For* $\lambda = \frac{r}{2} \in \mathbb{R}$ *and* $m = \frac{r}{2}(1 - 2e^{\frac{r\tau}{2}}) \in \mathbb{R}$, $(\lambda, m) \in \mathcal{S}_{12}$ *if* $m^2 = \Lambda$*; otherwise,* $\mathcal{S}_{12} = \varnothing$,
2. $\mathcal{S}_{12} \subset \mathcal{S}_1$ *and* $\mathcal{S}_{12} \subset \mathcal{S}_2$,
3. $\mathcal{S}_1 \subset \mathcal{S}[h'(\lambda; m)]$ *and* $\mathcal{S}_2 \subset \mathcal{S}[h'(\lambda; m)]$,

4. *$\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric with respect to* $\operatorname{Re}\lambda = \frac{r}{2}$*, that is,* $(\lambda, m) \in \mathcal{S}_1$ *if and only if* $(r - \lambda, m) \in \mathcal{S}_2$*,*

5. $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathcal{S}[h'(\lambda; m)]$.

Proof. *For the first part, note that for the solutions* $(\lambda, m) = (\eta + i\omega, \alpha + i\beta) \in \mathcal{S}_{12}$ *are characterized by*

$$\begin{aligned}
\cos\omega\tau &= \frac{r - \mu - \alpha}{re^{\mu\tau}}, \\
\sin\omega\tau &= \frac{-(\omega + \beta)}{re^{\mu\tau}}, \\
\cos\omega\tau &= \frac{\mu - \alpha}{re^{(r-\mu)\tau}}, \\
\sin\omega\tau &= \frac{\omega - \beta}{re^{(r-\mu)\tau}}, \\
m(r - re^{\tau\lambda} - re^{r\tau}e^{-\tau\lambda}) - m^2 &= \Lambda.
\end{aligned}$$

The rest follows. Parts (2)-(5) are obvious. ■

Parts (4) and (5) of Theorem (2.3) implies that to find the complete root distribution $\mathcal{S}[h'(\lambda; m)]$, it is sufficient to find the roots of $\mathcal{S}_1$ (or $\mathcal{S}_2$). Part (5) also implies that there are infinite number of stable and unstable roots due to symmetry around $\operatorname{Re}\lambda = \frac{r}{2}$ which implies that this economy is saddle path stable. This result is in accordance with Winkler *et al.* (2004): "[Characteristic equation] has [...] an infinite number of complex solutions with negative real parts and an infinite number of solutions with positive real parts."

Although we are short on completing the analysis towards identifying the root distribution, we believe that it is now possible to exploit existing tools to find the roots of $g(\lambda; m) = 0$, and the sensitivity of the distribution of the roots with respect to m. If such sensitivity is low enough, we can securely be sure to at least characterize the shape of the root distribution.

CHAPTER 3

CAPITAL DEPENDENT POPULATION GROWTH INDUCES CYCLES[24]

Kaleckian investment lag is historically important since Kalecki laid a mathematical foundation of the economic cycles as early as mid thirties. The main mathematical apparatus (namely the Hayes' Theorem) which analyzes the characteristic roots of quasi-polynomials emerged at fifties. Hayes (1950) gives a complete stability characterization for the first order linear delay differential equations. However, as Zak (1999) points out, the first thorough analysis of a general class delay differential equations is by Bellman and Cooke (1963) with later fundamental work by Hale (1977).

Kalecki (1935) introduces production lags, a time delay between the investment decisions and delivery of the capital goods, to show the generation of endogenous cycles. Kalecki (1935) employes a linear delay differential equation of the deviation of investment which is denoted by J. The investment equation is $\dot{J}(t) = AJ(t) - BJ(t-\theta)$. Model of Kalecki (1935) exhibits endogenous cycles by employing simple time lags in a linear delay differential equation. The dynamics of Kaldor-Kalecki type models

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is extensively studied on a series of papers by Krawiec and Szydłowski (1999, 2000, 2001, 2005) and Krawiec, *et al.* (2001). Kaldor-Kalecki models has two mechanisms which would lead to cyclic behaviour, one being the nonlinearity of the investment function and the other being the time delay in investment (Krawiec and Szydłowski, 2001).

Periodic solutions to dynamic systems are also analyzed extensively in control theory. One way to detect limit cycles is Hopf bifurcation. Hopf bifurcation discards tedious calculations and provides a powerful and easy tool to detect limit cycles. Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter and meanwhile a cycle either emerges from or collapses in to the fixed point (Asea and Zak, 1999). Under these circumstances the system can either have a stable fixed point surrounded by an unstable cycle (called a *subcritical* Hopf bifurcation); or a stable cycle loses its stability and a stable cycle appears (called a *supercritical* Hopf bifurcation) as the parameter(s) approaches to a critical value (Asea and Zak, 1999). Both cases can be economically significantly meaningful. Supercritical case which implies a stable cycle can be considered as a stylized business cycle or growth cycles and the subcritical case can correspond to the corridor stability (Kind, 1999). Hopf Bifurcation dominates the literature when the problem reduces to detect cycles in dynamic models. The analysis further boils down to finding a pair of pure imaginary roots, since the nonzero speed condition is not actually necessary for having Hopf bifurcation[25] (see Farkas, 1994, p. 418; Manfredi and Fanti, 2004). Zak (1999) and Szydłowski and Krawiec (2004) applied the improvements of Hopf theorem to the Solow-Kalecki type of growth models.

[25] To be more specific, let us quote Farkas (1994): "[The nonzero speed condition] is expressed by saying that the pair of complex conjugate eigenvalues crosses the imaginary axis with non-zero speed. This is also a generic requirement, though it is not absolutely necessary: the existence part of the Theorem remains valid even in the degenerate case when this derivative is zero [etc.]"

According to the model presented by Zak (1999), the capital becomes productive after a time period r. That is, the productive capital at time t is $k(t-r)$. Moreover, capital also depreciates through production. Therefore, the evolution of capital is governed by the following delay differential equation:

$$\dot{k}(t) = f(k(t-r)) - \delta k(t-r). \tag{12}$$

However, Brandt-Pollman *et al.* (2008) classifies the lag structure given in equation (12) as a *delivery lag*[26] rather than a *time-to-build lag*[27]. Yet, we will employ *time-to-build lag* structure, which is of the form

$$\dot{k}(t) = f(k(t-r)) - \delta k(t). \tag{13}$$

We show that the capital evolution with the lag structure in equation (13) will not yield Hopf cycles if the production function is of Cobb-Douglas type.

The population growth in Zak (1999) is assumed to be zero. However, the results will mostly remain if constant population growth is used. Cigno (1981) introduced a capital dependent (variable) population growth. The said population growth equation tries to link the growth of population with per capita consumption and degree of industrialization, where the relation is positive for the former, but negative for the later. That is, the dynamics reflect the positive effect of higher per capita consumption and the negative effect of higher degree of industrialization. Denoting the per capita consumption with $(1-s)Q/L$, the dynamics in the paper is given to be $n(t) = \{(1-s)(Q/L)\}^{v_1}(K/L)^{v_2}$, where $v_1, v_2 > 0$. Cigno (1981) found out the stability characterization of endogenous population growth in an exhaustible resource

[26] *Delivery lag* is such that investment for new capital goods is made at time t but the new capital goods need some time r to be delivered and, thus, to be productive (Brandt-Pollman *et al.*, 2008).

[27] *Time-to-build lag* is such that capital goods need some time r over which they require investments in order to be produced (Brandt-Pollman *et al.*, 2008).

framework. Cigno (1981) concludes that, for certain parameter settings the steady state is stable.

We show that constant population growth is not sufficient to obtain cyclical behaviour in certain type of capital accumulations, given that the production is Cobb-Douglas. However, a capital-dependent population growth rule leads to Hopf cycles.

This chapter is organized as follows. In Section 2 we show that Cobb-Douglas production function and constant population growth model doesn't contain Hopf cycles. We have introduced the theorem from Louisell (2001), which gives an easier method to detect pure imaginary roots. In section 3, we extend the model so that the population growth is now capital dependent. Employing similar techniques, we have found out that the latter model gives Hopf cycles. Section 3 is the conclusion.

3.1 Constant Population Growth

Finding pure imaginary roots has been widely discussed in the literature. The following theorem from Louisell (2001) constitutes a shortcut to detect the pure imaginary roots of certain type of difference-differential systems.

Let A_0, $A_1 \in \mathbb{R}^{n\times n}$, $h > 0$. Consider the following difference-differential equation

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h), \tag{14}$$

which has a characteristic function of

$$T(\lambda) = \lambda I - A_0 - A_1 e^{-\lambda h}, \tag{15}$$

Theorem 3.1 *(Louisell, 2001) Let A_0, $A_1 \in \mathbb{R}^{n\times n}$, $h > 0$ and let*

$$J = \begin{pmatrix} A_0 \otimes I & A_1 \otimes I \\ -I \otimes A_1 & -I \otimes A_0 \end{pmatrix}, \tag{16}$$

where $\otimes$ denotes the Kronecker product[28]*. Then, all imaginary axis eigenvalues of the delay equation (14) are the eigenvalues of J.*

Assume that we are faced with an economy endowed with Cobb-Douglas production function and capital lag[29] which is given as follows:

$$\dot{k}(t) = sk^{\alpha}\left(t - d\right) - \left(n(t) + \delta\right) k(t), \tag{17}$$

where $\alpha \in (0, 1)$ is the constant capital's share in production, $d > 0$ is the constant capital lag, $\delta > 0$ is the constant depreciation of capital and $s > 0$ is the constant rate of savings. Denote $n(t) = \frac{\dot{L}(t)}{L(t)}$. Under the standard growth model with time lag, where the rate of population growth is assumed to be constant, i.e. $n(t) = n$ for all $t > 0$, we will show that this Solow-Kalecki growth model does not induce any Hopf cycles.

The steady state level of capital is

$$k_{ss} = (\frac{s}{n + \delta})^{\frac{1}{1-\alpha}}, \tag{18}$$

[28] Let $W \in \mathbb{R}^{m\times n}$ and $Y \in \mathbb{R}^{p\times q}$.Then $W \otimes Y \in \mathbb{R}^{pm\times qn}$ is as follows

$$W \otimes Y = \begin{pmatrix} w_{11}Y & ... & w_{1n}Y \\ . & . & . \\ w_{m1}Y & ... & w_{mn}Y \end{pmatrix},$$

[29] Capital lagged Cobb-Douglas type production function is assumed to be

$$Y(t) = K^{\alpha}(t - d)L^{1-\alpha}(t)$$

and the linearization of the dynamic system around its steady state will yield

$$\dot{z}(t) = (\alpha s k_{ss}^{\alpha-1}) z(t-d) - (n+\delta) z(t), \tag{19}$$

with the change of variable $z(t) = k(t) - k_{ss}$. The matrix which should be used to employ the result of the theorem from Louisell (2001) is as follows[30]:

$$J = \begin{pmatrix} A_0 & A_1 \\ -A_1 & -A_0 \end{pmatrix},$$

where $A_0 = -(n+\delta)$ and $A_1 = \alpha s k_{ss}^{\alpha-1} = \alpha(n+\delta)$. In this case, we have $\lambda_{1,2} = \pm(n+\delta)\sqrt{1-\alpha^2} \in \mathbb{R}$ as eigenvalues. Since, this matrix does not possess any pure imaginary eigenvalues, the linearized system which is characterized by equation (19) has no pure imaginary eigenvalues Therefore, Kaleckian growth models with Cobb-Douglas type of production functions and capital lag do not induce any Hopf bifurcation.

3.2 Capital Dependent Population Growth

Note that any variation in the population growth rate within some certain limits does not change the above result. Suppose that the population growth is not constant but exogenously time dependent. Moreover, suppose that the $n(t)$ is convergent for some n_{ss}, that is $n(t) \to n_{ss}$ as time goes to infinity. Then neither the steady state values, nor the linearized system dynamics which is given by equation (19) is effected. Thus, time-varying population growth is not sufficient for cyclic behaviour[31], since the only mechanism that would give this kind of behaviour is a Hopf cycle.

[30] Note that $A \otimes I = A$ if $I \in \mathbb{R}^{1x1}$ for any $A \in \mathbb{R}^{1x1}$.

[31] Time-varying population growth case is exploited for the insights it presents. Other than that, the author is fully informed that this kind of population growth functions are not employed in the literature.

On the other hand, the behaviour can drastically change if we use wealth-induced population dynamics, even if we stick to the Cobb-Douglas production function. Cigno (1981) proposes the following population growth

$$n(t) = (1-s)^{v_1} k(t)^{\alpha v_1 - v_2},$$

where v_1 and v_2 are positive constants. For the ease of calculations, assume zero depreciation, i.e. $\delta = 0$. Substituting this into the capital accumulation equation, we obtain

$$\dot{k}(t) = sk^{\alpha}(t-d) - (1-s)^{v_1} k(t)^{1+\alpha v_1 - v_2}, \tag{20}$$

Steady state equation will adjust accordingly:

$$k_{ss} = \left(\frac{s}{(1-s)^{v_1}}\right)^{\frac{1}{1-\alpha(1-v_1)-v_2}}, \tag{21}$$

whence the linearized system around the steady state will be governed by

$$\dot{z}(t) = (\alpha s k_{ss}^{\alpha-1}) z(t-d) - (1-s)^{v_1}(1+\alpha v_1 - v_2) k_{ss}^{\alpha v_1 - v_2} z(t), \tag{22}$$

with the change of variable $z(t) = k(t) - k_{ss}$. The corresponding matrix J in accordance with Louisell (2001) is

$$J = \begin{pmatrix} A_0 & A_1 \\ -A_1 & -A_0 \end{pmatrix},$$

where $A_0 = -(1-s)^{v_1}(1+\alpha v_1 - v_2) k_{ss}^{\alpha v_1 - v_2}$ and $A_1 = (\alpha s k_{ss}^{\alpha - 1})$. The two eigenvalues of J are

$$\lambda_{1,2} = \pm\sqrt{A_0^2 - A_1^2}, \tag{23}$$

Proposition 3.1 *If $-\alpha < 1 + \alpha v_1 - v_2 < \alpha$, then the system undergoes Hopf bifur-*

cation.

Proof. The eigenvalues are pure imaginary given that $A_0^2 - A_1^2 < 0$. This is the case if and only if $|1 + \alpha v_1 - v_2| < |\alpha| = \alpha$. ■

We know from D-Subdivision method that the Hopf boundary is obtained in either the first or second quadrant of the coefficient space[32]. The sign of the coefficient of $z(t)$, which is $-(1-s)^{v_1}(1 + \alpha v_1 - v_2)$, determines on which quadrant the coefficients lie. If $(1 + \alpha v_1 - v_2) > 0$, the coefficients are on the second quadrant and otherwise they are on the first. We should also note that the saddle-path stability is sacrificed for a limit cycle. That is, endogenous population growth eliminates the unstable manifold, however we obtained a limit cycle.

The Hopf cycles exist when the parameters are in a relationship within some limits. To understand this, the following restatement of the previous proposition may help:

Proposition 3.2 *If $v_1 < 1$, then the system undergoes Hopf bifurcation if $\frac{1-v_2}{1-v_1} < \alpha$ and $\frac{1-v_2}{1+v_1} > -\alpha$.*

Proposition 3.3 *If $v_1 > 1$, then the system undergoes Hopf bifurcation if $\frac{1-v_2}{1-v_1} > \alpha$ and $\frac{1-v_2}{1+v_1} > -\alpha$.*

Both propositions keep the parameters v_1 and v_2 close enough to ensure nonexplosive dynamics[33] where cyclic behaviour is not possible. In the both propositions,

[32] The coefficients can lie on the first or second quadrant of the parameter space (a, b), since $b > 0$ and these quadrants are those on where the Hopf boundary (the boundary where the system loses it stability) lies (See Bambi, 2008). The parameters (a, b) are the coefficients of the characteristic equation $h(z) = z + a + be^{-z\tau} = 0$.

[33] The positivity constraint of the parameters v_1 and v_2 maintains the economic intuition as in Cigno (1981), that the population growth rate is positively related to per capita consumption and inversely related to the degree of industrialization. We do not see these explicitly since we are employing per capita variables. Yet, Cigno (1981) also finds a similar result that underlines that these parameters should be close to each other to obtain stable growth.

the relative ratio of distance to one should not exceed α given a lower bound to v_2 for a given v_1. Whereas, the other inequality is an upper bound to v_2. To be more illustrative, we can substitute for a common value for the constant of the share of capital in production, α, is $\alpha = \frac{1}{3}$ and further analyze the parameter combinations that allows for Hopf cycles.

Proposition 3.4 *Let $\alpha = \frac{1}{3}$. If $-4+3v_2 < v_1 < -2+3v_2$, then the system undergoes Hopf bifurcation.*

Proof. Plug $\alpha = \frac{1}{3}$. The rest is straightforward. ∎

This relation between parameters v_1 and v_2 is visualized in Figure (1).

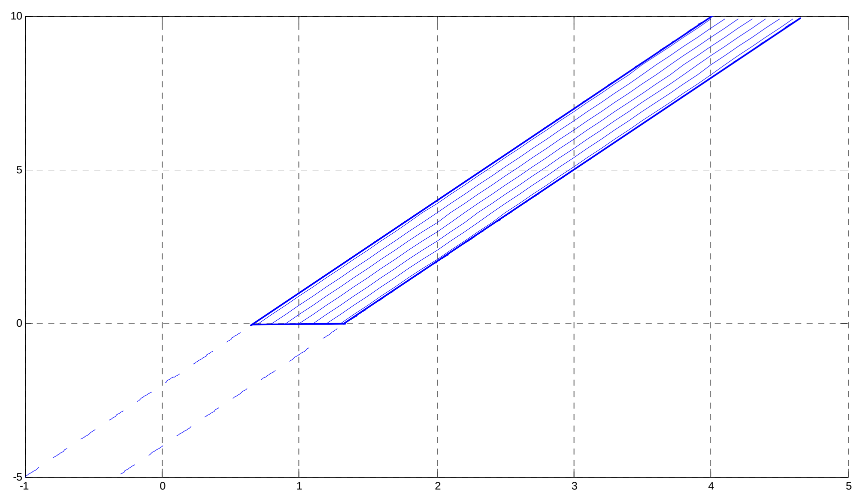

Figure 1: v_1 and v_2 combinations which allows for Hopf bifurcation when $\alpha = \frac{1}{3}$ (The horizontal axis is v_2 and the vertical axis is v_1).

The shaded region gives the v_1 and v_2's which induces Hopf cycles when $\alpha = \frac{1}{3}$, whereas the bold lines gives the boundaries of this region.

3.3 Conclusion

In this chapter, we have analyzed the effects of varying population growth in a Solow-Kalecki type of growth model. We show that Cobb-Douglas type production functions and time-delay are not sufficient for the economy to have cyclical behaviour. This is contrary to the common belief that delay is sufficient to obtain cyclic dynamics.

We extend the model so that population growth is endogenized. Then we show that capital dependent population dynamics can enforce Hopf bifurcation.

CHAPTER 4

HOPF BIFURCATION IN AN OVERLAPPING GENERATIONS RESOURCE ECONOMY WITH ENDOGENOUS POPULATION GROWTH RATE[34]

In response to a feedback mechanism between population growth and carrying capacity of the environment, scarce environmental resources have been advocated to put a constraint on population growth (Smith, 1974). Indeed, Smith (1974) postulates that population growth should possess the following properties for a more realistic growth model: "1) when population is small in proportion to environmental carrying capacity, then it grows at a positive constant rate, 2) when population is large in proportion to environmental carrying capacity, the resources become relatively more scarce and as result this must affect the population growth rate negatively" (see Accinelli and Brida, 2005; Brianzoni *et al.*, 2007, p. 2).

Motivated by Smith's idea, we analyze the dynamics of an overlapping generations economy that contemplates a feedback mechanism between population growth rate and resource availability. We adopt a Beverton Holt population growth function

[34] This essay is a joint work with Burcu Afyonoğlu Fazlıoğlu and Hüseyin Çağrı Sağlam.

(see Beverton and Holt, 1957) which is a discrete time version of the logistic population growth function (for the logistic population growth function see among others, Schtickzelle, 1981; Faria, 2004; Accinelli and Brida, 2005; for a discrete time Romer model, see Brianzoni *et al.*, 2007). However, we modify Beverton Holt population growth function in which the carrying capacity depends on the available resource stock in a convex-concave fashion. As carrying capacity of nature are nor fixed neither static (Arrow *et al.*, 1995), we consider that the carrying capacity increases with the available resource stock at an increasing rate at first and at a decreasing rate afterwards. This is simply more realistic because population growth rate responds to the changes in the available resource stock and population is bounded from above. Through this feedback mechanism between population and resource availability, we show that the introduction of endogenous population growth rate implies that Hopf bifurcation may emerge in an overlapping generations resource economy.

Nonlinear dynamics (such as multiplicity of the steady states or Hopf bifurcation) have been obtained in overlapping generations models with resources (see among others Koskela *et al.*, 2008; Antoci and Sodini, 2009). The dynamics in these studies mainly rest on the assumptions of logistic regeneration function or some assumptions on the intertemporal elasticity of substitution. Under logistic regeneration function of resources, it has been shown that further assumptions on the parameters of utility and production function bring dynamics such as local indeterminacy or bifurcations[35]. In particular, Koskela *et al.* (2008) examines whether renewable resource based overlapping generations economies may have different types of dynamics other than saddles and numerically show that flip bifurcation may arise if the intertemporal elasticity of substitution in utility is less than a certain critical value. Our setting allows us

[35] Under linear regeneration of renewable resources, the overwhelming majority of standart resource models in OLG framework (where population is constant or growing at a consant rate, see Farmer, 2000; Valente, 2008; Kemp and van Long, 1979) reveal that the dynamics converge to a single steady state or to a balanced growth path with saddle path stability (see Mourmouras, 1991).

to obtain Hopf bifurcation under the convex-concave dependence of carrying capacity on the resource availability without referring to logistic regeneration, shocks or constraints on parameter values. Thus, the novel feature of our study is to reveal Hopf bifurcation by incorporating endogenous population growth rate *à la* Smith (1974). In this regard, our study complements Koskela *et al.* (2008), Seegmuller and Verchère (2007) (overlapping generations economy with environment and endogenous labor supply) and Antoci and Sodini (2009) (an overlapping generations economy with negative environmental externalities) that provide additional channels for interesting dynamics in overlapping generations framework.

Hopf bifurcation is economically important as it provides a powerful and easy tool to detect limit cycles and justify the emergence of cycles endogenously (for further details, see Benhabib and Farmer, 1999; Kind, 1999). Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter and meanwhile a cycle either emerges from or collapses into the fixed point. The dynamic system can either have a stable fixed point surrounded by an unstable cycle; or a stable cycle loses its stability and a stable cycle appears as the parameter(s) approach(es) to a critical value (see Asea and Zak, 1999; Yüksel, 2011). Both cases can be economically significantly meaningful so that cycles or cyclical paths may turn out to be optimal via Hopf bifurcation (for further details see Kind, 1999).

The chapter is structured as follows. The model is introduced in the following section. The equilibrium dynamics and the local stability analysis are provided in Section 4.2. Section 4.3 concludes.

4.1 The Model

We consider a two period overlapping generations model with infinite horizon. We differ from the standard framework in two respects[36]. Firstly, we assume that the renewable resources are essential to production. Second, under the presence of limited resources, we allow for the growth rate of the population to depend on the per capita resource availability.

At each period t, a generation of agents appears and lives for two periods, young and old. The population in period t, consists of N_t young and N_{t-1} old individuals. We assume that the rate of population growth n_{t+1} is related with the total available resource stock e_t and the population growth rate n_t:

$$\begin{aligned} N_{t+1} &= (1+n_t)N_t, \text{ where} \\ n_{t+1} &= g(e_t, n_t)n_t. \end{aligned}$$

We consider a Beverton-Holt population growth rate function (see Beverton and Holt, 1957) in which the carrying capacity of the environment depends on the available per capita resource,e_t, stock in the following manner:

$$g(e_t, n_t) = \frac{rh(e_t)}{h(e_t) + (r-1)n_t},$$

where $h(e) : \mathbb{R}^+ \to \mathbb{R}^+$represents the carrying capacity of the environment and $r > 1$ denotes the inherent growth rate (this rate being determined by life cycle and demographic properties such as birth rates etc., see among others, Brianzoni *et al.*, 2007). We conjecture that for low values of the resource stock the carrying capacity of the environment increases with an increasing rate while for high levels of the resource

[36]For the standard framework, see de la Croix and Michel (2002).

stock at an increasing rate at first and at a decreasing rate afterwards with the following convex-concave function (Capasso *et al.*, 2012):

$$h(e) = \frac{\tau_1 e^{\rho}}{1 + \tau_2 e^{\rho}}, \tag{24}$$

with $\tau_1 > 0$, being *the population scale factor,* $\tau_2, \rho > 0$ being *the population curvature* parameters[37].

The resource can act as both stores of value and inputs to the production process. The economy is initially endowed with a positive amount of the natural resource E_0 which belongs to the first generation of old agents. We assume that at the beginning of each period t, the old agents (generation $t-1$) own the stock of the natural resource, E_t. Incurring no extraction costs (see Dasgupta and Heal, 1979), old agents decide on how much of this resource will be extracted for production X_t and how much would be sold to the young (generation t) as assets $A_t(= E_t - X_t)$. From period t to $t+1$, the assets bought by the young generation regenerates at a rate $\Pi \geqslant 1$[38]. Therefore, the law of motion of the resource stock writes as follows:

$$\begin{aligned} E_{t+1} &= \Pi A_t, \\ a_t &= e_t - x_t, \\ (1+n_t)e_{t+1} &= \Pi \left(e_t - x_t \right), \end{aligned}$$

[37]Note that $\tau_2 = \frac{\rho-1}{(\rho+1)e^{\rho}}$ determines the *inflection* point at which carrying capacity $h(e)$ function switches from convexity to concavity, or vice versa (this point is simply the one through which the second derivative changes sign). Also note that for the parameter combination (τ_2, ρ), e is on the convex portion of the function if it satisfies $\tau_2 < \frac{\rho-1}{(\rho+1)e^{\rho}}$; and e is on the concave portion of the function if it satisfies $\tau_2 > \frac{\rho-1}{(\rho+1)e^{\rho}}$. The second derivative of $h(e)$ with respect to e is

$$\frac{dh(e)}{de} = \frac{\tau_1 \rho e^{\rho-2}[\rho - 1 - \tau_2 e^{\rho}(\rho+1)]}{(1+\tau_2 e^{\rho})^3}.$$

[38]Note that if $\Pi = 1$, the resource turns out to be non-renewable.

where quantities of resource assets and extracted resources per young individual are denoted by, $e_t = \frac{E_t}{N_t}$, $a_t = \frac{A_t}{N_t}$ and $x_t = \frac{X_t}{N_t}$, respectively.

Each agent is endowed with one unit of labor when she is young and supplies it to firms inelastically. Young households receive wage w_t, which is allocated between consumption of the good produced by the representative firm and the purchase of the ownership rights for the natural resource. When old, they consume their entire income generated by selling their stock of natural resources X_{t+1} to the firms and their assets A_{t+1} to the young at prices P_{t+1} and Q_{t+1}, respectively. We assume that the life-time well-being of the representative individual is measured by the logarithmic function over young and old periods consumption, i.e., $U(c,d) = u(c) + \beta u(d)$, where $\beta \in (0,1)$ is the subjective discount factor. Accordingly, the representative agent born in period t, maximizes his utility with respect to the young and old periods' consumption, taking wages and the price of the natural resource as given.

$$\max_{\{c_t, d_{t+1}, a_t\}} \ln c_t + \beta \ln d_{t+1}$$

subject to

$$c_t + Q_t a_t = w_t, \tag{25}$$

$$d_{t+1} = P_{t+1}(1+n_t)x_{t+1} + Q_{t+1}(1+n_t)a_{t+1}, \tag{26}$$

$$(1+n_t)e_{t+1} = \Pi a_t, \tag{27}$$

$$a_t = e_t - x_t, \tag{28}$$

$$c_t \geq 0, d_{t+1} \geq 0,$$

$$e_{t+1} \geq 0, E_0 > 0, \text{ given.}$$

The first order conditions for an interior solution of this problem is as follows:

$$\frac{d_{t+1}}{\beta c_t} = \Pi \frac{Q_{t+1}}{Q_t}, \tag{29}$$

$$P_{t+1} = Q_{t+1}. \tag{30}$$

Equation (29) guarantees the equalization of the intertemporal marginal rate of substitution and the change in prices taking the regeneration factor into account. The latter is a no-arbitrage condition which is an equilibrium condition revealing from the maximization problem of the representative household.

Firms are owned by the old households and produce an homogenous consumption good under perfect competition. At each period, a single final good Y_t is produced in the economy by means of labor N_t and the natural resource X_t according to the following technology:

$$Y_t = {X_t}^{\alpha} N_t^{1-\alpha},\ 0 < \alpha < 1.$$

Under the perfectly competitive environment, the representative firm producing at period t maximizes its profit by choosing the amount of labor and the resource input that will be utilized in the production process. At an interior solution of the firm's optimization problem, where all variables are expressed in per capita terms $(y_t = \frac{Y_t}{N_t})$, profit maximization implies:

$$(1-\alpha) y_t = w_t, \tag{31}$$

$$\alpha y_t = P_t x_t. \tag{32}$$

Intertemporal equilibrium requires the clearing of the resource market, the clearing of the labor market and the clearing of the goods market for all t:

$$(1+n_t)e_{t+1} = \Pi\left(e_t - x_t\right), \tag{33}$$

$$y_t = c_t + d_t(1+n_{t-1})^{-1}. \tag{34}$$

4.2 Equilibrium Dynamics

The intertemporal equilibrium dynamics can be reduced to a three–dimensional linear system in terms of the law of motions of e_t, x_t and n_t.

From equations (25)-(31) and (34), we obtain that

$$c_t = \frac{w_t}{(1+\beta)} = \frac{(1-\alpha)y_t}{(1+\beta)}, \tag{35}$$

$$d_{t+1} = \frac{(1+n_t)(\alpha+\beta)}{(1+\beta)} y_{t+1}. \tag{36}$$

Plugging equations (30), (32), (35) and (36) into (29), we obtain the law of motion of the resource stock:

$$x_{t+1} = \frac{\Pi\beta(1-\alpha)}{(1+n_t)(\alpha+\beta)} x_t. \tag{37}$$

In addition, we have the dynamics of the natural resource stock and the population:

$$e_{t+1} = -\frac{\Pi x_t}{(1+n_t)} + \frac{\Pi e_t}{(1+n_t)}, \tag{38}$$

$$n_{t+1} = g(e_t, n_t)n_t. \tag{39}$$

Thus, the dynamics of the model economy is driven by (37), (38) and (39).

Lemma 4.1 *(**Steady States**) The steady states of the model economy are charac-*

terized by the following steady state equations:

$$
\begin{aligned}
x\left(\frac{\Pi\beta(1-\alpha)}{(1+n)(\alpha+\beta)}-1\right) &= 0, & (40)\\
\text{so that, } x &= 0 \text{ or } \frac{\Pi\beta(1-\alpha)}{(1+n)(\alpha+\beta)}=1, \\
\left(\frac{\Pi}{(1+n)}-1\right)e &= \frac{\Pi}{(1+n)}x, & (41)\\
n\left(g(e,n)-1\right) &= 0, & (42)\\
\text{so that, } n &= 0 \text{ or } g(e,n)=1.
\end{aligned}
$$

The Jacobian that governs this system of equations at the corresponding steady states is as follows:

$$
\left.\begin{bmatrix}
\frac{\Pi\beta(1-\alpha)}{(1+n)(\alpha+\beta)} & 0 & -\frac{\Pi\beta(1-\alpha)}{(\alpha+\beta)}\frac{x}{(1+n)^2} \\
-\frac{\Pi}{(1+n)} & \frac{\Pi}{(1+n)} & -\frac{\Pi(e-x)}{(1+n)^2} \\
0 & g_e n & g_n n+g
\end{bmatrix}\right|_{(x,e,n)}.
$$

Lemma 4.2 *(Locally Unique Steady States) Among the steady states characterized by equations (40), (41) and (42), the following are the ones that satisfy local uniqueness:*

1. ***Zero Steady State with No Population Growth*** *with* $n=e=x=0$;

2. ***Zero Steady State with Non-zero Population Growth*** *with* $e=x=0$, $g(0,n)=1$ *and* $g(0,0)\neq 1$;

3. ***Zero Extraction, Non-zero Resource Steady State*** *with* $\frac{\Pi}{(1+n)}=1$, $x=0$ *and* $g(e,n)=1$;

4. ***Non-Zero Steady State*** *with* $g(e,n)=1$, $\frac{\Pi\beta(1-\alpha)}{(1+n)(\alpha+\beta)}=1$ *and.*$\left(\frac{\Pi}{(1+n)}-1\right)e=\frac{\Pi}{(1+n)}x$.

The first, the second and the forth steady states in the above list exhibit unstable monotone dynamics [39]. However, the third steady state exhibits nonlinear dynamics under plausible parameters. In what follows we will show that the in *Zero Extraction, Non-zero Resource* steady state, we may encounter Hopf bifurcation leading to limit cycles. The analysis of Hopf bifurcation provides a powerful and easy tool to detect limit cycles that discards tedious calculations. Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter and meanwhile a cycle either emerges from or collapses into the fixed point. The dynamic system can either have a stable fixed point surrounded by an unstable cycle; or a stable cycle loses its stability and a stable cycle appears as the parameter(s) approach(es) to a critical value (see Asea and Zak, 1999; Yüksel, 2011). Both cases can be economically significantly meaningful (for further details see Kind, 1999). Yet, to prove the result, we need the following lemma.

Lemma 4.3 *(**Hopf Conditions** for the 2x2 discrete dynamic system) Let J be the 2×2 Jacobian matrix associated with the 2×2 discrete dynamic system and T and D be the trace and the determinant, respectively. Then, Hopf bifurcation occurs when*

$$\begin{aligned} D &= 1, \\ -2 &< T < 2. \end{aligned}$$

Proof. See Antoci and Sodini (2009, p. 1443). ■

Proposition 4.1 *(**Hopf Bifurcation** for the **Zero Steady State with non-zero Population Growth**) Following the Beverton-Holt specification, we employ the modified Beverton-Holf function $g(e_t, n_t) = \frac{rh(e_t)}{h(e)+(r-1)n_t}$, with convex-concave carrying ca-*

[39] The unstable character of the first steady state is analytically obvious, yet for the forth steady state, the unstable dynamics are demostrated numerically and the details of this analysis is skipped for the sake of compactness. The details can be provided upon request.

pacity $h(e) = \frac{\tau_1 e^\rho}{1+\tau_2 e^\rho}$ *for some* $\rho, \tau_1, \tau_2 > 0$. *Then, if there exists a parameter combination* $(\Pi, r, \rho, \tau_1, \tau_2)$ *such that*

$$\begin{aligned} h'(e)e &= 1+n, \\ \Pi &> 1, \\ 0 &< \frac{r-1}{r} < 4, \end{aligned}$$

then the steady state

$$x = 0, \quad n = \Pi - 1, \quad and \quad n = h(e),$$

undergoes Hopf bifurcation.

Proof. The Jacobian associated with this steady state is

$$\begin{bmatrix} \frac{\beta(1-\alpha)}{(\alpha+\beta)} & 0 & 0 \\ -\frac{\Pi}{(1+n)} & 1 & -\frac{e}{(1+n)} \\ 0 & g_e n & g_n n + 1 \end{bmatrix}.$$

The associated characteristic equation,

$$\left(\frac{\beta(1-\alpha)}{(\alpha+\beta)} - \lambda\right)\left((\lambda-1)(\lambda - g_n n - 1) + \frac{e}{(1+n)} n g_e\right) = 0,$$

reveals that one of the eigenvalues is $\lambda_1 = \frac{\beta(1-\alpha)}{(\alpha+\beta)}$. Note that $\lambda_1 = \frac{\beta(1-\alpha)}{(\alpha+\beta)} < 1$. Then, if the remaining second order polynomial has complex roots with unit magnitude, we can conclude that the steady state exhibits Hopf bifurcation (see Wen *et al.*, 2002).

Consider, the 2×2 matrix associated the quadratic polynomial,

$$\begin{bmatrix} 1 & -\frac{e}{(1+n)} \\ ng_e & g_n n + 1 \end{bmatrix}. \tag{43}$$

Denote

$$\begin{aligned} T &= 2 + g_n n, \\ D &= 1 + g_n n + \frac{e}{(1+n)} n g_e, \end{aligned}$$

as the trace and the determinant of matrix (43), respectively. Rewriting the Hopf conditions (see Lemma 4.3), we have

$$0 < \frac{e}{(1+n)} n g_e = -g_n n < 4. \tag{44}$$

From the steady state condition (42), we know that $g(e, n) = 1$. Since $\frac{rh(e)}{h(e)+(r-1)n} = 1$ and $r > 1$, we have

$$h(e) = n. \tag{45}$$

Suppose $h(e) = \frac{\tau_1 e^\rho}{1+\tau_2 e^\rho}$. We know from (45), $h(e) = n$. Moreover, suppose

$$h'(e)e = 1 + n.$$

Thus, the steady state can be recast as,

$$x = 0,$$
$$n = \Pi - 1 > 0,$$
$$\frac{\tau_1 e^\rho}{1+\tau_2 e^\rho} = n.$$

Now, we want to show that this steady state satisfies the Hopf conditions provided in (44). Note that,

$$\begin{aligned} g_e &= \frac{rh'(e)(r-1)n}{[h(e)+(r-1)n]^2}, \\ g_n &= \frac{-rh(e)(r-1)}{[h(e)+(r-1)n)]^2}. \end{aligned}$$

Furthermore, note that,

$$\frac{e}{(1+n)} n g_e = -g_n n = \frac{r-1}{r},$$

and

$$0 < \frac{r-1}{r} < 1 < 4.$$

Thus, the zero steady state with non-zero population growth exhibits Hopf bifurcation. ■

Example 4.1 *For the parameter combination*

$$\rho = 3, \tau_1 = 1, \tau_2 = .05, \Pi_{bif} = 1.5203,$$

the steady state

$$x = 0, \quad n = \Pi - 1 = .5203, \quad \textit{and} \quad e = h^{-1}(n) = 0.8114,$$

undergoes Hopf bifurcation.

The above example clearly shows that there exists Hopf bifurcation for plausible parameters. However, to fully comprehend the relationship of the parameters which causes Hopf dynamics, we further our analysis by constructing a Hopf boundary.

To maintain simplicity, without loss of generality we assume that $r = 2$. Then, the trace of the Jacobian is

$$1 < T = 2 - \frac{r-1}{r} = \frac{3}{2} < 2.$$

Thus, by the Lemma (4.3), Hopf condition reduces into a condition about the determinant, i.e., $D = 1$. Given the steady state equations, the determinant of the Jacobian in terms of ρ and Π is

$$D(\rho, \Pi) = 1 + \left(\frac{r-1}{r}\right)\left(\frac{\rho(\Pi-1)(\tau_1 - (\Pi-1)\tau_2)}{\tau_1 \Pi} - 1\right).$$

(ρ, Π) couples that maintain the condition that $D(\rho, \Pi) = 1$ imply that

$$\rho_{hopf} = \frac{\tau_1 \Pi}{(\Pi-1)(\tau_1 - (\Pi-1)\tau_2)}. \tag{46}$$

Equation (46) gives the Hopf boundary. Figure (2) presents this boundary. Note that inside the curve we have a fully unstable system, whereas outside the curve, the system becomes full stable. Thus, as the parameters (ρ, Π), the population parameter and regeneration rate respectively, vary, in the trace-determinant space, the system moves from the inside to the outside, or vice versa, of the boundary[40].

In figure (3), we present the trace-determinant space. Note that trace only depends on r, thus as parameters (ρ, Π), trace-determinant couple varies in the sense that trace stays as a constant. This can be clearly seen in the line on the graph. In the graph, $\Pi = 1.5203$ is kept as a constant and as ρ varies, one can keep track of the determinant. For the values of ρ that are close its lower physical levels, the trace-determinant couple stays in the stable region (triangle) where the eigenvalues are real.

[40] Note that $\Pi > 1$ by definition and $\Pi < 1 + \frac{\tau_1}{\tau_2}$ in order to obtain a positive ρ which is a physical condition of the model.

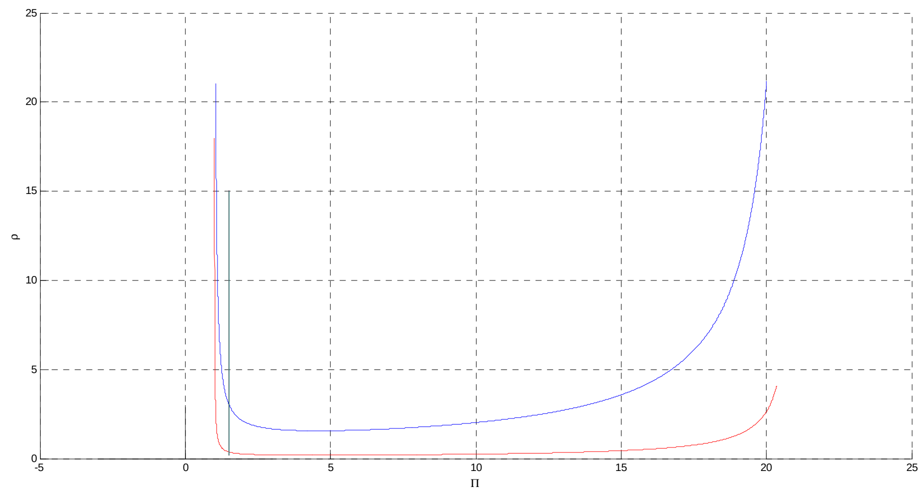

Figure 2: $\rho - \Pi$ couples at which the Hopf bifurcation occur

As ρ increases and exceeds ρ_{complex},

$$\rho_{\text{complex}} = \frac{1}{4}\left(\frac{r-1}{r}\right)\frac{\tau_1 \Pi}{(\Pi - 1)(\tau_1 - (\Pi - 1)\tau_2)} = 0.375,$$

trace-determinant couple crosses the complex eigenvalue border. Note that this complex eigenvalue boundary is also given by the red-curve in figure (2). As ρ further increases, trace-determinant couple crosses the Hopf boundary, i.e.,

$$\rho_{hopf} = \frac{\tau_1 \Pi}{(\Pi - 1)(\tau_1 - (\Pi - 1)\tau_2)} = 3.$$

The similar behaviour can be traced on the line in the figure (2). Note also that, these values are compatible with the numerical example.

One remark is that Hopf bifurcation supporting steady state is on the convex part of the $h(e; \tau, \rho)$ which is well expected. Hopf bifurcation arises as an interaction

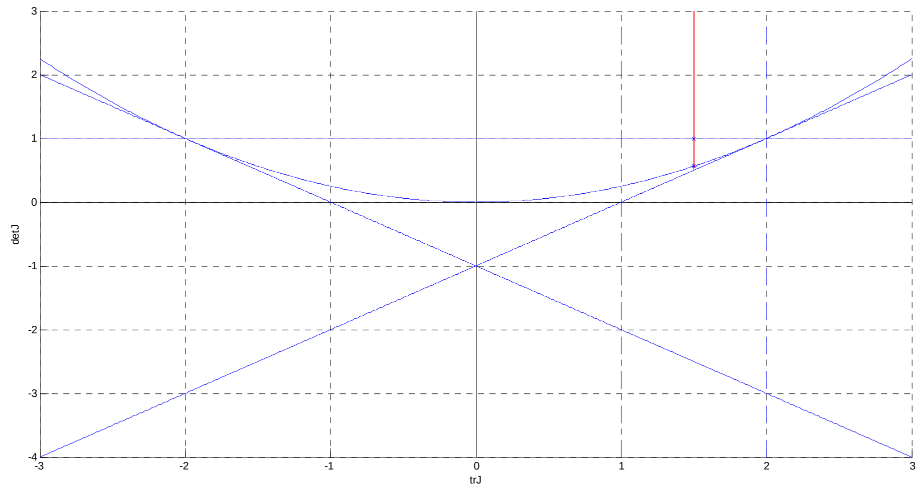

Figure 3: Trace-determinant space as parameters vary

between the curvature (convexity) of the population and the regeneration rate of the resources. In other words, limit cycle behaviour is observed because the economy fails to smooth out the counter cyclical effects of population change and the resource regeneration.

4.3 Conclusion

Through a feedback mechanism between population and resource availability, we show that Hopf bifurcation may emerge in an overlapping generations resource economy. It is worthwhile to point out that the linear regeneration specification in our model provokes the question of how the stability of the system would change under a nonlinear regeneration function. Allowing the renewable resource to regenerate nonlinearly (e.g. logistic) could bring even more complex dynamics.

CHAPTER 5

THE OPTIMAL GROWTH MODEL WITH ENDOGENOUS POPULATION GROWTH RATE AND THE EFFECT OF PAST GENERATIONS[41]

5.1 Introduction

Endogenized population growth is long been considered in economic models. Even Solow (1956) takes endogenized population growth into account. However, the accumulated literature fails to account for the relationship between the past fertility habits of older generations and the choice of fertility today. The aim of this chapter is to investigate and analyze the effects endogenous and past fertility dependent labor supply growth (i.e. population growth) on growth in an optimal growth model.

Within the Solowian framework, Fanti and Manfredi (2003) endogenizes labor supply growth with respect to per-capita capital and past fertility rates. Their endogenization depends on an introduction of the age structure prevailing in the population (p. 103). Fanti and Manfredi (2003) shows that 'globally stable oscillations

[41] This essay is a joint work with Burcu Afyonoğlu Fazlıoğlu and Hüseyin Çağrı Sağlam.

around the path of balanced growth' exist. We follow Fanti and Manfredi (2003) by extending their population growth framework into an optimal growth model.

We model a population structure that embodies the fact that older generations' fertility choices, to some extend, sustain today. Mathematically speaking, we employ an Erlangian process that takes past fertility choices into account in an exponentially fading memory fashion. That is to say, the older the generation, the smaller its effect on today's fertility choice. We further try to consider endogenizing population growth rate. In other words, population growth doesn't evolve exogenously, that is, follow a time path independent of economic variables. Compared to growth models with constant exogenous population growth, in this chapter, we specifically assume that the population growth rate is an integral term that depends on per-capita capital, k; the purpose of which is to mimic the effects of past wealth related fertility on current population.

Our model employs a continuous delay structure[42] in the process of recruitment in the population growth *á la* Fanti and Manfredi (2003) and hence obtain cyclic solutions. Cyclic solutions for optimal growth models with delay has been discussed in economics since Kalecki (1935). Detailed exposition of these discussions can be found in the Introduction and specifically in Chapter 2 of this thesis. The literature on optimal growth models with delay and concave production technology developed on three distinct lines of research. The complexity of the characteristic equation prevents to handle analytical results and thus, some researchers incline towards numerical simulations. Winkler *et al.* (2004), Collard *et al.* (2008) and Brandt-Pollmann *et al.* (2008) are those who try numerical simulations to comprehend the dynamic

[42] We may as well choose fixed delay instead of a distributed delay structure, yet "the former is better suited when there is no variability in the process of transmission of the past into the future: for instance when we assume that all individuals are recruited in the labour force at the same fixed age" (Fanti and Manfredi, 2003, p. 107) Moreover, an interested reader may consult to Invernizzi and Medio (1991) for an economically oriented discussion of distributed delays.

behaviour of optimal growth models with delay. The main findings is summarized by Winkler *et al.* (2004) who states that "both the frequency and the amplitude of the cycles depend on the length of the investment period," and by Collard et al. (2008) who states that "for a large delay the economy converges to steady state by oscillations, but consumption smoothing mitigates the induced echo effects through an advanced Euler-type differential equation." Furthermore, Collard *et al.* (2006) numerically shows that the advanced terms in Euler equations governing the dynamic system dampen the fluctuations caused by the lags through a kind of smoothing effect (They call this phenomenon 'time-to-build echo'). Short run dynamics of time-to-build echoes are further studied by Collard *et al.* (2008) in which one can find the associated numerical simulations. Winkler *et al.* (2004) provides numerical solutions of models of time delay optimal growth models for a linear limitational production function, while Winkler *et al.* (2005) gives a numerical analysis of time-lagged capital accumulation optimal growth model with Leontief type of production functions. Brandt-Pollmann *et al.* (2008) extends the numerical solutions to objective functions with state externalities.

Note that the deficiency of numerical simulations when it comes to Hopf bifurcation is that Hopf bifurcation depends on the precise calibration of the Hopf parameter and without such calibration it may be impossible to hit the limit cycle solution simply by the randomization of parameters. Moreover, the quasi-polynomial associated with the characteristic equation naturally contains infinitely many complex roots which would result in cyclic behaviour. Considering the conditions which exclude completely unstable solutions, like that of transversality condition, it is natural that a random choice of parameters would result in decaying cycles that is, for the most part, in accordance with the results and interpretation of Collard *et al.* (2008).

Another line is AK simplification. Assuming that the production schedule follows

AK production technology simplifies the $\dot{c}$ equation in the first order conditions. Note that, the resulting characteristic equation is easier to handle. Bambi (2008) exploits the simplified characteristic equation and finds Hopf cycles and Winkler (2008) solves $\dot{c}$ equation first and then using the solution solves $\dot{k}$ equation (See Barro and Sala-i Martin, 1995, Ch. 4.1).

Although 'AK simplification' approach enables some analytical results, the main question of whether there exists limit cycles under concave production remains unanswered. Though there is no clear justification, the third approach is to show the non-existence of such persistent cycles. Benhabib and Rustichini (1991), Caulkins *et al.* (2010) and Hartl and Kort (2010) represents the school of 'lack-in-faith in cycles'. Caulkins *et al.* (2010) states that "here we in some sense defend the traditional emphasis on models without delays by showing that an important class of models with delays can be transformed into equivalent optimal control problems without delays," and "the existence of an equivalent problem without delays implies that the optimal solution to the model with delays cannot involve oscillation." Thus, Caulkins *et al.* (2010) argues for the "non-oscillatory behaviour under exponential depreciation."

We offer a delayed model in which persistent cyclic solutions are analytically valid, thus there is no need for numerical simulations for searching the existence of cycles, but only for their validation and there is no need for economically binding mathematical simplifications such as AK production technology.

We specifically find Hopf bifurcation. "In 1942, Hopf published the groundbreaking work in which he presented the conditions necessary for the appearance of periodic solutions, represented in phase space by a limit cycle" (Szydłowski, 2002). With reference to the contributors of the study of the sufficient conditions under which periodic orbits occur from stationary states, these theorems are called Poincaré–Andronov–Hopf theorems. Hopf bifurcation discards tedious calculations and provides a powerful and easy tool to detect limit cycles. Kind (1999) confirms this by stating "in most cases the proof of a Hopf bifurcation is not difficult because it does not require any information on the nonlinear parts of the equation system. Moreover, in systems whose dimensions are higher than two, Hopf bifurcation theorem may constitute the only tool for the analysis of cyclical equilibria, since the Poincaré–Bendixson theorem is not applicable in these cases." Hopf cycles appear when a fixed point loses or gains stability due to a change in a parameter and meanwhile a cycle either emerges from or collapses in to the fixed point (Asea and Zak, 1999). Under the circumstances the system can either have a stable fixed point surrounded by an unstable cycle (called a *subcritical* Hopf bifurcation); or a stable cycle loses its stability and a stable cycle appears (called a *supercritical* Hopf bifurcation) as the parameter(s) approaches to a critical value (Asea and Zak, 1999). Both cases can be economically significantly meaningful. Supercritical case which implies a stable cycle can be considered as a stylized business cycle or growth cycles and the subcritical case can correspond to the corridor stability (Kind, 1999).

The chapter is structured as follows: In the first part we present the model and introduce the population structure along with the economy; in the following part, we show the existence of a pair of pure imaginary eigenvalues; next we simulate to show whether there exist plausible economic parameters that support Hopf bifurcation and the last part concludes.

5.2 Model

Suppose that the representative consumer maximizes lifetime utility, according to the constant discount factor $r > 0$, that is

$$
\begin{aligned}
&\max \qquad \int_0^\infty e^{-rt} u(c(t))dt \\
&\text{subject to} \\
&\qquad \dot{k}\,(t) = f\,(k(t)) - c(t) - (\delta + \hat{n}(t, k(t)))\,k(t), \\
&\qquad k(0) = k_0.
\end{aligned}
$$

where $\delta > 0$ is the constant depreciation. $\frac{\dot{L}(t)}{L(t)} = \hat{n}(t, k(t))$ is the endogenous time-varying population growth rate in the following form:

$$\hat{n}(t, k(t)) = \int_{-\infty}^{t} n(k(\tau))G(t - \tau)d\tau.$$

The population growth rate depends on the structure of the memory function G, which is a probability distribution over time. In other words, the memory function G satisfies

$$\int_0^\infty G(u)du = 1.$$

One advantage for the abovementioned population structure is that by the appropriate choice of functions the model can be transformed to various optimal growth models. Below we present one such example.

Example 5.1 *When $n(k(\tau)) = n \in \mathbb{R}_+$ and $G(t - \tau) = \delta(t - \tau)$, where $\delta(\cdot)$ is Dirac-delta function, $\hat{n}(t, k(t)) = n$, and the model reduces into the standard Ramsey optimal growth framework. Also note that this benchmark model has a unique steady state which is saddle path stable.*

For the purposes of this chapter, we assume that the function $n(\cdot)$ is linear[43] with respect to $k(t)$, i.e. $n(k(\cdot)) = nk(\cdot)$. is the Erlangian type of density function which is defined as

$$G_{j,a}(x) = \frac{a^j}{(j-1)!}x^{j-1}e^{-ax}; \quad x > 0,\ a > 0,\ j = 1, 2, ..., r$$

Varying r will give different densities. For $j = 1$, we have classical exponentially fading memory, whereas for larger j's, we have several "humped" memories of Gamma shape. The function $G_{j,a}(\cdot)$ is the delaying kernel, which is chosen to be a probability density. In particular, the mean delay implied by Erlangian density of parameters (j, a) is given by $\mu_\tau = \frac{j}{a}$ and its variance is $\sigma_\tau^2 = \frac{j}{a^2}$ (Manfredi and Fanti, 2003, p. 586). As $\mu_\tau \to 0$, that is $a \to \infty$ when $j = 1$, the underlying unlagged system is recovered, in other words, the capital accumulation equation reduces into

$$\dot{k}\,(t) = f\,(k(t)) - c(t) - (\delta + \hat{n}(k(t))\,k(t).$$

Thus a finite a implies that the population dynamics are affected by the past generation's fertility rate with special emphasis on the generation at $t - \mu_\tau$.

The idea is that the supply of labor is related to past fertility which is a function past levels of the per capita capital. We will assume $j = 1$ in this chapter[44]. Denote $\Pi_1(\tau, k) = \int\limits_{-\infty}^{t} nk(\tau)ae^{-a(t-\tau)}d\tau$ and $\Pi_2(\tau, k) = \int\limits_{-\infty}^{t} nae^{-a(t-\tau)}d\tau$. Note that $\frac{d}{dk}\Pi_1(x, k) = \Pi_2(x, k)$ by Leibniz Rule. In other words, the optimal growth model in

[43] Positive and nondecreasing with respect to per capita income is in accordance with Malthusian mechanism (see Manfredi and Fanti, 2004)

[44] Note that $G_{j,a}$ brings $2j$ additional equations. We will show that in the following sections.

question is

$$\max \qquad \int_0^{\infty} e^{-rt}u(c(t))dt$$

subject to

$$\dot{k}\,(t) = f\,(k(t)) - c(t) - \left(\delta + \int_{-\infty}^{t} n(k(\tau))G_{j,a}(t-\tau)d\tau\right)k(t),$$

$$k(0) = k_0.$$

Then, the dynamics governing the Hamiltonian system will be as follows:

$$\begin{aligned} \dot{c} &= \frac{u_c(c)}{u_{cc}(c)}\left(r + \delta - f'(k) + \Pi_1 + k\Pi_2\right), \\ \dot{k} &= f(k) - c - \delta k - k\Pi_1, \\ \Pi_1(\tau, k) &= \int_{-\infty}^{t} nk(\tau)ae^{-a(t-\tau)}d\tau, \\ \Pi_2(\tau, k) &= \int_{-\infty}^{t} nae^{-a(t-\tau)}d\tau. \end{aligned}$$

We will employ a *linear chain trick* and introduce auxiliary variables (see MacDonald, 1978; Manfredi and Fanti, 2004) to transform the integral terms into ordinary differential equations. The auxiliary variables are such that "these extra equations are linear and each links two successive numbers of a chain of variables" (Manfredi and Fanti, 2004). It has to be emphasized that, for the memory function to solve [$\dot{k}$, $\dot{c}$] from $t = 0$, with a given previous set of values of [$k(t)$] over the interval $(-\infty, 0)$ is equivalent to solving the set of equations [$\dot{k}$, $\dot{c}$ and the additional linear ordinary differential equations] from $t = 0$ with an appropriately interrelated set of initial values.

The the first order conditions are

$$\dot{c} = \frac{u_c(c)}{u_{cc}(c)} \left(r + \delta - f'(k) + \Pi_1 + k\Pi_2 \right), \tag{47}$$

$$\dot{k} = f(k) - c - \delta k - k\Pi_1, \tag{48}$$

$$\dot{\Pi}_1 = a(nk - \Pi_1), \tag{49}$$

$$\dot{\Pi}_2 = a(n - \Pi_2), \tag{50}$$

with the steady state equations

$$\begin{aligned} f'(k) &= r + \delta + \Pi_1 + k\Pi_2, \\ c &= f(k) - \delta k - k\Pi_1, \\ \Pi_1 &= nk, \\ \Pi_2 &= n. \end{aligned}$$

The 4×4 system has the following Jacobian around its steady state:

$$\begin{bmatrix} 0 & -\frac{u_c(c)}{u_{cc}(c)} \left(f''(k) - 2n \right) & \frac{u_c(c)}{u_{cc}(c)} & \frac{u_c(c)}{u_{cc}(c)} k \\ -1 & r & -k & 0 \\ 0 & -an & -a & 0 \\ 0 & 0 & 0 & -a \end{bmatrix}$$

A quick glance shows that the last row of the characteristic matrix gives the eigenvalue, $\lambda_4 = -a < 0$. To analyze the stability of the system, we have to focus on the remaining 3×3 matrix,

$$\det \begin{bmatrix} -\lambda & -\frac{u_c}{u_{cc}} \left(f''(k) - 2n \right) & \frac{u_c}{u_{cc}} \\ -1 & r - \lambda & -k \\ 0 & -an & -a - \lambda \end{bmatrix} = 0.$$

The characteristic equation of this system is

$$\lambda^3 + (a-r)\,\lambda^2 - \left(\frac{u_c(c)}{u_{cc}(c)}\left(f''(k)-2n\right) + akn + ar\right)\lambda - a\left(f''(k)-n\right)\frac{u_c(c)}{u_{cc}(c)} = 0.$$

We will check Liénard-Chipard (LC) conditions to test whether the system is stable or not[45]. A first glance shows that the system is not necessarily stable if the standard concavity assumptions on utility and production technology is assumed, since $b_n = b_3 < 0$.

$$\begin{aligned}
b_3 &= -a\left(f''(k)-n\right)\frac{u_c(c)}{u_{cc}(c)} < 0, \\
b_2 &= -\left(\frac{u_c(c)}{u_{cc}(c)}\left(f''(k)-2n\right) + akn + ar\right) < 0, \\
b_1 &= (a-r), \\
b_0 &= 1.
\end{aligned}$$

[45] See Gandolfo (1996) for the complete LC characterization of stability. To recall, given an $n-th$ *order* characteristic equation;

$$b_0\lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + ... + b_n = 0,\ b_0 > 0$$

one should have any one of the following four conditions

1. $$b_n > 0, b_{n-2} > 0...; \triangle_1 > 0, \triangle_3 > 0, ...$$
2. $$b_n > 0, b_{n-2} > 0...; \triangle_2 > 0, \triangle_4 > 0, ...$$
3. $$b_n > 0, b_{n-1} > 0, b_{n-3} > 0, ...; \triangle_1 > 0, \triangle_3 > 0, ...$$
4. $$b_n > 0, b_{n-1} > 0, b_{n-3} > 0, ...; \triangle_2 > 0, \triangle_4 > 0, ...$$

where

$$\triangle_n = \begin{vmatrix} b_1 & b_3 & b_5 & b_7 & ... & 0 \\ b_0 & b_2 & b_4 & b_6 & ... & 0 \\ 0 & b_1 & b_3 & b_5 & ... & 0 \\ 0 & b_0 & b_2 & b_4 & ... & 0 \\ 0 & 0 & b_1 & b_3 & ... & 0 \\ ... & ... & ... & ... & ... & 0 \\ 0 & 0 & 0 & 0 & ... & b_n \end{vmatrix}.$$

5.3 Pure Imaginary Roots

Below we present a theorem that gives the conditions on the coefficients of a third order polynomial to admit a pair of pure imaginary roots.

Theorem 5.1 *The characteristic equation* $\Delta(\lambda) := \lambda^3+b_1\lambda^2+b_2\lambda+b_3 = 0$, *has a pair of purely imaginary roots* $\pm hi$, $(i = \sqrt{-1}, h \neq 0)$ *if and only if* $b_1 > 0$ *and* $b_1b_2-b_3 = 0$ *are satisfied. In this case, we have the explicit solutions* $\lambda = -b_1, \pm\sqrt{b_2}i$.

Proof. *We have a third order equation as the characteristic equation. For this equation to have a couple of complex conjugate pure imaginary roots, we should have*

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = (\lambda - \chi)(\lambda - i\theta)(\lambda + i\theta) = 0,$$

where $\theta \in \mathbb{R}$. *That is, we have*

$$(\lambda - \chi)(\lambda - i\theta)(\lambda + i\theta) = (\lambda - \chi)(\lambda^2 + \theta^2) = \lambda^3 - \chi\lambda^2 + \theta^2\lambda - \chi\theta^2.$$

In other words, there should be a certain relation between the coefficients, which can be summarized as follows,

$$b_2 = \frac{b_3}{b_1}. \tag{51}$$

This relation holds if $b_1 = (a - r) > 0$ *since* b_2 *and* b_3 *are negative*[46]. *Immediately note that if equation (51) holds, then we have*

$$\begin{aligned} \lambda_1 &= \chi = -b_1 = -(a-r) < 0, \\ \lambda_{2,3} &= \pm i\theta = \pm i\sqrt{\left(\frac{u_c(c)}{u_{cc}(c)}\left(f''(k) - 2n\right) + akn + ar\right)} \in \mathbb{C}. \end{aligned}$$

■

[46]Note that the other real root $\chi = b_1 = (a - r) < 0$.

Given the theorem (5.1), we only need to show whether there exists a feasible value for the parameter a so that the economy admits Hopf bifurcation. Below we prove that result.

Theorem 5.2 *For any $n > 0$, there exists $a > r > 0$ such that the above system admits Hopf bifurcation.*

Proof. *Denoting $U \equiv \frac{u_c(c)}{u_{cc}(c)}$ and $F = f''(k)$, coefficient-wise, equation (51) enforces the following equation:*

$$a\,(F - n)\,U = (U\,(F - 2n) + akn + ar)\,(a - r)\,,$$

or, equivalently,

$$(kn + r)a^2 - (Un + rkn + r^2)a + Ur(2n - F) = 0. \tag{52}$$

Note that equation (52) has a root a such that $a > r$ if

$$A(a - r)^2 + B(a - r) + C = 0, \tag{53}$$

has a positive root. Reorganizing terms we obtain

$$Aa^2 + (B - 2Ar)a + (Ar^2 + C) = 0. \tag{54}$$

Thus, equation (54) has a positive root, if

$$\begin{aligned} A &= (kn + r) > 0, \\ B &= 2r(kn + r) - (Un + rkn + r^2), \\ C &= Ur(2n - F) - (kn + r)r^2 < 0. \end{aligned}$$

Now, once again, note that equation (54) has a positive root since for any n, we have $A > 0$ and $C < 0$; and Intermediate Value Theorem[47] *applies.* ■

Theorem (5.2) implies that for any choice of n, one can find a plausible value for a such that the Jacobian that derives the system of equations at the steady state admits a pair of pure imaginary eigenvalues and that is to say, the system admits Hopf bifurcation. In the section that follows we simulate to show that for commonly accepted values of economic parameters, one can obtain parameters that support Hopf bifurcation.

5.4 Simulation

We can further simulate[48] to show explicitly that such a exists. Let

$$\begin{aligned} u(c) &= \ln(c), \\ f(k) &= k^{\alpha}. \end{aligned}$$

Then,

$$\begin{aligned} U &\equiv \frac{u_c(c)}{u_{cc}(c)} = -c, \\ F &= f''(k) = \alpha(\alpha-1)k^{\alpha-2}. \end{aligned}$$

where the steady state values are

$$\begin{aligned} c &= k^{\alpha} - \delta k - nk^2, \\ \alpha k^{\alpha-1} - 2nk &= r + \delta. \end{aligned}$$

[47] Let $f(x) = Ax^2 + Bx + C$ where $A > 0$and $C < 0$. Then, $f(0) = C < 0$ and $\lim_{x\to\infty} f(x) > 0$. Moreover, f is continuous. Thus, by IVT, there exists a $\tilde{x} > 0 : f(\tilde{x}) = 0$.

[48] We use *Matlab R2010a* and *Mathematica 7* for the simulations.

Let the parameters be as follows

$$\begin{aligned}
\alpha &= 0.3, \\
r &= 0.03, \\
\delta &= 0.1, \\
n &= 1.
\end{aligned}$$

Then, we have $a_{hopf} = 0.1006$ (and $a = -0.8185$) that solves[49] equation (52), which satisfies $a > r$. Thus, we have a couple of pure imaginary roots. a is a parameter that is incorporated into the model by the Erlangian setup. A finite a_{hopf} implies when the population dynamics are affected by the past generation's fertility rate with special emphasis on the generation at $t - \mu_\tau$, the economy is driven into persistent cycles. Moreover, it is totally consistent with the existing literature that we choose a to be the Hopf parameter (see Fanti and Manfredi, 2003). However, Fanti and Manfredi (2003) is short on interpreting the Hopf parameter with respect to the economy they are considering. In Fanti and Manfredi (2003), Hopf bifurcation arises for a very large value of elasticity of capital $\alpha = 0.88$. The authors avoid it with a rather unconvincing argument[50]. In an optimal growth model where Erlangian population is incorporated, we obtain Hopf bifurcation with rather more plausible set of parameters.

The contribution is that under standard neoclassical assumptions, the optimal

[49] The steady state levels are,

$$\begin{aligned}
c &= 0.6048 > 0, \\
k &= 0.3160 > 0.
\end{aligned}$$

[50] "We notice that, although persistent oscillations seem to require a rather large value of the elasticity of capital α (even higher than those estimated by using a broad definition of capital stock including the human capital), this is just a feature of the low-order kernel ($r = 2$) considered here for purposes of analytical simplicity. Indeed by resorting to higher order distributions of the delaying kernel (e.g., Erlangian densities of order $r = 3, 4, 5...$), which are more realistic, we obtained, via simulation, more realistic "critical" values of the elasticity of capital" (p. 111ff).

growth model with a population growth structure of (distributed) time delay admits Hopf bifurcation (limit cycle behaviour).

5.5 Conclusion

In this chapter we incorporate a per capita capital dependent age structure in the population growth mechanism in an optimal growth model. Through this mechanism, not only the population is considered as a function of per capita capital, or in other words, population growth is endogenized, but also the current level of population growth is linked with those of older generations. We show that the interaction between population growth that takes the fertility choices of past generations into account and capital accumulation may drive the economy into a persistent cycles.

CHAPTER 6

CONCLUDING REMARKS

In the introduction of the thesis, we try to summarize the development and history of the use of delay in economic models. Mainly, it is Kalecki (1935) who introduces the delay in economic models to show that the crises (cycles) are intrinsic to economic behaviour. The development of mathematical apparatus makes the reproduce the results of Kalecki (1935) in more elaborated models possible. In the second chapter, we try to sharpen the analysis of one sector optimal growth model with one control and one state variables and time delay. We firstly give a brief outline of the mathematical history and 'know-how' of delays in economic models, as well as its interpretation, and then, we further the analysis set of the model of Asea and Zak (1999) and try to introduce of a new technique for the exposition of the eigenvalues of the characteristic equation of these type of models in a generalized framework. However, it is necessary to further the analysis to lay out the exact distribution of eigenvalues in the complex plane either via more elaboration in the analytical techniques or via numerical tools.

We also develop a new method that could be used to evaluate the relationship between the magnitude of delay and the frequency of cycles in a Ramsey-Kalecki model. Louisell (2001) gives an example in which there is no correlation between the two. We approve this and yet, it seems that for some interval of delay, there is a positive correlation. The results should be numerically simulated.

In the forth chapter, we show that Hopf bifurcation may emerge in an overlapping generations resource economy through a feedback mechanism between population and resource availability. In overlapping generations resource economy models, the cycle inducing factor is mainly the nonlinearity of the regeneration of the resources. On the contrary, we assume linear regeneration and yet, endogenize the population growth rate. We show that the interaction between instantenous population growth and regeneration rate triggers persistent cycles in the economy. Allowing the renewable resource to regenerate nonlinearly (e.g. logistic) could bring even more complex dynamics. This is in our research agenda.

In the fifth chapter, we employ a continuous delay structure in the process of recruitment in the population growth in an optimal growth model and hence obtain cyclic solutions. We exploit Erlangian process in the population growth mechanism. We find out that the interaction between the effect of older generations' fertility choices and the accumulation of capital induces cyclic behaviour in the economy. We are in the process of further numerical simulations that shows the economy in action.

We also study the wealth effect in utility by way of Erlangian process. That is to say, we are working on a model where the representative agent takes utility from his or her ancestors' wealth level through an exponentially fading memory and we intend to contribute to the existing literature of social status.

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