Part III. Applications to the Collisionless Plasma in Fluid Model

A

Plasma Waves in the Long Wave Approximation

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(Received January 29, 1974)

The reductive perturbation method developed in Part I of this series is applied to plasma waves with long wavelength, where the fluid model will be used throughout. As typical examples, we consider the ion-acoustic, magneto-acoustic, and Alfvén waves. The ion-acoustic wave belongs to the ordinary case to which the general theory can directly be applied, while the magneto-acoustic and Alfvén waves belong to the exceptional cases noted in Part I.

§ 1. Introduction

Amongst various nonlinear waves sustained in plasma, the ion-acoustic wave in a collisionless plasma is a typical example to which the reductive perturbation method developed in I §4 (Section 4 in Part I of this series will be referred to as I§4 and so on) can directly be applied. Such an application was first made by Washimi and Taniuti[1] who succeeded in reducing the original system of equations for the ion-acoustic wave into a single Korteweg-de Vries equation under the long wave approximation. Another example to which the reductive perturbation method can also be applied is the hydromagnetic wave in a cold collisionless plasma under a uniform magnetic field. It is well known that there exist two types of low frequency modes in hydromagnetic waves; one is the magneto-acoustic wave and the other the Alfvén wave. As will be shown in §3, both modes belong to the exceptional cases noted in Part I, more precisely, for magneto-acoustic wave the coefficient of dispersion term, μ, becomes zero, while for Alfvén wave the coefficient of nonlinear term, α, vanishes. Hence one needs some modifications of the general method in Part I for each case. Historically speaking, the magneto-acoustic wave is nothing but the first example for which Gardner and Morikawa[2] derived the classical Korteweg-de Vries equation which was originally obtained at the end of the last century by Korteweg and de Vries[3] for the shallow water wave.

In this review, we first consider the ion-acoustic wave in §2 and show how the reduction to the Korteweg-de Vries equation can be made along the general method. Brief discussion will then be given on comparison between the Korteweg-de Vries soliton and the observed ion-acoustic soliton found experimentally by Ikezi et al.[4),5] Finally a modification of the Korteweg-de Vries equation will be remarked when we include effective electron-ion collisions.[6]

Section 3 of this paper will be devoted to the hydromagnetic waves including magneto-acoustic and Alfvén waves. We show how the general method in Part I should be modified for these exceptional cases. It is found that the magneto-acoustic wave is again described by the Korteweg-de Vries equation[7] except for the critical propagation angle relative to the external magnetic field. For the critical angle the Korteweg-de Vries equation is modified by an analogous equation with the dispersion term replaced by the fifth order derivative.[8] It should be noted that this equation has a solitary wave solution with oscillatory structure[9] instead of the usual "monotone" Korteweg-de Vries soliton. On the other hand, the Alfvén wave is found to be governed by the modified Korteweg-de Vries equation,[8] which has two types of solitary wave solution, one being compressive the other rarefactive. Some relating extensions will be mentioned such as inclusion of a finite temperature effect,[10),11] an inhomogeneity effect,[12] and an effect of electron-ion collisions.[13]

§ 2. Ion-acoustic wave

We consider a fully ionized collisionless plasma which consists of cold ions ($T_i=0$) and isothermal warm electrons ($T_e=\text{const}\neq 0$). There is assumed to be no external magnetic field. In such a plasma, Landau damping due to ions cannot occur, while effect of Landau damping due to electrons may be of the order of $\sqrt{m_e/m_i}$, m_i and m_e being, respectively, masses of ion and of electron. In fact, Andersen et al.[14] confirmed experimentally that the effect of Landau damping is very small provided that $T_e/T_i \gtrsim 10$. Therefore if we ignore the effect of Landau damping, the behaviour of the plasma under consideration may be described by the following two-fluid equations, equations of continuity:

$$\frac{\partial n_i}{\partial t}+\operatorname{div}(n_i \boldsymbol{v}_i)=0, \tag{2·1}$$

$$\frac{\partial n_e}{\partial t}+\operatorname{div}(n_e \boldsymbol{v}_e)=0, \tag{2·2}$$

equations of momentum:

$$\frac{\partial \boldsymbol{v}_i}{\partial t}+(\boldsymbol{v}_i\cdot\text{grad})\boldsymbol{v}_i=R_{pi}^2\boldsymbol{E}, \tag{2·3}$$

$$\frac{m_e}{m_i}\left\{\frac{\partial \boldsymbol{v}_e}{\partial t}+(\boldsymbol{v}_e\cdot\text{grad})\boldsymbol{v}_e\right\}=-\frac{1}{M^2 n_e}\text{grad } n_e-R_{pi}^2\boldsymbol{E}, \tag{2·4}$$

together with Maxwell's equations:

$$\text{rot } \boldsymbol{E}=0, \tag{2·5}$$

$$\frac{\partial \boldsymbol{E}}{\partial t}=n_e\boldsymbol{v}_e-n_i\boldsymbol{v}_i, \tag{2·6}$$

$$\text{div } \boldsymbol{E}=n_i-n_e, \tag{2·7}$$

where ions are assumed, for simplicity, to be monovalent. In the above equations, $\boldsymbol{E}$ is the electric field, n_i and n_e are the number densities of ion and of electron, respectively, $\boldsymbol{v}_i$ and $\boldsymbol{v}_e$ are the velocities of ion-fluid and of electron-fluid, respectively. All quantities have been normalized with respect to characteristic number density n_0^*, characteristic speed U_0^*, and characteristic length L_0^*, whereby characteristic electric field is defined as $E_0^*=4\pi e n_0^* L_0^*$, e being the unit electronic charge, and characteristic frequency is defined as $\omega_0^*=U_0^*/L_0^*$. The non-dimensional parameter R_{pi} denotes the normalized frequency of ion-plasma oscillation, i.e., $R_{pi}=\omega_{pi}/\omega_0^*$, $\omega_{pi}=\sqrt{4\pi e^2 n_0^*/m_i}$, and the Mach number M is defined as $M=U_0^*/a$, $a=\sqrt{\kappa T_e/m_i}$ being the ion-acoustic speed, κ the Boltzmann constant.

Since $m_e/m_i\ll 1$ for most practical plasmas, the electron inertia (the left-hand side of Eq. (2·4)) is negligible, which is compatible with the neglect of Landau damping. It should be noted here that the Mach number M can also be expressed in terms of R_{pi} and the normalized Debye length, that is, $M=(R_{pi}l_D)^{-1}$, where $l_D=L_D/L_0^*$ and $L_D=\sqrt{\kappa T_e/(4\pi e^2 n_0^*)}$. By taking the divergence of Eq.(2·6) and by employing Eq.(2·7), it is shown that only three equations are independent of each other out of four equations (2·1), (2·2), (2·6) and (2·7).

Let us now consider the one-dimensional plane waves, in which all physical quantities such as $\boldsymbol{E}(E, 0, 0)$, n_i, n_e, $\boldsymbol{v}_i(u_i, 0, 0)$, and $\boldsymbol{v}_e(u_e, 0, 0)$ are functions of one space coordinate, say x, and the time t only. Then the system of Eqs. (2·1)∼(2·7) can be written in component form as follows:

$$\frac{\partial n_i}{\partial t}+\frac{\partial(n_i u_i)}{\partial x}=0, \tag{2·1′}$$

$$\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x}=R_{pi}^2 E, \tag{2·3′}$$

$$\frac{1}{M^2 n_e}\frac{\partial n_e}{\partial x}+R_{pi}^2 E=0, \tag{2·4′}$$

$$\frac{\partial E}{\partial t}=n_e u_e - n_i u_i, \tag{2·6'}$$

$$\frac{\partial E}{\partial x}=n_i - n_e. \tag{2·7'}$$

It should be noted that Eq. (2·5) is automatically satisfied by virtue of the one-dimensionality and that the redundant equation (2·2) has been discarded. The electron inertia term in Eq. (2·4′) has been omitted. We assume here that the plasma is in undisturbed uniform state upstream at infinity, so that we impose the following boundary conditions:

$$\left.\begin{aligned} E &\longrightarrow 0, \\ n_i &\longrightarrow 1, \\ n_e &\longrightarrow 1, \\ u_i &\longrightarrow 0, \\ u_e &\longrightarrow 0 \end{aligned}\right\} \quad \text{as} \quad x \longrightarrow -\infty. \tag{2·8}$$

Assuming now a sinusoidal wave proportional to $\exp[i(kx-\omega t)]$, k and ω being, respectively, wavenumber and frequency, and linearizing the system of Eqs. (2·1′)~(2·7′), we can obtain the following linear dispersion relation:

$$V_p=\frac{\omega}{k}=\pm V_0(1+l_D^2 k^2)^{-1/2}, \qquad V_0=\frac{1}{M}, \tag{2·9}$$

where V_p denotes the phase velocity. It is evident from this that the dispersive character of the ion-acoustic wave is due to the charge separation, since the Debye length l_D is a measure of local non-neutrality. For long wavelength ($k \ll 1$), Eq. (2·9) can be expanded as

$$V_p=\pm V_0\left\{1-\frac{l_D^2}{2}k^2+O(k^4)\right\}, \tag{2·10}$$

which satisfies the necessary condition under which the system may be reduced to the Korteweg-de Vries equation (cf. Eq. (I.4·10) with $p=3$; Eq. (4·10) in Part I will be referred to as Eq. (I.4·10) and so on). In fact, eliminating E, n_i and u_e from the system of Eqs. (2·1′)~(2·7′), we have the following set of equations for n_e and u_i:[15]

$$\left.\begin{aligned} &\frac{\partial n_e}{\partial t}+\frac{\partial}{\partial x}(n_e u_i)-l_D^2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u_i\frac{\partial}{\partial x}\right)\left(\frac{1}{n_e}\frac{\partial n_e}{\partial x}\right)=0, \\ &\frac{\partial u_i}{\partial t}+u_i\frac{\partial u_i}{\partial x}+\frac{1}{M^2 n_e}\frac{\partial n_e}{\partial x}=0, \end{aligned}\right\} \tag{2·11}$$

which can be written in matrix form:

$$\frac{\partial U}{\partial t}+A(U)\frac{\partial U}{\partial x}+\prod_{\alpha=1}^{3}\left(H_\alpha\frac{\partial}{\partial t}+K_\alpha\frac{\partial}{\partial x}\right)U=0, \tag{2·12}$$

with

$$\left.\begin{aligned}
U&=\begin{pmatrix} n_e \\ u_i \end{pmatrix},\\
A&=\begin{pmatrix} u_i & n_e \\ M^{-2}n_e^{-1} & u_i \end{pmatrix},\\
H_1&=0,\\
K_1&=\begin{pmatrix} -l_D^2 & 0 \\ 0 & 0 \end{pmatrix},\\
H_2&=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\\
K_2&=\begin{pmatrix} u_i & 0 \\ 0 & 0 \end{pmatrix},\\
H_3&=0,\\
K_3&=\begin{pmatrix} n_e^{-1} & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned}\right\} \tag{2·13}$$

which is nothing but a special case of Eq.(I.4·1) with $p=3$ and $s=1$, and all the conditions presumed in I§4 are satisfied for the matrix form given by (2·12) and (2·13). Therefore it is a straight-forward matter to reduce Eq.(2·12) into the Korteweg-de Vries equation along the prescription described in I§4. In fact, $U_0=\binom{1}{0}$, so that eigenvalues of A_0 are given by $\lambda_0=\pm 1/M=\pm V_0$, which shows that the eigenvalues of A_0 represent the phase velocity of the "dispersionless" wave in the long wave limit $k\to 0$. The right and left eigenvectors for $\lambda_0=V_0$ become

$$R=\begin{pmatrix} 1 \\ 1/M \end{pmatrix} \quad \text{and} \quad L=(1/M \quad 1), \tag{2·14}$$

respectively. The other root $\lambda_0=-V_0$, which corresponds to the wave propagating along negative x-direction, leads to essentially the same results. Introducing the coordinate-stretching defined by (note that $p=3$ in Eq. (I.4·3))

$$\xi=\varepsilon^{1/2}(x-V_0t), \qquad \tau=\varepsilon^{3/2}t, \tag{2·15}$$

and expanding

$$\begin{aligned}
U&=U_0+\varepsilon U_1+\varepsilon^2 U_2+\cdots\\
&=\begin{pmatrix} 1 \\ 0 \end{pmatrix}+\varepsilon\begin{pmatrix} n_e^{(1)} \\ u_i^{(1)} \end{pmatrix}+\varepsilon^2\begin{pmatrix} n_e^{(2)} \\ u_i^{(2)} \end{pmatrix}+\cdots,
\end{aligned} \tag{2·16}$$

we obtain

$$\frac{\partial\varphi^{(1)}}{\partial\tau}+\alpha\varphi^{(1)}\frac{\partial\varphi^{(1)}}{\partial\xi}+\mu\frac{\partial^3\varphi^{(1)}}{\partial\xi^3}=0, \tag{2·17}$$

where

$$\left.\begin{aligned}
&\begin{pmatrix} n_e^{(1)} \\ u_i^{(1)} \end{pmatrix}=\varphi^{(1)}\begin{pmatrix} 1 \\ M^{-1} \end{pmatrix},\\
&\alpha=[L(R\cdot\nabla_u A_0)R]/(LR)=M^{-1}=V_0,\\
&\mu=(LK_0R)/(LR)=\frac{l_D^2}{2M}=\frac{V_0 l_D^2}{2},\\
&K_0=\prod_{\alpha=1}^{3}(-V_0H_{\alpha 0}+K_{\alpha 0})=\begin{pmatrix} V_0 l_D^2 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}\right\} \tag{2·18}$$

Soliton solution to Eq.(2·17) was first found experimentally by Ikezi et al.[4] in the UCLA double plasma device. After refining the preliminary experiment, Ikezi[5] confirmed recently under the experimental condition $T_e/T_i \gtrsim 30$ that the observed relations among width, speed, and amplitude of soliton and the observed number of emerged solitons show fairly good agreement with those obtained theoretically from Eq.(2·17). He observed, however, there are some systematic deviations of the observed results from those due to theoretical prediction. For example, the observed width of soliton *versus* its amplitude always shows lower value than the corresponding theoretical value, whilst the observed speed of soliton *versus* its amplitude always exceeds the corresponding theoretical value. They argued that these discrepancies are due to finite ion-temperature effects such as ion reflection by the wave potential and the Landau damping due to ions. In order to clarify this point, one must give up the fluid model and appeal to kinetic-theoretic approach. Such a treatment will be discussed in Part IV B.

In concluding this section, let us mention that an effect of electron-ion collisions modifies the Eq. (2·17) as

$$\frac{\partial\varphi^{(1)}}{\partial\tau}+\alpha\varphi^{(1)}\frac{\partial\varphi^{(1)}}{\partial\xi}+\mu\frac{\partial^3\varphi^{(1)}}{\partial\xi^3}-\nu M l_D^2\frac{\partial}{\partial\xi}\left(\frac{\partial\varphi^{(1)}}{\partial\tau}+\frac{1}{M}\varphi^{(1)}\frac{\partial\varphi^{(1)}}{\partial\xi}\right)=0, \tag{2·19}$$

where ν is a parameter characterizing the effective collision frequency.[6] It is found that, instead of the soliton or cnoidal wave solution to Eq.(2·17), Eq.(2·19) has either oscillatory or quasi-monotone shock wave solution according to dispersion-dominant or dissipation-dominant case. It is interesting to note that the larger the effect of dissipation becomes, the steeper becomes the shock profile. This unusual character must be due to the fact that the dissipation term (the last member of Eq. (2·19)) contains the nonlinear term which contributes to steepen the profile.

§ 3. Hydromagnetic wave

We consider a cold collisionless plasma under a uniform magnetic field. Since the velocity distribution function takes a form of the delta function, we may describe the behaviour of such a plasma by the following two-fluid model, equations of continuity:

$$\frac{\partial n_i}{\partial t}+\operatorname{div}(n_i\boldsymbol{v}_i)=0, \tag{3·1}$$

$$\frac{\partial n_e}{\partial t}+\operatorname{div}(n_e\boldsymbol{v}_e)=0, \tag{3·2}$$

equations of momentum:

$$\frac{\partial \boldsymbol{v}_i}{\partial t}+(\boldsymbol{v}_i\cdot\operatorname{grad})\boldsymbol{v}_i=R_i(\boldsymbol{E}+\boldsymbol{v}_i\times\boldsymbol{B}), \tag{3·3}$$

$$\frac{\partial \boldsymbol{v}_e}{\partial t}+(\boldsymbol{v}_e\cdot\operatorname{grad})\boldsymbol{v}_e=-R_e(\boldsymbol{E}+\boldsymbol{v}_e\times\boldsymbol{B}), \tag{3·4}$$

together with Maxwell's equations:

$$\operatorname{rot}\boldsymbol{B}=\left(\frac{U_0^*}{c}\right)^2\frac{\partial \boldsymbol{E}}{\partial t}+\frac{M_A^2R_iR_e}{R_i+R_e}(n_i\boldsymbol{v}_i-n_e\boldsymbol{v}_e), \tag{3·5}$$

$$\operatorname{rot}\boldsymbol{E}=-\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3·6}$$

$$\operatorname{div}\boldsymbol{B}=0, \tag{3·7}$$

$$\operatorname{div}\boldsymbol{E}=\frac{R_{pe}^2}{R_e}(n_i-n_e), \tag{3·8}$$

where ions are assumed, as before, to be monovalent. In the above equations, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{E}$ the electric field, n_i and n_e the number densities of ion and of electron, respectively, $\boldsymbol{v}_i$ and $\boldsymbol{v}_e$ the velocities of ion-fluid and of electron-fluid, respectively. All quantities have been normalized with respect to characteristic number density n_0^*, characteristic speed U_0^*, characteristic length L_0^*, and characteristic magnetic field B_0^*, whereby characteristic frequency is defined as $\omega_0^*=U_0^*/L_0^*$ and characteristic electric field as $E_0^*=U_0^*B_0^*$. The nondimensional parameters R_i, R_e, and R_{pe} are, respectively, the normalized ion-cyclotron, electron-cyclotron, and electron plasma frequencies, i.e., $R_i=\omega_i/\omega_0^*$, $R_e=\omega_e/\omega_0^*$, and $R_{pe}=\omega_{pe}/\omega_0^*$, where $\omega_i=eB_0^*/(m_ic)$, $\omega_e=eB_0^*/(m_ec)$, and $\omega_{pe}=\sqrt{4\pi n_0^*e^2/m_e}$; m_i and m_e being, respectively, masses of ion and of electron, e the unit electronic charge, and c the speed of light. Another nondimensional parameter M_A denotes the Alfvén Mach number defined as $M_A=U_0^*/V_A$, V_A being the Alfvén speed, i.e., $V_A=B_0^*/\sqrt{4\pi n_0^*(m_i+m_e)}$.

Since we are concerned with hydromagnetic waves, for which $U_0^*/c\ll 1$

and $R_{pe}\gg 1$, we may neglect the displacement current in Eq. (3·5) and assume the quasi-neutrality $n_i\approx n_e\equiv n$ by virtue of Eq. (3·8). It should be noted that Eq.(3·6) implies that Eq.(3·7) remains valid when it is valid initially. Thus Eqs.(3·1)~(3·8) constitute the consistent system of fundamental equations for the present problem. Eliminating $\boldsymbol{E}$ and $\boldsymbol{v}_e$ from Eqs.(3·1)~(3·8), we have[7)]

$$\frac{\partial n}{\partial t}+\operatorname{div}(n\boldsymbol{v}_i)=0, \tag{3·9}$$

$$M_A^2\frac{d\boldsymbol{v}_i}{dt}=\frac{1}{n}\operatorname{rot}\boldsymbol{B}\times\boldsymbol{B}+\frac{1}{R_e}\left\{\left(\frac{1}{n}\operatorname{rot}\boldsymbol{B}\cdot\operatorname{grad}\right)\boldsymbol{v}_i+\frac{d}{dt}\left(\frac{1}{n}\operatorname{rot}\boldsymbol{B}\right)\right\}$$
$$-\frac{1}{M_A^2R_iR_e}\left(1+\frac{m_e}{m_i}\right)\left(\frac{1}{n}\operatorname{rot}\boldsymbol{B}\cdot\operatorname{grad}\right)\left(\frac{1}{n}\operatorname{rot}\boldsymbol{B}\right), \tag{3·10}$$

$$\frac{\partial\boldsymbol{B}}{\partial t}=\operatorname{rot}(\boldsymbol{v}_i\times\boldsymbol{B})-\frac{1}{R_i}\operatorname{rot}\left(\frac{d\boldsymbol{v}_i}{dt}\right), \tag{3·11}$$

where $d/dt\equiv\partial/\partial t+(\boldsymbol{v}_i\cdot\operatorname{grad})$. Thus we arrive at a one-fluid model, that is, we can formulate the problem in terms of the magnetic field and the variables of ion-fluid only. In this sense, one may call it "magneto-ion-dynamics"[16)] in analogy with the usual magnetohydrodynamics. It is interesting to note that, in the limit of $R_i\to\infty$ and $R_e\to\infty$, the system of equations (3·9)~(3·11) takes the same form as that of the ideal (lossless) magnetohydrodynamics of negligible pressure (note that we have assumed cold plasma). Moreover R_i and R_e play a similar role to that of the hydrodynamical and magnetic Reynolds numbers, in the sense that their roles become appreciable for high frequency and short wavelength. The essential difference is that the hydrodynamical and magnetic Reynolds numbers are measures of dissipation while R_i and R_e are those of dispersion. In this sense, the magneto-ion-dynamics is a sort of "dispersive magnetohydrodynamics" and R_i and R_e may be called, respectively, dispersive ion and electron Reynolds numbers. A systematic derivation of the magneto-ion-dynamics was first made by Kawahara, Taniuti and the present author with including an effect of isothermal electron pressure.[16)] The conditions for the validity of the equations, in particular, the conditions for the validity of the neglect of the displacement current and of the charge separation are established in their paper.

Let us now consider one-dimensional plane waves and take Cartesian coordinates (x, y, z) with the x-axis parallel to the wave normal under the presence of a uniform magnetic field $\boldsymbol{B}_0(\cos\theta, \sin\theta, 0)$, where $\theta(0<\theta\leq\pi/2)$ is an angle between the wave normal and the applied uniform magnetic field. Then all quantities n, $\boldsymbol{v}_i(u, v, w)$, and $\boldsymbol{B}(B_x, B_y, B_z)$ may be assumed to be functions of one space coordinate x and the time t, except for $B_x(=\cos\theta)$ which, by virtue of Maxwell's equations, can be shown to be identically constant. The basic system of equations (3·9)~(3·11) then reduces to

$$\frac{\partial n}{\partial t}+\frac{\partial(nu)}{\partial x}=0, \tag{3·9'}$$

$$\frac{du}{dt}+\frac{1}{M_A^2 n}\frac{\partial}{\partial x}\left\{\frac{1}{2}(B_y^2+B_z^2)\right\}=0, \tag{3·10'a}$$

$$\frac{dv}{dt}-\frac{B_x}{M_A^2 n}\frac{\partial B_y}{\partial x}+\frac{1}{M_A^2 R_e}\frac{d}{dt}\left(\frac{1}{n}\frac{\partial B_z}{\partial x}\right)=0, \tag{3·10'b}$$

$$\frac{dw}{dt}-\frac{B_x}{M_A^2 n}\frac{\partial B_z}{\partial x}-\frac{1}{M_A^2 R_e}\frac{d}{dt}\left(\frac{1}{n}\frac{\partial B_y}{\partial x}\right)=0, \tag{3·10'c}$$

$$\frac{dB_y}{dt}-B_x\frac{\partial v}{\partial x}+B_y\frac{\partial u}{\partial x}-\frac{1}{R_i}\frac{\partial}{\partial x}\left(\frac{dw}{dt}\right)=0, \tag{3·11'a}$$

$$\frac{dB_z}{dt}-B_x\frac{\partial w}{\partial x}+B_z\frac{\partial u}{\partial x}+\frac{1}{R_i}\frac{\partial}{\partial x}\left(\frac{dv}{dt}\right)=0, \tag{3·11'b}$$

where $d/dt\equiv\partial/\partial t+u\partial/\partial x$. We assume, for simplicity, that the plasma is in a uniform state upstream at infinity, so that we impose the following boundary conditions:

$$\left.\begin{array}{l} n \longrightarrow 1, \\ u \longrightarrow 0, \\ v \longrightarrow 0, \\ w \longrightarrow 0, \\ B_y \longrightarrow \sin\theta, \\ B_z \longrightarrow 0 \end{array}\right\} \quad \text{as} \quad x \longrightarrow -\infty. \tag{3·12}$$

As for the case of ion-acoustic waves, it is instructive to investigate first the dispersion relation in the linearized limit. Linearizing the system of equations (3·9')~(3·11') and assuming a sinusoidal wave proportional to $\exp[i(kx-\omega t)]$, we obtain the following dispersion relation:

$$\begin{aligned} V_p^{\pm}&=\frac{\omega}{k} \\ &=\frac{1}{2M_A(1+M_A^{-2}R_i^{-1}R_e^{-1}k^2)} \\ &\times\left[\sqrt{(1+\cos\theta)^2+\left\{\left(\frac{R_e}{R_i}+\frac{R_i}{R_e}\right)\cos^2\theta+\sin^2\theta+2\cos\theta\right\}M_A^{-2}R_i^{-1}R_e^{-1}k^2}\right. \\ &\left.\pm\sqrt{(1-\cos\theta)^2+\left\{\left(\frac{R_e}{R_i}+\frac{R_i}{R_e}\right)\cos^2\theta+\sin^2\theta-2\cos\theta\right\}M_A^{-2}R_i^{-1}R_e^{-1}k^2}\right], \end{aligned} \tag{3·13}$$

where $V_p^{\pm}$ are the phase velocities, k the wavenumber, and ω the angular freuqency, and where we have discarded the wave propagating along the negative x-direction, because it leads, as for the case of ion-acoustic wave, to essentially the same results. Inspection of the relation (3·13) shows that there are three cases for which the phase velocity coincides with the group velocity. Two of them occur at $k\to 0$ and the other at $k=k_0$, say. Since we are concerned here with long waves, we shall consider the former only, while the latter has been dealt with by Mizutani and Taniuti[17] for steady oblique propagation and by Taniuti and Washimi[18] for parallel propagation including unsteady regime(see also Part III B).

For long waves with small values of k, we can expand (3·13) as power series in k^2 as follows:

$$V_p^{\pm}=V_0^{\pm}\{1+\beta^{\pm}k^2+O(k^4)\}$$

with

$$V_0^{+}=\frac{1}{M_A},\qquad V_0^{-}=\frac{\cos\theta}{M_A},$$

$$\beta^{\pm}=-\frac{1}{2M_A^2R_iR_e}\left\{1\mp\left(\sqrt{\frac{R_e}{R_i}}-\sqrt{\frac{R_i}{R_e}}\right)^2\cot^2\theta\right\}. \tag{3·13'}$$

We may interprete V_p^+ as the phase velocity of the magneto-acoustic wave while V_p^- as that of the Alfvén wave, because V_0^+ and V_p^- represent, respectively, the phase velocities of the magneto-acoustic and Alfvén waves for ideal (non-dispersive) magnetohydrodynamics.

It is obvious from (3·13') that the coefficient of k^2 for the magneto-acoustic wave, $V_0^+\beta^+$, vanishes at the critical angle θ_c given by

$$\theta_c=\tan^{-1}\left(\sqrt{\frac{R_e}{R_i}}-\sqrt{\frac{R_i}{R_e}}\right)=\tan^{-1}\left(\sqrt{\frac{m_i}{m_e}}-\sqrt{\frac{m_e}{m_i}}\right), \tag{3·14}$$

while the coefficient for the Alfvén wave, $V_0^-\beta^-$, is negative definite. Therefore the Alfvén wave has a uniform negative dispersion irrespective of the propagation direction relative to the external magnetic field, whilst the magneto-acoustic wave has positive or negative dispersion depending upon the propagation angle. We have adopted here the convention that a dispersion effect is called to be positive if the phase velocity increases with the wave number. When $\theta=\theta_c$, the phase velocity for the magneto-acoustic wave, V_p^+, may be expressed as follows:

$$V_p^+=V_0^+\{1+\beta_ck^4+O(k^6)\},$$

$$\beta_c=-\frac{1}{2M_A^4R_i^2R_e^2\sin^2\theta_c}. \tag{3·15}$$

It should be remarked here that the expansions (3·13') are valid under the condition

$$(1-\cos\theta)^2 \gg \frac{1}{M_A^2 R_i R_e}\left\{\left(\frac{R_e}{R_i}+\frac{R_i}{R_e}\right)\cos^2\theta+\sin^2\theta-2\cos\theta\right\}k^2, \tag{3·16}$$

which is roughly equivalent to

$$\theta^4 \gg \frac{4m_i k^2}{m_e M_A^2 R_i R_e} \tag{3·16'}$$

for $\theta \ll 1$ and $m_e/m_i \ll 1$. Therefore as θ approaches zero, the wave becomes arbitrarily "smooth". In this sense one must regard the limit $\theta\to 0$ as an asymptotic one.

In view of the relation

$$\frac{d}{dt}\left(\frac{\partial}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{d}{dt}\right)-\frac{\partial u}{\partial x}\frac{\partial}{\partial x}, \tag{3·17}$$

together with Eq. (3·9'), the system of equations (3·9')~(3·11') can be transformed into the matrix form:[7]

$$\frac{\partial U}{\partial t}+A(U)\frac{\partial U}{\partial x}+\prod_{\alpha=1}^{2}\left(H_\alpha\frac{\partial}{\partial t}+K_\alpha\frac{\partial}{\partial x}\right)U=0 \tag{3·18}$$

with

$$\left.\begin{aligned}
U&=\begin{pmatrix} n\\ u\\ v\\ B_y\\ w\\ B_z\end{pmatrix},\\
A&=\begin{pmatrix} A^+ & B\\ C & A^-\end{pmatrix},\\
A^+&=\begin{pmatrix} u & n & 0 & 0\\ 0 & u & 0 & M_A^{-2}n^{-1}B_y\\ 0 & 0 & u & -M_A^{-2}n^{-1}B_x\\ 0 & B_y & -B_x & u\end{pmatrix},\\
A^-&=\begin{pmatrix} u & -M_A^{-2}n^{-1}B_x\\ -B_x & u\end{pmatrix},\\
B&=\begin{pmatrix} 0 & 0\\ 0 & M_A^{-2}n^{-1}B_z\\ 0 & 0\\ 0 & 0\end{pmatrix},\\
C&=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & B_z & 0 & 0\end{pmatrix},\\
H_1&=0,
\end{aligned}\right\} \tag{3·19}$$

$$\left.\begin{aligned}
K_1 &= \begin{pmatrix} 0 & K_1^- \\ K_1^+ & 0 \end{pmatrix}, \\
K_1^+ &= \begin{pmatrix} 0 & 0 & 0 & -M_A^{-2}R_e^{-1}n^{-1} \\ 0 & 0 & R_i^{-1} & 0 \end{pmatrix}, \\
K_1^- &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & M_A^{-2}R_e^{-1}n^{-1} \\ -R_i^{-1} & 0 \end{pmatrix}, \\
H_2 &= I, \\
K_2 &= uI,
\end{aligned}\right\}$$

where I stands for the unit matrix. This is of the standard form of Eq.(I.4·1) with $p=2$ and $s=1$.

Let us now consider a solution in the neighbourhood of the uniform state

$$U_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \sin\theta \\ 0 \\ 0 \end{pmatrix} \tag{3·20}$$

(cf., the boundary conditions (3·12)). Then the matrix A_0 takes the following irreducible form:

$$A_0 = \begin{pmatrix} A_0^+ & 0 \\ 0 & A_0^- \end{pmatrix}, \tag{3·21}$$

and the eigenvalues of A_0 become

$$\lambda_0 = 0, \quad \pm V_0^+; \quad \pm V_0^-, \tag{3·22}$$

which correspond, respectively, to the contact surface, magneto-acoustic wave, and Alfvén wave. Since we consider here the progressive waves, we shall discard the contact surface specified by $\lambda_0=0$.

3.1 *Magneto-acoustic wave* ($\theta \neq \theta_c$)

Let us first consider the magneto-acoustic wave specified by $\lambda_0 = V_0^+$ under the condition $\theta \neq \theta_c$. Historically speaking, this is the first example for which Gardner and Morikawa[2] obtained the Korteweg-de Vries equation for the special case of transverse propagation $\theta=\pi/2$. Then Berezin-Karpman,[19] Morton[20] and Kakutani-Ono-Taniuti-Wei[7] have extended the results to general propagation angle $\theta \neq \pi/2$. In their analysis Berezin-Karpman restricted themselves to the propagation angle which satisfies $\sqrt{m_e/m_i} \ll \pi/2-\theta \ll 1$, for which electron inertia is negligible. In this connection, it should be noted that when $\cot\theta \lesssim m_e/m_i$ electron inertia cannot be neglected even if $m_e/m_i \ll 1$ (cf., (3·13′) and (3·14)). On the other hand, Morton did not discriminate clearly between magneto-acoustic and Alfvén modes.

Now that the system has been reduced to the standard matrix form given by Eq.(I.4·1), let us try to proceed along the prescription described in I§4. It should be noted, however, that

$$\left.\begin{aligned} &K_0=\prod_{\alpha=1}^{2}(-V_0^+H_{\alpha 0}+K_{\alpha 0})=\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}, \\ &D^+=\begin{pmatrix} 0 & 0 & 0 & M_A^{-3}R_e^{-1} \\ 0 & 0 & -M_A^{-1}R_i^{-1} & 0 \end{pmatrix}, \\ &D^-=\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -M_A^{-3}R_e^{-1} \\ M_A^{-1}R_i^{-1} & 0 \end{pmatrix}, \end{aligned}\right\} \tag{3·23}$$

which represents the interchange between the two invariant subspaces associated with A_0^+ and A_0^- (cf., (3·21)). On the other hand, the right and left eigenvectors for the magneto-acoustic wave, $\lambda_0 = V_0^+$, become respectively

$$R=\begin{pmatrix} R^+ \\ 0 \end{pmatrix} \quad \text{and} \quad L=(L^+ \ \ 0),$$

where

$$R^+=\begin{pmatrix} 1 \\ M_A^{-1} \\ -M_A^{-1}\cot\theta \\ \operatorname{cosec}\theta \end{pmatrix}, \quad L^+=(0 \ \ M_A\sin\theta \ \ -M_A\cos\theta \ \ 1). \tag{3·24}$$

Therefore a direct application of the general theory developed in I§4 leads to

$$\mu=(LK_0R)/(LR)=0. \tag{3·25}$$

It is thus found that the magneto-acoustic wave belongs to the exceptional case for which the coefficient of dispersion term becomes zero, hence, for magneto-acoustic wave, the ordering specified by the expansion (I.4·2) and the coordinate-stretching (I.4·3) with $p=2$ is not valid. In fact, the dispersion relation (I.4·10) with $p=2$ shows that the lowest order dispersion first appears from the order of $O(k^{2p-1})$, that is, the dispersion relation takes the explicit form given by (3·13′), which suggests the following coordinate-stretching

$$\xi=\varepsilon^{1/2}(x-V_0^+t), \qquad \tau=\varepsilon^{3/2}t, \tag{3·26}$$

instead of (I.4·3) with $p=2$. On the other hand, a simple dimensional argument then suggests that the expansion of the dependent variables should be

$$U^{+}=\sum_{j=0}^{\infty}\varepsilon^{j}U_{j}^{+}, \tag{3·27 a}$$

$$U^{-}=\varepsilon^{1/2}\sum_{j=1}^{\infty}\varepsilon^{j}U_{j}^{-}, \tag{3·27 b}$$

instead of (I.4·2), where

$$U^{+}=\begin{pmatrix} n \\ u \\ v \\ B_y \end{pmatrix} \quad \text{with} \quad U_0^{+}=\begin{pmatrix} 1 \\ 0 \\ 0 \\ \sin\theta \end{pmatrix} \tag{3·27 c}$$

and

$$U^{-}=\begin{pmatrix} w \\ B_z \end{pmatrix} \quad \text{with} \quad U_0^{-}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3·27 d}$$

In terms of the new variables (ξ, τ), Eq. (3·18) takes the form:

$$\varepsilon\frac{\partial U}{\partial \tau}+(-V_0^{+}I+A)\frac{\partial U}{\partial \xi}+\varepsilon^{1/2}\prod_{\alpha=1}^{2}\left\{(-V_0^{+}H_\alpha+K_\alpha)\frac{\partial}{\partial \xi}+\varepsilon H_\alpha\frac{\partial}{\partial \tau}\right\}U=0. \tag{3·28}$$

In view of the property of H_α and K_α given by (3·19), the last term of Eq.(3·28) can be expressed as

$$\varepsilon^{1/2}\prod_{\alpha=1}^{2}\left\{(-V_0^{+}H_\alpha+K_\alpha)\frac{\partial}{\partial \xi}+\varepsilon H_\alpha\frac{\partial}{\partial \tau}\right\}U=\varepsilon^{1/2}\sum_{q=0}^{2}\varepsilon^{q}D_q U, \tag{3·29}$$

where D_q takes the form

$$D_q=\begin{pmatrix} 0 & D_q^{-} \\ D_q^{+} & 0 \end{pmatrix}$$

with

$$D_0^{\pm}=K_1^{\pm}\left[\frac{\partial}{\partial \xi}\left\{(-V_0^{+}+u)\frac{\partial}{\partial \xi}\right\}\right],$$

$$D_1^{\pm}=K_1^{\pm}\frac{\partial^2}{\partial \xi \partial \tau},$$

$$D_2^{\pm}=0.$$

Hence Eq. (3·28) splits into the following two parts:

$$\varepsilon\frac{\partial U^{+}}{\partial \tau}+(-V_0^{+}I+A^{+})\frac{\partial U^{+}}{\partial \xi}+B\frac{\partial U^{-}}{\partial \xi}+\varepsilon^{1/2}\sum_{q=0}^{2}\varepsilon^{q}D_q^{-}U^{-}=0, \tag{3·30 a}$$

$$\varepsilon\frac{\partial U^-}{\partial\tau}+(-V_0^+I+A^-)\frac{\partial U^-}{\partial\xi}+C\frac{\partial U^+}{\partial\xi}+\varepsilon^{1/2}\sum_{q=0}^{2}\varepsilon^q D_q^+U^+=0. \qquad (3\cdot30\,\mathrm{b})$$

By virtue of the expansions (3·27), we may expand A, B, C, and D_q as the power series in ε:

$$A^\pm=\sum_{j=0}^{\infty}\varepsilon^j A_j^\pm, \qquad (3\cdot31\,\mathrm{a})$$

$$B=\varepsilon^{1/2}\sum_{j=1}^{\infty}\varepsilon^j B_j, \qquad (3\cdot31\,\mathrm{b})$$

$$C=\varepsilon^{1/2}\sum_{j=1}^{\infty}\varepsilon^j C_j, \qquad (3\cdot31\,\mathrm{c})$$

$$D_q^\pm=\sum_{j=0}^{\infty}\varepsilon^j d_{qj}^\pm, \qquad (3\cdot31\,\mathrm{d})$$

in which

$$A_1^\pm=U_1^+\cdot(\nabla_{u^+}A^\pm)_0, \qquad d_{q1}^\pm=U_1^+\cdot(\nabla_{u^+}D_q^\pm)_0,$$

$$B_1=U_1^-\cdot(\nabla_{u^-}B)_0, \qquad d_{00}^\pm=\hat{d}_{00}^\pm\partial^2/\partial\xi^2,$$

$$C_1=U_1^-\cdot(\nabla_{u^-}C)_0, \qquad \hat{d}_0^\pm=-V_0^+K_{10}^\pm.$$

Substituting (3·27) and (3·31) into Eqs. (3·30) and equating terms with the same powers in ε, we obtain the following sets of equations for each power in ε. First we have for ε^1:

$$(-V_0^+I+A_0^+)\frac{\partial U_1^+}{\partial\xi}=0, \qquad (3\cdot32\,\mathrm{a})$$

$$(-V_0^+I+A_0^-)\frac{\partial U_1^-}{\partial\xi}+d_{00}^+U_1^+=0. \qquad (3\cdot32\,\mathrm{b})$$

Equation (3·32a) can be integrated at once, giving rise to

$$U_1^+=R^+\varphi^{(1)+}, \qquad (3\cdot33)$$

where $\varphi^{(1)+}$ is one of the components of U_1^+ (here we set $\varphi^{(1)+}=n^{(1)}$), and where the boundary conditions (3·12) have been used to specify an arbitrary function of τ resulting from the integration. Substitution of (3·33) into Eq. (3·32b) gives an expression for U_1^- in terms of $\varphi^{(1)+}$, that is,

$$U_1^-=-(-V_0^+I+A_0^-)^{-1}\hat{d}_{00}^+R^+\frac{\partial\varphi^{(1)+}}{\partial\xi}. \qquad (3\cdot34)$$

Next we have for ε^2:

$$(-V_0^+I+A_0^+)\frac{\partial U_2^+}{\partial\xi}=-\frac{\partial U_1^+}{\partial\tau}-A_1^+\frac{\partial U_1^+}{\partial\xi}-d_{00}^-U_1^-, \qquad (3\cdot35\,\mathrm{a})$$

$$(-V_0^+ I + A_0^-)\frac{\partial U_2^-}{\partial \xi} + d_{00}^+ U_2^+$$
$$= -\frac{\partial U_1^-}{\partial \tau} - A_1^- \frac{\partial U_1^-}{\partial \xi} - C_1 \frac{\partial U_1^+}{\partial \xi} - d_{10}^+ U_1^+ - d_{01}^+ U_1^+. \qquad (3\cdot35\,\text{b})$$

Multiplying (3·35a) by the left eigenvector L^+ of A_0^+, and remembering the relation (3·34) we obtain finally

$$\frac{\partial \varphi^{(1)+}}{\partial \tau} + \alpha\varphi^{(1)+}\frac{\partial \varphi^{(1)+}}{\partial \xi} + \mu \frac{\partial^3 \varphi^{(1)+}}{\partial \xi^3} = 0, \qquad (3\cdot36)$$

where

$$\left.\begin{aligned} \alpha &= [L^+\{R^+\cdot(\nabla_{u^+}A^+)_0\}R^+]/(L^+R^+) = \frac{3}{2M_A} = \frac{3}{2}V_0^+, \\ \mu &= -[L^+\hat{d}_{00}^-(-V_0^+ I + A_0^-)^{-1}\hat{d}_{00}^+ R^+]/(L^+R^+) \\ &= -V_0^+\beta^+ = \frac{1}{2M_A^3 R_i R_e}\left\{1 - \left(\sqrt{\frac{R_e}{R_i}} - \sqrt{\frac{R_i}{R_e}}\right)^2 \cot^2\theta\right\}. \end{aligned}\right\} \qquad (3\cdot37)$$

Thus we arrive at a conclusion that the magneto-acoustic wave can also be governed by the Korteweg-de Vries equation under the long wave approximation, and that the coefficient of the dispersion term, μ, exactly corresponds to the coefficient of k^2 in the dispersion relation (3·13′).

The modification of the general method adopted in this section can be extended to more general case satisfying the following conditions:[7),15)]

(1) For the equation given by Eq.(I.4·1) with $s=1$, the matrix A has the representation

$$A = \begin{pmatrix} A^+ & B \\ C & A^- \end{pmatrix}, \qquad (3\cdot38)$$

in which A^+ and A^- are respectively $m \times m$ and $(n-m)\times(n-m)$ matrices, which are functions of U^+ only, while the elements of the matrices B and C are linear combinations of the components of U^- multiplied by functions of U^+ only, where $U^\pm$ are defined by

$$U = \begin{pmatrix} U^+ \\ U^- \end{pmatrix}; \quad U^+ = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}, \quad U^- = \begin{pmatrix} u_{m+1} \\ u_{m+2} \\ \vdots \\ u_n \end{pmatrix}. \qquad (3\cdot39)$$

(2) There exists a uniform state given by

$$U_0 = \begin{pmatrix} U_0^+ \\ U_0^- \end{pmatrix} = \begin{pmatrix} U_0^+ \\ 0 \end{pmatrix}, \qquad (3\cdot40)$$

and $U^{\pm}$ can be expanded around the uniform state as power series in the scale parameter ε:

$$U^{+}=\sum_{j=0}^{\infty}\varepsilon^{j}U_{j}^{+}, \tag{3·41 a}$$

$$U^{-}=\varepsilon^{1/2}\sum_{j=1}^{\infty}\varepsilon^{j}U_{j}^{-}. \tag{3·41 b}$$

(3) A_0 has an irreducible representation:

$$A_0=\begin{pmatrix} A_0^{+} & 0 \\ 0 & A_0^{-} \end{pmatrix}, \tag{3·42}$$

where the eigenvalues of A_0^{+} are real, at least one of them is nondegenerate, finite, and nonzero and is not equal to any eigenvalue of A_0^{-}.
(4) H_α and K_α are functions of U^{+} only, and the operator in the last term of Eq. (I.4·1) takes the following form:

$$\prod_{\alpha=1}^{p}\left(H_\alpha\frac{\partial}{\partial t}+K_\alpha\frac{\partial}{\partial x}\right)=\begin{pmatrix} 0 & D^{-} \\ D^{+} & 0 \end{pmatrix},\quad \text{say.} \tag{3·43}$$

(5) U tends to the uniform state U_0 as $x\to-\infty$.
If the above five conditions are satisfied then the original system given by Eq. (I.4·1) can be reduced to the Korteweg-de Vries type equation under the coordinate-stretching defined by

$$\xi=\varepsilon^{b}(x-\lambda_0^{+}t),\qquad \tau=\varepsilon^{b+1}t,\qquad b=\frac{1}{2(p-1)}, \tag{3·44}$$

where λ_0^{+} is the nondegenerate, finite, nonzero eigenvalue of A_0^{+}, which is not equal to any eigenvalue of A_0^{-}. The procedure of the reduction is quite parallel to that given in this subsection for the magneto-acoustic wave ($\theta\neq\theta_c$).

3.2 *Magneto-acoustic wave at the critical angle* $\theta=\theta_c$[8)]

As was already remarked, when the propagation angle θ of the wave relative to the external magnetic field becomes the critical angle θ_c given by (3·14), the dispersion parameter $V_0^{+}\beta^{+}$ becomes zero, so that one needs a further modification similar to that employed in the preceding subsection, §3.1.

Since the dispersion relation at the critical angle is given by (3·15), which shows that the lowest order dispersion first appears in the order of $O(k^4)$ instead of $O(k^2)$, we should stretch our coordinates as follows:

$$\xi=\varepsilon^{1/2}(x-V_0^{+}t),\qquad \tau=\varepsilon^{5/2}t. \tag{3·45}$$

Equation (3·18) then takes an analogous but slightly different form from that of Eqs. (3·30). Employing the same expansions as those given by (3·27), and carrying out a similar procedure to that in the preceding subsection, one finds that

$$U_1^+ = U_1^- = 0, \tag{3·46}$$

and that

$$\frac{\partial\varphi^{(2)+}}{\partial\tau} + \frac{3}{2} V_0^+ \varphi^{(2)+} \frac{\partial\varphi^{(2)+}}{\partial\xi} + V_0^+ \beta_c \frac{\partial^5 \varphi^{(2)+}}{\partial\xi^5} = 0, \tag{3·47}$$

where $\varphi^{(2)+}$ is one of the components of U_2^+ (we set here $\varphi^{(2)+} = n^{(2)}$). Equation (3·47) is of a generalized form of the Korteweg-de Vries equation in the sense that the dispersion term is replaced by the fifth order derivative.

It was found by Kawahara[9] that Eq. (3·47) has an oscillating solitary wave solution instead of the "monotone" Korteweg-de Vries soliton. He considered the following type of equation:

$$\frac{\partial\varphi}{\partial\tau} + \alpha\varphi \frac{\partial\varphi}{\partial\xi} + \mu_1 \frac{\partial^3\varphi}{\partial\xi^3} + \mu_2 \frac{\partial^5\varphi}{\partial\xi^5} = 0, \tag{3·48}$$

and found that the solitary wave solution becomes to have an oscillatory structure if an effect of μ_2 term dominates over that of μ_1 term.

A generalized system in which the lowest order dispersion first appears in the higher order in k^2 can also be dealt with by a similar way if we modify the corresponding coordinate-stretching appropriately.[8]

3.3 *Alfvén wave*[8]

For the case of Alfvén wave specified by $\lambda_0 = V_0^-$, it is natural to introduce the coordinate-stretching defined as

$$\xi = \varepsilon^{1/2}(x - V_0^- t), \qquad \tau = \varepsilon^{3/2} t, \tag{3·49}$$

in analogy with that for the magneto-acoustic wave (cf., (3·26)). We have then an analogous system of equations to that of Eqs. (3·30a) and (3·30b) in which V_0^+ is replaced by V_0^-. If we attempt a similar procedure to that used in §3.1 based on the expansions of the dependent variables defined by (3·27a) and (3·27b), then with the boundary conditions (3·12) we have

$$U_1^+ = 0, \tag{3·50}$$

and we obtain the following linear equation:

$$\frac{\partial\varphi^{(1)-}}{\partial\tau} - V_0^- \beta^- \frac{\partial^3 \varphi^{(1)-}}{\partial\xi^3} = 0, \tag{3·51}$$

where $\varphi^{(1)-}$ is one of the components of U_1^-, i.e., $\varphi^{(1)-}$ is either $w^{(1)}$ or $B_z^{(1)}$. It is thus found that the Alfvén wave belongs to the exceptional case for which nonlinear term vanishes under the same ordering to that of the magneto-acoustic wave. Physically speaking, this may reflect the fact that in this ordering the Alfvén wave is essentially "transverse" wave and is not accompanied by the density change. On the other hand, however, earlier investigations on steady regime by Kazantsev[21] and also by Kellogg[22] have shown that even the Alfvén mode has a solitary wave solution associated with it, though this is obviously not possible for a linear equation such as Eq. (3·51). Therefore in order to take a nonlinear effect into account we must modify the ordering of the dependent variables. A dimensional argument concerning the balance between nonlinear and dispersion effects suggests that the appropriate expansions of the dependent variables should be

$$U^+=\sum_{j=0}^{\infty}\varepsilon^{j/2}U_j^+, \tag{3·52 a}$$

$$U^-=\sum_{j=0}^{\infty}\varepsilon^{j/2}U_j^-, \tag{3·52 b}$$

where

$$U_0^+=\begin{pmatrix} n^{(0)} \\ u^{(0)} \\ v^{(0)} \\ B_y^{(0)} \end{pmatrix} \quad \text{and} \quad U_0^-=\begin{pmatrix} w^{(0)} \\ B_z^{(0)} \end{pmatrix},$$

which are not constant but functions of (ξ, τ). The boundary conditions (3·12) now imply that

$$U_0^+\to\begin{pmatrix} 1 \\ 0 \\ 0 \\ \sin\theta \end{pmatrix}, \quad U_0^-\to\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad U_j^\pm\to 0 \;(j=1, 2, \cdots) \quad \text{as} \quad \xi\to-\infty. \tag{3·53}$$

The essential difference between the expansions (3·27) and (3·52) is that we take into account in (3·52) "zeroth" order deviations of the dependent variables from the uniform state. It should also be noted that the series expansions (3·52) are not in integral powers of ε but in half-integral powers of ε, and that the perturbations of all quantities are of the same order of magnitude with respect to one another.

Substituting (3·52) into the system of equations analogous to that of Eqs. (3·30a) and (3·30b) with V_0^+ replaced by V_0^-, and equating terms with the same powers in $\varepsilon^{1/2}$, we obtain the following sets of equations for each half-integral power of ε.

For ε^0, we have

$$(-V_0^-I+A_0^+)\frac{\partial U_0^+}{\partial\xi}+B_0\frac{\partial U_0^-}{\partial\xi}=0, \tag{3·54 a}$$

$$(-V_0^- I + A_0^-)\frac{\partial U_0^-}{\partial \xi} + C_0 \frac{\partial U_0^+}{\partial \xi} = 0. \tag{3·54 b}$$

Since $A_0^{\pm}$, B_0, and C_0 are functions of $U_0^{\pm}$ which are now not constant but functions of (ξ, τ), there seems to be no elegant method to integrate Eqs. (3·54) unlike the case for magneto-acoustic wave. It may, therefore, be more convenient to deal with Eqs. (3·54) in component form rather than in matrix form. Expressing the system of Eqs. (3·54) in component form, and remembering the boundary conditions (3·53), we obtain

$$\left.\begin{aligned}
&n^{(0)} = 1,\\
&u^{(0)} = 0,\\
&B_y^{(0)2} + B_z^{(0)2} = \sin^2\theta,\\
&M_A v^{(0)} + B_y^{(0)} = \sin\theta,\\
&M_A w^{(0)} + B_z^{(0)} = 0.
\end{aligned}\right\} \tag{3·55}$$

Next for $\varepsilon^{1/2}$, we obtain

$$\left.\begin{aligned}
&n^{(1)}\cos\theta - M_A u^{(1)} = 0,\\
&M_A u^{(1)}\cos\theta - (B_y^{(0)} B_y^{(1)} + B_z^{(0)} B_z^{(1)}) = 0,\\
&M_A v^{(1)} + B_y^{(1)} = -\frac{1}{M_A R_e}\frac{\partial B_z^{(0)}}{\partial \xi},\\
&M_A w^{(1)} + B_z^{(1)} = \frac{1}{M_A R_e}\frac{\partial B_y^{(0)}}{\partial \xi},\\
&M_A v^{(1)} + B_y^{(1)} - B_y^{(0)} n^{(1)} = \frac{1}{R_i}\frac{\partial w^{(0)}}{\partial \xi},\\
&M_A w^{(1)} + B_z^{(1)} - B_z^{(0)} n^{(1)} = -\frac{1}{R_i}\frac{\partial v^{(0)}}{\partial \xi},
\end{aligned}\right\} \tag{3·56}$$

which give

$$\left.\begin{aligned}
&B_y^{(0)} n^{(1)} = \frac{1}{M_A}\left(\frac{1}{R_i} - \frac{1}{R_e}\right)\frac{\partial B_z^{(0)}}{\partial \xi},\\
&B_z^{(0)} n^{(1)} = -\frac{1}{M_A}\left(\frac{1}{R_i} - \frac{1}{R_e}\right)\frac{\partial B_y^{(0)}}{\partial \xi},
\end{aligned}\right\} \tag{3·57}$$

for ε^1,

$$\begin{aligned}
&n^{(2)}\cos\theta - M_A u^{(2)} = M_A n^{(1)} u^{(1)},\\
&M_A u^{(2)}\cos\theta - (B_y^{(0)} B_y^{(2)} + B_z^{(0)} B_z^{(2)}) = \frac{1}{2}(B_y^{(1)2} + B_z^{(1)2}),\\
&M_A \frac{\partial v^{(2)}}{\partial \xi} + \frac{\partial B_y^{(2)}}{\partial \xi} = \frac{1}{V_0^-}\frac{\partial B_y^{(0)}}{\partial \tau} - \frac{1}{M_A R_e}\frac{\partial^2 B_z^{(1)}}{\partial \xi^2}
\end{aligned}$$

$$\left.\begin{aligned}
&\qquad\qquad\qquad\qquad+\frac{1}{M_AR_e}\frac{\partial}{\partial\xi}\left(n^{(1)}\frac{\partial B_z^{(0)}}{\partial\xi}\right),\\
&M_A\frac{\partial w^{(2)}}{\partial\xi}+\frac{\partial B_z^{(2)}}{\partial\xi}=\frac{1}{V_0^-}\frac{\partial B_z^{(0)}}{\partial\tau}+\frac{1}{M_AR_e}\frac{\partial^2 B_y^{(1)}}{\partial\xi^2}\\
&\qquad\qquad\qquad\qquad-\frac{1}{M_AR_e}\frac{\partial}{\partial\xi}\left(n^{(1)}\frac{\partial B_y^{(0)}}{\partial\xi}\right),\\
&M_A\frac{\partial v^{(2)}}{\partial\xi}+\frac{\partial B_y^{(2)}}{\partial\xi}-\frac{\partial}{\partial\xi}(B_y^{(0)}n^{(2)})=\frac{1}{V_0^-}\frac{\partial B_y^{(0)}}{\partial\tau}+\frac{\partial}{\partial\xi}(n^{(1)}B_y^{(1)})\\
&\qquad+\frac{1}{R_i}\frac{\partial^2 w^{(1)}}{\partial\xi^2}+\frac{1}{M_AR_i}\frac{\partial}{\partial\xi}\left(n^{(1)}\frac{\partial B_z^{(0)}}{\partial\xi}\right)-\frac{\partial}{\partial\xi}(B_y^{(0)}n^{(1)2}),\\
&M_A\frac{\partial w^{(2)}}{\partial\xi}+\frac{\partial B_z^{(2)}}{\partial\xi}-\frac{\partial}{\partial\xi}(B_z^{(0)}n^{(2)})=\frac{1}{V_0^-}\frac{\partial B_z^{(0)}}{\partial\tau}+\frac{\partial}{\partial\xi}(n^{(1)}B_z^{(1)})\\
&\qquad-\frac{1}{R_i}\frac{\partial^2 v^{(1)}}{\partial\xi^2}-\frac{1}{M_AR_i}\frac{\partial}{\partial\xi}\left(n^{(1)}\frac{\partial B_y^{(0)}}{\partial\xi}\right)-\frac{\partial}{\partial\xi}(B_z^{(0)}n^{(1)2}),
\end{aligned}\right\}\quad(3\cdot58)$$

where (3·55)~(3·57) have been used. Eliminating the higher order terms in (3·58) by means of the relations (3·55)~(3·57), we have

$$\frac{1}{V_0^-}\left(B_y^{(0)}\frac{\partial\hat{B}_z^{(0)}}{\partial\tau}-B_z^{(0)}\frac{\partial\hat{B}_y^{(0)}}{\partial\tau}\right)-\frac{1}{2M_A^2}\left(\frac{1}{R_e}-\frac{1}{R_i}\right)^2\cos^2\theta\frac{\partial}{\partial\xi}\left(\frac{1}{B_z^{(0)}}\frac{\partial B_y^{(0)}}{\partial\xi}\right)$$
$$-\frac{1}{2M_A^2R_iR_e}\left(B_z^{(0)}\frac{\partial^2 B_y^{(0)}}{\partial\xi^2}-B_y^{(0)}\frac{\partial^2 B_z^{(0)}}{\partial\xi^2}\right)=0,\qquad(3\cdot59)$$

where

$$\hat{B}_{y,z}^{(0)}\equiv\int^\xi B_{y,z}^{(0)}d\xi.$$

It is now convenient to introduce new variables ($B^{(0)}$, $\Phi^{(0)}$) defined as

$$B_y^{(0)}=B^{(0)}\cos\Phi^{(0)},\qquad B_z^{(0)}=B^{(0)}\sin\Phi^{(0)}.\qquad(3\cdot60)$$

The third line of (3·55) then gives

$$B^{(0)2}=\sin^2\theta=\text{const.}\qquad(3\cdot61)$$

In view of (3·53) and (3·60), the boundary condition for $\Phi^{(0)}$ is given by $\Phi^{(0)}\to0$ as $\xi\to-\infty$.

In terms of the new dependent variable $\Phi^{(0)}$, we can rewrite Eq. (3·59) as

$$\left\{\frac{1}{V_0^-}\frac{\partial\Phi^{(0)}}{\partial\tau}-\beta^-\frac{\partial^3\Phi^{(0)}}{\partial\xi^3}+\beta^-\left(\frac{\partial\Phi^{(0)}}{\partial\xi}\right)^3\right\}\frac{\partial^2\Phi^{(0)}}{\partial\xi^2}$$
$$=\frac{\partial\Phi^{(0)}}{\partial\xi}\frac{\partial}{\partial\xi}\left(\frac{1}{V_0^-}\frac{\partial\Phi^{(0)}}{\partial\tau}-\beta^-\frac{\partial^3\Phi^{(0)}}{\partial\xi^3}\right),$$

which can be integrated once with respect to ξ, giving rise to

$$\frac{\partial \Phi^{(0)}}{\partial \tau} - \frac{1}{2} V_0^- \beta^- \left(\frac{\partial \Phi^{(0)}}{\partial \xi} \right)^3 - V_0^- \beta^- \frac{\partial^3 \Phi^{(0)}}{\partial \xi^3} = 0, \tag{3·62}$$

where the boundary condition for $\Phi^{(0)}$ and the linear dispersion relation (3·13′) have been used. Since $n^{(1)} = M_A^{-1}(R_i^{-1} - R_e^{-1}) \partial \Phi^{(0)} / \partial \xi$ by virtue of (3·57) and (3·60), we can rewrite Eq. (3·62) as an equation for $n^{(1)}$:

$$\frac{\partial n^{(1)}}{\partial \tau} + \alpha n^{(1)2} \frac{\partial n^{(1)}}{\partial \xi} + \mu \frac{\partial^3 n^{(1)}}{\partial \xi^3} = 0, \tag{3·63}$$

where

$$\left. \begin{aligned} \alpha &= -\frac{3}{2} V_0^- \beta^- M_A^2 (R_i^{-1} - R_e^{-1})^{-2}, \\ \mu &= -V_0^- \beta^-. \end{aligned} \right\} \tag{3·64}$$

Thus we arrive at a conclusion that the Alfvén wave is governed by the modified Korteweg-de Vries equation. It should be noted that not only the dispersion term but also the nonlinear term is affected by the dispersion effect, since α contains the dipersion parameter β^-. It is also noted that there are two types of solitary wave solution to Eq. (3·63), one being compressive and the other rarefactive, and that they coincide exactly with the approximate steady solutions obtained by Kazantsev[21] and roughly coincide with the exact steady solutions obtained numerically by Kellogg.[22]

The problem of hydromagnetic waves considered here has been extended to various cases. One involves the inclusion of the plasma temperature. Starting from the Vlasov equation, Kever-Morikawa[10] showed that the magneto-acoustic wave is governed by the Korteweg-de Vries equation even in warm plasma. By taking account of the electron temperature, Kawahara[11] also showed that the magneto-acoustic wave is governed by the Korteweg-de Vries equation, whereas the Alfvén wave is governed by the modified Korteweg-de Vries equation similar to Eq. (3·63). In these cases, the coefficients of the resulting equations are, of course, functions of plasma temperature. The second extension involves the examination of an effect of electron-ion collisions. In this case, the governing equations describing the magneto-acoustic wave can be reduced to the Korteweg-de Vries-Burgers equation,[13] which has either oscillatory or monotone shock wave solution according to dispersion-dominant or dissipation-dominant case. Another extension is concerned with the examination of an effect of inhomogeneity of the plasma density and that of the applied magnetic field.[12] Effect of inhomogeneity of the wave media has been discussed in Part II B of this series from a generalized point of view.

In concluding this paper, we refer the interested reader to Jeffrey and Kakutani[23] who reviewed other examples, such as water waves and lattice waves, for which the reductive perturbation method can effectively be applied.

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