INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **18** (2005) 1249–1267 doi:10.1088/0951-7715/18/3/016

A phase field formulation of the Willmore problem

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Received 30 August 2004, in final form 31 August 2004
Published 23 February 2005
Online at stacks.iop.org/Non/18/1249

Recommended by Weinan E

Abstract

In this paper, we demonstrate, through asymptotic expansions, the convergence of a phase field formulation to model surfaces minimizing the mean curvature energy with volume and surface area constraints. Under the assumption of the existence of a smooth limiting surface, it is shown that the interface of a phase field, which is a critical point of the elastic bending energy, converges to a critical point of the surface energy. Further, the elastic bending energy of the phase field converges to the surface energy and the Lagrange multipliers associated with the volume and surface area constraints remain uniformly bounded. This paper is a first step to analytically justify the numerical simulations performed by Du, Liu and Wang in 2004 to model equilibrium configurations of vesicle membranes.

Mathematics Subject Classification: 35B40, 74K15, 92B05

1. Introduction

1.1. The Willmore problem

The Willmore problem is the classical problem from differential geometry to find Γ in an admissible class of surfaces embedded in $\mathbb{R}^3$, which minimizes the mean curvature energy

$$\int_\Gamma H^2 \, \mathrm{d}S, \tag{1}$$

where $H = (k_1 + k_2)/2$ is the mean curvature and k_1 and k_2 are the principal curvatures of Γ. If Γ is a critical point of (1), then the mean curvature and Gaussian curvature $K = k_1 k_2$ of Γ will satisfy the Euler–Lagrange equation

$$\Delta_\Gamma H + 2H(H^2 - K) = 0. \tag{2}$$

A good reference for the derivation of this equation and an in-depth description of the problem can be found in [2].

0951-7715/05/031249+19$30.00

1.2. Dynamic surface models

In the last few years, the study of surface motion has attracted much attention. Traditionally, there are several well-established methods of analytically and computationally modelling surfaces. Most notably, these include direct methods, the front tracking [3, 4], volume of fluid (VOF) [5] and level set methods [6].

The most straightforward way of handling a moving surface is the direct method. One employs a discretization with grid points on the surface itself, using finite differences, finite elements, and boundary-integral techniques. Although conceptually convenient, this method inherits the trappings of a moving mesh scheme. Large deformations in the surface may lead to mesh entanglement, and keeping track of the mesh requires a great deal of algorithmic complexity. Most importantly though, it is difficult to couple the surface motion with the field equation of a body force, making interface motion through a fluid difficult to model.

Alternatively, one may fix a discretization of the domain, and represent the surface motion as a vector field distributed along a thin band within which the surface resides. Methods of this type include the level set, VOF and front tracking methods. The advantage here is that the surface motion, although distributed over a small region, is a bulk quantity and couples easily with other fields. Further, there is no algorithmic overhead in keeping track of the quality of the domain discretization. The above-mentioned schemes, however, do not treat the discretization uniformly on the whole domain. Front tracking requires the solution of an auxiliary Riemann problem to extrapolate the difference scheme at the interface. In the other models, the indicator function must be renormalized at each time step, introducing artificial damping to the surface motion.

1.3. Energetic phase field models

The phase field method is also a level-set method except that the surface motion can be viewed as due to the physical energy dissipation $\phi_t = -\delta E_\epsilon/\delta\phi$. E_ϵ is the phase field's free energy functional, which depends on the interface transitional thickness ϵ. Although resolution of the interface for small ϵ becomes difficult, the phase field motion is dictated by a bulk field over the whole domain. Therefore, it inherits all of the aforementioned qualities; ease of coupling with a fluid, indifference to morphological singularities in the interface, physical dissipation, and a spatially uniform discretization. In the context of the surface elasticity, the value $E_\epsilon(\phi)$ can represent different interfacial energies associated to the phase field. The most basic energy functional one may introduce is

$$E_\epsilon(\phi) = \int_\Omega \frac{1}{\epsilon^2} W(\phi) + \frac{1}{2}|\nabla\phi|^2 \,\mathrm{d}x, \tag{3}$$

which approximates the (normalized) surface area of the interface. $W(\phi)$ penalizes for values of ϕ that are out of phase, while the gradient term penalizes for large transition interfaces. If ϕ obeys a steepest descent law with respect to E_ϵ, i.e. $\phi_t = -\delta E_\epsilon/\delta\phi$, and we choose the admissible space to be $L^2(\Omega)$, we recover the Allen–Cahn/Ginzburg–Landau equation,

$$\phi_t = \Delta\phi - \frac{1}{\epsilon^2} W'(\phi). \tag{4}$$

We may also recover the Cahn–Hilliard equation,

$$\phi_t = -\Delta\left(\Delta\phi - \frac{1}{\epsilon^2} W'(\phi)\right) \tag{5}$$

if the admissible space is $H^{-1}(\Omega)$, the dual space of $H^1(\Omega)$; see [7]. A solution of the Cahn–Hilliard equation is volume preserving, which can immediately be seen by integrating (5) over Ω. On the other hand, one may also modify (4) by the addition of a Lagrange multiplier

$$\lambda_\epsilon(t) = -\frac{1}{\epsilon^2}\int_\Omega W'(\phi)\mathrm{d}x,$$

as a simple alternative to retaining volume preserving solutions; see [7, 8]. The modification is known as the non-local Ginzburg–Landau equation due to the non-local dependence of the surface velocity V on the mean curvature H. The analogous, equilibrium problem to the non-local Ginzburg–Landau equation is to minimize (3) with the constraint

$$\int_\Omega \phi = \alpha. \tag{6}$$

The asymptotic behaviour of minimizers of (3) and (6) has been studied in [9, 10], and that of general critical points in [11].

A more fundamental study than that of the dynamic surface is the stationary problem to minimize the curvature energy with density $(a(H-c)+bK)^2$ among a certain class of surfaces. In [1], the authors presented the interfacial balance density

$$\frac{\epsilon}{2}\left(\Delta\phi - \frac{1}{\epsilon^2}W'(\phi)\right)^2 \tag{7}$$

for a double-welled potential W in order to describe the Willmore problem, i.e. when $a = 1$ and $b, c = 0$. They produced numerical evidence that the limiting interface of minimizers of the energy functional with density (7) (also constrained by volume and surface area functionals) converges to a stable surface. For different volume and surface areas, these surfaces resemble many physical surfaces found in nature [1], i.e. spheres, tori, dimpled discs and double bubbles.

Questions remain as to whether, analytically, the convergence of the free energy to the mean curvature energy is stable for a perturbation η around the phase function $q(d/\epsilon)$ and whether the interface (the zero level set of ϕ) of a minimizer of (8)–(10) converges to a minimizer of (1). Here, d is the signed distance function to the interface and $q(\cdot) = \tanh(\cdot/\sqrt{2})$ [1]. Indeed, it is not immediately apparent that this is true when making a small perturbation of the form $\eta = \epsilon h$ for some well-behaved function h. Expanding (7) by $\phi = q(d/\epsilon) + \epsilon h$, we see that

$$\frac{\epsilon}{2}\left(\Delta\phi - \frac{1}{\epsilon^2}W'(\phi)\right)^2 = \frac{1}{2\epsilon^3}(q''|\nabla d|^2 - W'(q))^2 + \frac{1}{2\epsilon^2}(q'\Delta d + W''(q)h) \times (q''|\nabla d|^2 - W'(q)) + \frac{1}{2\epsilon}(q'\Delta d - W''(q)h)^2 + \mathrm{O}(1).$$

The terms that multiply the factors ϵ^{-3} and ϵ^{-2} are zero if q satisfies $q'' = W'(q)$. However, *a priori*, nothing can be said about the term multiplying ϵ^{-1}, in particular, h.

In this paper, we study the stability of the elastic bending energy with respect to a perturbation of the form $\epsilon h + \mathrm{o}(\epsilon)$. It is shown that if $\phi = q(d/\epsilon) + \epsilon h + g$ is a minimizer of the constrained elastic energy, then h is of order ϵ and g is of order ϵ^2. We also show that the Lagrange multipliers for the volume and surface area constraints remain bounded as $\epsilon \to 0$. In comparison to [7, 12], where the initial datum and energy estimates of the non-local Ginzburg–Landau/Cahn–Hilliard equation produce a bound for the constrained volume multiplier, we rely on the *a priori* assumption that ϕ is almost a phase function and the integro-algebraic consequences of this to produce bounds. Second, we show that under the assumption of the existence of a smooth interface as the critical points of the elastic bending energy, and stronger assumption $g = 0$, that the interface converges to a critical point of (1). In other words, the Euler–Lagrange equation for the unconstrained elastic bending energy

$$E_\epsilon(\phi) = \int_\Omega \frac{\epsilon}{2}\left(\Delta\phi - \frac{1}{\epsilon^2}W(\phi)\right)^2 \mathrm{d}x$$

converges to (2).

The approach taken is as follows. For fixed ϵ, ϕ_ϵ is the constrained minimizer of the elastic bending energy $E_\epsilon(\phi)$. The $\{\phi_\epsilon\}_{\epsilon>0}$ form a one-parameter family of functions in the admissible class $\mathcal{L}$, which approach a singular limit. For each ϵ there corresponds an Euler–Lagrange equation of the constrained problem and variational form $\delta E_\epsilon/\delta\phi$ of the unconstrained problem. Using energy estimates derived from the Euler–Lagrange equation and various test functions, we show that the Lagrange multipliers for the volume and surface area constraints are uniformly bounded. Although ϕ_ϵ and $\delta E_\epsilon/\delta\phi$ become singular as ϵ approaches zero, they do converge to surface functions in an appropriate sense. This allows one to show the convergence of $E_\epsilon(\phi_\epsilon)$ and $\delta E_\epsilon/\delta\phi$ to (1) and the left-hand side of (2), respectively.

This paper is divided into sections as follows. The phase field formulation and some basic consequences of the asymptotic assumption are discussed in section 2. In particular, we use the boundedness of the elastic bending energy (see appendix) to show that the minimizer is stable around the phase function with respect to first-order perturbations. These results are used in section 3 to show that the Lagrange multipliers' lowest order is zero. Finally, the asymptotic convergence of the surface and phase field variational problems is developed in section 4.

2. Phase field formulation

Consider the variational problem

$$\min_{\phi\in L}\int_\Omega \frac{\epsilon}{2}\left(\Delta\phi - \frac{1}{\epsilon^2}W'(\phi)\right)^2 \mathrm{d}x \tag{8}$$

with constraints

$$A(\phi) = \int_\Omega \phi\,\mathrm{d}x = \alpha, \tag{9}$$

$$B_\epsilon(\phi) = \int_\Omega \frac{\epsilon}{2}|\nabla\phi|^2 + \frac{1}{\epsilon}W(\phi)\mathrm{d}x = \beta \tag{10}$$

over the admissible set

$$\mathcal{L} = \{\phi \in H^2 | \phi|_{\partial\Omega} = 1, \nabla\phi\cdot n|_{\partial\Omega} = 0\}. \tag{11}$$

Let the double-welled potential be $W(\phi) = (\phi^2-1)^2/4$. Constraints (9) and (10) fix the volume and surface area, respectively, for level sets of phase functions in $\mathcal{L}$. In order that α and β be realistic constraints, we require that

$$\beta > \frac{|\mathbf{S}^2|}{c_0}\left(\frac{|\Omega|-\alpha}{2|\mathbf{B}_1^3|}\right)^{2/3}, \tag{12}$$

where $\mathbf{S}^2$ is the two-dimensional unit sphere and $\mathbf{B}_1^3$ is the three-dimensional unit ball. $c_0 = \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{sech}^4(t)\mathrm{d}t$ is a scaling of (10) in terms of surface area. See definition A.1 for an explanation of (12).

Let ϕ_ϵ solve the constrained minimization (8)–(10). Suppose the following assumptions on ϕ_ϵ hold:

$\{\Gamma_\epsilon\}_{\epsilon\geqslant 0}$ is a family of class C^4, compact surfaces converging uniformly to Γ_0. Denote $d(x) = \mathrm{dist}(x, \Gamma_\epsilon)$. (A1)

$\phi_\epsilon(x) = q(d(x)/\epsilon) + \epsilon h + g$ where $q \in C^2(\mathbb{R})$ and $h \in C^2(\Omega)$ are independent of ϵ, and $\|\nabla^k g\|_{L^\infty} = \mathrm{o}(\epsilon)$ for $k = 0, \ldots, 4$. (A2)

The assumption (A2) indicates that ϕ_ϵ is a perturbed profile to mark the free interface. (A1) states that the limiting interface is in fact a smooth surface. A more specific formulation is as follows; Γ_0 is a compact, class C^4 surface. There exists a family of C^4 mappings $\{\Psi_\epsilon\}_{\epsilon\geqslant 0}$ from Ω to itself, differentiable in ϵ, with $\|\nabla^k\Psi_\epsilon\|_{L^\infty}$ bounded independently of ϵ for $k = 0, \ldots, 4$ and

$$\Psi_0 = \mathrm{id}, \qquad \Gamma_\epsilon = \Psi_\epsilon(\Gamma_0).$$

The function d is the signed distance and determines the geometric properties Γ_ϵ.

The more general (A2) is used in all results except theorem 4.2 and corollary 3.4, where we need the following stronger assumption:

$$\phi_\epsilon(x) = q(d(x)/\epsilon) + \epsilon h \text{ where } q \in C^2(\mathbb{R}) \text{ and } h \in C^2(\Omega) \text{ are independent of } \epsilon. \tag{A2a}$$

Under (A2a), we show that the curvatures H and K of the level set $\{\phi_\epsilon = 0\}$ converge to a weak solution of (2). Since $\{\phi_\epsilon = 0\}$ converges uniformly to $\{d = 0\} = \Gamma_0$, it follows that Γ_0 is a critical point of (1). Interpreting differential values of d for $\epsilon \to 0$ is the key to ascertaining convergence of the respective Euler–Lagrange equations. We do not denote the dependence of d on ϵ. O and o denote, as usual, quantities bounded independently of and vanishing with ϵ, respectively.

First, we present two lemmas regarding functions of d on an ϵ scale. The first lemma demonstrates that certain integrable functions of d/ϵ, which vanish rapidly at $\pm\infty$, can be viewed as a δ-sequence of the limiting surface Γ_0. The second lemma demonstrates that integrals of functions of d/ϵ, with an anti-derivative, are in fact $\mathrm{O}(\epsilon^2)$. This class consists of either odd functions with an anti-derivative or functions whose anti-derivative vanishes rapidly at infinity. The second lemma is frequently used to justify raising the order of terms that formally appear $\mathrm{O}(\epsilon^{-2})$.

Lemma 2.1. *Suppose that Γ_ϵ and d are given as above. Suppose further that $f \in C^0(\Omega)$ and $p \in L^1(\mathbb{R})$ satisfies*

$$\max_{|t|>s} |p(t)t| \leqslant \frac{C}{s^m}, \qquad m > 1. \tag{13}$$

Then

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_\Omega p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x = \int_{-\infty}^{\infty} p(t)\mathrm{d}t \int_{\Gamma_0} f(z)\mathrm{d}S(z). \tag{14}$$

Proof. Let O be a neighbourhood of Γ_ϵ within which ∇d is Lipschitz. Let $\eta(t, x)$ be the integral curves of ∇d with initial datum $z \in \Gamma_\epsilon$, i.e.

$$\dot\eta(t, z) = \nabla d(\eta(t, z)). \tag{15}$$

Note that

$$\frac{\mathrm{d}}{\mathrm{d}t} d(\eta(t, z)) = \nabla d(\eta(t, z))\dot\eta(t, z) = |\nabla d(\eta(t, z))|^2 = 1 \tag{16}$$

and that $d(\eta(0, z)) = d(z) = 0$. Thus,

$$d(\eta(t, z)) = t$$

for all $z \in \Gamma_\epsilon$. Further, let

$$J(t, z) = \det(\nabla_{t,z}\eta(t, z)).$$

It is clear from the fact that $\nabla d(z) = \mathbf{n}$, where $\mathbf{n}$ is the unit normal at z, that

$$J(0, z) = \det(\mathbf{n}, \mathbf{z_1}, \mathbf{z_2}) = 1,$$

where $\mathbf{z_1}$ and $\mathbf{z_2}$ are an orthonormal pair in the tangent space of Γ_ϵ at z. Consider thus the change of coordinates $(t, z) \to \eta(t, z)$ and let $U = \eta(\{(-\delta, \delta), \Gamma_\epsilon\}) \subset \mathrm{O}$ for sufficiently small but fixed δ:

$$\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\,\mathrm{d}x = \int_{-\delta}^{\delta}\int_{\Gamma_\epsilon} p\left(\frac{d(\eta(t,z))}{\epsilon}\right) f(\eta(t,z))J(t,z)\mathrm{d}S(z)\mathrm{d}t$$
$$= \int_{-\delta}^{\delta} p\left(\frac{t}{\epsilon}\right)\int_{\Gamma_\epsilon} f(\eta(t,z))J(t,z)\mathrm{d}S(z)\mathrm{d}t.$$

Changing coordinates $\epsilon s = t$, we find

$$\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x = \epsilon\int_{-\delta/\epsilon}^{\delta/\epsilon} p(s)\int_{\Gamma_\epsilon} f(\eta(\epsilon s, z))J(\epsilon s, z)\mathrm{d}S(z)\mathrm{d}s.$$

By continuity and dominated convergence,

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x = \lim_{\epsilon\to 0}\int_{-\delta/\epsilon}^{\delta/\epsilon} p(s)\int_{\Gamma_\epsilon} f(\eta(\epsilon s, z))J(\epsilon s, z)\mathrm{d}S(z)\mathrm{d}s$$
$$= \int_{-\infty}^{\infty} p(s)\int_{\Gamma_0} f(\eta(0,z))J(0,z)\mathrm{d}S(z)\mathrm{d}s$$
$$= \int_{-\infty}^{\infty} p(s)\int_{\Gamma_0} f(z)\mathrm{d}S(z)\mathrm{d}s.$$

Equation (14) now follows because

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{\{d>\delta\}=U^c} p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x \leqslant \lim_{\epsilon\to 0}\max_{|s|>\delta}\frac{1}{\epsilon} p\left(\frac{s}{\epsilon}\right)\int_\Omega |f(x)|\mathrm{d}x = 0.$$

□

Lemma 2.2. *In addition to lemma (2.1), assume that* $f \in C^1(\Omega)$ *and* $d \in C^2(U)$. *For* $P' = p$, *let* $P \in L^1(\mathbb{R})$. *If (i) p is odd and or (ii)* $\lim_{s\to\infty}\max_{|t|>s}|P(t)t| = 0$, *then*

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_\Omega p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x < \infty.$$

Proof. We resume exactly as in the proof of lemma (2.1) to find that

$$\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x = \int_{-\delta}^{\delta} p\left(\frac{t}{\epsilon}\right)\int_{\Gamma_\epsilon} f(\eta(t,z))J(t,z)\mathrm{d}S(z)\mathrm{d}t.$$

$$F_\epsilon(t) = \int_{\Gamma_\epsilon} f(\eta(t,z))J(t,z)\mathrm{d}S(z).$$

$F_\epsilon(t)$ is continuous in ϵ and continuously differentiable for $t \in (-\delta, \delta)$.

If (i) holds, let $G_\epsilon(t) = (F_\epsilon(t) - F_\epsilon(0))/t$. Then,

$$\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x = \int_{-\delta}^{\delta} p\left(\frac{t}{\epsilon}\right) F_\epsilon(t)\mathrm{d}t$$
$$= \int_{-\delta}^{\delta} p\left(\frac{t}{\epsilon}\right) tG_\epsilon(t)\mathrm{d}t.$$

Changing coordinates $t = \epsilon s$ and taking absolute values we find

$$\left|\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x\right| \leqslant \epsilon^2 \max_{t\in(-\delta,\delta)} G_\epsilon(t) \int_{-\infty}^{\infty} |p(s)s|\mathrm{d}s.$$

Dividing by ϵ^2, the remaining terms are finite.

If (ii) holds, changing coordinates $t = \epsilon s$

$$\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x = \epsilon \int_{-\delta/\epsilon}^{\delta/\epsilon} p(s)F_\epsilon(\epsilon s)\mathrm{d}s$$

$$= \epsilon P(s)F_\epsilon(\epsilon s)|_{\delta/\epsilon}^{\delta/\epsilon} - \epsilon^2 \int_{\delta/\epsilon}^{\delta/\epsilon} P(s)F_\epsilon'(\epsilon s)\mathrm{d}s.$$

Dividing both sides by ϵ^2 and taking the limit $\epsilon \to 0$ we find

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\int_U p\left(\frac{d(x)}{\epsilon}\right) f(x)\mathrm{d}x = \lim_{\epsilon\to 0}\frac{1}{\epsilon} P(s)F_\epsilon(\epsilon s)|_{-\delta/\epsilon}^{\delta/\epsilon} - \int_{-\delta/\epsilon}^{\delta/\epsilon} P(s)F_\epsilon'(\epsilon s)\mathrm{d}s$$

$$= 0 - F_\epsilon'(0)\int_{-\infty}^{\infty} P(s)\mathrm{d}s$$

$$< \infty.$$

□

Theorem 2.1. *Suppose ϕ_ϵ solves (8)–(10) and satisfies (A1) and (A2). Then*

$$q(t) \equiv \tanh\left(\frac{t}{\sqrt{2}}\right). \tag{17}$$

Proof. If α and β satisfy (12), then the minimum energy (8) is uniformly bounded by a constant M for sufficiently small ϵ (see the appendix).

$$M \geqslant E_\epsilon(\phi_\epsilon) \geqslant \int_\Omega \frac{\epsilon}{2}\left(\Delta\phi_\epsilon - \frac{1}{\epsilon^2}W'(\phi_\epsilon)\right)^2 \mathrm{d}x$$

$$\geqslant \frac{1}{2\epsilon^3}\int_\Omega \left(q'' - W'(q)\right)^2 \mathrm{d}x + \mathrm{O}\left(\frac{1}{\epsilon^2}\right).$$

It follows, using a change of variables, that $q'' - W'(q) = 0$ everywhere. Since $\mathrm{d}(x)/\epsilon$: $\Omega \times [0,\infty) \to \mathbb{R}$ is onto, we must have

$$q''(t) = W'(q(t)), \qquad \forall t \in \mathbb{R}. \tag{18}$$

Since $\alpha < |\Omega|$, q must have a zero and (10) implies that $|q(\pm\infty)| = 1$. It follows that $q(t) = \tanh(t/\sqrt{2})$. □

Theorem 2.2. *Suppose ϕ_ϵ solves (8)–(10) and satisfies (A1) and (A2). Then,*

$$h \equiv 0. \tag{19}$$

Proof. We continue expanding $E_\epsilon(\phi_\epsilon)$ from theorem 2.1;

$$M \geqslant E_\epsilon(\phi_\epsilon) = \frac{1}{2\epsilon}\int_\Omega (q'\Delta d - W''(q)h)^2 \mathrm{d}x + \mathrm{O}(1)$$

$$= \frac{1}{2\epsilon}\int_\Omega (q')^2(\Delta d)^2 - 2W''(q)q'h\Delta d + (W''(q))^2h^2\mathrm{d}x + \mathrm{O}(1).$$

With $q(t) = \tanh(t/\sqrt{2})$, by lemma 2.1 with $p = (q')^2$, the first integral converges to

$$2\int_{-\infty}^{\infty}(q'(s))^2\mathrm{d}s \int_{\Gamma_0} H^2\mathrm{d}z,$$

since $\Delta d(x) = 2H(x)$ for $x \in \Gamma_0$. By lemma 2.2, with $p = W''(q)q'$ and $P = W'(q)$, the second integral is $O(\epsilon)$. Further, since $W''(q) \to 2$ as $\epsilon \to 0$, lemma 2.1 implies that

$$\frac{1}{\epsilon}\int_\Omega (W''(q))^2 h^2\,\mathrm{d}x \longrightarrow \left(\frac{4}{\epsilon} + \int_{-\infty}^{\infty} (W''(q(t)))^2 - 4\,\mathrm{d}t\right)\int_\Omega h^2\,\mathrm{d}x < \infty.$$

But h is independent of ϵ so h must be identically zero. □

In the next section we derive from the Euler–Lagrange equation for (8)–(10) expressions for the Lagrange multipliers for general ϕ, i.e. *without* (A1) and (A2). Using theorems 2.1 and 2.2 we show that the Lagrange multipliers are bounded. Then, we derive an asymptotic expression for the variation of the unconstrained elastic bending energy, which we show in section 4 to be equivalent to the variation of the Willmore energy functional. We also drop the ϵ from ϕ_ϵ, B_ϵ and E_ϵ. Later we need

$$\beta_1 = \int_\Omega \frac{\epsilon}{2}|\nabla\phi|^2\,\mathrm{d}x, \qquad \beta_2 = \int_\Omega \frac{1}{\epsilon}W(\phi)\mathrm{d}x.$$

3. Euler–Lagrange equation

The Euler–Lagrange equation of (8)–(10) reads

$$\epsilon\Delta A - \frac{1}{\epsilon}AW''(\phi) + \epsilon\lambda A + \mu = 0, \tag{20}$$

where

$$A = \Delta\phi - \frac{1}{\epsilon^2}W'(\phi). \tag{21}$$

λ and μ are the Lagrange multipliers corresponding to the constraints (9) and (10), respectively.

Lemma 3.1.

$$\mu|\Omega| - \lambda c_1 = \frac{1}{\epsilon}\int_\Omega AW''(\phi)\mathrm{d}x, \tag{22}$$

$$3\mu(\alpha - |\Omega|) - \lambda(\beta + 2\beta_2) = E(\phi) + \frac{2}{\epsilon}\int_\Omega AW'(\phi)\,\mathrm{d}x. \tag{23}$$

where $c_1 = (1/\epsilon)\int_\Omega W'(\phi)\mathrm{d}x$.

Proof. Equation (22) follows immediately by integrating (20) over Ω; (23) follows from a variation of the domain.

Let $x(s) : \Omega \times [0,\infty) \to \Omega$ be a diffeomorphism with

$$x(0) = \mathrm{id}, \quad x'(0) = y, \quad \nabla x(0) = I \quad \text{and} \quad \nabla x = F.$$

Consider the variation of ϕ by transforming its argument with $x(s)$, i.e. let $\tilde\phi(s, X) = \phi(x(s, X))$. Denote by δ the operator $\mathrm{d}/\mathrm{d}s|_{s=0}$. Then, a few applications of the chain rule will give

$$\begin{aligned}
\delta\tilde\phi &= \nabla_i\phi y_i,\\
\delta\nabla_i\tilde\phi &= \nabla_i\nabla_k\phi y_k + \nabla_j\phi\nabla_i y_j,\\
\delta\Delta\tilde\phi &= \nabla_l\Delta\phi y_l + 2\nabla_i\nabla_j\phi\nabla_i y_j + \nabla_j\phi\Delta y_j.
\end{aligned}$$

X and x are the original and transformed coordinates, respectively. We calculate the variation of the energies with respect to s. (The following sign on λ is negative to be consistent with (20).)

$$
\begin{aligned}
0 &= \delta E(\tilde{\phi}) + \mu\delta A(\tilde{\phi}) - \lambda\delta B(\tilde{\phi}) \\
&= \int_\Omega \delta\frac{\epsilon}{2}A^2\,\mathrm{d}X - \lambda\int_\Omega \delta\frac{\epsilon}{2}|\nabla\tilde{\phi}|^2 + \frac{1}{4\epsilon}\delta W(\tilde{\phi})\mathrm{d}X + \mu\int_\Omega \delta\tilde{\phi}\,\mathrm{d}X \\
&= \int_\Omega \epsilon A\left(\delta\Delta\tilde{\phi} - \frac{1}{\epsilon^2}W''(\phi)\delta\tilde{\phi}\right)\mathrm{d}X - \lambda\int_\Omega \epsilon\nabla\phi\delta\nabla\tilde{\phi} + \frac{1}{4\epsilon}W'(\phi)\delta\tilde{\phi}\,\mathrm{d}X \\
&\quad + \mu\int_\Omega \delta\tilde{\phi}\,\mathrm{d}X \\
&= \int_\Omega \epsilon A\left(\nabla_l\Delta\phi y_l + 2\nabla_i\nabla_j\phi\nabla_i y_j + \nabla_j\phi\Delta y_j - \frac{1}{\epsilon^2}W''(\phi)\nabla_i\phi y_i\right)\mathrm{d}X \\
&\quad - \lambda\int_\Omega \epsilon\nabla_i\phi(\nabla_i\nabla_k\phi y_k + \nabla_j\phi\nabla_i y_j) + \frac{1}{4\epsilon}W'(\phi)\nabla_i\phi y_i\,\mathrm{d}X \\
&\quad + \mu\int_\Omega \nabla_i\phi y_i\,\mathrm{d}X.
\end{aligned}
$$

Many of the terms in each integrand are total derivatives. We integrate these and integrate by parts ($\phi = 1$, $\partial\phi/\partial n = 0$ on $\partial\Omega$),

$$
\begin{aligned}
0 &= \int_\Omega -\frac{\epsilon}{2}A^2\nabla\cdot y + \epsilon A(2\nabla_i\nabla_j\phi\nabla_i y_j + \phi\Delta y_j)\mathrm{d}X \\
&\quad - \lambda\int_\Omega -\frac{\epsilon}{2}|\nabla\phi|^2\nabla\cdot y + \epsilon\nabla_i\phi\nabla_j\phi\nabla_i y_j - \frac{1}{4\epsilon}W(\phi)\nabla\cdot y\,\mathrm{d}X \\
&\quad + \mu\left(\int_{\partial\Omega} y\cdot n\,\mathrm{d}S - \int_\Omega \phi\nabla\cdot y\,\mathrm{d}X\right).
\end{aligned}
$$

Setting $y = x$, $\nabla\cdot y = 3$, $\nabla_i y_j = \delta_i^j$ and $\Delta y = 0$, the variational equation is

$$
\begin{aligned}
0 &= \int_\Omega -3\frac{\epsilon}{2}A^2 + 2\epsilon A\Delta\phi\,\mathrm{d}X - \lambda\int_\Omega -3\frac{\epsilon}{2}|\nabla\phi|^2 + \epsilon|\nabla\phi|^2 \\
&\quad - \frac{3}{4\epsilon}W(\phi)\mathrm{d}X + \mu\left(3|\Omega| - 3\int_\Omega \phi\,\mathrm{d}X\right) \\
&= E(\phi) + \frac{2}{\epsilon}\int_\Omega AW'(\phi)\mathrm{d}X + \lambda(\beta + 2\beta_2) + 3\mu(|\Omega| - \alpha). \qquad \square
\end{aligned}
$$

Remark 3.1. Readers may note that (23) may also be recovered by multiplying (20) by the test function $\nu = x\cdot\nabla\phi$. Viewing (23) this way is a variant of what is known as the Pohozaev identity. It can be shown that for a sufficiently smooth solution to a variational problem, variation of the domain is equivalent to the Pohozaev identity. See section 9.4 of [13] for a reference.

Recall from theorems 2.1 and 2.2, if ϕ_ϵ satisfies (A1) and (A2), then $\phi = q(d/\epsilon) + g$ where $q(t) = \tanh(t/\sqrt{2})$ and $q'' = W'(q)$. An equivalent identity is $W'(q) = qq'/\sqrt{2}$. First, we demonstrate that (22) and (23) can be solved for λ and μ.

Lemma 3.2. *If ϕ_ϵ satisfies (A1) and (A2), then*

$$
J = \begin{pmatrix} |\Omega| & c_1 \\ 3(\alpha - |\Omega|) & -(\beta + 2\beta_2) \end{pmatrix} \tag{24}
$$

is invertible. Again $c_1 = (1/\epsilon)\int_\Omega W'(\phi)\mathrm{d}x$.

Proof.

$$\det(J) = -|\Omega|(\beta + 2\beta_2) + 3(\alpha - |\Omega|)c_1.$$

Using lemma 2.2 with $p = W'(q) = qq'/\sqrt{2}$

$$c_1 = \frac{1}{\epsilon}\int_\Omega W'(\phi)\mathrm{d}x = \frac{1}{\epsilon}\int_\Omega W'(q) + W''(q)g + W'''(q)g^2\,\mathrm{d}x = \mathrm{o}(1).$$

Thus,

$$\det(J) = -|\Omega|(\beta + 2\beta_2) + \mathrm{o}(1) \neq 0$$

for sufficiently small ϵ. □

Returning to expressions (22) and (23), we see that the right-hand side contains terms of the form $AP(\phi)$ for $p(\phi) = W''(\phi)$, $W'(\phi)$. Note that

$$A = \Delta\phi - \frac{1}{\epsilon^2}W'(\phi) = \frac{1}{\epsilon}q'\Delta d - \frac{1}{\epsilon^2}W''(q)g + \mathrm{o}(1). \tag{25}$$

For $p = W'(q)$, $P = W(q)$ and $p = W''(q)$, $P = W'(q)$ in lemma 2.2, it follows that

$$\begin{aligned}\frac{1}{\epsilon}\int_\Omega Ap(\phi)\mathrm{d}x &= \frac{1}{\epsilon^2}\int_\Omega p(q)q'\Delta d\,\mathrm{d}x - \frac{1}{\epsilon^3}\int_\Omega p(q)W''(q)g\,\mathrm{d}x \\ &= -\frac{1}{\epsilon^3}\int_\Omega p(q)W''(q)g\,\mathrm{d}x + \mathrm{O}(1).\end{aligned} \tag{26}$$

In order to show that λ and μ are uniformly bounded in ϵ, we must control the multiples of $1/\epsilon^3$. We invert J and expand (20) up to lowest order to derive an integral equation in g and q. The desired bound will then follow. Some arithmetic will show that

$$\epsilon\Delta A = -\frac{1}{\epsilon^3}6((q')^2 + qq'')g + \frac{1}{\epsilon^2}q'''\Delta d + \mathrm{O}\left(\frac{1}{\epsilon}\right), \tag{27}$$

$$\frac{1}{\epsilon}AW''(\phi) = -\frac{1}{\epsilon^3}(W''(q))^2 g + \frac{1}{\epsilon^2}W''(q)q'\Delta d + \mathrm{O}\left(\frac{1}{\epsilon}\right). \tag{28}$$

Rather than write out the $\mathrm{O}(1/\epsilon)$ terms explicitly, we note that they take the form

$$\frac{1}{\epsilon}q'F(\nabla^k d, \nabla^l g), \qquad k, l = 0, \ldots, 4,$$

where F is polynomial. Note that $q''' = (W'(q))' = W''(q)q'$ and $6((q')^2 + qq'') - (W''(q))^2 = -2W''(q)$. Subtracting (28) from (27),

$$\epsilon\Delta A - \frac{1}{\epsilon}AW''(\phi) = \frac{1}{\epsilon^3}2W''(q)g + \frac{1}{\epsilon}q'F + \mathrm{O}(1).$$

Let $\tilde{g} = W''(q)g$.

Corollary 3.1. *If ϕ_ϵ satisfies (A1) and (A2), then, up to lowest order*

$$\lambda = \frac{1}{\epsilon^3}\int_\Omega N(q)\tilde{g}\,\mathrm{d}x + \mathrm{O}(1), \qquad \mu = \frac{1}{\epsilon^3}\int_\Omega M(q)\tilde{g}\,\mathrm{d}x + \mathrm{O}(1), \tag{29}$$

where

$$M(q) = \frac{2c_1 W'(q) - (\beta + 2\beta_2)W''(q)}{\det(J)},$$

$$N(q) = \frac{2|\Omega|W'(q) - 3(\alpha - |\Omega|)W''(q)}{\det(J)}.$$

Proof. Simply invert J and replace p in (26) by M and N. □

Corollary 3.2. *If ϕ_ϵ satisfies (A1) and (A2), then the Euler–Lagrange equation (20) is equivalent to*

$$2\tilde{g} + \left(q'\Delta d - \frac{1}{\epsilon}\tilde{g}\right)\int_\Omega N(q)\tilde{g}\,\mathrm{d}x + \int_\Omega M(q)\tilde{g}\,\mathrm{d}x = -\epsilon^2 q' F + \mathrm{O}(\epsilon^3), \qquad (30)$$

where $M(q)$ and $N(q)$ are given in corollary 3.1. Further,

$$\|\tilde{g}\|_{L^\infty(\Omega)} = \mathrm{O}(\epsilon^2)$$

and

$$\int_\Omega M(q)\tilde{g}\,\mathrm{d}x, \qquad \int_\Omega N(q)\tilde{g}\,\mathrm{d}x = \mathrm{O}(\epsilon^3).$$

Proof. Equation (30) follows from the above observations. We point out here that M and N are both linear combinations of the polynomials W'' and W', and so we may apply lemma 2.2. Multiplying (30) by $M(q)$, and integrating over Ω,

$$\left(2 + \int_\Omega M(q)\,\mathrm{d}x\right)\int_\Omega M(q)\tilde{g}\,\mathrm{d}x + \left(\int_\Omega q'M(q)\Delta d\,\mathrm{d}x + \frac{1}{\epsilon}\int_\Omega M(q)\tilde{g}\,\mathrm{d}x\right)\int_\Omega N(q)\tilde{g}\,\mathrm{d}x$$
$$= -\epsilon^2\int_\Omega q'M(q)F\,\mathrm{d}x + \mathrm{O}(\epsilon^3).$$

Performing the same operation with $N(q)$ in place of $M(q)$, and collecting like orders yields

$$\left(2 + \int_\Omega M(q)\mathrm{d}x\right)\int_\Omega M(q)\tilde{g}\,\mathrm{d}x + \mathrm{o}(1)\int_\Omega N(q)\mathrm{d}x = \mathrm{O}(\epsilon^3),$$
$$(2 + \mathrm{o}(1))\int_\Omega N(q)\tilde{g}\,\mathrm{d}x + \int_\Omega N(q)\mathrm{d}x\int_\Omega M(q)\tilde{g}\,\mathrm{d}x = \mathrm{O}(\epsilon^3).$$

The corollary will follow if $2+\int_\Omega M(q)\mathrm{d}x$ or $2/|\Omega| + M(q)$ are bounded uniformly away from zero. Note that $M(q) = W''(q)/|\Omega| + \mathrm{o}(1)$ and since clearly

$$2 + W''(q) = 3q^2 + 1 > 1,$$

the result follows. □

Corollary 3.3. *If ϕ_ϵ satisfies (A1) and (A2), then λ and μ are uniformly bounded in the limit $\epsilon \to 0$.*

Proof. It is a simple consequence of combining corollaries 3.1 and 3.2. □

Corollary 3.4. *If ϕ_ϵ satisfies (A1) and (A2a), then the Euler–Lagrange equation (20) is equivalent to*

$$q'\Delta^2 d + \frac{1}{\epsilon}q''((\Delta d)^2 + 2\nabla d\nabla\Delta d) + \lambda q'\Delta d + \mu = 0. \qquad (31)$$

Proof. This follows from a calculation with $\phi = q(d/\epsilon)$.

$$\epsilon\Delta A = \Delta(q'\Delta d)$$
$$= \frac{1}{\epsilon^2}q'''\Delta d + \frac{1}{\epsilon}q''((\Delta d)^2 + 2\nabla d\nabla\Delta d) + q'\Delta^2 d,$$
$$\frac{1}{\epsilon}AW''(\phi) = \frac{1}{\epsilon^2}q'W''(q)\Delta d.$$

Equation (31) follows after noting that $q''' = W''(q)q'$. □

4. Convergence to the Willmore problem

The first corollary of theorem (2.2) is that the elastic bending energy is stable with respect to perturbations of the form $\epsilon h + g$ around solutions of (8)–(10). A second, more profound result is the convergence of the respective Euler–Lagrange equations. The limiting interface is a critical point of the mean curvature energy in the space of surfaces.

Theorem 4.1. *If ϕ_ϵ solves (8)–(10) and satisfies (A2) and (A1), then,*

$$E(\phi) \longrightarrow 4c_0 \int_{\Gamma_0} H^2 \, \mathrm{d}s,$$

where H is the mean curvature of the limit interface Γ_0 and $c_0 = \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sech}^4(s/\sqrt{2})\mathrm{d}s$.

Proof. From theorems 2.1 and 2.2, $\phi = q(d/\epsilon)$ where $q(t) = \tanh(t/\sqrt{2})$. Then, by lemma 2.1,

$$\begin{aligned}\lim_{\epsilon\to 0} E(\phi) &= \lim_{\epsilon\to 0} \frac{1}{2\epsilon}\int_\Omega \operatorname{sech}^4\left(\frac{\mathrm{d}(x)}{\epsilon}\right)(\Delta d)^2 \mathrm{d}x \\ &= \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{sech}^4(t)\,\mathrm{d}t \int_{\Gamma_0} (\Delta d(s))^2 \mathrm{d}s = 4c_0 \int_{\Gamma_0} H^2\,\mathrm{d}s.\end{aligned}$$ □

Before proceeding with the final theorem, we define what is meant by the convergence of a function defined on Ω to a function defined on a surface.

Definition 4.1. *Let $\Gamma \subset \Omega$ be smooth, compact surface and η a function defined on Γ. An* extension *of η is a function $\tilde\eta$, whose restriction to Γ equals η and is locally constant along the integral curves of ∇d.*

Definition 4.2. *Let $\Gamma \subset \Omega$ be a smooth surface and $h \in C(\Gamma; \mathbb{R})$. We say a family of functions $\{\sigma_\epsilon\}_{\epsilon>0}$ on Ω converges to h if*

$$\lim_{\epsilon\to 0} \int_\Omega \sigma_\epsilon \tilde\eta \,\mathrm{d}x = \int_\Gamma h\eta\,\mathrm{d}S \tag{32}$$

for all smooth functions η on Γ and compactly supported extensions $\tilde\eta$.

The existence of an extension $\tilde\eta$ follows from the construction of the integral curves $y(t, x)$ of ∇d, laid out in lemma 2.1. Let $\delta > 0$ be such that $y(t, x)$ is defined for $|t| < \delta$ and all $x \in \Gamma$. For $|d(x)| < \delta$, define $\tilde\eta(x) = \eta(z)$, where $x = y(d(x), z)$. Then, smoothly extend $\tilde\eta$ to the rest of Ω. We have the additional property that

$$\nabla\tilde\eta(x)\cdot\nabla d(x) = 0, \qquad \forall |d(x)| < \delta. \tag{33}$$

This follows by noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde\eta(y(t,z)) = \frac{\mathrm{d}}{\mathrm{d}t}\eta(z) = 0, \qquad \forall |t| < \delta,$$

while on the other hand

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde\eta(y(t,z)) = \nabla\tilde\eta(y(t,z))\cdot\dot y(t,z) = \nabla\tilde\eta(y(t,z))\cdot\nabla d(y(t,z)).$$

Theorem 4.2. *If ϕ_ϵ solves (8)–(10) and satisfies (A2a) and (A1), then the (scaled) variation of the elastic bending energy, $\delta E/\delta\phi$, converges to the Euler–Lagrange function of the Willmore energy, i.e.*

$$\frac{1}{\epsilon} q' \frac{\delta E}{\delta\phi} \longrightarrow 4c_0(\Delta_\Gamma H + 2H(H^2 - K))$$

in the sense of (32).

Proof. We prove that

$$\lim_{\epsilon\to 0}\frac{\delta E}{\delta\phi}\left[\frac{1}{\epsilon}q'\tilde{\eta}\right] = 4c_0\int_\Gamma(\Delta_{\Gamma_0}H + 2H(H^2-K))\eta\,\mathrm{d}S$$

for all smooth η with compactly supported extension $\tilde{\eta}$. In view of corollary (3.4),

$$\frac{\delta E}{\delta\phi}[\nu] = \int_\Omega\left(q'\Delta^2 d + \frac{1}{\epsilon}q''((\Delta d)^2 + 2\nabla d\nabla\Delta d)\right)\nu\,\mathrm{d}x \tag{34}$$

for any $\nu\in C_c^\infty(\Omega)$.

Some useful identities are

$$\Delta d = 2H, \tag{35}$$

$$H^2 - K = \tfrac{1}{2}\mathrm{tr}\,(\nabla^2 d)^2 - \tfrac{1}{4}(\Delta d)^2. \tag{36}$$

Consider the splitting of the gradient operator into its normal ($n\,n\cdot\nabla$) and surface (∇_Γ) components:

$$\nabla^i_\Gamma = \nabla^i - n_i n_j\nabla^j, \tag{37}$$

$$\Delta_\Gamma = \Delta - 2Hn_i\nabla^i - n_i n_j\nabla^i\nabla^j. \tag{38}$$

The reader should also note that $n_i\nabla_i n_k = n_i\nabla_k n_i = 0$ because $n = \nabla d$, $|n| = 1$. The corresponding terms of the Willmore equation in terms of the volumic variable d are derived in the following lemma.

Lemma 4.1.

$$\Delta_\Gamma H = \tfrac{1}{2}(\Delta^2 d - \nabla\cdot(\nabla d\nabla d\cdot\nabla\Delta d)), \tag{39}$$

$$2H(H^2-K) = -\tfrac{1}{4}\nabla\cdot(\nabla d(\Delta d)^2). \tag{40}$$

Proof. Using identities (35) and (38),

$$\begin{aligned}\Delta^2 d = 2\Delta H &= 2\Delta_\Gamma H + 4Hn_k\nabla^k H + 2n_i n_k\nabla^i\nabla^k H\\ &= 2\Delta_\Gamma H + 2\nabla^i n_i n_k\nabla^k H + 2n_i n_k\nabla^i\nabla^k H\\ &= 2\Delta_\Gamma H + 2\nabla^i(n_i n_k\nabla^k H)\\ &= 2\Delta_\Gamma H + \nabla\cdot(\nabla d\nabla d\cdot\nabla\Delta d).\end{aligned}$$

This is (39). Now for (40), we use identity (36),

$$\begin{aligned}2H(H^2-K) &= \tfrac{1}{4}\Delta d(2\mathrm{tr}(\nabla^2 d)^2 - (\Delta d)^2)\\ &= \tfrac{1}{2}\nabla\cdot(\Delta d\nabla d\nabla^2 d) - \tfrac{1}{2}\nabla\Delta d\nabla d\nabla^2 d - \tfrac{1}{2}\Delta d\nabla d\nabla\Delta d - \tfrac{1}{4}(\Delta d)^3\\ &= -\tfrac{1}{4}\nabla d\nabla(\Delta d)^2 - \tfrac{1}{4}(\Delta d)^3 = -\tfrac{1}{4}\nabla\cdot(\nabla d(\Delta d)^2).\end{aligned}$$

□

Using the previous identities and referring to (39) and (40), we derive

$$\begin{aligned}2(q')^2(\Delta_\Gamma H + 2H(H^2-K)) &= (q')^2(\Delta^2 d - \nabla\cdot(\nabla d\nabla d\cdot\nabla\Delta d)) - \frac{(q')^2}{2}\nabla\cdot(\nabla d(\Delta d)^2)\\ &= (q')^2\Delta^2 d + \nabla(q')^2\nabla d(\nabla d\cdot\nabla\Delta d + \tfrac{1}{2}(\Delta d)^2) - \nabla\cdot((q')^2R)\\ &= (q')^2\Delta^2 d + \frac{1}{\epsilon}q'q''|\nabla d|^2((\Delta d)^2 + 2\nabla d\cdot\nabla\Delta d) - \nabla\cdot((q')^2R)\\ &= q'\left(q'\Delta^2 d + \frac{1}{\epsilon}q''((\Delta d)^2 + 2\nabla d\cdot\nabla\Delta d)\right) - \nabla\cdot((q')^2R),\end{aligned}$$

where $R = \nabla d(\nabla d \cdot \nabla \Delta d + (\Delta d)^2/2)$. Thus, dividing through by ϵ, multiplying by the test function ν and integrating over Ω,

$$\int_\Omega \frac{2(q')^2}{\epsilon}(\Delta_\Gamma H + 2H(H^2 - K))\nu \,\mathrm{d}x = \frac{\delta E}{\delta \phi}\left[\frac{1}{\epsilon}q'\nu\right] - \int_\Omega \frac{1}{\epsilon}\nabla \cdot ((q')^2 R)\nu \,\mathrm{d}x$$

for all $\nu \in C_c^\infty(\Omega)$. We now restrict the class of ν to extensions of functions on Γ, thereby limiting the test space only to functions defined on the surface. Let $\eta \in C^\infty(\Gamma_0)$ and $\tilde{\eta}$ a compactly supported extension of η. Define the function $\tilde{\eta}_\epsilon(x) = \tilde{\eta}(\Psi_\epsilon^{-1}(x))$, where $\{\Psi_\epsilon\}_{\epsilon>0}$ are given in (A1). Let $\nu = \tilde{\eta}_\epsilon$. Note that R is a multiple of ∇d and so by (33), $R \cdot \nabla \tilde{\eta} = 0$ for $|d(x)| < \delta$. Thus, after integrating by parts,

$$\left|\int_\Omega \frac{1}{\epsilon}\nabla \cdot ((q')^2 R)\tilde{\eta}_\epsilon \,\mathrm{d}x\right| = \left|\int_{\{|d(x)|>\delta\}} \frac{1}{\epsilon}(q')^2 R \cdot \nabla \tilde{\eta}_\epsilon \,\mathrm{d}x\right| \leqslant \frac{1}{\epsilon}|\Omega|\,\|R\nabla\tilde{\eta}_\epsilon\|_{L^\infty(\Omega)}(q')^2\left(\frac{\delta}{\epsilon}\right).$$

The latter quantity vanishes exponentially in ϵ since $\|R \cdot \nabla \tilde{\eta}_\epsilon\|_{L^\infty(\Omega)}$ is bounded independently of ϵ. Thus, by lemma 2.1,

$$\begin{aligned} 4c_0 \int_{\Gamma_0} (\Delta_{\Gamma_0} H + 2H(H^2 - K))\eta \,\mathrm{d}S &= \lim_{\epsilon\to 0} \frac{1}{\epsilon}\int_\Omega 2(q')^2(\Delta_{\Gamma_\epsilon} H + 2H(H^2 - K))\tilde{\eta}_\epsilon \,\mathrm{d}S \\ &= \lim_{\epsilon\to 0} \frac{\delta E}{\delta \phi}\left[\frac{1}{\epsilon}q'\tilde{\eta}\right]. \end{aligned}$$

Hence, convergence in the sense of (32) follows for $\sigma_\epsilon = q'(\delta E/\delta\phi)/\epsilon$. □

Remark 4.1. Theorem 4.2 demonstrates the convergence of the respective *unconstrained* Euler–Lagrange equations. The theorem may be modified to show convergence of the constrained Euler–Lagrange equations as explained below. For simplicity we have elected to demonstrate only the former.

The Lagrange multipliers λ and μ are uniformly bounded so all that remains to be shown is convergence of the constraint variations. Following theorem 4.2,

$$\frac{\delta B}{\delta \phi}\left[\frac{1}{\epsilon}q'\tilde{\eta}\right] = \frac{1}{\epsilon}\int_\Omega (q')^2 \Delta \,\mathrm{d}\tilde{\eta}\,\mathrm{d}x \longrightarrow 4c_0 \int_{\Gamma_0} H\eta \,\mathrm{d}S$$

and

$$\frac{\delta A}{\delta \phi}\left[\frac{1}{\epsilon}q'\tilde{\eta}\right] = \frac{1}{\epsilon}\int_\Omega q'\tilde{\eta}\,\mathrm{d}x \longrightarrow c_1 \int_{\Gamma_0} \eta \,\mathrm{d}S,$$

where $c_1 = \int_{-\infty}^{\infty} q'(t)\mathrm{d}t$. It is, however, well known that H and 1 are the variational forms for minimal surfaces and volume constrained surfaces, respectively.

5. Conclusion

We have demonstrated through rigorous asymptotic expansions that the phase field variational problem formally converges to the Willmore variational problem. In particular, define $E_0(\Gamma_0)$ as (1). The contents of theorems 4.1 and 4.2 can then be summarized as follows: if the zero set of ϕ_ϵ converges to Γ_0, then $E_\epsilon(\phi_\epsilon)$ converges to $E_0(\Gamma_0)$. Further, if the above holds, $\delta E_\epsilon/\delta\phi = 0$ implies that $\delta E_0/\delta\Gamma = 0$ when evaluated at ϕ_ϵ and Γ_0, respectively. However, we have not demonstrated that this holds true for general ϕ_ϵ and Γ_0 nor that $\delta E_0/\delta\Gamma = 0$ holds uniquely for $E_0(\Gamma_0) = \min_{S\in\mathcal{A}} E_0(S)$ for the admissable surface class $\mathcal{A}$. For smooth surfaces atleast, our results give the construction of the lower bound ($\exists\phi_\epsilon \to \Gamma_0 : \limsup_{\epsilon\to 0} E_\epsilon(\phi_\epsilon) \leqslant E_0(\Gamma_0)$), in the Γ-convergence framework. Generalizing our problem to this framework will be the subject of subsequent papers.

Our main goal is to develop a dynamic model of curvature driven surface motion based on energetic formulations. We will compare our phase field approach with other established curvature and interface dynamic models; see [14–16]. Then, we will couple the phase field with other fields, e.g. velocity and electric fields, to ultimately extend the work in [1] to model membrane dynamics in electro-rheological fluids.

Appendix

A.1. Uniform energy bound

An important part of the convergence argument is that the minimal elastic bending energy is uniformly bounded for sufficiently small ϵ. This fact is developed in the theorems below.

Theorem A.1. *Let $\Gamma \subset \Omega$ be a smooth, compact surface. Let $\alpha = |\Omega| - 2\mathrm{Vol}(\Gamma)$ and $\beta = c_0|\Gamma|$. Then, for $\delta > 0$ sufficiently small and $\epsilon > 0$, there exists $u_\epsilon \in H^2(\Omega)$ satisfying*

$$\begin{aligned}
&|A(u_\epsilon) - \alpha| \leqslant C(1+\delta)\epsilon,\\
&|B_\epsilon(u_\epsilon) - \beta| \leqslant C\epsilon^m\delta^{-n},\\
&\left|E_\epsilon(u_\epsilon) - 4c_0\int_\Gamma H^2\,\mathrm{d}S\right| \leqslant C(\epsilon + \epsilon^m\delta^{-n})
\end{aligned}$$

for $C = C(\Gamma, \delta)$ and $m, n \in \mathbb{N}$. Also, $c_0 = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sech}^4(t)\mathrm{d}t$.

Proof. Let U be a neighbourhood of Γ within which $d(x)$, the signed distance from x to Γ, is smooth. Choose δ such that $\{|d| < 2\delta\} \subset U$. Define $U_- = \{d < -2\delta\}$, $U_+ = \{d > 2\delta\}$, $U_1 = \{|d| < \delta\}$, and $U_2 = \{\delta < |d| < 2\delta\}$. Define u as follows:

$$u(x) = \begin{cases} +1 & \text{for } x \in U_+,\\ -1 & \text{for } x \in U_-,\\ q\left(\dfrac{h(d(x))}{\epsilon}\right) & \text{for } x \in U_1 \cup U_2, \end{cases}$$

where $q(t) = \tanh(t/\sqrt{2})$, $h \in C^2((-2\delta, 2\delta))$ is monotone, $h(t) = t$ for $|t| \leqslant \delta$, $\lim_{t\to\pm 2\delta} h(t) = \pm\infty$, and

$$\left|\frac{(h^{(m)}(t))^n}{(h(t))^p}\right| \leqslant \frac{C}{\delta^r}, \qquad \forall t \in (-2\delta, 2\delta)$$

for some $p > 1, r > 0$ and all $0 \leqslant mn \leqslant p$, and some $C > 0$. Letting $h(t) = t + 1/(4\delta^2 - t^2)$ for $|t| \geqslant 3\delta/2$ demonstrates the existence of such a function. First, note that $q'(t) = \mathrm{sech}^2(t) \leqslant (\exp(-t))^2 \leqslant p^2(p-1)^2 t^{-2p}$. Some preliminary pointwise estimates are

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t} q\left(\frac{h(t)}{\epsilon}\right)\right| &= \left|q'\left(\frac{h(t)}{\epsilon}\right)h'(t)\right|\\
&\leqslant \epsilon^{2p}p^2(p-1)^2\left|\frac{h^{(1)}(t)}{(h(t))^{2p}}\right|\\
&\leqslant \frac{C\epsilon^{2p}}{\delta^r}
\end{aligned}$$

and

$$\left|\frac{\mathrm{d}^2}{\mathrm{d}t^2}q\left(\frac{h(t)}{\epsilon}\right)\right| = \left|q''\left(\frac{h(t)}{\epsilon}\right)\frac{(h'(t))^2}{\epsilon^2} + q'\left(\frac{h(t)a}{\epsilon}\right)\frac{h''(t)}{\epsilon}\right|$$
$$\leqslant \epsilon^{2p}p^2(p-1)^2\left(\left|\frac{(h^{(1)}(t))^2 + h^{(2)}(t)}{(h(t))^{2p}}\right|\right)$$
$$\leqslant \frac{C\epsilon^{2p}}{\delta^r}$$

for some $C > 0$ independent of ϵ and δ. u is the usual phase function in U_1. In U_+ and U_-, u takes the values ± 1 identically. The patching of these two functions takes place in U_2. Since U_2 is distance δ away from the interface, we may pick ϵ_0 small enough so that the energy densities do not 'see' the patch. We will frequently use the smoothness of Γ to guarantee the existence of a constant $C > 0$ such that $U_i \leqslant C\delta$, for $i = 1, 2$. With this inequality in mind, consider the first constraint inequality. Following the notation of lemma 2.1,

$$|A(u) - \alpha| = \left|\int_\Omega u\,\mathrm{d}x - \int_{\{d>0\}}\mathrm{d}x + \int_{\{d<0\}}\mathrm{d}x\right|$$
$$= \int_{U_2\cup U_1} q\left(\frac{h(d)}{\epsilon}\right) - \mathrm{sign}(d)\mathrm{d}x$$
$$\leqslant A_1 + A_2,$$

where

$$A_1 = \left|\int_{-\delta}^{\delta}\left(q\left(\frac{t}{\epsilon}\right) - \mathrm{sign}(t)\right)\int_\Gamma J(t,z)\mathrm{d}S(z)\mathrm{d}t\right|, \qquad A_2 = \max_{\delta\leqslant|t|\leqslant 2\delta}\left|q\left(\frac{h(t)}{\epsilon}\right) - 1\right||U_2|.$$

A calculation shows that $\int_{-\delta}^{\delta}|\tanh(t/\epsilon) - \mathrm{sign}(t)|\mathrm{d}t \leqslant \epsilon(\log(2) + \exp(-2\delta/\epsilon))$. Therefore, $A_1 \leqslant \epsilon|\Gamma|C_0(\log(2) + \exp(-2\delta/\epsilon)) \leqslant C\epsilon$ where $C_0 = \|J\|_{L^\infty(\Gamma\times(-\delta,\delta))}$. Also, $A_2 \leqslant C\exp(-2\delta/\epsilon)\delta$. Thus $|A(u) - \alpha| \leqslant C(1+\delta)\epsilon$ for some $C = C(\Gamma)$. For the second constraint,

$$|B(u) - c_0\beta| = \left|\int_\Omega \frac{\epsilon}{2}|\nabla u|^2\,\mathrm{d}x - c_0\int_\Gamma \mathrm{d}S\right|$$
$$= \left|\int_\Omega \frac{1}{2\epsilon}|q'\left(\frac{h(d)}{\epsilon}\right)h'(d)\nabla d|^2\,\mathrm{d}x - c_0\int_\Gamma \mathrm{d}S\right|$$
$$\leqslant A_1 + A_2,$$

where

$$A_1 = \left|\frac{1}{2\epsilon}\int_{U_1}\left(q'\left(\frac{d}{\epsilon}\right)\right)^2 - c_0\int_\Gamma \mathrm{d}S\right|, \qquad A_2 = \left|\frac{1}{2\epsilon}\int_{U_2}\left(q'\left(\frac{h(d)}{\epsilon}\right)h'(d)\right)^2\,\mathrm{d}x\right|.$$

By the above considerations, $A_2 \leqslant C\epsilon^{4p-1}\delta^{-2r+1}$. Also, following the notation of lemma 2.1,

$$A_1 = \left|\frac{1}{2}\int_{-\delta/\epsilon}^{\delta/\epsilon}(q'(s))^2\int_\Gamma (J(\epsilon s, z) - 1)\mathrm{d}S(z)\mathrm{d}s + \frac{1}{2\epsilon}|\Gamma|\int_{|t|\geqslant\delta/\epsilon}(q'(s))^2\mathrm{d}s\right|$$
$$\leqslant \frac{C_1|\Gamma|\epsilon}{2}\int_{-\infty}^{\infty}(q'(s))^2\mathrm{d}s + \frac{p^2(p-1)^2|\Gamma|\epsilon^{2p-1}}{2\delta^{2r}}$$
$$\leqslant C\left(\epsilon + \frac{\epsilon^{2p-1}}{\delta^{2r}}\right)$$

for some $C > 0$ and where $C_1 = \|J_t(t, z)\|_{L^\infty(\Gamma\times(-\delta,\delta))}$. Thus, we have $|B(u) - c_0\beta| \leqslant C\epsilon^m\delta^{-n}$ for some $m, n > 0$. The uniform energy bound follows in similar fashion;

$$
\begin{aligned}
E(u) &= \int_\Omega \frac{\epsilon}{2}\left(\Delta u - \frac{1}{\epsilon^2}W'(u)\right)^2 \mathrm{d}x \\
&= \frac{1}{4\epsilon}\int_{U_1}\left(q'\left(\frac{d}{\epsilon}\right)\Delta d\right)^2 \mathrm{d}x + \int_{U_2}\epsilon\left(\frac{q''(h(d))((h'(d))^2-1)}{\epsilon^2}\right. \\
&\quad \left. + \frac{q'(h(d))(h''(d)+h'(d)\Delta d)}{\epsilon}\right)^2 \mathrm{d}x \\
&\leqslant A_1 + A_2,
\end{aligned}
$$

where

$$
A_1 = \frac{1}{2\epsilon}\int_{U_1}\left(q'\left(\frac{d}{\epsilon}\right)\Delta d\right)^2 \mathrm{d}x, \qquad A_2 = \int_{U_2}\epsilon\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}q + \frac{\mathrm{d}}{\mathrm{d}t}q\Delta d - \frac{1}{\epsilon^2}W'(q)\right)^2 \mathrm{d}x
$$

and where $\mathrm{d}/\mathrm{d}t$ is taken with respect to $d(x)$. Clearly,

$$
A_2 \leqslant \delta\epsilon\left(\frac{C\epsilon^{2p}}{\delta^r}(1 + C_1) + \frac{p^2(p-1)^2\epsilon^{2p-2}}{\delta^{2p}}\right)^2 \leqslant C\left(\frac{\epsilon^{4p+1}}{\delta^{2r-1}} + \frac{\epsilon^{4p-4}}{\delta^{4p-1}}\right),
$$

where $C_1 = \|\Delta d\|_{L^\infty(U_1)}$. Also, following lemma 2.1,

$$
\begin{aligned}
\left|A_1 - 4c_0\int_\Gamma H^2\,\mathrm{d}S\right| &= \frac{1}{2}\int_{-\delta/\epsilon}^{\delta/\epsilon}(q'(s))^2\int_\Gamma(\Delta d(\eta(\epsilon s, z)))^2 J(\epsilon s, z) - (\Delta d(z))^2 \mathrm{d}S(z)\mathrm{d}s \\
&\leqslant \epsilon c_0 C_2|\Gamma|,
\end{aligned}
$$

where $C_2 = \max\{\|J\|_{L^\infty(\Gamma\times(-\delta,\delta))}, \|\Delta d\|_{L^\infty(U_1)}, \|\nabla\Delta d\|_{L^\infty(U_1)}\}$. □

Remark A.1. Theorem A.1 demonstrates the existence of a sequence of functions u_ϵ whose energies $A(u_\epsilon)$, $B_\epsilon(u_\epsilon)$, and $E_\epsilon(u_\epsilon)$ converge to α, β, and $\int_\Gamma H^2\,\mathrm{d}S$, respectively. In addition, if Γ is not a sphere, there exists $\tilde{u}_\epsilon$ with $E_\epsilon(\tilde{u}_\epsilon)$ converging to $\int_\Gamma H^2\,\mathrm{d}S$ and $A(\tilde{u}_\epsilon) = \alpha$ and $B_\epsilon(\tilde{u}_\epsilon) = \beta$ identically for all sufficiently small ϵ. In particular, $\min_{u\in\mathcal{L}} E_\epsilon(u) \leqslant E_\epsilon(\tilde{u}_\epsilon) \leqslant M$ for some M independent of ϵ.

Definition A.1. *Let*

$$
P := \{(\nu, \sigma) : \sigma > |\mathbf{S}^2||\mathbf{B}_1^3|^{-2/3}\nu^{2/3}\}.
$$

The closure of P is the set of admissible volume, surface area doubles for compact surfaces in $\mathbb{R}^3$. Note that the volume, surface area doubles for all spheres forms the boundary of P.

Lemma A.1. *Let $(\nu, \sigma) \in P$, $\alpha = |\Omega| - 2\nu$ and $\beta = c_0\sigma$. Then, there exists $\eta > 0$ and $\Gamma_{s,t}$ for $s, t > 0$ such that the mapping*

$$
z : (s, t) \longrightarrow (|\Omega| - 2Vol(\Gamma_{s,t}), c_0 SA(\Gamma_{s,t}))
$$

is onto $(\alpha - \eta, \alpha + \eta) \times (\beta - \eta, \beta + \eta)$. Further, $w : (s, t) \to \int_{\Gamma_{s,t}} H^2\,\mathrm{d}S$ is continuous.

Proof. Consider the tube surface $\Gamma_{s,t}$; a cylinder of radius s and length t, capped at both ends by a hemisphere of radius s. Then, $\mathrm{Vol}(\Gamma_{s,t}) = s^3|\mathbf{B}_1^3| + st|\mathbf{B}_1^2|$ and $\mathrm{SA}(\Gamma_{s,t}) = s^2|\mathbf{S}^2| + st|\mathbf{S}^1|$. By definition of P, there exists (s, t) so that $z(s, t) = (\alpha, \beta)$. The existence of η follows from the fact that z is open and the equivalence of the l^∞ and l^2 topologies in $\mathbb{R}^2$. Further, $\int_{\Gamma_{s,t}} H^2\,\mathrm{d}S = |\mathbf{S}^2| + (t/4s)|\mathbf{S}^1|$ so that w is continuous for $s > 0$. □

Corollary A.1. *Let* $(\mu, \sigma) \in P$, $\alpha = |\Omega| - 2\nu$ *and* $\beta = c_0\sigma$. *Then, there exist* $\epsilon_0 > 0$, $M > 0$ *and* $\{\tilde{u}_\epsilon\}_{\epsilon<\epsilon_0}$ *such that*

$$A(\tilde{u}_\epsilon) = \alpha, \quad B_\epsilon(\tilde{u}_\epsilon) = \beta, \quad E_\epsilon(\tilde{u}_\epsilon) \leqslant M, \qquad \forall \epsilon \leqslant \epsilon_0.$$

Proof. The corollary will follow from a continuity argument. We must show that approximately satisfying the constraint equations for a family of functions is enough to ensure the existence of a function which satisfies the constraint equations exactly. Let 2η, $\Gamma_{s,t}$, z, and w be given by lemma A.1. Let $A = (\alpha - \eta, \alpha + \eta) \times (\beta - \eta, \beta + \eta)$. The closure of $z^{-1}(A)$ is compact and contained in $\{t > 0\}$. Therefore, let $M = \max_{(s,t)\in z^{-1}(A)} w(s, t) + 1$.

Let $\delta_{s,t}$ and $C_{s,t}$ be the constants given by theorem A.1. Note that their dependence on (s, t) is also continuous so that we may define $\delta, C = \max_{(s,t)\in z^{-1}(A)} \delta_{s,t}, C_{s,t}$. Let $u_{\epsilon,s,t}$ also be given by theorem A.1. Choose $\epsilon_0 > 0$ sufficiently small (w.r.t. δ and C) so that

$$|A(u_{\epsilon_0,s,t}) - z_1(s, t)| \leqslant \theta, \qquad |B_\epsilon(u_{\epsilon_0,s,t}) - z_2(s, t)| \leqslant \theta$$

and $E_{\epsilon_0}(u_{\epsilon_0,s,t}) \leqslant M$, where $\theta = \eta/2$. Choose s_1, s_2, t_1, and t_2 so that $z(s_i, t_j) = (\alpha + (-1)^i\theta, \beta + (-1)^j\theta)$. Define a continuous path $\gamma_{\epsilon,t}(s)$ in $\mathbb{R}^2$ by

$$\gamma_{\epsilon,t}(s) = (A(u_{\epsilon,s,t}), B(u_{\epsilon,s,t})), \qquad s \in (s_1, s_2), \quad t \in (t_1, t_2).$$

By the definition of θ and theorem A.1, for all $\epsilon \leqslant \epsilon_0$,

$$\gamma_{\epsilon,t}(s_i) \in ((\alpha + (-1)^i\theta - \eta, \alpha + (-1)^i\theta + \eta) \times (\beta - \eta, \beta + \eta), \qquad \forall t \in (t_1, t_2).$$

Similarly,

$$\gamma_{\epsilon,t_j}(s) \in (\alpha - \eta, \alpha + \eta) \times (\beta + (-1)^j\theta - \eta, \beta + (-1)^j + \eta), \qquad \forall s \in (s_1, s_2).$$

The above inclusions show that $\gamma_{\epsilon,t}(s)$ lies in A for all $\epsilon \leqslant \epsilon_0$. In particular, the first inclusion shows that the endpoints of $\gamma_{\epsilon,t}(\cdot)$ lie on either side of the line $\{s = \alpha\}$. By continuity, $\gamma_{\epsilon,t}(s_{**}(t)) = (\alpha, a(t))$ for some $a(t)$ and $s_{**}(t) \in (s_1, s_2)$. The second inclusion shows $a(t_1) \geqslant \beta$ and $a(t_2) \leqslant \beta$, and by the continuous dependence of $a(t)$ on t, $a(t_*) = \beta$ for some $t_* \in (t_1, t_2)$. Let $s_* = s_{**}(t_*)$. Let $\tilde{u}_\epsilon = u_{\epsilon,s,t}$. Then,

$$(A(u_\epsilon), B_\epsilon(u_\epsilon)) = \gamma_{\epsilon,t_*}(s_*) = (\alpha, \beta)$$

and $E_\epsilon(u_\epsilon) \leqslant c_0 \max_{(s,t)\in z^{-1}(A)} w(s, t) + C(\epsilon_0^m\delta^{-n}) \leqslant M$ for another sufficiently small ϵ_0. (For simplicity we have not denoted the dependence on ϵ of s_*, s_{**}, a and t_*.) □

Acknowledgments

The research of QD and XW was supported in part by NSF-DMS 0196522 and NSF-ITR 0205232. The research of CL was supported in part by American Chemical Society and NSF-DMS 0405850.

References

[1] Du Q, Liu C and Wang X 2004 A phase field approach in the numerical study of the elastic bending energy for vesicle membranes *J. Comput. Phys.* **198** 450–68

[2] Willmore T J 1993 *Riemannian Geometry* (New York: Oxford Science Publications, The Clarendon Press, Oxford University Press)

[3] Glimm J, Marchesin D and McBryan O 1981 A numerical method for two phase flow with unstable interface *J. Comput. Phys.* **39** 179–200

[4] Unverdi S O and Tryggvason G 1992 A front-tracking method for viscous, incompressible multi-fluid flows *J. Comput. Phys.* **100** 25–37

[5] Li J and Renardy Y 2000 Numerical study of flows of two immiscible liquids at low reynolds number *SIAM Rev.* **42** 417–39

[6] Chang Y C, Hou T Y, Merriman B and Osher S 1996 A level set formulation of Eulerian interface capturing methods for incompressible fluid flows *J. Comput. Phys.* **124** 449–64

[7] Bronsard L and Stoth B 1997 Volume-preserving mean curvature flow as a limit of a non-local Ginzburg–Landau equation *SIAM J. Math. Anal.* **28** 769–807

[8] Rubinstein J and Sternberg P 1992 Nonlocal reaction diffusion equations and nucleation *IMA J. Appl. Math.* **48** 249–64

[9] Luckhaus S and Modica L 1989 The Gibbs–Thomson relation within the gradient theory for phase transitions *Arch. Ration. Mech. Anal.* **107** 71–83

[10] Modica L 1987 Gradient theory of phase transitions and minimal interface criteria *Arch. Ration. Mech. Anal.* **98** 123–42

[11] Hutchinson J and Tonegawa Y 2000 Convergence of phase interfaces in the van der Waals–Cahn–Hilliard theory *Calc. Var. Partial Diff. Eqns* **10** 49–84

[12] Caffarelli L A and Muler N E 1995 An L^∞ bound for solutions of the Cahn–Hilliard equation *Arch. Ration. Mech. Anal.* **133** 129–44

[13] Evans 1998 *Partial Differential Equations* (Graduate Studies in Mathematics) (Providence, RI: AMS)

[14] Evans L C, Soner M and Souganidis P E 1992 Phase transitions and generalized motion by mean curvature *Commun. Pure Appl. Math.* **45** 1097–123

[15] Glimm J, Grove J W, Li X L and Tan D C 2000 Robust computational algorithms for dynamic interface tracking in three dimensions *SIAM J. Sci. Comput.* **21** 2240–56

[16] Osher S and Sethian J 1988 Fronts propagating with curvature dependent speed: Algorithms based on Hamilton Jacobi formulations *J. Comput. Phys.* **79** 12–49