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A Phase Space Approach to the Projective Representation of the Canonical Commutation Relations

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The projective nature of the representations of canonical commutation relations on the space of states is probed via cohomological considerations. Some interesting geometrical pictures arise, which associate themselves with the phase descriptions of quantum systems.

The distinction between faithful and unfaithful representations of groups, acting on quantum spaces, is as old as Wigner's foundational work on symmetries.[1] A prime example of a group which attains only projective realizations on the space of states is the Weyl-Heisenberg group.[2),3)] Recall that the Lie algebra of this group furnishes the canonical commutation relations ($\hat{1}$ denotes the unit operator, $\hbar=1$):

$$[\hat{q}_i, \hat{q}_j]=[\hat{p}_i, \hat{p}_j]=[\hat{q}_i, \hat{1}]=[\hat{p}_i, \hat{1}]=0\,, \quad [\hat{q}_i, \hat{p}_j]=i\delta_{ij}\,. \tag{1}$$

For related reasons the Galilei group also attains projective representations on the space of states.[2)~4)]

The subject of projective representations was revisited in more recent years by Faddeev[5] who employed cohomological methods of analysis in his study. This new language was subsequently employed[6] for the purpose of reassessing the situation with the Weyl-Heisenberg group. It was determined,[6] within a discrete setting, that the two-cocycle entering the projective unitary representations of this group provides the symplectic structure necessary to define a (discrete) "quantum phase space".

In this paper we intend to gain further insight on the aforementioned phase-space construction drawing, in part, from the work in Refs. 7)~9). The latter, being much closer in spirit to Faddeev's original viewpoint, will prove to be a valuable guide in our effort. At a subsequent stage we shall extend our considerations to the Galilei group.

Consider the following situation.[7)~9)] A charged, non-relativistic quantum mechanical particle is coupled to an external electromagnetic field. Let $A_i(\boldsymbol{x})$ denote the components of the (static) vector potential. Working in the coordinate representation we proceed to effect a gauge-invariant action of translations in the wave function $\Psi(\boldsymbol{r})$. The need to employ the covariant derivative as the correct generator of translations automatically leads to a ray representation:

$$U(\boldsymbol{a})\Psi(\boldsymbol{r})=e^{-i\boldsymbol{a}\cdot\boldsymbol{D}}\Psi(\boldsymbol{r})=e^{-i\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{a}}\boldsymbol{A}(\boldsymbol{x})\cdot d\boldsymbol{x}}\Psi(\boldsymbol{r}+\boldsymbol{a})\,, \tag{2}$$

where D denotes covariant derivation ($\boldsymbol{D}\equiv\boldsymbol{\nabla}+i\boldsymbol{A}$).

On the basis of Eq. (2), one obtains a direct connection between the one-cochain

$\alpha_1(\boldsymbol{r};\boldsymbol{a})$ entering the ray representation and the one-form $A_i\,dx_i$. One simply writes

$$\alpha_1(\boldsymbol{r};\boldsymbol{a})=\int_{\boldsymbol{r}}^{\boldsymbol{r}+\boldsymbol{a}}\boldsymbol{A}\cdot d\boldsymbol{x}\,. \tag{3}$$

Note that $\boldsymbol{a}$ serves as the coordinate label for the translation group element.

Following the rules of cohomology theory, a two-cocycle is constructed according to

$$\alpha_2(\boldsymbol{r};\boldsymbol{a}_1,\boldsymbol{a}_2)=\alpha_1(\boldsymbol{r}+\boldsymbol{a}_1;\boldsymbol{a}_2)+\alpha_1(\boldsymbol{r};\boldsymbol{a}_1)-\alpha_1(\boldsymbol{r};\boldsymbol{a}_1+\boldsymbol{a}_2)=\int_S\boldsymbol{F}\cdot d\boldsymbol{s}\,, \tag{4}$$

where S is a surface bounded by the vectors $\boldsymbol{r}$, $\boldsymbol{r}+\boldsymbol{a}_1$ and $\boldsymbol{r}+\boldsymbol{a}_1+\boldsymbol{a}_2$, $\boldsymbol{F}=\boldsymbol{\nabla}\times\boldsymbol{A}$, and $d\boldsymbol{s}$ is an oriented surface element. For definiteness, S is chosen so that it corresponds to the minimal triangular surface bounded by the aforementioned three vectors.

It becomes obvious that as long as the flux through the triangle is not zero, the two-cocycle is non-trivial, and a genuine projective representation emerges.

With the above observations in hand we now depart company with Refs. 7)~9) in that we shall insist on associativity for the group action, i.e., we shall not be interested in 3-cocycles and monopoles. At the same time, we backtrack to the more primitive situation within which translations are regarded as elements of the Weyl-Heisenberg group.

In order to utilize the work of Ref. 6) we adopt a discrete set of coordinate states $\{|n\rangle\}$ and a corresponding dual basis $\{|\tilde{n}\rangle\}$ which carries a discrete "momentum" label. Both sets are periodic, with period N.*) According to Schwinger's fundamental construction,[10] there exists a pair of unitary operators U and V with the following mutual action on members of the above sets:

$$U|n\rangle=|n+1\rangle,\ U|\tilde{n}\rangle=e^{2\pi i\tilde{n}/N}|n\rangle;\ V|n\rangle=e^{2\pi in/N}|n\rangle,\ V(\tilde{n})=|\tilde{n}+1\rangle\,. \tag{5}$$

It is evident that U and V operate as generators of translations along the (discrete) q and p-directions, respectively. They can be viewed as special elements of the Weyl-Heisenberg group acting on the dual collection of discrete quantum states.

There is, now, a further aspect of Schwinger's scheme which involves the following set of operators:

$$W_{n\tilde{n}}=e^{i\pi n\tilde{n}/N}U^nV^{\tilde{n}}\,. \tag{6}$$

As it turns out, the $W_{n\tilde{n}}$ form a complete set in terms of which any dynamical operator can be constructed.[10] This occurrence parallels the role of classical monomials, in phase space variables, which provide a basis of expansion for any physical quantity.

Consider now a generic wave function $\Psi(n)$ in our discrete coordinate representation. $W_{m\tilde{m}}$ acts as a translation operator on $\Psi(n)$ via a ray representation:

$$W_{m\tilde{m}}\Psi(n)=e^{i\pi(2n+m)\tilde{m}/N}\Psi(n+m) \tag{7}$$

with one co-cycle

*) For simplicity we assume the same number of sites per direction.

$$\alpha_1(n;\, W_{m\tilde{m}})=\frac{\pi}{N}(2n+m)\tilde{m}\,. \tag{8}$$

A straightforward application of the co-boundary operation produces the following two-cocycle:

$$\alpha_2(n;\, W_{m\tilde{m}},\, W_{l\tilde{l}})=\frac{\pi}{N}(m\tilde{l}-l\tilde{m})\,. \tag{9}$$

Observe that α_2 is globally defined, i.e., it does not depend on n. Its existence, nevertheless, underlines the projective character of Weyl-Heisenberg group representations on our set of states. Independently, one verifies that

$$W_{m\tilde{m}}W_{l\tilde{l}}=e^{i\pi(m\tilde{l}-l\tilde{m})/N}W_{(m+l)(\tilde{m}+\tilde{l})}\,. \tag{10}$$

Had we worked within a restricted framework, we would have defined conventional translation operators U^m which furnish a "ray-free" representation:

$$U^m\Psi(n)=\Psi(n+m)\,. \tag{11}$$

It follows that only once the full phase space description of the system is taken into account does a non-vanishing two-cocycle and, hence, a projective representation emerge. Intuitively, we understand projectiveness to result from a translation accompanied by a "drift" in the momentum direction. Given that in the coordinate representation momentum fluctuations cannot be controlled, such drifts do have a natural, albeit implicit, presence. In a phase space context they acquire more direct meaning as they reflect localizability problems that arise on account of the canonical commutation relations.

To summarize, translations, when studied in isolation, do not give any signal of projectiveness. As part of the wider Weyl-Heisenberg group structure, on the other hand, they most certainly do. Similar comments apply to "translations" in momentum space as well.

Let us now return to the two-cocycle $\alpha_2(n; W_{m\tilde{m}},\, W_{l\tilde{l}})$ and "read" the right-hand side of (9) as a two-form, in analogy to (3). This two-form corresponds to a given area formed on a certain toroidal section of the phase space grid. We are referring to the triangular area with vertices at the points $(n, 0)$, $(n+m,\, \tilde{m})$ and $(n+m+l,\, \tilde{m}+\tilde{l})$ lying on the two-dimensional toroidal surface formed by a conjugate pair, labeled by j, of (discrete) phase space variables.

But $1/2\ |m\tilde{l}-l\tilde{m}|$ furnishes the area of the above-mentioned triangle. Therefore, we are in a position to make the following identification. Upon comparison with (4),

$$B_j=\frac{2\pi}{N_j}\,, \tag{12}$$

where j denotes the particular pair of conjugate variables whose toroidal surface is being surveyed. To retrieve physical units, let us introduce a lattice spacing a, whereupon the previous relation assumes the form

$$B_j = \frac{2\pi}{aN_j}\,. \tag{13}$$

For each conjugate pair (q_i, p_j) we have a "magnetic field"*) B_j, referred to as the "geometrical magnetic field", whose magnitude controls passage to the continuous limit:

$$\lim_{\substack{a\to 0\\ N_j\to\infty}} \frac{2\pi}{aN_j} = B_j\,. \tag{14}$$

We have thereby reached a picture, associated with the projective implementation of the Weyl-Heisenberg group on the space of states, according to which the non-vanishing of each commutator $[\hat{q}_j, \hat{p}_j]$, $j=i, \cdots, n$, induces an equivalent "geometric magnetic field" B_j. Moreover, the global nature of the two-cocycle (9) leads to the conclusion that B_j is a (scalar) constant on the jth phase-space toroidal.

The fact that our geometrical picture has been attained within a two-dimensional phase space "slice" automatically puts us in touch with the work of Dunne, Jackiw and Trugenberger[11] which addresses a similar situation. We shall come back to this point once we extend our considerations to the Galilei group.

The situation appears somewhat reversed when our reference group shifts from the Weyl-Heisenberg to that of Galilei. Recall, first, that the latter is a ten-dimensional group whose natural action takes place in $R^3 \otimes R$. It consists of time translations, space traslations, (Galilean) velocity boosts, and rotations. Adopting this particular order, we denote the generic group element by $g=(b, \boldsymbol{a}, \boldsymbol{v}, R)$. The corresponding generators are T, P_i, K_i and L_i, $i=1, 2, 3$. These generators obey the commutation relations:

$$[L_i, L_j]=\varepsilon_{ijk}L_k,\ [L_i, P_j]=\varepsilon_{ijk}P_k,\ [L_i, K_j]=\varepsilon_{ijk}K_k,\ [K_i, T]=K_i\,,$$
$$[L_i, T]=[P_i, T]=[P_i, P_j]=[K_i, K_j]=[P_i, K_j]=0\,. \tag{15}$$

We focus our attention on the commutator $[P_i, K_j]=0$ for the system of a single, free quantum mechanical particle, classically speaking, the P_i generate space and the K_i velocity translations. For free particles, of course, $P_j=mu_j$, and thus the K_j can be thought of as generators of momentum translations as well.**) Some key ingredient is thereby missing when this particular algebraic commutator is set to assume its quantum mechanical form.

As already mentioned, the situation has reversed itself in that the problem is **not** how to accomodate, representation-wise, the given algebraic commutator, but how to **alter** it as it is being transferred to the quantum domain. In dealing with this problem, Bargmann[2] realized that the unitary implementation of the Galilei group on the one-particle (with mass m) quantum system must attain projective status. In particular, he determined

*) Our two-dimensional (q, p) arrangement implies that the "magnetic field" is a scalar.

) Note, on the other hand, that in the presence of an electromagnetic field, the covariant derivative, which was used earlier as a generator of translations, corresponds to particle velocity and **not canonical momentum.

$$U(g_1)U(g_2)=e^{i\xi(g_1,g_2)}U(g_1, g_2)\,, \tag{16}$$

where $g_i=(b_i, \boldsymbol{a}_i, \boldsymbol{v}_i, R_i)$, i=1, 2, and the function ξ is given by

$$\xi(g_1, g_2)=\frac{m}{2}(\boldsymbol{a}_1\cdot R_1\boldsymbol{v}_2-\boldsymbol{v}_1\cdot R_1\boldsymbol{a}_2+b_2\boldsymbol{v}_1-R_1\boldsymbol{v}_2)\,. \tag{17}$$

From the cohomological point of view, ξ is a (global) two-cocycle which would be removable, i.e., trivial, if there existed some function $\alpha(g)$ of a single group element such that

$$\xi(g_1, g_2)=\alpha(g_1)+\alpha(g_2)-\alpha(g_{12})\,. \tag{18}$$

Let us verify, by counterexample, that such a function does not exist. Recall first the composition law for the Galilei group:

$$g_{12}=(b_1, \boldsymbol{a}_1, \boldsymbol{v}_1, R_1)(b_2, \boldsymbol{a}_2, \boldsymbol{v}_2, R_2)=(b_1+b_2,\ \boldsymbol{a}_1+R_1\boldsymbol{a}_2+b_2\boldsymbol{v}_1,\ \boldsymbol{v}_1+R_1\boldsymbol{v}_2, R_1R_2)\,. \tag{19}$$

Consider the following two cases: i) $\boldsymbol{a}_2=0$, $b_2=0$ and ii) $\boldsymbol{v}_2=0$, $b_2=0$. The triviality condition (18) gives for the first case

$$\frac{m}{2}\boldsymbol{a}_1\cdot R_1\boldsymbol{v}_2=\alpha(g_1)+\alpha(g_2)-\alpha(g_{12})\,, \tag{20}$$

and for the second

$$-\frac{m}{2}\boldsymbol{v}_1\cdot R_1\boldsymbol{a}_2=\alpha(g_1)+\alpha(g_2)-\alpha(g_{12})\,. \tag{21}$$

The mixed terms entering the left-hand sides of the above equations can only be eliminated from corresponding terms coming from $\alpha(g_{12})$. An inspection of (19) leads to the following choice:

$$\alpha(g)=\lambda\frac{m}{2}\boldsymbol{a}\cdot\boldsymbol{v}\,, \tag{22}$$

where λ is an as yet undertermined parameter. Substituting in (20) we find

$$\frac{m}{2}\boldsymbol{a}_1\cdot R_1\boldsymbol{v}_2=\lambda\frac{m}{2}\boldsymbol{a}_1\cdot\boldsymbol{v}_1-\lambda\frac{m}{2}\boldsymbol{a}_1\cdot(\boldsymbol{v}_1+R_1\boldsymbol{v}_2)\,, \tag{23}$$

while substitution in (21) gives

$$-\frac{m}{2}\boldsymbol{v}_1\cdot R_1\boldsymbol{a}_2=\lambda\frac{m}{2}\boldsymbol{a}_1\cdot\boldsymbol{v}_1-\lambda\frac{m}{2}\boldsymbol{v}_1\cdot(\boldsymbol{a}_1+R_1\boldsymbol{a}_2)\,. \tag{24}$$

We observe that $\lambda=+1$ solves the first triviality condition, whereas $\lambda=-1$ solves the second. The incompatibility between the two solutions implies that $\xi(g_1, g_2)$ is nontrivial, therefore we have a genuine two-cocycle. We note in passing that what has been determined here is the occurrence of an **anomaly**, in a similar sense by which it arises in massless fermionic field theories.[12] The novelty of the present anomaly is that it pertains to a space-time symmetry group.

On the basis of (16), Bargmann recognized[2] that a faithful unitary representation, for the single particle system, of the Galilei group **is** attainable provided that the latter is extended by a phase factor. At the infinitesimal level, Hammermesh determined[3] that the called for extension amounts to the restoration of the canonical commutation relations via the introduction of mass:

$$[K_j, P_j]=0 \xrightarrow[\text{extension}]{\text{Bargmann}} [K_i, P_j]=\mathrm{im}\delta_{ij}\,. \tag{25}$$

Once again, the canonical commutation relation has induced a new quantity, this time mass, into quantum algebraic structure. We conjecture that the (scalar) "geometrical magnetic field" introduced via the more primitive setting of the Weyl-Heisenberg group is equivalent to the "mass extension" induced on the Galilei group. We base our conjecture on the work of Ref 11), where a first order Lagrangian in two dimensions using the Lorentz force term as a "kinetic term" leads to a commutator of the form

$$[q, p]=i/B\,, \tag{26}$$

where B is a single-component magnetic field for the 2-d space spanned by the conjugate pair (q, p). The similarity with (25), per conjugate pair, is self-evident. We shall not elaborate any further on this connection in the present note. We do intend, however, to make it the focal point of a forthcoming study.

For our closing subject of discussion we wish to consider the implications of the Galilei group action on a phase space representation of **states** (for the free particle system). We adopt a scheme that we have developed in a series of papers (see, e.g., Refs. 13) and 14)), according to which a wave function $\Psi(\boldsymbol{q}, \boldsymbol{p})$ is constructed as follows:[15]

$$\Psi(\boldsymbol{q}, \boldsymbol{p})=\int d^3x\xi_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{x})\Psi(\boldsymbol{x})\,, \tag{27}$$

where $\Psi(\boldsymbol{x})$ is a conventional configuration space wave function, and the coefficient functions $\xi_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{x})$ satisfy the completeness property

$$\int\xi^*_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{x})\xi_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{x}')d^3q d^3p=\delta(\boldsymbol{x}-\boldsymbol{x}')\,. \tag{28}$$

The specific features of our "phase-space representation" scheme can be summarized by the conditions:

i) Position and momentum operators take the Van Hove form[16] (or variations there of)

$$\hat{Q}_j=i\frac{\partial}{\partial p_j}+q_j\,, \quad \hat{P}_j=-i\frac{\partial}{\partial q_j}\,, \tag{29}$$

which obey canonical commutation relations.

ii) The phase space integration measure which furnishes the inner product is specified in terms of the $\xi_{\boldsymbol{q},\boldsymbol{p}}(\boldsymbol{x})$. If we write

$$(\Psi, \Phi)=\int \Psi^*(\boldsymbol{q}, \boldsymbol{p})\Phi(\boldsymbol{q}, \boldsymbol{p})\rho(\boldsymbol{q}, \boldsymbol{p})d^3q d^3p\,, \tag{30}$$

then one consistent choice[13] is $\rho(\boldsymbol{q}, \boldsymbol{p})=e^{-1/2(q^2+p^2)}$.

iii) The wave function $\Psi(\boldsymbol{q}, \boldsymbol{p})$ can serve simultaneously as an eigenstate of a certain version of Van Hove's $\hat{Q}$ and $\hat{P}$, i.e.,

$$\hat{Q}_j\Psi(\boldsymbol{q}, \boldsymbol{p})=q_j\Psi(\boldsymbol{q}, \boldsymbol{p})\,, \quad \hat{P}_i\Psi(\boldsymbol{q}, \boldsymbol{p})=p_i\Psi(\boldsymbol{q}, \boldsymbol{p})\,. \tag{31}$$

However, under the inner product (30), **Van Hove's P and Q cannot be simultaneously defined as hermitian operators.**[14]

We wish to assess the action of Galilean translations and velocity boosts on the phase space wave function $\Psi(\boldsymbol{q}, \boldsymbol{p})$. Our first task is to characterize P_i and K_i as differential operators independent of the Van Hove choice. Consider translations first. If $U(\boldsymbol{a})$ denotes translation by the constant vector $\boldsymbol{a}$, then

$$U(\boldsymbol{a})\Psi(\boldsymbol{p}, \boldsymbol{q})=e^{ia_j\hat{P}_j}\Psi(\boldsymbol{p}, \boldsymbol{q})\,. \tag{32}$$

The infinitesimal version of the above relation reads

$$\Psi(\boldsymbol{p}, \boldsymbol{q})+a_j\frac{\partial}{\partial q_j}\Psi(\boldsymbol{p}, \boldsymbol{q})=\Psi(\boldsymbol{p}, \boldsymbol{q})+ia_j\hat{P}_j\Psi(\boldsymbol{p}, \boldsymbol{q})\,, \tag{33}$$

which implies the identification

$$\hat{P}_j=-i\frac{\partial}{\partial q_j}\,. \tag{34}$$

In a similar manner, we find for an infinitesimal boost transformation

$$\Psi(\boldsymbol{p}, \boldsymbol{q})-mv_j\frac{\partial}{\partial p_j}\Psi(\boldsymbol{p}, \boldsymbol{q})-tv_j\frac{\partial}{\partial q_j}\Psi(\boldsymbol{p}, \boldsymbol{q})=\Psi(\boldsymbol{p}, \boldsymbol{q})+imv_j\hat{K}_j\Psi(\boldsymbol{p}, \boldsymbol{q})\,. \tag{35}$$

A very specific circumstance under which $\hat{\boldsymbol{K}}$ could be indentified with the position operator can be read off the above relation. Namely, if $(\partial/\partial q_j)\Psi(\boldsymbol{p}, \boldsymbol{q})=0$, then

$$\hat{K}_j=i\frac{\partial}{\partial p_j}\,. \tag{36}$$

We recognize, in the above solution, the position operator in the momentum representation. In this case, of course, (34) is devoid of any meaning, an occurence which serves to underline the impossibility of simultaneously defining the position and momentum operators in a quantisation scheme based on faithful unitary irreducible representations of the Galilei group.

To identify a bonafide position operator, we proceed as follows. Consider the time slice $t=0$ on which (36) holds automatically. Now form the operators

$$\hat{Q}_i=\hat{K}_i+q_i\hat{I}\,. \tag{37}$$

From (35) we deduce

$$(\hat{Q}_i-q_i)\Psi(\boldsymbol{q}, \boldsymbol{p})=i\frac{\partial}{\partial p_i}\Psi(\boldsymbol{p}, \boldsymbol{q})\,. \tag{38}$$

In the position representation, where $(\partial/\partial p_i)\Psi(\boldsymbol{p}, \boldsymbol{q})=0$ and where (34) retains its validity, the above equation gives

$$\hat{Q}_i\Psi(\boldsymbol{q})=q_i\Psi(\boldsymbol{q}), \tag{39}$$

i.e., $\hat{Q}$ behaves as the true position operator in the coordinate representation, while $\hat{\boldsymbol{P}}$ is, according to (34), the corresponding momentum operator. In the general case, i.e., in a bonafide phase space representation of the Galilei group, $\hat{\boldsymbol{Q}}$ can be read off (39) as Van Hove's position operator:

$$\hat{Q}_j=i\frac{\partial}{\partial p_j}+q_j\,. \tag{40}$$

It is certainly satisfying that we have arrived at Van Hove's solution of the canonical quantisation mapping (cf. Eqs. (34) and (40)) via considerations based on the unitary implementation of the Galilei group.

One way of interpreting the above result is that, upon quantisation, the Galilean boost generators K_j gain the status of Van Hove's position operators $\hat{Q}_j$. This alternative version of the Galilean anomaly further emphasizes the importance of phase space structures as far as our understanding of the quantisation process is concerned.

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