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# A PLANE SYMMETRIC COSMOLOGICAL MODEL

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*Summary*

A plane symmetric perfect fluid cosmological model has been derived from class-one considerations. The pressure and density are found to be decreasing functions of time and lines of flow are geodesics. The coordinate system used turns out to be a co-moving system. The spatial cross-section of the universe at any given instant of time is flat. When the model represents a universe filled with disordered radiation it reduces to a particular case of the Lemaître universe. From the consideration of geodesic deviation we conclude that the Universe is expanding with time. The fourteen scalar invariants of the second order for the line-element have been evaluated and it is found that only two of them are independent. The Lemaître universe is conformal to flat space-time while this model is not so in general and hence it is not a special case of the Lemaître universe. The line-element admits a four-parameter group of motions.

1. *Derivation of the line-element.* It was shown by Singh & Pandey (1) that the Lemaître universe

$$ds^2 = -\frac{\exp[g(t)]}{\left(1+\dfrac{r^2}{4R_0^2}\right)^2}(dx^2+dy^2+dz^2)+dt^2 \tag{1}$$

is of class one, i.e. the line-element (1) can be imbedded in a flat space of one higher dimension. Also it is well-known that the Lemaître universe is conformally flat. It may be noted here that out of the fourteen scalar invariants of the second order only four are non-vanishing for (1), and of these four only two, viz. the pressure and density are independent. For Einstein and de Sitter universes which are special cases of (1) the four non-zero invariants can be expressed in terms of the radius of the universe.

In the following we derive a cosmological model of plane symmetry which is of class one but not conformal to flat space-time. The most general cylindrically symmetric metric can be written as equation (2)

$$ds^2 = A^2(dt^2-dx^2)-B^2dy^2-C^2dz^2, \tag{2}$$

where $A$, $B$, $C$ are functions of $x$ and $t$ only. In general the class of equation (2) is not greater than three. Here we confine ourselves to the case where $A$, $B$, $C$ are functions of $t$ alone. The condition that equation (2) may be of class one is given by (3)

$$R_{hijk} = e(b_{ik}b_{hj}-b_{ij}b_{hk}) \tag{3}$$

and

$$b_{ij,k}-b_{ik,j} = 0, \tag{4}$$

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where $b_{ij}$ are the coefficients of the second fundamental form. The conditions (3) and (4) when applied to equation (2) lead to

$$\begin{aligned} C_4B_{44}-C_{44}B_4 &= 0, \\ B_4A_{44}-B_{44}A_4 &= 0. \end{aligned} \tag{5}$$

Equations (5) ultimately yield the solution

$$A = k_1C+k_2, \qquad B = k_3C+k_4$$

determining the metric form (2) as

$$ds^2 = (1+\alpha_1C)^2(dt^2-dx^2)-(1+\alpha_2C)^2dy^2-C^2dz^2, \tag{6}$$

where $\alpha$'s are arbitrary constants.

The non-vanishing components of Riemann tensor are as follows:

$$\begin{aligned} R_{1212} &= -\frac{1+\alpha_2C}{1+\alpha_1C}\alpha_1\alpha_2C_4{}^2, \\ R_{1313} &= -\frac{C}{1+\alpha_1C}\alpha_1C_4{}^2, \\ R_{2323} &= -\frac{C(1+\alpha_2C)}{(1+\alpha_1C)^2}\alpha_2C_4{}^2, \\ R_{1414} &= \alpha_1\{(1+\alpha_1C)C_{44}-\alpha_1C_4{}^2\}, \\ R_{2424} &= \frac{1+\alpha_2C}{1+\alpha_1C}\{\alpha_2(1+\alpha_1C)C_{44}-\alpha_1\alpha_2C_4{}^2\}, \\ R_{3434} &= \frac{C}{1+\alpha_1C}\{(1+\alpha_1C)C_{44}-\alpha_1C_4{}^2\}. \end{aligned} \tag{7}$$

The surviving components of Ricci tensor are given by

$$\begin{aligned} R_1{}^1 &= \frac{\alpha_1}{(1+\alpha_1C)^3}\left\{C_{44}+\left(\frac{\alpha_2}{1+\alpha_2C}+\frac{1}{C}-\frac{\alpha_1}{1+\alpha_1C}\right)C_4{}^2\right\}, \\ R_2{}^2 &= \frac{\alpha_2}{(1+\alpha_1C)^2(1+\alpha_2C)}\left\{C_{44}+\frac{C_4{}^2}{C}\right\}, \\ R_3{}^3 &= \frac{1}{C(1+\alpha_1C)^2}\left\{C_{44}+\frac{\alpha_2C_4{}^2}{1+\alpha_2C}\right\}, \\ R_4{}^4 &= \frac{1}{(1+\alpha_1C)^2}\left\{\frac{\alpha_1}{1+\alpha_1C}+\frac{\alpha_2}{1+\alpha_2C}+\frac{1}{C}\right\}\left\{C_{44}-\frac{\alpha_1C_4{}^2}{1+\alpha_1C}\right\}. \end{aligned} \tag{8}$$

The scalar curvature $R$ is given by

$$R = R_1{}^1+R_2{}^2+R_3{}^3+R_4{}^4. \tag{9}$$

The surviving components of energy-momentum tensor are:

$$\begin{aligned} -8\pi T_1{}^1 &= \tfrac{1}{2}(R_1{}^1-R_2{}^2-R_3{}^3-R_4{}^4), \\ -8\pi T_2{}^2 &= \tfrac{1}{2}(R_2{}^2-R_1{}^1-R_3{}^3-R_4{}^4), \\ -8\pi T_3{}^3 &= \tfrac{1}{2}(R_3{}^3-R_1{}^1-R_2{}^2-R_4{}^4), \\ -8\pi T_4{}^4 &= \tfrac{1}{2}(R_4{}^4-R_1{}^1-R_2{}^2-R_3{}^3). \end{aligned} \tag{10}$$

In order that equation (6) may admit perfect fluid distribution, we must have

$$T_1^1 = T_2^2 = T_3^3. \tag{11}$$

The above conditions imply that

$$R_1^1 = R_2^2 = R_3^3. \tag{12}$$

Equation (12) requires

$$\alpha_1 = \alpha_2. \tag{13}$$

Equations (12) and (13) give $\alpha_1 C_{44} = 0$. $\alpha_1 = 0$ leads to flat space-time. When $C_{44} = 0$, we get

$$C = \alpha_3 t + \alpha_4,$$

$$A = B = (1 + \alpha_1\alpha_4) + \alpha_1\alpha_3 t, \tag{14}$$

where $\alpha_3$ and $\alpha_4$ are constants of integration. The metric (6) ultimately reduces to the form

$$ds^2 = (1 + \alpha t)^2(dt^2 - dx^2 - dy^2) - (1 + \beta t)^2 dz^2. \tag{15}$$

2. *Physical features.* The components of $T_j^i$, for equation (15) are given by

$$-8\pi T_1^1 = -8\pi T_2^2 = -8\pi T_3^3 = \frac{\alpha^2}{(1 + \alpha t)^4},$$

$$8\pi T_4^4 = \frac{\alpha^2}{(1 + \alpha t)^4} + \frac{2\alpha\beta}{(1 + \alpha t)^3(1 + \beta t)}. \tag{16}$$

For the perfect fluid distribution we have

$$8\pi T_j^i = (\rho + p)v^i v_j - g_j^i p, \tag{17}$$

where $v^i v_i = 1$. Equation (17) leads to

$$v^1 = v^2 = v^3 = v_1 = v_2 = v_3 = 0,$$

$$v^4 = \frac{1}{1 + \alpha t}, \qquad v_4 = 1 + \alpha t,$$

$$p = \frac{\alpha^2}{(1 + \alpha t)^4}, \tag{18}$$

and

$$\rho = \frac{\alpha^2}{(1 + \alpha t)^4} + \frac{2\alpha\beta}{(1 + \alpha t)^3(1 + \beta t)}.$$

Density and pressure are positive provided that $\alpha$ and $\beta$ are both positive or both negative. Here we take $\alpha$, $\beta$ both positive. The perfect fluid condition $\rho > 3p$ demands that $\beta > \alpha$. The equation of state for the model is given by

$$\beta\left(\frac{\alpha^2}{p}\right)^{1/4}(\rho - 3p) = (\rho - p)(\beta - \alpha) \tag{19}$$

which is consistent with the perfect fluid hypothesis. It follows that if $p = 0$, $\rho$ also vanishes. Hence the model does not admit of a distribution of discrete particles.

The flow vector indicates that in the coordinate system used the matter in the model is permanently at rest. $v^i$ also satisfies the equations of the geodesics, viz. $v^i v^j{}_{,i} = 0$, indicating that the lines of flow are geodesics. Here a comma (,) preceding a suffix denotes covariant differentiation. If $\alpha = \beta$ then $\rho = 3p$, implying that the space-time is filled with disordered radiation.

The equations of the geodesics for the metric (15) are given by

$$\frac{d^2x}{ds^2}+\frac{2\alpha}{1+\alpha t}\frac{dx}{ds}\frac{dt}{ds} = 0,$$

$$\frac{d^2y}{ds^2}+\frac{2\alpha}{1+\alpha t}\frac{dy}{ds}\frac{dt}{ds} = 0,$$

$$\frac{d^2z}{ds^2}+\frac{2\beta}{1+\beta t}\frac{dz}{ds}\frac{dt}{ds} = 0,$$

$$\frac{d^2t}{ds^2}+\frac{\alpha}{1+\alpha t}\left\{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2+\left(\frac{dt}{ds}\right)^2\right\}+\frac{\beta(1+\beta t)}{(1+\alpha t)^2}\left(\frac{dz}{ds}\right)^2 = 0. \tag{20}$$

If a particle is initially at rest, i.e. if

$$\frac{dx}{ds} = \frac{dy}{ds} = \frac{dz}{ds} = 0, \tag{21}$$

we get

$$\frac{dt}{ds} = \frac{1}{1+\alpha t}.$$

From equation (20) we find that for all such particles the components of spatial acceleration would vanish, viz.

$$\frac{d^2x}{ds^2} = \frac{d^2y}{ds^2} = \frac{d^2z}{ds^2} = 0,$$

and the particle would remain permanently at rest.

The path of light in the model (15) is obtained by setting

$$ds^2 = 0,$$

i.e.

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{1+\beta t}{1+\alpha t}\right)^2\left(\frac{dz}{dt}\right)^2 = 1 \tag{22}$$

and for the case when the velocity is along the $z$-axis equation (22) gives

$$\frac{dz}{dt} = \pm\frac{1+\alpha t}{1+\beta t} = \pm\phi(t).$$

Hence the light pulse leaving a particle at (0, 0, $z$) at time $t_1$ would arrive at the origin at a later time $t_2$ given by

$$\int_{t_1}^{t_2}\phi(t)dt = \int_0^z dz.$$

Hence

$$\phi_2(t)\delta t_2 = \phi_1(t)\delta t_1+\frac{dz}{dt}\delta t_1 = \phi_1(t)\delta t_1+U_z\delta t_1, \tag{23}$$

where $dz/dt = U_z$ is the component of the velocity of the particle along the $z$-axis at the time of emission and $\phi_1(t)$ and $\phi_2(t)$ are the values of $\phi(t)$ for $t = t_1$ and $t = t_2$ respectively. From the above equation we get

$$\delta t_2 \doteq \frac{\phi_1(t)+U_z}{\phi_2(t)}.$$

The proper time interval $\delta t_1{}^0$ between two successive wave crests as measured by the local observer moving with the source is given by

$$\delta t_1{}^0 = \left[(1+\alpha t)^2-(1+\alpha t)^2\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}-(1+\beta t)^2\left(\frac{dz}{dt}\right)^2\right]^{1/2}\delta t_1.$$

This can be written as

$$\delta t_1{}^0 = [(1+\alpha t)^2-U^2]^{1/2}\delta t_1,$$

where $U$ is the velocity of the source at the time of emission. Similarly we may write

$$\delta t_2{}^0 = (1+\alpha t)\delta t_2$$

as the proper time interval between the reception of two successive wave crests by an observer at rest at the origin.

Hence the red shift in this case is given by (4)

$$\frac{\lambda+\delta\lambda}{\lambda} = \frac{\delta t_2{}^0}{\delta t_1{}^0} = \frac{(1+\alpha t)\{(1+\alpha t_1)(1+\beta t_1)^{-1}+U_z\}}{(1+\alpha t_2)(1+\beta t_2)^{-1}\{(1+\alpha t)^2-U^2\}^{1/2}}. \tag{24}$$

3. *Some geometrical features.* The metric (15) is a plane symmetric metric as it obviously admits of the group of motions

$$X = x+a,$$

$$Y = y+b,$$

and also rotation about the $z$-axis. Further it admits the motion

$$Z = z+c.$$

Thus it admits of a four-parameter group of motions.

The transformation

$$(1+\alpha t)^2 = 2\alpha T \tag{25}$$

reduces equation (15) to

$$ds^2 = dT^2-2\alpha T(dx^2+dy^2)-\left(1+\frac{\beta}{\alpha}\{(2\alpha T)^{1/2}-1\}\right)^2 dz^2, \tag{26}$$

which is of the geodesic form indicating that the lines of parameter $T$ are geodesics orthogonal to the hypersurface $T$ = constant. The components of $C_{hijk}$ for equation (15) are given by

$$C_{1212} = \frac{2}{3}\frac{\alpha(\beta-\alpha)}{1+\beta t},$$

$$C_{1414} = C_{2424} = \frac{1}{2}C_{1212},$$

$$C_{3434} = -\frac{2}{3}\frac{\alpha(1+\beta t)}{(1+\alpha t)^2}(\beta-\alpha),$$

$$C_{1313} = C_{2323} = \frac{1}{2}C_{3434}, \tag{27}$$

showing that equation (15) is in general not conformally flat. However, if $\alpha = \beta$ it becomes conformal to flat space-time and reduces to a particular case of the Lemaître universe (1).

It is obvious that the spatial cross-section of equation (15) by $t = k$, a constant, is a flat space-time given by

$$ds^2 = -(1+\alpha k)^2(dx^2+dy^2)-(1+\beta k)^2 dz^2. \tag{28}$$

Equations of the time curves for the metric (15) are $x$ = constant, $y$ = constant, $z$ = constant, so that the components of the unit tangent vector $u^i$ are given by $(0, 0, 0, dt/ds)$, i.e. $(0, 0, 0, (1+\alpha t)^{-1})$. The unit tangent vector satisfies the equation $u^i u^j{}_{,i} = 0$. Hence the time curves are geodesics. The geodesic deviation vector $\eta^i$ is given by (2)

$$\frac{\delta^2\eta^h}{\delta s^2}+R_{ijk}{}^h\frac{dx^i}{ds}\frac{dx^k}{ds}\eta^j = 0. \tag{29}$$

In this case the equation (29) after some simplification can be written down as

$$\begin{aligned}
&\frac{d^2\eta^1}{dt^2}+\frac{\alpha}{1+\alpha t}\frac{d\eta^1}{dt} = 0,\\
&\frac{d^2\eta^2}{dt^2}+\frac{\alpha}{1+\alpha t}\frac{d\eta^2}{dt} = 0,\\
&\frac{d^2\eta^3}{dt^2}+\left(\frac{2\beta}{1+\beta t}-\frac{\alpha}{1+\alpha t}\right)\frac{d\eta^3}{dt} = 0,\\
&\frac{d^2\eta^4}{dt^2}+\frac{\alpha}{1+\alpha t}\frac{d\eta^4}{dt}+\frac{\alpha^2}{(1+\alpha t)^2}\eta^4 = 0.
\end{aligned} \tag{30}$$

The solution of the equation (30) are given by

$$\begin{aligned}
\eta^1 &= \frac{k_1}{\alpha}\log(1+\alpha t)+C_1,\\
\eta^2 &= \frac{k_2}{\alpha}\log(1+\alpha t)+C_2,\\
\eta^3 &= k_3\left\{\frac{\alpha}{\beta^2}\log(1+\beta t)-\frac{1+\alpha t}{1+\beta t}\right\}+C_3,\\
\eta^4 &= k_4(1+\alpha t)^2+C_4,
\end{aligned} \tag{31}$$

where $k$'s and $C$'s are constants of integration. Since $\eta^i$ is orthogonal to the tangent vector we have

$$\eta^4 \equiv 0 \tag{32}$$

so that $k_4 = C_4 = 0$. In order to show that the universe is expanding we suppose that initially $\eta^1$ and $d\eta^1/dt$ are non-zero, and $\eta^2$, $\eta^3$ and their first derivatives are zero. This gives

$$\begin{aligned}
k_1 &= \left(\frac{d\eta^1}{dt}\right)_0,\\
C_1 &= (\eta^1)_0,
\end{aligned} \tag{33}$$

and

$$C_2 = C_3 = k_2 = k_3 = 0.$$

Hence the magnitude of the vector $\eta^i$ is

$$\left|(1+\alpha t)\left\{\frac{k_1}{\alpha}\log(1+\alpha t)+C_1\right\}\right|. \tag{34}$$

It is clear from the expression (34) that the magnitude of the deviation vector is an increasing function of $t$ from and after a certain time $t$. This shows that the universe represented by the metric (15) is expanding.

An explicit enumeration of the fourteen scalar invariants for a $V_4$ was given by Narlikar & Karmarkar (5) as follows:

$$\begin{aligned}
I_1 &= R_i{}^i, & I_2 &= R_j{}^i R_i{}^j, \\
I_3 &= R_j{}^i R_i{}^k R_k{}^j, & I_4 &= R_i{}^j R_j{}^k R_k{}^l R_e{}^i, \\
J_1 &= A_{hijk} g^{hj} g^{ik}, & J_2 &= B_{hijk} g^{hj} g^{ik}, \\
J_3 &= E_{hijk} g^{hj} g^{ik}, & J_4 &= F_{hijk} g^{hj} g^{ik}, \\
K_1 &= C_{hijk} R^{hj} R^{ik}, & K_2 &= A_{hijk} R^{hj} R^{ik} \\
K_3 &= \bar{D}_{hijk} R^{hj} R^{ik}, & K_4 &= C_{hijk} Q^{hj} Q^{ik}, \\
K_5 &= A_{hijk} Q^{hj} Q^{ik}, & K_6 &= D_{hijk} Q^{hj} Q^{ik},
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
A_{hijk} &= C_{hipq} C_{jkrs} g^{pr} g^{qs}, \\
B_{hijk} &= C_{hipq} A_{jkrs} g^{pr} g^{qs}, \\
D_{hijk} &= B_{hijk} - \frac{J_2}{12}(g_{hj} g_{ik} - g_{hk} g_{ij}) - \frac{J_1}{4} C_{hijk}, \\
\bar{D}_{hijk} &= (J_3)^{-1/2} D_{hijk}, \\
E_{hijk} &= C_{hipq} D_{jkrs} g^{pr} g^{qs}, \\
F_{hijk} &= C_{hipq} E_{jkrs} g^{pr} g^{qs}, \\
Q_j{}^i &= R_k{}^i R_j{}^k.
\end{aligned} \tag{36}$$

The fourteen invariants for the metric (15) are as follows:

$$J_3 = J_4 = K_1 = K_4 = K_6 = 0, \tag{37}$$

$$I_1 = 3R_1{}^1 + R_4{}^4 = \frac{2\alpha(\beta-\alpha)}{(1+\alpha t)^4 (1+\beta t)}, \tag{38}$$

$$I_2 = 3(R_1{}^1)^2 + (R_4{}^4)^2, \tag{39}$$

$$I_3 = 3(R_1{}^1)^3 + (R_4{}^4)^3, \tag{40}$$

$$I_4 = 3(R_1{}^1)^4 + (R_4{}^4)^4, \tag{41}$$

$$J_1 = \frac{4}{3} I_1{}^2, \tag{42}$$

$$J_2 = \frac{4}{9} I_1{}^3, \tag{43}$$

$$K_2 = \frac{1}{8} J_1 R_1{}^1 (R_1{}^1 + R_4{}^4), \tag{44}$$

$$K_5 = \frac{1}{8} J_1 (R_1{}^1)^2 ([R_1{}^1]^2 + [R_4{}^4]^2). \tag{45}$$

$K_3$ is indeterminate. It is found that the tensor relation

$$D_{hijk} = 0 \tag{46}$$

is necessarily satisfied by the metric (15). It is also obvious from the equations (37)–(45) that all the invariants can be expressed in terms of $\rho$ and $p$.

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