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A pointwise regularity theory for the two-obstacle problem

by

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Contents

Introduction

A detailed study of the boundary regularity for solutions of the Dirichlet problem in an open region D of $\mathbf{R}^N$, $N\geqslant 3$, was carried out by H. Lebesgue and others: this investigation culminated in the celebrated *Wiener criterion*. By relying on a fundamental notion of potential theory, namely that of *capacity* of an arbitrary subset of $\mathbf{R}^N$, N. Wiener was able to characterize the boundary *regular points*—as classically defined by H. Lebesgue and N. Wiener himself—in terms of an intrinsic condition which must be satisfied by the domain D in the neighbourhood of the given point x_0 (see [23]).

A further interesting contribution was given by V. G. Maz'ja, who showed that the pointwise modulus of continuity of solutions of Dirichlet problems, with arbitrary continuous boundary datum h, is related to the rate of divergence of the integral appearing in the Wiener's criterion (see [18], [19]).

At the same time, in the framework of the theory of *variational inequalities*, H. Brézis, H. Lewy, G. Stampacchia, and others initiated the study of the regularity of solutions of a class of free boundary problems, the so called unilateral *obstacle problems,* involving a second order elliptic operator L (see [21], [3], [16]). This study was pursued by L. A. Caffarelli, J. Frehse, D. Kinderlehrer, and others (see [4], [8], [9]. Most of these results are of global or local nature, in the sense that, for example, the solutions are shown to be continuous at a given point, provided the obstacle is continuous on a neighbourhood of that point.

The methods used are primarily "a priori" estimates like in the usual P.D.E. theory. However, the connection with potential theory and related methods were explicitly also taken into account, in particular by H. Lewy and G. Stampacchia and later on by L. Caffarelli and D. Kinderlehrer.

Related to both P.D.E. and potential theory is the approach taken by J. Frehse and U. Mosco to study the pointwise regularity of local solutions of obstacle problems for a class of quite general *irregular obstacles*, i.e. obstacles not necessarily continuous (see [10], [11], [12]). These authors introduced the notion of *regular point* of an obstacle and, by relying on capacity methods as in the classical theory, they established a criterion for regularity of the type of the Wiener criterion. Moreover they proved estimates of the modulus of continuity of the solutions of the type of the Maz'ja estimate.

In this paper we consider a more general class of variational inequalities, the so called *two-obstacle problems* (see Definition II.1.1), and we carry out the study of the pointwise behaviour of the local solutions. This theory provides a unified framework for the study of regular points both for Dirichlet problems and for unilateral obstacle problems.

The point x_0, at which the regularity is tested, may indeed be a point of a *fixed boundary,* as in the Dirichlet problems, as well as a point of a *free boundary* in a two-obstacle problem, that is a point of the boundary where the solution leaves one of the two obstacles. It may even happen that the "geometry" of the obstacles at the given point x_0 is more complicated: the two obstacles may "touch" each other at x_0, while both oscillate very much in an arbitrarily small neighbourhood of the point, interpenetrating each other.

We will consider *variational solutions* in Part II and *generalized solutions* in Part III. The former are solutions in the Sobolev space H^1, which exist provided the two given obstacles ψ_1, ψ_2 are separated by some H^1 function w. The latter can be defined, more generally, as limit of variational solutions, by only requiring the separating function w to be quasi continuous in the capacity sense.

The notion of *regular point* for two given obstacles ψ_1, ψ_2 is first introduced in Part II in terms of continuity at a given x_0 of all variational solutions in a neighbourhood of x_0 and then extended in Part III in terms of generalized solutions. A *Wiener criterion,* which characterizes the regular points, is proved for variational solutions in Theorem II.1.1 and for generalized solutions in Theorem III.2.1. In particular, such a criterion shows that even a point x_0 where the two obstacles touch one each other will be regular for the two-obstacle problem, provided it is regular separately for each of the (one-sided) obstacles. This qualitative result follows indeed from the one-sided criterion by means of suitable comparison arguments.

We are also concerned in establishing a priori estimates for *local solutions*, to be satisfied at an arbitrary point of the domain. A peculiar interesting feature of all these estimates is their structural nature. By this we mean that they depend only on the dimension of the space and on the structural constants of the operator L, such as its ellipticity constants.

Estimates of the *modulus of continuity* are given in Theorem II.2.1 for variational solutions and in Theorem III.2.2 for generalized solutions. More general estimates of *energy type* for variational solutions are given in Theorem II.2.2, by assuming the separating function w to be in a suitable Kato class. The estimated energies of Dirichlet problems must be replaced, in general obstacle problems, by some *potential* seminorms as given in Section I.8.

Let us point out finally a special interesting case of our theory, namely Dirichlet problems in which a non-homogeneous condition $u=h$ is prescribed on an *arbitrary Borel set* E of positive capacity in $\mathbf{R}^N$. Such a problem can indeed be formulated as a two-obstacle problem, with the obstacles ψ_1 and ψ_2 defined to be $\psi_1=\psi_2=h$ on E and $\psi_1=-\infty, \psi_2=+\infty$ on $\mathbf{R}^N-E$. Unless E is compact, the regularity at a point $x_0 \in \partial E$ can not be reduced to the classical boundary regularity theory in $\mathbf{R}^N-E$, nor the pointwise results of the potential theory can be applied (except in the case, typical in potential theory, where h is a constant on E).

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Part I. Notation and preliminary results

In this part of the paper we fix the notation and state some preliminary results.

I.1. Capacity notions

If K is a compact subset of $\mathbf{R}^N$, $N \geq 3$, we define

$$\operatorname{Cap}(K) = \inf\left\{ \int_{\mathbf{R}^N} (|\nabla\varphi|^2 + \varphi^2)\,dx \colon \varphi \in C_0^1(\mathbf{R}^N),\ \varphi \geq 1 \text{ on } K \right\},$$

where $\nabla\varphi$ denotes the gradient of φ. If A is an open subset of $\mathbf{R}^N$ we put

$$\operatorname{Cap}(A) = \sup\{\operatorname{Cap}(K) \colon K \text{ compact}, K \subseteq A\}.$$

If E is an arbitrary subset of $\mathbf{R}^N$ we put

$$\operatorname{Cap}(E) = \inf\{\operatorname{Cap}(A) \colon A \text{ open}, A \supseteq E\}.$$

Let Ω be a bounded open subset of $\mathbf{R}^N$; if K is a compact subset of Ω we define

$$\operatorname{cap}(K, \Omega) = \inf\left\{ \int_{\Omega} |\nabla\varphi|^2 dx \colon \varphi \in C_0^1(\Omega),\ \varphi \geq 1 \text{ on } K \right\}. \tag{1.1}$$

We then extend this definition to an arbitrary $E \subseteq \Omega$ as in the previous case. We refer to [3], [5], and [9] for details and properties.

We say that a function u defined on a subset $E \subseteq \mathbf{R}^N$ is *quasi continuous* (in the capacity sense) if for every $\varepsilon > 0$ there exists an open subset A of $\mathbf{R}^N$ with $\operatorname{Cap}(A) < \varepsilon$, such that $u|_{E-A}$ is continuous on $E-A$.

If a statement depending on $x \in \mathbf{R}^N$ holds for every $x \in E$ except for a subset N of E with $\operatorname{Cap}(N) = 0$, then we say that it holds *quasi everywhere (q.e.)* on E.

We say that a sequence of functions $\psi_h \colon E \to [-\infty, +\infty]$ converges *quasi uniformly* (in the capacity sense) to a function $\psi \colon E \to [-\infty, +\infty]$ if for every $\varepsilon > 0$ there exists an open subset A of $\mathbf{R}^N$, with $\operatorname{Cap}(A) < \varepsilon$, such that $\psi_h - \psi \to 0$ uniformly on $E - A$ (with the convention $+\infty - (+\infty) = -\infty - (-\infty) = 0$). If each ψ_h is quasi continuous on E, then ψ also is quasi continuous on E.

Let v be a function $E \to [-\infty, +\infty]$, then we denote by $\sup_E v$ the *essential supremum* of v on E taken in the *capacity sense*; in the same way we define $\inf_E v$. Boundedness from above and below (in the capacity sense) are defined accordingly. For every $x \in E$ we define

$$\bar{v}(x) = \inf_{\varrho>0} \left\{ \sup_{B_\varrho(x)\cap E} v \right\} \quad \text{and} \quad \underline{v}(x) = \sup_{\varrho>0} \left\{ \inf_{B_\varrho(x)\cap E} v \right\} \tag{1.2}$$

where $B_\varrho(x) = \{y \in \mathbf{R}^N : |x-y| < \varrho\}$, $\varrho > 0$. We have $\bar{v}(x) < +\infty$ (resp. $\underline{v}(x) > -\infty$) if and only if v is locally bounded from above (resp. below) in some neighbourhood of x.

If $\operatorname{Cap}(E) > 0$, $\sup_E v > -\infty$, and $\inf_E v < +\infty$, the oscillation of v on E is defined by

$$\operatorname*{osc}_E v = \sup_E v - \inf_E v. \tag{1.3}$$

We set $\operatorname{osc}_E v = 0$ in any other case. We say that v is *continuous* at x_0 on E if

$$\lim_{\varrho\to 0} \left(\operatorname*{osc}_{E\cap B_\varrho(x_0)} v \right) = 0.$$

I.2. The Sobolev spaces

Let Ω be an arbitrary open subset of $\mathbf{R}^N$. By $H^1(\Omega)$ we denote the space of all functions u of $L^2(\Omega)$ whose distribution derivatives are in $L^2(\Omega)$, endowed with the norm

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

By $H^1_{\text{loc}}(\Omega)$ we denote the set of all functions $u \in L^2_{\text{loc}}(\Omega)$ such that $u_{\Omega'} \in H^1(\Omega')$ for every open set $\Omega' \Subset \Omega$ (i.e. $\bar{\Omega}'$ compact and $\bar{\Omega}' \subseteq \Omega$). By $H^1_0(\Omega)$ we denote the closure of $C^1_0(\Omega)$ in $H^1(\Omega)$, and by $H^{-1}(\Omega)$ we denote the dual space of $H^1_0(\Omega)$. The dual pairing is denoted by $\langle\,,\rangle$.

It is known that for every $u \in H^1_{\text{loc}}(\Omega)$ the limit

$$\lim_{\varrho\to 0^+} \frac{1}{|B_\varrho(x)|} \int_{B_\varrho(x)} u(y)\,dy$$

exists and is finite quasi everywhere in Ω, where $|B_\varrho(x)|$ denotes the Lebesgue measure of the ball $B_\varrho(x)$.

We make the following convention about the pointwise values of functions $u \in H^1_{\text{loc}}(\Omega)$: for every $x \in \Omega$ we always require that

$$\liminf_{\varrho\to0^+}\frac{1}{|B_\varrho(x)|}\int_{B_\varrho(x)}u(y)\,dy\le u(x)\le\limsup_{\varrho\to0^+}\frac{1}{|B_\varrho(x)|}\int_{B_\varrho(x)}u(y)\,dy. \tag{2.1}$$

With this convention, the pointwise value $u(x)$ is determined quasi everywhere in Ω and the function u is quasi continuous.

Note that for a function $u\in H^1_{\mathrm{loc}}(\Omega)$ the condition $u\geq0$ a.e. in Ω and $u\geq0$ q.e. in Ω are equivalent. A function $u\in H^1_0(\Omega)$ can be extended to a quasi continuous function $u\in H^1(\mathbf{R}^N)$ by simply putting $u=0$ q.e. in $\mathbf{R}^N-\Omega$.

It can be proved that for every $E\subseteq\mathbf{R}^N$

$$\mathrm{Cap}(E)=\min\left\{\int_{\mathbf{R}^N}(|\nabla u|^2+u^2)\,dx\colon u\in H^1(\mathbf{R}^N),\,u\ge1\text{ q.e. on }E\right\}.$$

Moreover

$$\mathrm{cap}(E,\Omega)=\min\left\{\int_\Omega|\nabla u|^2\,dx\colon u\in H^1_0(\Omega),\,u\ge1\text{ q.e. on }E\right\},$$

provided that Ω is bounded and contains E.

By a *non-negative Radon measure* on Ω we mean a non-negative distribution on Ω. By a (signed) *Radon measure* we mean the difference of two non-negative Radon measures.

If $\mu\in H^{-1}$ is a non-negative Radon measure, then the equality

$$\langle\mu,v\rangle=\int_\Omega v\,d\mu$$

holds for every $v\in H^1_0(\Omega)$, where the pointwise values of v are determined q.e. in Ω by the convention (2.1). For the preceding properties of $H^{-1}(\Omega)$ see e.g. [5].

Given two functions u and v defined in Ω, we denote by $u\wedge v$ and $u\vee v$ the functions defined in Ω by

$$(u\wedge v)(x)=\min\{u(x),v(x)\},\quad (u\vee v)(x)=\max\{u(x),v(x)\}.$$

The function u^+ and u^- are defined by $u^+=u\vee0$ and $u^-=-(u\wedge0)$.

It is well known that if u and v belong to $H^1(\Omega)$ (resp. $H^1_{\mathrm{loc}}(\Omega), H^1_0(\Omega)$), then $u\wedge v$ and $u\vee v$ belong to $H^1(\Omega)$ (resp. $H^1_{\mathrm{loc}}(\Omega), H^1_0(\Omega)$).

Definition 2.1. We say that two functions $u,v\in H^1(\Omega)$ satisfy the inequality $u\leq v$ on $\partial\Omega$ (in the sense of $H^1(\Omega)$) or equivalently that $v\geq u$ on $\partial\Omega$, if $(v-u)\wedge0$ belongs to $H^1_0(\Omega)$.

Note that in the previous definition we do not assume that u and v can be extended to quasi continuous functions defined on $\bar{\Omega}$.

If, $u, v \in H^1(\mathbf{R}^N)$, then $u \leqslant v$ on $\partial\Omega$ in the sense of $H^1(\Omega)$ if and only if $u \leqslant v$ q.e. on $\partial\Omega$, where the values of u and v on $\partial\Omega$ are defined according to our convention (2.1). If $u \in H^1(\Omega)$, $v \in H_0^1(\Omega)$, and $u \leqslant v$ a.e. on Ω, then clearly $u \leqslant 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$. More generally, we can prove the following lemma.

LEMMA 2.1. *Let Ω be a bounded open subset of $\mathbf{R}^N$ and let $u \in H^1(\Omega)$. Assume that there exists a quasi continuous function $\psi: \bar{\Omega} \to \bar{\mathbf{R}}$ such that $u \leqslant \psi$ q.e. on Ω and $\psi = 0$ q.e. on $\partial\Omega$. Then $u \leqslant 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$.*

Proof. It is enough to prove the lemma under the additional assumption $0 \leqslant u \leqslant \psi \leqslant 1$ q.e. on Ω. Since ψ is quasi continuous on Ω, for every $h \in N$ there exists an open set A_h such that $\psi|_{\bar{\Omega}-A_h}$ is continuous and $\mathrm{Cap}(A_h) < 1/h$. Since $\psi = 0$ q.e. on $\partial\Omega$, we may assume that $\psi(x) = 0$ for every $x \in \partial\Omega - A_h$. Therefore the set

$$K_h = \{x \in \bar{\Omega} : \psi(x) \geqslant 1/h\} - A_h$$

is compact and contained in Ω.

For every $h \in N$ we denote by v_h the solution of the minimum problem

$$\min\left\{\int_{\mathbf{R}^N} (|\nabla v|^2 + v^2)\, dx : v \in H^1(\mathbf{R}^N),\, v \geqslant 1 \text{ q.e. on } A_h\right\}.$$

Since $\mathrm{Cap}(A_h) < 1/h$, the sequence $\{v_h\}$ converges to 0 strongly in $H^1(\mathbf{R}^N)$. Let

$$u_h = \left(u - \frac{1}{h} - v_h\right) \vee 0.$$

Then $u_h \in H^1(\Omega)$ and $u_h = 0$ q.e. on $\Omega - K_h$. Therefore $u_h \in H_0^1(\Omega)$. Since u_h converges to $u \vee 0$ strongly in $H^1(\Omega)$, it follows that $u \vee 0 \in H_0^1(\Omega)$, hence $u \leqslant 0$ on $\partial\Omega$ in the sense of $H^1(\Omega)$. □

I.3. The Kato spaces

Let Ω be a bounded open subset of $\mathbf{R}^N$. By $K(\Omega)$ we denote the set of all signed Radon measures μ on Ω such that

$$\lim_{\varrho \to 0^+} \left\{ \sup_{x \in \Omega} \int_{\Omega \cap B_\varrho(x)} |y - x|^{2-N}\, d|\mu|(y) \right\} = 0,$$

where $|\mu|$ denotes the total variation of μ. We define a norm in $K(\Omega)$ by setting

$$\|\mu\|_{K(\Omega)} = \sup_{x\in\Omega} \int_\Omega |y-x|^{2-N}\, d|\mu|(y)$$

For every $\mu \in K(\Omega)$ and every $x \in \Omega$ we have

$$\lim_{\varrho\to 0^+} \|\mu\|_{K(B_\varrho(x))} = 0 \tag{3.1}$$

(see e.g. [14], [2], [5]). Moreover $K(\Omega) \subseteq H^{-1}(\Omega)$ with continuous imbedding. In fact

$$\int_\Omega \int_\Omega |y-x|^{2-N}\, d|\mu|(y)\, d|\mu|(x) \leq \operatorname{diam}(\Omega)^{2-N} \|\mu\|^2_{K(\Omega)}$$

for every $\mu \in K(\Omega)$.

I.4. The operator L

In the whole paper we shall denote by L a linear second order partial differential operator in $\mathbf{R}^N$ in divergence form

$$Lu = -\sum_{i,j=1}^N (a_{ij} u_{x_j})_{x_i} \tag{4.1}$$

with coefficients $a_{ij} \in L^\infty(\mathbf{R}^N)$, $i,j = 1, \dots, N$, and satisfying the uniform ellipticity conditions

$$\sum_{i,j=1}^N a_{ij}(x)\, \xi_j \xi_i \geq \lambda |\xi|^2, \quad |a_{ij}(x)| \leq \Lambda \quad \text{for a.e. } x \in \mathbf{R}^n \tag{4.2}$$

for some constants $0 < \lambda \leq \Lambda$.

Let Ω be a bounded open subset of $\mathbf{R}^N$. We define the bilinear form a on $H^1(\Omega)$ by

$$a(u,v) = a_\Omega(u,v) = \sum_{i,j=1}^N \int_\Omega a_{ij} u_{x_j} v_{x_i}\, dx. \tag{4.3}$$

According to [22] we say that u is a (local) *solution* in Ω of the equation

$$Lu = f \tag{4.4}$$

for a given $f \in H^{-1}(\Omega)$ if $u \in H^1(\Omega)$ and

$$a_\Omega(u,\varphi)=\langle f,\varphi\rangle \quad \text{for every } \varphi\in H_0^1(\Omega); \tag{4.5}$$

we say that u is a *subsolution* of the equation (4.4) if

$$a_\Omega(u,\varphi)\leqslant\langle f,\varphi\rangle \quad \text{for every } \varphi\in H_0^1(\Omega),\ \varphi\geqslant 0;$$

the *supersolutions* are defined similarly.

If u is a subsolution (supersolution) of the equation $Lu=0$ we also say that u is a subsolution (supersolution) of the operator L.

I.5. A priori estimates for solutions and subsolutions

In this section we give some estimates for solutions and subsolutions of the equation $Lu=\nu$, where ν is a measure of the Kato space $K(\Omega)$.

Let $B_R=B_R(x_0), x_0\in\mathbf{R}^N, R>0$, let $0<s<1$, and let $\nu\in K(B_R)$.

PROPOSITION 5.1. *Let $u\in H^1(B_R)$ be a solution, or a non-negative subsolution, of the equation $Lu=\nu$. Then there exists a constant $c=c(\lambda,\Lambda,N,s)>0$ such that*

$$\|u\|^2_{L^\infty(B_{sR})}\leqslant c\left[R^{-N}\|u\|^2_{L^2(B_R-B_{sR})}+\|\nu\|^2_{K(B_R)}\right], \tag{5.1}$$

$$\int_{B_{sR}}|\nabla u|^2|x-x_0|^{2-N}\,dx\leqslant c\left[R^{-N}\|u\|^2_{L^2(B_R-B_{sR})}+\|\nu\|^2_{K(B_R)}\right]. \tag{5.2}$$

Proof. Lemmas 6.7 and 6.8 of [5], applied with $\mu=0$, give the result when u is a solution. The same proofs can be easily adapted to the case of non-negative subsolutions. □

For every $u\in H^1(B_R)$ we put

$$u_R=\frac{1}{|B_R|}\int_{B_R}u(x)\,dx.$$

PROPOSITION 5.2. *Let $u\in H^1(B_R)$ be a solution of the equation $Lu=\nu$. Then*

$$\left(\operatorname*{osc}_{B_{sR}}u\right)^2\leqslant c\left[R^{-N}\|u-u_R\|^2_{L^2(B_R-B_{sR})}+\|\nu\|^2_{K(B_R)}\right],$$

where c is a constant depending only on λ, Λ, N, and s.

Proof. Since $\operatorname{osc}_{B_{sR}}u\leqslant 2\|u-u_R\|_{L^\infty(B_{sR})}$, it is enough to apply the estimate (5.1) to the function $u-u_R$. □

PROPOSITION 5.3. *Let $u \in H^1(B_R)$ be a solution of the equation $Lu=\nu$. Then*

$$\|u\|_{L^\infty(B_R)} \leq \sup_{\partial B_R} |u| + c\|\nu\|_{K(B_R)},$$

where c is a constant depending only on λ, Λ, and N.

Proof. For every $y \in B_R$ we set

$$w(y) = \int_{B_R} G^y(x)\, d|\nu|(x),$$

where G^y is the Green function for the Dirichlet problem relative to the operator L in B_{2R} with singularity at y. Then $Lw=|\nu|$ on B_R and

$$\sup_{B_R} w \leq c\|\nu\|_{K(B_R)} \tag{5.3}$$

by the well known estimates of the Green function (see [17], [22], [13]). Since $|u|$ is a subsolution of the equation $Lv=|\nu|$ (see, for instance, [5], Proposition 2.6), the function $z=|u|-w$ is a subsolution of the equation $Lv=0$. By the maximum principle we have

$$\sup_{B_R} z \leq \sup_{\partial B_R} z,$$

hence

$$\sup_{B_R} |u| \leq \sup_{B_R} z + \sup_{B_R} w \leq \sup_{\partial B_R} z + \sup_{B_R} w \leq \sup_{\partial B_R} |u| + \sup_{B_R} w.$$

The conclusion follows now from (5.3). □

I.6. H^1-dominated quasi uniform convergence

In this section we introduce a convergence which will be used in Part III, in connection with our definition of *generalized solutions*. Let E be an arbitrary subset of $\mathbf{R}^N$.

Definition 6.1. We say that a function $\psi: E \to \bar{\mathbf{R}}$ is *H^1-dominated* (on E) if there exists $v \in H^1(\mathbf{R}^N)$ such that $|\psi| \leq v$ q.e. on E.

We refer to [1] for a characterization of the H^1-dominated functions ψ in terms of the capacities of the level sets of $|\psi|$.

Definition 6.2. Let $\psi_h\colon E\to\bar{\mathbf{R}}$ be a sequence of functions converging *quasi uniformly* to $\psi\colon E\to\bar{\mathbf{R}}$. If in addition there exists $v\in H^1(\mathbf{R}^N)$ such that $|\psi_h-\psi|\leqslant v$ q.e. on E then we say that ψ is the *H^1-dominated quasi uniform limit of* ψ_h (on E).

PROPOSITION 6.1. *A function $\psi\colon E\to\bar{\mathbf{R}}$ is the H^1-dominated quasi uniform limit of a sequence of functions $\psi_h\colon E\to\bar{\mathbf{R}}$ if and only if there exists a decreasing sequence v_h in $H^1(\mathbf{R}^N)$ converging to 0 strongly in $H^1(\mathbf{R}^N)$ such that $|\psi_h-\psi|\leqslant v_h$ q.e. on E for every $h\in\mathbf{N}$. If E is bounded, we may assume in addition that each function v_h is a supersolution of the operator L in a neighbourhood Ω of E.*

Proof. Assume that ψ is the H^1-dominated quasi uniform limit of ψ_h. Then there exists $v\in H^1(\mathbf{R}^N)$ such that $|\psi_h-\psi|\leqslant v$ q.e. on E for every $h\in\mathbf{N}$. For every $k\in\mathbf{N}$ there exist $\sigma(k)\in\mathbf{N}$ and an open set A_k such that $\mathrm{Cap}(A_k)<2^{-k}$ and $|\psi_h-\psi|<1/k$ on $E-A_k$ for every $h\geqslant\sigma(k)$. We may assume that the sequence A_k is decreasing and that $\sigma\colon\mathbf{N}\to\mathbf{N}$ is strictly increasing. For every $k\in\mathbf{N}$ we may consider the solution w_h of the minimum problem.

$$\min\left\{\int_{\mathbf{R}^N}(|\nabla w|^2+w^2)\,dx;\, w\in H^1(\mathbf{R}^N),\, w\geqslant v \text{ q.e. on } A_k\right\}.$$

Then the sequence w_k is decreasing. Since $v\in H^1(\mathbf{R}^N)$ and $\mathrm{Cap}(A_k)\to 0$ as $k\to+\infty$, the sequence w_k converges to 0 strongly in $H^1(\mathbf{R}^N)$.

We define

$$v_h=\begin{cases} v & \text{for } h<\sigma(1),\\ w_k+\left(\dfrac{1}{k}\wedge v\right) & \text{for } \sigma(k)\leqslant h<\sigma(k+1).\end{cases}$$

Then v_h is decreasing and converging to 0 strongly in $H^1(\mathbf{R}^N)$. Since for $\sigma(k)\leqslant h$

$$|\psi_h-\psi|\leqslant\begin{cases}\dfrac{1}{k}\wedge v & \text{q.e. on } E-A_k,\\ v\leqslant w_k & \text{q.e. on } E\cap A_k,\end{cases}$$

we have $|\psi_h-\psi|\leqslant v_h$ q.e. on E for every $h\in\mathbf{N}$.

Let us suppose that E is contained in a bounded open set Ω of $\mathbf{R}^N$. Then we can replace the functions v_h by the solutions z_h of the variational inequality

$$\begin{cases} z_h-v_h\in H^1_0(\Omega), \quad z_h\geqslant v_h \text{ q.e. in } \Omega.\\ a_\Omega(z_h, z-z_h)\geqslant 0\\ \forall z\in H^1(\Omega), \quad z-v_h\in H^1_0(\Omega), \quad z\geqslant v_h \text{ q.e. in } \Omega.\end{cases}$$

It is then clear that each function z_h is a supersolution of L on Ω and satisfies the inequality $|\psi_h-\psi|\leq v_h$ q.e. on E. Moreover the sequence z_h is decreasing and converges to 0 strongly in $H^1(\Omega)$.

Converserly, assume that there exists a decreasing sequence v_h in $H^1(\mathbf{R}^N)$ converging to 0 strongly in $H^1(\mathbf{R}^N)$ such that $|\psi_h-\psi|\leq v_h$ q.e. on E for every $h\in\mathbf{N}$. Since $|\psi_h-\psi|\leq v_h$ q.e. on E for every $h\in\mathbf{N}$, to prove ψ is the H^1-dominated quasi uniform limit of ψ_h it is enough to show that ψ_h convergs to ψ quasi uniformly (in the capacity sense).

For every $k\in\mathbf{N}$ there exists $\sigma(k)\in\mathbf{N}$ such that

$$\|v_{\sigma(k)}\|_{H^1(\mathbf{R}^N)}\leq 1/k^2.$$

Let $A_k=\{v_{\sigma(k)}>1/k\}$. Then $\mathrm{Cap}(A_k)<1/k^2$ and $|\psi_h-\psi|\leq v_h\leq v_{\sigma(k)}\leq 1/k$ q.e. on $E-A_k$ for every $h\geq\sigma(k)$. This proves that ψ_h converges to ψ quasi uniformly. □

I.7. The Wiener moduli

In this section functions $\psi\colon\mathbf{R}^N\to\bar{\mathbf{R}}$ will be considered which will play the role either of a *lower obstacle* or of an *upper obstacle* for our problem. The *variational* behavior of these one-sided obstacles at a given point $x_0\in\mathbf{R}^N$ will be described in terms of a function

$$\omega_\sigma(r,R)=\omega_\sigma(\psi,x_0;r,R),\quad 0<r<R,\ \sigma>0, \tag{7.1}$$

called the *Wiener modulus* of ψ at x_0.

For a lower obstacle ψ, this will be done in terms of suitable one-sided level sets of ψ, namely

$$E(\varepsilon,\varrho)=E(\psi,x_0;\varepsilon,\varrho)=\left\{x\in B_\varrho(x_0)\colon \psi(x)\geq \sup_{B_\varrho(x_0)}\psi-\varepsilon\right\} \tag{7.2}$$

and their relative capacities

$$\delta(\varepsilon,\varrho)=\delta(\psi,x_0;\varepsilon,\varrho)=\frac{\mathrm{cap}(E(\varepsilon,\varrho),B_{2\varrho}(x_0))}{\mathrm{cap}(B_\varrho(x_0),B_{2\varrho}(x_0))}. \tag{7.3}$$

We then define the (lower) Wiener modulus (7.1) by setting

$$\omega_\sigma(r,R)=\inf\left\{\omega>0\colon \omega\exp\left(\int_r^R\delta(\sigma\omega,\varrho)\frac{d\varrho}{\varrho}\right)\geq 1\right\}.$$

The *modulus* ω, for a fixed "*scaling factor*" $\sigma>0$, can be regarded as implicitly defined by

$$\omega=\exp\left(-\int_r^R \delta(\sigma\omega,\varrho)\frac{d\varrho}{\varrho}\right).$$

More precisely we have the following lemma (for the proof of the lemmas of this section see [20], Section 4).

LEMMA 7.1. *Let* $0<r\leq R$ *be fixed. Then* $\varepsilon>0$ *and* $\sigma>0$ *verify*

$$\sigma=\varepsilon\exp\left(\int_r^R \delta(\varepsilon,\varrho)\frac{d\varrho}{\varrho}\right)$$

if and only if

$$\omega_\sigma(r,R)=\exp\left(-\int_r^R \delta(\varepsilon,\varrho)\frac{d\varrho}{\varrho}\right) \quad \textit{and} \quad \sigma\omega_\sigma(r,R)=\varepsilon.$$

In addition to the integrals

$$\int_r^R \delta(\varepsilon,\varrho)\frac{d\varrho}{\varrho} \tag{7.4}$$

we shall also consider the integrals

$$\int_r^R \delta^*(\varepsilon,\varrho)\frac{d\varrho}{\varrho}, \tag{7.5}$$

where now $\delta^*(\varepsilon,\varrho), \varepsilon>0, \varrho>0$, is defined to be

$$\delta^*(\varepsilon,\varrho)=\frac{\operatorname{cap}(E^*(\varepsilon,\varrho),B_{2\varrho}(x_0))}{\operatorname{cap}(B_\varrho(x_0),B_{2\varrho}(x_0))}$$

and

$$E^*(\varepsilon,\varrho)=\{x\in B_\varrho(x_0)\colon \psi(x)\geq \bar{\psi}(x_0)-\varepsilon\}.$$

The Wiener modulus $\omega_\sigma^*(r,R)$ is defined as $\omega_\sigma(r,R)$ with $\delta(\varepsilon,\varrho)$ replaced by $\delta^*(\varepsilon,\varrho)$.

Remark 7.1. It follows immediately from the definitions that $\delta(\varepsilon,\varrho)\leq\delta^*(\varepsilon,\varrho)$ for every $\varepsilon>0, \varrho>0$, hence $\omega_\sigma(r,R)\geq\omega_\sigma^*(r,R)$ for every $0<r\leq R$ and for every $\sigma>0$. Moreover Lemma 7.1 continues to hold with $\delta(\varepsilon,\varrho)$ and $\omega_\sigma(r,R)$ replaced by $\delta^*(\varepsilon,\varrho)$ and $\omega_\sigma^*(r,R)$ respectively.

The vanishing of $\omega_\sigma(r,R)$ as $r\to0^+$ is clearly related to the divergence as $r\to0^+$ of each one of the (lower) Wiener integrals (7.4) and (7.5). In fact we can prove the following lemma.

LEMMA 7.2. *Assume that* $-\infty<\bar{\psi}(x_0)<+\infty$. *Then the following conditions are equivalent:*

(a) *for every* $\varepsilon>0$ *there exists* $R>0$ *such that* $\lim_{r\to 0^+}\omega_\sigma(r,R)=0$ *for suitable* $\sigma=\sigma(\varepsilon,R;r)$ *such that* $\sigma\omega_\sigma(r,R)\leq\varepsilon$ *for all* $0<r\leq R$;

(b) *for every* $\varepsilon>0$ *there exists* $R>0$ *such that*

$$\int_0^R \delta(\varepsilon,\varrho)\frac{d\varrho}{\varrho}=+\infty;$$

(c) *for every* $\varepsilon>0$ *there exists* $R>0$ such that

$$\int_0^R \delta^*(\varepsilon,\varrho)\frac{d\varrho}{\varrho}=+\infty.$$

The vanishing of $\omega_\sigma(r,R)$ as $r\to 0^+$ is also related to the *regularity* of the set

$$F=F_\psi=\{x:\psi(x)>-\infty\}$$

at the point x_0 (in the sense of the classical potential theory) and to the continuity at x_0 of the restriction of ψ to F (in the sense of Section 1). For every $\varrho>0$ we set

$$B_\varrho^F(x_0)=\begin{cases}B_\varrho(x_0) & \text{if } \operatorname{Cap}(B_\varrho(x_0)\cap F)=0,\\ B_\varrho(x_0)\cap F & \text{if } \operatorname{Cap}(B_\varrho(x_0)\cap F)>0,\end{cases}$$

and for every $0<r\leq R$ we define

$$W_F(r,R)=\exp\left(-\int_r^R \frac{\operatorname{cap}(B_\varrho^F(x_0),B_{2\varrho}(x_0))}{\operatorname{cap}(B_\varrho(x_0),B_{2\varrho}(x_0))}\,\frac{d\varrho}{\varrho}\right). \tag{7.6}$$

Then the following estimate holds.

LEMMA 7.3. *For arbitrary* $\sigma>0$ *we have*

$$\frac{r}{R}\leq\omega_\sigma(r,R)\leq\min\left\{1,\max\left[W_F(r,R),\frac{1}{\sigma}\operatorname*{osc}_{B_\varrho(x_0)\cap F}\psi\right]\right\}$$

for every $0<r\leq R$.

For an upper obstacle ψ, the (upper) *Wiener modulus* of ψ at the point x_0 and the corresponding (upper) *Wiener integrals* (7.4) are defined similarly by just replacing in (7.2) the supremum with the infimum, $-\varepsilon$ with ε, and $\geq$ with $\leq$; that is by taking

$$E(\varepsilon,\varrho)=\left\{x\in B_\varrho(x_0)\colon \psi(x)\leqslant \inf_{B_\varrho(x_0)}\psi+\varepsilon\right\};$$

accordingly, the sets E^* in the upper Wiener integrals (7.5) will be defined as

$$E^*(\varepsilon,\varrho)=\{x\in B_\varrho(x_0)\colon \psi(x)\leqslant \underline{\psi}(x_0)+\varepsilon\}.$$

I.8. Potential estimates for one-sided obstacle problems

In this section the function $\psi\colon\mathbf{R}^N\to\bar{\mathbf{R}}$ plays the role of a lower obstacle. Let $\Omega=B_R(x_0), x_0\in\mathbf{R}^N, R>0$, and let $\mu\in K(\Omega)$. We consider a *local variational solution u* of the *one-sided obstacle problem*

$$\tag{8.1}\begin{cases} u\in H^1(\Omega), \quad u\geqslant\psi \text{ q.e. in } \Omega,\\ a_\Omega(u,v-u)\geqslant \displaystyle\int_\Omega (v-u)\,d\mu\\ \forall v\in H^1(\Omega), \quad v\geqslant \psi \text{ q.e. in } \Omega, \quad v-u\in H^1_0(\Omega).\end{cases}$$

For every $0<r\leqslant R$ we set $B_r=B_r(x_0)$ and we consider the potential seminorm $\mathcal{V}(r)$ of u defined by

$$\tag{8.2}\mathcal{V}^2(r)=\left(\operatorname*{osc}_{B_r} u\right)^2+\int_{B_r}|\nabla u|^2|x-x_0|^{2-N}\,dx.$$

The decay of $\mathcal{V}(r)$ to zero as $r\to0^+$ can be estimated according to the following proposition.

PROPOSITION 8.1. *There exist two constants $c=c(\lambda,\Lambda,N)>0$ and $\beta=\beta(\lambda,\Lambda,N)>0$ such that for every solution u of* (8.1) *we have*

$$\tag{8.3}\mathcal{V}(r)\leqslant c\left[R^{-N/2}\|u-d\|_{L^2(B_R)}\,\omega_\sigma(r,R)^\beta+\sigma\omega_\sigma(r,R)+\|\mu\|_{K(B_R)}\right]$$

for every $0<r\leqslant R/2$, for every $\sigma>0$, and for every constant $d\geqslant\sup_{B_R}\psi$.

Proof. The results for $\mu=0$ are proved in [20], Theorems 6.1 and 6.2. Let us discuss now the case $\mu\neq0$. We shall denote by c and β various positive constants, depending only on λ,Λ, and N, whose value can change from one line to the other.

Let us consider the unique solution $w\in H^1_0(B_R)$ of the equation $Lw=\mu$. By Proposition 5.3 we have

$$\|w\|_{L^\infty(B_R)} \leq c\|\mu\|_{K(B_R)}. \tag{8.5}$$

The function $z=u-w$ is a solution in B_R of the variational inequality

$$\begin{cases} z \in H^1(B_R), \quad z \geq \psi - w \text{ q.e. in } B_R, \\ a_\Omega(z, v-z) \geq 0, \quad \Omega = B_R \\ \forall v \in H^1(B_R), \quad v \geq \psi - w \text{ q.e. in } B_R, \quad v - z \in H_0^1(B_R). \end{cases} \tag{8.6}$$

For every $\eta>0$, $0<\varrho<R$ we define

$$E_w(\eta, \varrho) = \left\{ x \in B_\varrho \colon \psi(x) - w(x) \geq \sup_{B_\varrho} (\psi - w) - \eta \right\}$$

and

$$\delta_w(\eta, \varrho) = \frac{\operatorname{cap}(E_w(\eta, \varrho), B_{2\varrho})}{\operatorname{cap}(B_\varrho, B_{2\varrho})}.$$

Let $\mathcal{V}_w(r)$, $0<r\leq R$, be the potential seminorm defined as in (8.2) with u replaced by $z=u-w$. Let us fix $0<r\leq R/2$. By applying Theorem 6.1 of [20] to the obstacle problem (8.6), we obtain

$$\mathcal{V}_w(r) \leq c\mathcal{V}_w(R/2) \exp\left(-\beta \int_r^{R/2} \delta_w(\eta, \varrho) \frac{d\varrho}{\varrho} \right) + c\eta$$

for every $\eta>0$ (see also Lemma 7.1). Take $\eta = \varepsilon + \operatorname{osc}_{B_{R/2}} w$ with $\varepsilon>0$. Since $E(\varepsilon, \varrho) \subseteq E_w(\eta, \varrho)$, we have

$$\mathcal{V}_w(r) \leq c\mathcal{V}_w(R/2) \exp\left(-\beta \int_r^{R/2} \delta(\varepsilon, \varrho) \frac{d\varrho}{\varrho} \right) + c\varepsilon + c \operatorname*{osc}_{B_{R/2}} w. \tag{8.7}$$

By (8.5) and by Proposition 5.1 we obtain

$$\begin{aligned} |\mathcal{V}(\varrho) - \mathcal{V}_w(\varrho)| &\leq \left[\left(\operatorname*{osc}_{B_{R/2}} w \right)^2 + \int_{B_{R/2}} |\nabla w|^2 |x - x_0|^{2-N} \, dx \right]^{1/2} \\ &\leq c\left[\|w\|_{L^\infty(B_R)} + \|\mu\|_{K(B_R)} \right] \leq c\|\mu\|_{K(B_R)} \end{aligned} \tag{8.8}$$

for every $0<\varrho\leq R/2$, and Theorem 6.2 of [20] implies

$$\mathcal{V}(R/2) \leq c\left[R^{-N/2} \|u - d\|_{L^2(B_R)} + \|\mu\|_{K(B_R)} \right] \tag{8.9}$$

for every constant $d>\sup_{B_R}\psi$. Therefore (8.5), (8.7), (8.8), and (8.9) yield

$$\mathcal{V}(r)\leq c2^{\beta}R^{-N/2}\|u-d\|_{L^2(B_R)}\exp\left(-\beta\int_r^R \delta(\varepsilon,\varrho)\frac{d\varrho}{\varrho}\right)+c\varepsilon+c\|\mu\|_{K(B_R)}.$$

Let us fix $\sigma>0$. For every $\varepsilon>\sigma\omega_\sigma(r,R)$ the previous inequality implies

$$\mathcal{V}(r)\leq c\left[R^{-N/2}\|u-d\|_{L^2(B_R)}\left(\frac{\varepsilon}{\sigma}\right)^{\beta}+\varepsilon+\|\mu\|_{K(B_R)}\right]$$

and taking the limit as $\varepsilon\downarrow\sigma\omega_\sigma(r,R)$ we obtain (8.3). □

Part II. Variational solutions

Throughout this part of the paper, ψ_1 and ψ_2 are two arbitrary given functions from $\mathbf{R}^N$ into $\bar{\mathbf{R}}$ and x_0 is an arbitrary fixed point of $\mathbf{R}^N$. We shall write B_r instead of $B_r(x_0)$, $r>0$, and we shall freely use the notation from Part I. In the proofs we shall denote by c and β various positive constants, depending only on the ellipticity constants λ and Λ of the operator L, on the dimension N of the space, and, possibly, on a parameter $0<s<1$. The value of these constants can change from one line to the other.

II.1. Statement of the main results

Definition 1.1. For every open subset $\Omega\subseteq\mathbf{R}^N$, we say that a function u is a *(local) variational solution* in Ω of the two-obstacle problem $\{\psi_1,\psi_2\}$ if

$$(1.1)\qquad \begin{cases} u\in H^1(\Omega), \quad \psi_1\leq u\leq\psi_2 \text{ q.e. in } \Omega,\\ a_\Omega(u,v-u)\geq 0\\ \forall v\in H^1(\Omega),\quad \psi_1\leq v\leq\psi_2 \text{ q.e. in } \Omega,\quad v-u\in H_0^1(\Omega).\end{cases}$$

In all this section we shall only consider *variational solutions* and we shall omit in the following the term *variational.*

By $\mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ we denote the set of all functions u which are local solutions of the problem $\{\psi_1,\psi_2\}$ on some open neighbourhood Ω of x_0 (depending on u).

Definition 1.2. We say that x_0 is a *regular point* of problem $\{\psi_1,\psi_2\}$ if the set $\mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is not empty and every $u\in\mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is finite and continuous at x_0.

With the lower obstacle ψ_1 we associate the (lower) Wiener moduli $\omega_{1,\sigma}(r,R)$, $\omega_{1,\sigma}^*(r,R)$, $0<r\leq R$, $\sigma>0$, and the (lower) Wiener integrals $\int\delta_1(\varepsilon,\varrho)\,d\varrho/\varrho$, $\int\delta_1^*(\varepsilon,\varrho)\,d\varrho/\varrho$

defined as in Section I.7. With the upper obstacle ψ_2 we associate the (upper) Wiener moduli $\omega_{2,\sigma}(r,R)$, $\omega^*_{2,\sigma}(r,R)$, $0<r\leq R$, $\sigma>0$, and the corresponding (upper) Wiener integrals $\int \delta_2(\varepsilon,\varrho)\,d\varrho/\varrho$, $\int \delta^*_2(\varepsilon,\varrho)\,d\varrho/\varrho$ also defined in Section I.7.

Definition 1.3. We say that x_0 is a *Wiener point* of the problem $\{\psi_1,\psi_2\}$ if

$$\int_0^R \delta^*_1(\varepsilon,\varrho)\frac{d\varrho}{\varrho}=+\infty \quad\text{and}\quad \int_0^R \delta^*_2(\varepsilon,\varrho)\frac{d\varrho}{\varrho}=+\infty \tag{1.2}$$

for every $\varepsilon>0$ and for every $R>0$.

Remark 1.1. By Lemma I.7.2. we have that x_0 is a Wiener point of the problem $\{\psi_1,\psi_2\}$ if and only if x_0 is a Wiener point, according to Definition 3.1 of [20], both for the lower obstacle problem determined by ψ_1 and the upper obstacle problem determined by ψ_2.

The following *Wiener criterion* holds.

THEOREM 1.1. *The point x_0 is a regular point of $\{\psi_1,\psi_2\}$ if and only if all the following conditions* (1.3), (1.4), *and* (1.5) *are satisfied:*

(1.3) $\bar{\psi}_1(x_0)<+\infty$, $\underline{\psi}_2(x_0)>-\infty$, and $\bar{\psi}_1(x_0)\leq\underline{\psi}_2(x_0)$,

(1.4) *there exists $R>0$ and $w\in H^1(B_R)$ such that $\psi_1\leq w\leq\psi_2$ q.e. in B_R,*

(1.5) *x_0 is a Wiener point of $\{\psi_1,\psi_2\}$.*

Remark 1.2. By Remark 1.1 and by the characterization of the regular points of the unilateral problems (Theorem 5.1 of [20]) we have that x_0 is a regular point of $\{\psi_1,\psi_2\}$ if and only if all the following conditions (1.6), (1.7), and (1.8) are satisfied:

$$\bar{\psi}_1(x_0)\leq\underline{\psi}_2(x_0), \tag{1.6}$$

(1.7) there exists $R>0$ and $w\in H^1(B_R)$ such that $\psi_1\leq w\leq\psi_2$ q.e. on B_R,

(1.8) x_0 is a regular point, according to Definition 2.1 of [20], both for the lower obstacle problem determined by ψ_1 and for the upper obstacle problem determined by ψ_2.

The proof of the Wiener criterion will be given in Section 3 after the estimates for the solutions of the obstacle problem $\{\psi_1,\psi_2\}$ presented in the next section.

II.2. Oscillation and energy estimates

In this section we give an estimate of the oscillation and of the energy at x_0 of an arbitrary solution u of the two-obstacle problem

$$\begin{cases} u\in H^1(B_R), & \psi_1\leqslant u\leqslant\psi_2 \text{ q.e. in } B_R, \\ a_\Omega(u,v-u)\geqslant \displaystyle\int_\Omega (v-u)\,d\mu, & \Omega=B_R \\ \forall v\in H^1(B_R), & \psi_1\leqslant v\leqslant\psi_2 \text{ q.e. in } B_R, \quad v-u\in H_0^1(B_R), \end{cases} \tag{2.1}$$

where μ is a measure of the class $K(B_R)$, $R>0$. We shall always assume that

$$\sup_{B_R}\psi_1<+\infty \quad\text{and}\quad \inf_{B_R}\psi_2>-\infty. \tag{2.2}$$

In order to estimate the modulus of continuity of u at a Wiener point x_0 of $\{\psi_1,\psi_2\}$, for every $\varepsilon_1>0$, $\varepsilon_2>0$ we define

$$\begin{aligned}\Psi(\varepsilon_1,\varepsilon_2,R)&=\Big\{\sup_{B_R}\psi_1\vee[\underline{\psi}_2(x_0)+\varepsilon_2]\Big\}-\Big\{\inf_{B_R}\psi_2\wedge[\bar{\psi}_1(x_0)-\varepsilon_1]\Big\}\\ &=[\underline{\psi}_2(x_0)-\bar{\psi}_1(x_0)]+\Big\{\Big[\sup_{B_R}\psi_1-\underline{\psi}_2(x_0)\Big]\vee\varepsilon_2\Big\}+\Big\{\Big[\bar{\psi}_1(x_0)-\inf_{B_R}\psi_2\Big]\vee\varepsilon_1\Big\}\end{aligned}$$

and

$$Z(R)=\Big[\sup_{B_R}\psi_1-\inf_{B_R}\psi_2\Big]^+ + R^{-N/2}\|u-d_R\|_{L^2(B_R)},$$

where

$$d_R=\begin{cases}\sup_{B_R}\psi_1\wedge\inf_{B_R}\psi_2 & \text{if } u_R<\sup_{B_R}\psi_1\wedge\inf_{B_R}\psi_2,\\ u_R=\dfrac{1}{|B_R|}\displaystyle\int_{B_R}u(x)\,dx & \text{if } \sup_{B_R}\psi_1\wedge\inf_{B_R}\psi_2\leqslant u_R\leqslant\sup_{B_R}\psi_1\vee\inf_{B_R}\psi_2,\\ \sup_{B_R}\psi_1\vee\inf_{B_R}\psi_2 & \text{if } \sup_{B_R}\psi_1\vee\inf_{B_R}\psi_2,<u_R.\end{cases}$$

Moreover for every $0<r\leqslant R$, $\sigma_1>0$, $\sigma_2>0$ we set

$$\Psi_{\sigma_1,\sigma_2}(r,R)=\Psi(\varepsilon_1,\varepsilon_2,R),\quad\text{where}\quad \varepsilon_1=\sigma_1\omega^*_{1,\sigma_1}(r,R)\ \text{and}\ \varepsilon_2=\sigma_2\omega^*_{2,\sigma_2}(r,R).$$

Note that $\Psi(\varepsilon_1,\varepsilon_2,R)\geqslant0$ for every $\varepsilon_1>0,\varepsilon_2>0$.

THEOREM 2.1. *Assume* (2.2) *and let* $0<s<1$. *Then there exist two constants* $c=c(\lambda, \Lambda, N, s)>0$ *and* $\beta=\beta(\lambda, \Lambda, N)>0$ *such that for every solution u* of (2.1) *we have*

$$\operatorname*{osc}_{B_r} u \leq \Psi_{\sigma_1,\sigma_2}(r, R)+c\left\{Z(R)\left[\omega^*_{1,\sigma_1}(r, R)+\omega^*_{2,\sigma_2}(r, R)\right]^\beta+\|\mu\|_{K(B_R)}\right\} \tag{2.3}$$

for every $0<r\leq sR$ *and for every* $\sigma_1>0, \sigma_2>0$.

Remark 2.1. If $-\infty<\bar{\psi}_1(x_0)=\underline{\psi}_2(x_0)<+\infty$, then

$$\Psi_{\sigma_1,\sigma_2}(r, R)=\left\{\left[\sup_{B_R}\psi_1-\bar{\psi}_1(x_0)\right]\vee\sigma_2\,\omega^*_{2,\sigma_2}(r, R)\right\}+\left\{\left[\underline{\psi}_2(x_0)-\inf_{B_R}\psi_2\right]\vee\sigma_1\,\omega^*_{1,\sigma_1}(r, R)\right\}.$$

Therefore, if x_0 is a Wiener point of $\{\psi_1, \psi_2\}$, then the Wiener moduli at the right hand side of (2.3) vanish as $r\to 0^+$ by Lemma I.7.2. and we get

$$\limsup_{r\to 0^+}\operatorname*{osc}_{B_r} u\leq\left[\sup_{B_R}\psi_1-\bar{\psi}_1(x_0)\right]+\left[\underline{\psi}_2(x_0)-\inf_{B_R}\psi_2\right]+c\|\mu\|_{K(B_R)}.$$

Since the right hand side of this inequality tends to 0 as $R\to 0^+$ (see I.1.2) and (I.3.1)), we obtain that the function u is continuous at x_0.

If $\bar{\psi}_1(x_0)<\underline{\psi}_2(x_0)$, as for instance in the one-obstacle problem or in the free equation, then the estimate (2.3) will be improved in Theorem 2.2.

We shall see that Theorem 2.1 follows easily from the following propositions.

PROPOSITION 2.1. *Assume* (2.2). *Then there exist two constants* $c=c(\lambda, \Lambda, N)>0$ *and* $\beta=\beta(\lambda, \Lambda, N)>0$ *such that for every solution u of* (2.1) *we have*

$$\operatorname*{osc}_{B_r} u\leq\Psi_{\sigma_1,\sigma_2}(r,R)+c\left\{\left(\operatorname*{osc}_{B_R} u\right)\left[\omega^*_{1,\sigma_1}(r, R)+\omega^*_{2,\sigma_2}(r, R)\right]^\beta+\|\mu\|_{K(B_R)}\right\} \tag{2.4}$$

for every $0<r\leq R$ *and for every* $\sigma_1>0$, $\sigma_2>0$.

PROPOSITION 2.2. *Assume* (2.2) *and let* $0<s<1$. *Then there exists a constant* $c=c(\lambda, \Lambda, N, s)>0$ *such that*

$$\operatorname*{osc}_{B_{sR}} u\leq\left[\sup_{B_R}\psi_1-\inf_{B_R}\psi_2\right]^+ +c\left[R^{-N/2}\|u-d_R\|_{L^2(B_R)}+\|\mu\|_{K(B_R)}\right] \tag{2.5}$$

for every solution u of (2.1).

In Section III.3, in the more general setting of *generalized solutions,* we shall describe how the estimate (2.4) can be applied to obtain the *Maz'ja estimate* of the modulus of continuity of solutions of Dirichlet problems at a regular boundary point x_0 of the domain by just choosing σ_1 and σ_2 suitably.

Proof of Theorem 2.1. Let u be a solution of (2.1) and let $0<r\leq sR$. By Remark I.7.1. and by Proposition 2.1 we have

$$\operatorname*{osc}_{B_r} u \leq \Psi(\varepsilon_1, \varepsilon_2, sR) + c\left(\operatorname*{osc}_{B_{sR}} u\right)\left\{\sum_{i=1}^{2} \exp\left(-\int_r^{sR} \delta_i^*(\varepsilon_i, \varrho)\frac{d\varrho}{\varrho}\right)\right\}^{\beta} + c\|\mu\|_{K(B_R)}$$

for every $\varepsilon_1>0$, $\varepsilon_2>0$. Taking the inequality $\delta_i^*(\varepsilon_i, \varrho)\leq 1$ into account, the estimate (2.5) of Proposition 2.2 yields

$$\operatorname*{osc}_{B_r} u \leq \Psi(\varepsilon_1, \varepsilon_2, R) + c\,s^{-\beta} Z(R)\left\{\sum_{i=1}^{2} \exp\left(-\int_r^{R} \delta_i^*(\varepsilon_i, \varrho)\frac{d\varrho}{\varrho}\right)\right\}^{\beta} + c\|\mu\|_{K(B_R)}$$

for every $\varepsilon_1>0$, $\varepsilon_2>0$. From this inequality we obtain easily (2.3) using the definition of $\omega_{1,\sigma_1}^*(r, R)$. □

In order to prove Propositions 2.1 and 2.2 we need some preliminary results. We begin with an elementary comparison principle. Let Ω be a bounded open subset of $\mathbf{R}^N$, let $\varphi_1, \varphi_2, \chi_1, \chi_2$ be functions from Ω into $\bar{\mathbf{R}}$, let μ_1, μ_2 be two measures of the class $K(\Omega)$, and let u_1, u_2 be solutions of the problems $(i=1,2)$

$$\tag{2.6} \begin{cases} u_i \in H^1(\Omega), \quad \varphi_i \leq u_i \leq \chi_i \text{ q.e. in } \Omega, \\ a_\Omega(u_i, v-u_i) \geq \displaystyle\int_\Omega (v-u_i)\, d\mu_i, \\ \forall v \in H^1(\Omega), \quad \varphi_i \leq v \leq \chi_i \text{ q.e. in } \Omega, \quad v-u_i \in H_0^1(\Omega). \end{cases}$$

LEMMA 2.1. *Assume that* $\mu_1\leq\mu_2$ *on* Ω *(in the sense of measures) and that* $\varphi_1\leq\varphi_2$ *and* $\chi_1\leq\chi_2$ *q.e. on* Ω. *If* $u_1\leq u_2$ *on* $\partial\Omega$ *(in the sense of* $H^1(\Omega)$*), then* $u_1\leq u_2$ *q.e. on* Ω.

Proof. The function $v=u_1\vee u_2=u_2+(u_1-u_2)^+$ is admissible in (2.6) for $i=2$, so we have

$$\tag{2.7} a_\Omega(u_2, (u_1-u_2)^+) \geq \int_\Omega (u_1-u_2)^+ d\mu_2.$$

On the other hand, the function $v=u_1\wedge u_2=u_1-(u_1-u_2)^+$ is admissible in (2.6) for $i=1$, so we have

$$a_\Omega(u_1,-(u_1-u_2)^+)\geq -\int_\Omega (u_1-u_2)^+ d\mu_1. \tag{2.8}$$

Since $\mu_1\leq\mu_2$, by adding (2.7) and (2.8) we obtain

$$a_\Omega((u_1-u_2)^+,(u_1-u_2)^+)\leq 0,$$

which yields $(u_1-u_2)^+=0$, hence $u_1\leq u_2$ q.e. on Ω. □

We now estimate the supremum and the infimum of a solution u of (2.1) in terms of the quantities

$$\Psi_1(\varepsilon,R)=\inf_{B_R}\psi_2\wedge[\bar{\psi}_1(x_0)-\varepsilon],\quad \Psi_2(\varepsilon,R)=\sup_{B_R}\psi_1\vee[\underline{\psi}_2(x_0)+\varepsilon].$$

Note that $\Psi_1(\varepsilon_1,R)\leq\Psi_2(\varepsilon_2,R)$ and $\Psi(\varepsilon_1,\varepsilon_2,R)=\Psi_2(\varepsilon_2,R)-\Psi_1(\varepsilon_1,R)$ for every $\varepsilon_1>0$, $\varepsilon_2>0$. Moreover (2.2) implies that $\Psi_1(\varepsilon,R)<+\infty$ and $\Psi_2(\varepsilon,R)>-\infty$ for every $\varepsilon>0$.

In the following lemma we make the convention $+\infty-(+\infty)=-\infty-(-\infty)=0$.

LEMMA 2.2. *Assume* (2.2). *There exist two constants* $c=c(\lambda,\Lambda,N)$ *and* $\beta=\beta(\lambda,\Lambda,N)>0$ *such that for every solution* u *of* (2.1) *we have*

$$\inf_{B_r}u\geq\Psi_1(\varepsilon_1,R)-c\left\{\left[\inf_{B_R}u-\Psi_1(\varepsilon_1,R)\right]^-\exp\left(-\beta\int_r^R\delta_1^*(\varepsilon_1,\varrho)\frac{d\varrho}{\varrho}\right)+c\|\mu\|_{K(B_R)}\right\}, \tag{2.9}$$

$$\sup_{B_r}u\leq\Psi_2(\varepsilon_2,R)+c\left\{\left[\sup_{B_R}u-\Psi_2(\varepsilon_2,R)\right]^+\exp\left(-\beta\int_r^R\delta_2^*(\varepsilon_2,\varrho)\frac{d\varrho}{\varrho}\right)+c\|\mu\|_{K(B_R)}\right\} \tag{2.10}$$

for every $0<r\leq R$ *and for every* $\varepsilon_1>0$, $\varepsilon_2>0$.

Proof. Let u be a solution of (2.1). We shall prove only the estimate (2.10), the other being analogous. For the sake of simplicity we assume $\mu=0$. The case $\mu\neq0$ can be treated arguing as in the proof of Proposition I.8.1. Given $\varepsilon_2>0$, we set $t=\Psi_2(\varepsilon_2,R)$ and

$$E_t=\{x\in B_{R/2}\colon \psi_2(x)\leq t\}.$$

If $t=+\infty$, then (2.10) is trivial. If $t<+\infty$, then $t>-\infty$ by (2.2) and we can consider the solution w of the problem

$$\left\{\begin{array}{l} w\in H^1(B_R), \quad w\leqslant t \text{ q.e. on } E_t, \quad w=u\vee t \text{ on } \partial B_R, \\ a_\Omega(w,v-w)\geqslant 0, \quad \Omega=B_R \\ \forall v\in H^1(B_R), \quad v\leqslant t \text{ q.e. on } E_t, \quad v=u\vee t \text{ on } \partial B_R. \end{array}\right. \tag{2.11}$$

By the comparison principle (Lemma 2.1) we have $w\geqslant t$ and $w\geqslant u$ in B_R. By applying to (2.11) the unilateral results of [20] (Theorem 6.2 and Corollary of Theorem 6.1) for every $0<r\leqslant R/2$ we obtain

$$\begin{aligned} \sup_{B_r} u \leqslant \sup_{B_r} w = \inf_{B_r} w + \operatorname{osc}_{B_r} w \\ \leqslant t + c R^{-N/2}\left(\int_{B_R} |w-t|^2 dx\right)^{1/2} \exp\left(-\beta\int_r^{R/2} \frac{\operatorname{cap}(E_t\cap B_\varrho, B_{2\varrho})}{\operatorname{cap}(B_\varrho, B_{2\varrho})}\frac{d\varrho}{\varrho}\right). \end{aligned} \tag{2.12}$$

Since $E_2^*(\varepsilon_2,\varrho)\subseteq E_t\cap B_\varrho$ for every $0<\varrho\leqslant R/2$, from (2.12) we obtain

$$\sup_{B_r} u \leqslant t + c\, 2^\beta R^{-N/2}\left(\int_{B_R} |w-t|^2 dx\right)^{1/2} \exp\left(-\beta\int_r^{R} \delta_2^*(\varepsilon_2,\varrho))\frac{d\varrho}{\varrho}\right). \tag{2.13}$$

As $Lw\leqslant 0$ in B_R and $w=u\vee t$ on ∂B_R, by the maximum principle we deduce that

$$w\leqslant \sup_{\partial B_R} u\vee t \quad \text{q.e. on } B_R,$$

hence

$$0\leqslant w-t\leqslant \left(\sup_{B_R} u - t\right)^+. \tag{2.14}$$

The estimate (2.10) follows now easily from (2.13) and (2.14), provided $0<r\leqslant R/2$. In the case $R/2\leqslant r\leqslant R$ the estimate (2.10) is trivial: it is enough to take $c\geqslant 2^\beta$. □

Proof of Proposition 2.1. Let u be a solution of (2.1) and let $0<r\leqslant R$, $\sigma_1>0$, $\sigma_2>0$. Given $\varepsilon_1>\sigma_1\omega_{1,\sigma_1}^*(r,R)$ and $\varepsilon_2>\sigma_2\omega_{2,\sigma_2}^*(r,R)$, we set $t_1=\Psi_1(\varepsilon_1,R)$ and $t_2=\Psi_2(\varepsilon_2,R)$. From (2.9) and (2.10) of Lemma 2.2 we derive

$$\operatorname{osc}_{B_r} u\leqslant t_2-t_1+c\left\{\left[\left(\sup_{B_R} u-t_2\right)^+\vee\left(\inf_{B_R} u-t_1\right)^-\right]\sum_{i=1}^2 \exp\left(-\beta\int_r^R \delta_i^*(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)+\|\mu\|_{K(B_R)}\right\}.$$

Since $t_1\leqslant t_2$, $\inf_{B_R} u\leqslant t_2$, and $\sup_{B_R} u\geqslant t_1$, we have

$$\left(\sup_{B_R} u-t_2\right)^+\vee\left(\inf_{B_R} u-t_1\right)^-\leqslant \operatorname{osc}_{B_R} u.$$

Therefore

$$\operatorname*{osc}_{B_r} u \leq \Psi(\varepsilon_1, \varepsilon_2, R) + c\left\{\left(\operatorname*{osc}_{B_R} u\right) \sum_{i=1}^{2} \exp\left(-\beta \int_r^R \delta_i^*(\varepsilon_i, \varrho) \frac{\delta\varrho}{\varrho}\right) + c\|\mu\|_{K(B_R)}\right\}$$

and we can easily conclude the proof by taking the limit as $\varepsilon_1 \to \sigma_1 \omega^*_{1,\sigma_1}(r,R)$ and $\varepsilon_2 \to \sigma_2 \omega^*_{2,\sigma_2}(r,R)$.

To prove Proposition 2.2 we need the following lemma. We denote by μ^+ and μ^- the positive and the negative part of the measure μ.

LEMMA 2.3. *Let u be a solution of (2.1) and let $d \in \mathbf{R}$. If $d \geq \psi_1$ q.e. on B_R, then $(u-d)^+$ is a subsolution of the equation $Lv = \mu^+$. If $d \leq \psi_2$ q.e. on B_R then $(u-d)^-$ is a subsolution of the equation $Lv = \mu^-$.*

Proof. Assume that $d \geq \psi_1$ q.e. on B_R and define $z = u - d$. Let ψ_h, $h \in \mathbf{N}$, be a sequence of functions belonging to $C^2(\mathbf{R})$ such that

$$\lim_{h \to \infty} \psi_h(t) = t^+, \quad 0 \leq \psi_h(t) \leq t^+, \quad 0 \leq \psi_h'(t) \leq 1, \quad 0 \leq \psi_h''(t) \leq h$$

for every $t \in \mathbf{R}$. Let $\varphi \in H_0^1(B_R)$ with $\varphi \geq 0$. Since $\psi_1 \leq d$ q.e. on B_R, for every $0 < \varepsilon < 1$ the function $v = u - \varepsilon \psi_h'(z)(\varphi \wedge (z^+/\varepsilon))$, is admissible in (2.1). Therefore

$$a_\Omega\left(u, \varepsilon \psi_h'(z)\left(\varphi \wedge \frac{z^+}{\varepsilon}\right)\right) \leq \int_\Omega \left(\varphi \wedge \frac{z^+}{\varepsilon}\right) d\mu, \quad \Omega = B_R,$$

hence

$$\sum_{i,j=1}^{N} \int_{B_R} a_{ij} z_{x_j} \psi_h''(z) z_{x_i} \left(\varphi \wedge \frac{z^+}{\varepsilon}\right) dx + \sum_{i,j=1}^{N} \int_{E_\varepsilon} a_{ij} z_{x_j} \psi_h'(z) \varphi_{x_i} dx$$
$$+ \frac{1}{\varepsilon} \sum_{i,j=1}^{N} \int_{B_R - E_\varepsilon} a_{ij} z_{x_j} \psi_h'(z) (z^+)_{x_i} dx \leq \int_{B_R} \varphi \, d\mu^+.$$

where $E_\varepsilon = \{x \in B_R \colon \varphi(x) \leq z^+(x)/\varepsilon\}$. Since $\psi_h''(z) \geq 0$ and $\psi_h'(z) \geq 0$ we obtain

$$\sum_{i,j=1}^{N} \int_{E_\varepsilon} a_{ij} z_{x_j} \psi_h'(z) \varphi_{x_i} dx \leq \int_{B_R} \varphi \, d\mu^+.$$

As $z_{x_j} \psi_h'(z) = (\psi_h(z))_{x_j}$ and $\psi_h(z)$ converges to z^+ weakly in $H^1(B_R)$, passing to the limit as $h \to +\infty$ we obtain

$$\sum_{i,j=1}^{N} \int_{E_\varepsilon} a_{ij}(z^+)_{x_j} \varphi_{x_i}\, dx \leq \int_{B_R} \varphi\, d\mu^+. \tag{2.15}$$

Since $(z^+)_{x_j}=0$ a.e. on $\{z\leq 0\}$ and $E_\varepsilon \uparrow \{z>0\}$, by taking the limit in (2.18) as $\varepsilon\to 0$ we obtain that

$$a_\Omega(z^+,\varphi)\leq \int_\Omega \varphi\, d\mu^+, \quad \Omega = B_R,$$

hence $z^+=(u-d)^+$ is a subsolution of the equation $Lv=\mu^+$. The proof for $(u-d)^-$ is analogous. □

Proof of Proposition 2.2. Let us define

$$t_1 = \sup_{B_R} \psi_1 \vee d_R \quad \text{and} \quad t_2 = \inf_{B_R} \psi_2 \wedge d_R.$$

By Lemma 2.3 the function $(u-t_1)^+$ is a non-negative subsolution of the equation $Lv=\mu^+$. Therefore Proposition I.5.1 implies that

$$\sup_{B_{sR}} u \leq t_1 + \sup_{B_{sR}} (u-t_1)^+ \leq t_1 + c\left[R^{-N/2}\|(u-d_R)^+\|_{L^2(B_R)} + \|\mu^+\|_{K(B_R)}\right].$$

In the same way we prove that

$$\inf_{B_{sR}} u \geq t_2 - \sup_{B_{sR}} (u-t_2)^- \geq t_2 - c\left[R^{-N/2}\|(u-d_R)^-\|_{L^2(B_R)} + \|\mu^-\|_{K(B_R)}\right].$$

From these inequalities we obtain

$$\operatorname*{osc}_{B_{sR}} u \leq (t_1-t_2) + c\left[R^{-N/2}\|u-d_R\|_{L^2(B_R)} + \|\mu\|_{K(B_R)}\right]. \tag{2.16}$$

Since

$$\sup_{B_R} \psi_1 \wedge \inf_{B_R} \psi_2 \leq d_R \leq \sup_{B_R} \psi_1 \vee \inf_{B_R} \psi_2,$$

we have

$$t_1 - t_2 \leq \left[\sup_{B_R} \psi_1 - \inf_{B_R} \psi_2\right]^+,$$

so the estimate (2.5) follows easily from (2.16) □

We now consider the potential seminorm $\mathcal{V}(r)$ of the solution u introduced in Section I.8. The decay of $\mathcal{V}(r)$ to zero as $r \to 0^+$ can be estimated according to Theorem 2.2 below, under the following separation assumption: there exists a function w such that

$$w \in H^1(B_R), \quad Lw \in K(B_R), \quad \text{and} \quad \psi_1 \leq w \leq \psi_2 \quad \text{q.e. in } B_R. \tag{2.17}$$

By Theorem 4.11 of [5] the function w is continuous on B_R. Note that, if $\bar{\psi}_1(x_0) < \underline{\psi}_2(x_0)$, then (2.17) is satisfied by a suitable constant w, provided that R is small enough.

For every $r>0$ and for every $v \in H^1(B_R)$ we put

$$v_r = \frac{1}{|B_r|} \int_{B_r} v\,dx.$$

THEOREM 2.2. *Assume* (2.17) *and let* $0<s<1$. *Then there exist two constants* $c=c(\lambda, \Lambda, N, s)>0$ *and* $\beta=c(\lambda, \Lambda, N)>0$ *such that for every solution* u *of* (2.1) *we have*

$$\mathcal{V}(r) \leq c\Big\{R^{-N/2}\|u-w\|_{L^2(B_R)}[\omega_{1,\sigma_1}(r,R)+\omega_{2,\sigma_2}(r,R)]^\beta+\sigma_1\,\omega_{1,\sigma_1}(r,R)+\sigma_2\,\omega_{2,\sigma_2}(r,R)$$
$$+R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|Lw\|_{K(B_R)}+\|\mu\|_{K(B_R)}\Big\}$$

for every $0<r\leq sR$ *and for every* $\sigma_1>0, \sigma_2>0$.

Theorem 2.2 follows immediately from the following propositions.

PROPOSITION 2.3. *Assume* (2.17). *Then there exist two constants* $c=c(\lambda, \Lambda, N)>0$ *and* $\beta=\beta(\lambda, \Lambda, N)>0$ *such that for every solution* u *of* (2.1) *we have*

$$\begin{aligned}\mathcal{V}(r) \leq c\Big\{&\mathcal{V}(R)\,[\omega_{1,\sigma_1}(r,R)+\omega_{2,\sigma_2}(r,R)]^\beta+\sigma_1\,\omega_{1,\sigma_1}(r,R)+\sigma_2\,\omega_{2,\sigma_2}(r,R)\\&+R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|Lw\|_{K(B_R)}+\|\mu\|_{K(B_R)}\Big\}\end{aligned} \tag{2.18}$$

for every $0<r\leq R$ *and for every* $\sigma_1>0, \sigma_2>0$.

PROPOSITION 2.4. *Assume* (2.17) *and let* $0<s<1$. *Then there exists a constant* $c=c(\lambda, \Lambda, N, s)>0$ *such that*

$$\mathcal{V}(sR) \leq c\Big\{R^{-N/2}\|u-w\|_{L^2(B_R)}+\|Lw\|_{K(B_R)}+R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|\mu\|_{K(B_R)}\Big\},$$

for every solution u *of* (2.1).

To prove Proposition 2.3 and 2.4 we need some preliminary results. Let u be a solution of (2.1) and let $0<R\leq R_0$. We define $z=u-w$, $\varphi_1=\psi_1-w$, $\varphi_2=\psi_2-w$, $\nu=\mu-Lw$. Then the function z satisfies the following variational inequality on B_R:

$$\begin{cases} z\in H^1(B_R), \quad \varphi_1\leq z\leq \varphi_2 \text{ q.e. on } B_R, \\ a_\Omega(z,v-z)\geq \int_\Omega (v-z)\,d\nu, \quad \Omega=B_R, \\ \forall v\in H^1(B_R), \quad \varphi_1\leq v\leq \varphi_2 \text{ in } B_R, \quad z-v\in H_0^1(B_R). \end{cases} \tag{2.19}$$

Note that $\varphi_1\leq 0\leq\varphi_2$ q.e. on B_R and $\nu\in K(B_R)$.

LEMMA 2.4. *Assume* (2.17) *and let* $0<s<1$. *Then there exists a constant* $c=c(\lambda,\Lambda,N,s)>0$ *such that*

$$\begin{aligned} &\Big[\operatorname*{osc}_{B_{sR}}(u-w)\Big]^2+\int_{B_{sR}}|\nabla(u-w)|^2\,|x-x_0|^{2-N}\,dx \\ &\qquad\leq c\big[R^{-N}\,\|u-w\|^2_{L^2(B_R-B_{sR})}+\|\mu\|^2_{K(B_R)}+\|Lw\|^2_{K(B_R)}\big] \end{aligned} \tag{2.20}$$

for every solution u *of* (2.1).

Proof. Since $u-w$ is a solution of the obstacle problem (2.19), by Lemma 2.3 the functions $(u-w)^\pm$ are non-negative subsolutions of the equations $Lv=\nu^\pm$ in B_R, where $\nu=\mu-Lw$. Therefore Proposition I.5.1 gives

$$\begin{aligned} &\|(u-w)^\pm\|^2_{L^\infty(B_{sR})}+\int_{B_{sR}}|\nabla(u-w)^\pm|^2\,|x-x_0|^{2-N}\,dx \\ &\qquad\leq c\big[R^{-N}\|(u-w)^\pm\|^2_{L^2(B_R-B_{sR})}+\|\nu\|^2_{K(B_R)}\big], \end{aligned}$$

which implies easily (2.20). □

LEMMA 2.5. *Assume* (2.17). *Then there exist two constants* $c=c(\lambda,\Lambda,N)>0$ and $\beta=\beta(\lambda,\Lambda,N)>0$ *such that*

$$\begin{aligned} \inf_{B_r}(u-w)\geq \sup_{B_r}(\psi_1-w)-c\Big\{&\operatorname*{osc}_{B_R}(u-w)\exp\Big(-\beta\int_r^R \delta_1(\varepsilon_1,\varrho)\frac{d\varrho}{\varrho}\Big) \\ &+\varepsilon_1+R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|Lw\|_{K(B_R)}+\|\mu\|_{K(B_R)}\Big\} \end{aligned} \tag{2.21}$$

and

$$
\begin{aligned}
\sup_{B_r}(u-w) \le \inf_{B_r}(\psi_2-w)+c\Big\{&\operatorname*{osc}_{B_R}(u-w)\exp\Big(-\beta\int_r^R \delta_2(\varepsilon_2,\varrho)\frac{d\varrho}{\varrho}\Big) \\
&+\varepsilon_2+R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|\mu\|_{K(B_R)}\Big\}
\end{aligned} \tag{2.22}
$$

for every $0<r\le R$ and for every $\varepsilon_1>0, \varepsilon_2>0$.

Proof. We prove only (2.22), the proof of (2.21) being analogous. Let us fix $0<r\le R/4$ and let $\nu=\mu-Lw$ and $t=\inf_{B_{R/2}}(\psi_2-w)$. We consider the solution u_2 of the variational inequality

$$
\begin{cases}
u_2\in H^1(B_{R/2}), \quad u_2\le\psi_2 \text{ q.e. on } B_{R/2}, \quad u_2=u\vee(w+t) \text{ on } \partial B_{R/2}, \\
a_\Omega(u_2, v-u_2)\ge\langle Lw, v-u_2\rangle+\int_\Omega (v-u_2)\,d\nu^+, \quad \Omega=B_{R/2} \\
\forall v\in H^1(B_{R/2}), \quad v\le\psi_2, \text{ q.e. on } B_{R/2}, \quad v=u\vee(w+t) \text{ on } \partial B_{R/2},
\end{cases} \tag{2.23}
$$

and the solution w_2 of the Dirichlet problem

$$
Lw_2=Lw+\nu^+ \text{ in } B_{R/2}, \quad w_2=u\vee(w+t) \text{ on } \partial B_{R/2}.
$$

By the comparison principle (Lemma 2.1) we have

$$
\psi_1\le w+t\le u_2\le w_2 \quad \text{q.e. on } B_{R/2} \tag{2.24}
$$

and

$$
u\le u_2 \quad \text{q.e. on } B_{R/2}.
$$

Therefore

$$
\begin{aligned}
\sup_{B_r}(u-w) &\le \sup_{B_r} u_2-\inf_{B_r} w\le\inf_{B_r}\psi_2+\operatorname*{osc}_{B_r} u_2-\inf_{B_r} w \\
&\le\inf_{B_r}(\psi_2-w)+\operatorname*{osc}_{B_r} u_2+\operatorname*{osc}_{B_{R/4}} w
\end{aligned}
$$

and Proposition I.5.2 yields

$$
\sup_{B_r}(u-w)\le\inf_{B_r}(\psi_2-w)+\operatorname*{osc}_{B_r} u_2+c\big[R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|Lw\|_{K(B_R)}\big]. \tag{2.25}
$$

We now apply to (2.23) the estimates for the one-sided problems proved in Proposition I.8.1. By Lemma I.7.1, given $\varepsilon_2>0$ we obtain

(2.26)
$$\operatorname*{osc}_{B_r} u_2 \leq c\left\{R^{-N/2}\|u_2-d\|_{L^2(B_{R/2})}\exp\left(-\beta\int_r^{R/2}\delta_2(\varepsilon_2,\varrho)\frac{d\varrho}{\varrho}\right)+\varepsilon_2+\|\nu^+\|_{K(B_{R/2})}+\|Lw\|_{K(B_R)}\right\},$$

where $d=\inf_{B_{R/2}} w+t$. By (2.24) we have $d\leq u_2\leq w_2$ q.e. on $B_{R/2}$, and by the definition of t we have also

$$\inf_{B_{R/2}} u\leq \inf_{B_{R/2}} \psi_2 \leq \sup_{B_{R/2}}(w+t).$$

Therefore, by applying Proposition I.5.2 to w and Proposition I.5.3 to w_2-d we get

$$\begin{aligned}
R^{-N/2}\|u_2-d\|_{L^2(B_{R/2})} &\leq \sup_{B_{R/2}} w_2-d \\
&\leq \sup_{\partial B_{R/2}}[u\vee(w+t)]-\inf_{B_{R/2}}(w+t)+c\left[\|\nu^+\|_{K(B_{R/2})}+\|Lw\|_{K(B_{R/2})}\right] \\
&\leq \left[\sup_{B_{R/2}} u-\sup_{B_{R/2}}(w+t)\right]^+ +\operatorname*{osc}_{B_{R/2}} w+c\left[\|\mu\|_{K(B_R)}+\|Lw\|_{K(B_R)}\right] \\
&\leq \operatorname*{osc}_{B_{R/2}} u+\operatorname*{osc}_{B_{R/2}} w+c\left[\|\mu\|_{K(B_R)}+\|Lw\|_{K(B_R)}\right] \\
&\leq \operatorname*{osc}_{B_{R/2}}(u-w)+c\left[\|\mu\|_{K(B_R)}+R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|Lw\|_{K(B_R)}\right].
\end{aligned} \tag{2.27}$$

Since $\delta_2(\varepsilon_2,\varrho)\leq 1$, we have

$$\exp\left(-\beta\int_r^{R/2}\delta_2(\varepsilon_2,\varrho)\frac{d\varrho}{\varrho}\right)\leq 2^\beta\exp\left(-\beta\int_r^{R}\delta_2(\varepsilon_2,\varrho)\frac{d\varrho}{\varrho}\right). \tag{2.28}$$

From (2.25), (2.26), (2.27), and (2.28) we obtain easily (2.22) in the case $0<r\leq R/4$. If $R/4<r\leq R$, then

$$\exp\left(-\beta\int_r^{R}\delta_2(\varepsilon_2,\varrho)\frac{d\varrho}{\varrho}\right)\geq 4^{-\beta}.$$

Therefore

$$\begin{aligned}
\sup_{B_r}(u-w) = \inf_{B_r}(u-w)+\operatorname*{osc}_{B_r}(u-w) &\leq \inf_{B_r}(\psi_2-w) \\
&\quad +4^\beta \operatorname*{osc}_{B_R}(u-w)\exp\left(-\beta\int_r^{R}\delta_2(\varepsilon_2,\varrho)\frac{d\varrho}{\varrho}\right),
\end{aligned}$$

which implies (2.22) for every $c\geq 4^\beta$.

Proof of Proposition 2.3. Let us fix a solution u of (2.1). For every $0<r\leq R$ we denote by $\mathscr{V}_w(r)$ the potential seminorm of $z=u-w$, defined by

$$\mathscr{V}_w^2(r)=\left[\operatorname*{osc}_{B_r}(u-w)\right]^2+\int_{B_r}|\nabla(u-w)^2|x-x_0|^{2-N}\,dx$$

To estimate $\mathscr{V}_w(r)$, for every $0<r\leq R$ we define

$$\mathscr{E}(r)=\|\mu\|_{K(B_r)}+r^{-N/2}\|w-w_r\|_{L^2(B_r)}+\|Lw\|_{K(B_r)},$$

$$m_r=\frac{1}{|B_r-B_{r/2}|}\int_{B_r-B_{r/2}}(u-w)\,dx,$$

$$a_r=\begin{cases}\sup\limits_{B_r}(\psi_1-w) & \text{if } \ m_r<\sup\limits_{B_r}(\psi_1-w),\\ m_r & \text{if } \ \sup\limits_{B_r}(\psi_1-w)\leq m_r\leq\inf\limits_{B_r}(\psi_2-w),\\ \inf\limits_{B_r}(\psi_2-w) & \text{if } \ \inf\limits_{B_r}(\psi_2-w)<m_r.\end{cases}$$

Let us fix $\varepsilon_1>0$ and $\varepsilon_2>0$. By Lemma 2.5 for every $0<r\leq R/2$ we have

$$\begin{aligned}|a_r-m_r|&\leq c\left[\mathscr{V}_w(R/2)\sum_{i=1}^{2}\exp\left(-\beta\int_r^{R/2}\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)+\varepsilon_1+\varepsilon_2+\mathscr{E}(R/2)\right]\\ &\leq c2^\beta\left[\mathscr{V}_w(R/2)\sum_{i=1}^{2}\exp\left(-\beta\int_r^{R}\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)+\varepsilon_1+\varepsilon_2+\mathscr{E}(R)\right]\end{aligned}\tag{2.29}$$

Since $\psi_1\leq w+a_r\leq\psi_2$ q.e. on B_r, we can apply Lemma 2.4 with R replaced by r and w replaced by $w+a_r$. Therefore

$$\mathscr{V}_w^2(r/2)\leq c\left[r^{-N}\int_{B_r-B_{r/2}}(u-w-a_r)^2\,dx+\mathscr{E}^2(r)\right].\tag{2.30}$$

By the Poincaré inequality we have

$$\begin{aligned}r^{-N}\int_{B_r-B_{r/2}}(u-w-a_r)^2\,dx&\leq 2\left[r^{-N}\int_{B_r-B_{r/2}}(u-w-m_r)^2\,dx+(a_r-m_r)^2\right]\\ &\leq c\left[\int_{B_r-B_{r/2}}|\nabla(u-w)|^2|x-x_0|^{2-N}\,dx+(a_r-m_r)^2\right].\end{aligned}\tag{2.31}$$

From (2.29), (2.30), and (2.31), for every $0<r\leq R/2$ we obtain

$$\mathcal{V}_w^2(r/2)\leq c_1\int_{B_r-B_{r/2}}|\nabla(u-w)|^2|x-x_0|^{2-N}\,dx+c_2\,\mathcal{A}^2(r,R), \tag{2.32}$$

where

$$\mathcal{A}^2(r,R)=\left[\mathcal{V}_w^2(R/2)\sum_{i=1}^{2}\exp\left(-\beta\int_r^R\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)+(\varepsilon_1+\varepsilon_2)^2+\mathcal{E}^2(R)\right].$$

Let us fix $0<r\leq R/4$ and $1\leq\tau\leq R/2r$. If $\tau>2$ and

$$\mathcal{V}_w^2(r)\geq 2c_2\,\mathcal{A}^2(\tau r,R), \tag{2.33}$$

then by (2.32)

$$\mathcal{V}_w^2(\varrho/2)\leq 2c_1\int_{B_\varrho-B_{\varrho/2}}|\nabla(u-w)|^2|x-x_0|^{2-N}\,dx$$

for every $2r\leq\varrho\leq\tau r$. By adding $2c_1\mathcal{V}_w^2(\varrho/2)$ to both sides we obtain

$$(1+2c_1)\mathcal{V}_w^2(\varrho/2)\leq 2c_1\mathcal{V}_w^2(\varrho)$$

for every $2r\leq\varrho\leq\tau r$, and by a standard iteration argument this implies

$$\mathcal{V}_w^2(r)\leq c\tau^{-\beta}\mathcal{V}_w^2(\tau r). \tag{2.34}$$

The same inequality holds trivially if $1\leq\tau\leq 2$ by choosing $c\geq 2^\beta$. If (2.33) is not satisfied, then

$$\mathcal{V}_w^2(r)\leq 2c_2\,\mathcal{A}^2(\tau r,R). \tag{2.35}$$

In any case, from (2.34) or (2.35) we obtain

$$\mathcal{V}_w^2(r)\leq c\left\{\mathcal{V}_w^2(R/2)\left[\tau^{-\beta}+\sum_{i=1}^{2}\exp\left(-\beta\int_{\tau r}^R\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)\right]+(\varepsilon_1+\varepsilon_2)^2+\mathcal{E}^2(R)\right\} \tag{2.36}$$

for every $0<r\leq R/2$ and every $1\leq\tau\leq R/2r$.

Since $\delta_i(\varepsilon_i,\varrho)\leq 1$, we have

$$\tau^\beta\exp\left(-\beta\int_r^{\tau r}\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)\geq 1,$$

therefore (2.36) yields

$$\mathscr{V}_w^2(r)\leq c\left\{\mathscr{V}_w^2(R/2)\left[\tau^{-\beta}+\tau^{\beta}\sum_{i=1}^{2}\exp\left(-\beta\int_r^R\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)\right]+(\varepsilon_1+\varepsilon_2)^2+\mathscr{E}^2(R)\right\}.$$

By taking

$$\tau^{\beta}=\left\{\frac{1}{2}\sum_{i=1}^{2}\exp\left(-\beta\int_r^R\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)\right\}^{-1/2}$$

we obtain

$$\mathscr{V}_w^2(r)\leq c\left\{\mathscr{V}_w^2(R/2)\sum_{i=1}^{2}\exp\left(-\frac{\beta}{2}\int_r^R\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)+(\varepsilon_1+\varepsilon_2)^2+\mathscr{E}^2(R)\right\},$$

therefore

$$\mathscr{V}_w(r)\leq c\left\{\mathscr{V}_w(R/2)\sum_{i=1}^{2}\exp\left(-\beta\int_r^R\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)+\varepsilon_1+\varepsilon_2+\mathscr{E}(R)\right\}. \tag{2.37}$$

By Proposition I.5.2 we have

$$|\mathscr{V}(\varrho)-\mathscr{V}_w(\varrho)|\leq\left[\left(\operatorname*{osc}_{B_{R/2}}w\right)^2+\int_{B_{R/2}}|\nabla w|^2|x-x_0|^{2-N}dx\right]^{1/2}\leq c\,\mathscr{E}(R),$$

for every $0\leq\varrho\leq R/2$, thus (2.37) implies

$$\mathscr{V}(r)\leq c\left\{\mathscr{V}(R/2)\sum_{i=1}^{2}\exp\left(-\beta\int_r^R\delta_i(\varepsilon_i,\varrho)\frac{d\varrho}{\varrho}\right)+\varepsilon_1+\varepsilon_2+\mathscr{E}(R)\right\}. \tag{2.38}$$

Let us fix $\sigma_1>0$ and $\sigma_2>0$. For every $\varepsilon_1>\sigma_1\,\omega_{1,\sigma_1}(r,R)$ and $\varepsilon_2>\sigma_2\,\omega_{2,\sigma_2}(r,R)$ the inequality (2.38) implies

$$\mathscr{V}(r)\leq c\left\{\mathscr{V}(R)\left[\frac{\varepsilon_1}{\sigma_1}+\frac{\varepsilon_2}{\sigma_2}\right]^{\beta}+\varepsilon_1+\varepsilon_2+\mathscr{E}(R)\right\}.$$

Taking the limit as $\varepsilon_1\downarrow\sigma_1\,\omega_{1,\sigma_1}(r,R)$ and $\varepsilon_2\downarrow\sigma_2\,\omega_{2,\sigma_2}(r,R)$ we obtain (2.18), provided $0<r\leq R/4$. In the case $R/4\leq r\leq R$ the estimate (2.18) is trivial: it is enough to take $c\geq 4^{\beta}$. □

Proof of Proposition 2.4. Let u be a solution of (2.1). By Lemma 2.4 we have

$$\mathscr{V}_w(sR)\leq c\left[R^{-N/2}\|u-w\|_{L^2(B_R)}+\|\mu\|_{K(B_R)}+\|Lw\|_{K(B_R)}\right], \tag{2.39}$$

and from Proposition I.5.2 we obtain

$$
\begin{aligned}
\mathscr{V}(sR) &\leq \mathscr{V}_w(sR)+\left[\left(\operatorname*{osc}_{B_{sR}} w\right)^2+\int_{B_{sR}} |\nabla w|^2\,|x-x_0|^{2-N}\,dx\right]^{1/2} \\
&\leq \mathscr{V}_w(sR)+c\left[R^{-N/2}\|w-w_R\|_{L^2(B_R)}+\|Lw\|_{K(B_R)}\right].
\end{aligned}
\tag{2.40}
$$

The inequality to be proved follows now from (2.39) and (2.40).

II.3. Proof of the Wiener criterion

In this section we prove the Wiener criterion stated in Theorem 1.1. To this aim we introduce the $\{\psi_1,\psi_2\}$-potentials at x_0, relative to the operator L, defined as the solutions $u=u_{r,\tau}$, $r>0, \tau\in\mathbf{R}$, of the problem

$$
\begin{cases}
u\in H^1(B_{2r}), \quad \psi_1\leq u\leq\psi_2 \text{ q.e. in } B_r, \quad u-\tau\in H_0^1(B_{2r}),\\
a_\Omega(u,v-u)\geq 0, \quad \Omega=B_{2r}\\
\forall v\in H^1(B_{2r}), \quad \psi_1\leq v\leq\psi_2 \text{ q.e. in } B_r, \quad v-\tau\in H_0^1(B_{2r}).
\end{cases}
\tag{3.1}
$$

Note that, if condition (1.4) of Theorem 1.1 is satisfied, then $u_{r,\tau}$ is well defined for every $\tau\in\mathbf{R}$ and every $0<r<R$.

The following proposition gives another characterization of the regular points, which represents the analogue for obstacle problems of the classical de la Vallèe Poussin criterion.

PROPOSITION 3.1. *Assume* (1.3) *and* (1.4). *Then x_0 is a regular point of problem $\{\psi_1,\psi_2\}$ if and only if both the following conditions hold:*

$$
\text{if } \bar{\psi}_1(x_0)>-\infty, \text{ then } \inf_{r>0} u_{r,\tau_1}(x_0)=\bar{\psi}_1(x_0) \text{ for every } \tau_1<\bar{\psi}_1(x_0),
\tag{3.2}
$$

$$
\text{if } \underline{\psi}_2(x_0)<+\infty, \text{ then } \sup_{r>0} u_{r,\tau_2}(x_0)=\underline{\psi}_2(x_0) \text{ for every } \tau_2>\underline{\psi}_2(x_0),
\tag{3.3}
$$

The proof of Theorem 1.1 and Proposition 3.1 will be achieved by means of the following steps.

Step 1. If x_0 is a regular point of $\{\psi_1,\psi_2\}$, then (1.3), (1.4), (3.2), and (3.3) hold.

Step 2. If (1.3), (1.4), (3.2), and (3.3) hold, then x_0 is a Wiener point of $\{\psi_1,\psi_2\}$.

Step 3. If (1.3), (1.4), and (1.5) hold, then x_0 is a regular point of $\{\psi_1,\psi_2\}$.

Proof of Step 1. Assume that x_0 is a regular point for $\{\psi_1, \psi_2\}$. Then (1.4) follows easily from the fact that $\mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is not empty. To prove (1.3) we pick up an arbitrary $u \in \mathcal{U}_{\psi_1}^{\psi_2}(x_0)$. Since u is finite and continuous at x_0 we have

$$\bar{\psi}_1(x_0) \leq \bar{u}(x_0) = u(x_0) < +\infty, \quad \underline{\psi}_2(x_0) \geq \underline{u}(x_0) = u(x_0) > -\infty,$$
$$\bar{\psi}_1(x_0) \leq \bar{u}(x_0) = \underline{u}(x_0) \leq \underline{\psi}_2(x_0),$$

which imply (1.3).

To prove (3.2) we may assume that $\bar{\psi}_1(x_0) > -\infty$. Let us fix $\tau_1 < \bar{\psi}_1(x_0)$. Since the $\{\psi_1, \psi_2\}$-potentials u_{r,τ_1} belong to $\mathcal{U}_{\psi_1}^{\psi_2}(x_0)$, they are continuous at x_0, hence

$$\inf_{r>0} u_{r,\tau_1}(x_0) = \inf_{r>0} \bar{u}_{r,\tau_1}(x_0) \geq \bar{\psi}_1(x_0).$$

By the definition of $\bar{\psi}_1(x_0)$, for every $\varepsilon>0$ there exists $R>0$ such that $\bar{\psi}_1(x_0)+\varepsilon \geq \sup_{B_R} \psi_1$, thus the comparison principle (Lemma 2.1) implies $u_{R,\tau_1} \leq w = \bar{\psi}_1(x_0)+\varepsilon$ q.e. on B_{2R}. Therefore $\inf_{r>0} u_{r,\tau_1}(x_0) \leq \bar{\psi}_1(x_0)+\varepsilon$ for every $\varepsilon>0$, and this concludes the proof of (3.2).

The proof of (3.3) is analogous. □

Proof of Step 2. Assume (1.3), (1.4), (3.2), and (3.3). Let us consider the ψ_1-potentials at x_0, relative to the operator L, introduced in Definition 2.1 of [20] as the solutions $w = w_{r,\tau}, r>0, \tau \in \mathbf{R}$, of the problem

$$\begin{cases} w \in H^1(B_{2r}), \quad \psi_1 \leq w \text{ q.e. on } B_r, \quad w - \tau \in H_0^1(B_{2r}), \\ a_\Omega(w, v-w) \geq 0, \quad \Omega = B_{2r} \\ \forall v \in H^1(B_{2r}), \quad \psi_1 \leq v \text{ q.e. on } B_r, \quad v - \tau \in H_0^1(B_{2r}). \end{cases}$$

By the comparison principle (Lemma 2.1) we have $u_{r,\tau} \leq w_{r,\tau}$ q.e. on B_{2r}, thus condition (3.2) implies $\inf_{r>0} w_{r,\tau}(x_0) \geq \bar{\psi}_1(x_0)$ for every $\tau < \bar{\psi}_1(x_0)$. Since the opposite inequality is always satisfied (see the proof of Theorem 5.2 of [20]), we have $\inf_{r>0} w_{r,\tau}(x_0) = \bar{\psi}_1(x_0)$ for every $\tau < \bar{\psi}_1(x_0)$. By the Theorem 5.2 of [20] we then have either $\bar{\psi}_1(x_0) = -\infty$ or

$$\int_0^1 \delta_1(\varepsilon, \varrho) \frac{d\varrho}{\varrho} = +\infty,$$

so Remark I.7.1. implies that

$$\int_0^1 \delta_1^*(\varepsilon, \varrho) \frac{d\varrho}{\varrho} = +\infty.$$

Since the same property holds for δ_2^*, condition (1.2) is satisfied, hence x_0 is a Wiener point of $\{\psi_1, \psi_2\}$. □

Proof of Step 3. Assume (1.3), (1.4), and (1.5). If $\bar{\psi}_1(x_0)=\underline{\psi}_2(x_0)$, then Theorem 2.1 and Lemma I.7.2 imply that every $u \in \mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is continuous at x_0 (see Remark 2.1).

If $-\infty<\bar{\psi}_1(x_0)<\underline{\psi}_2(x_0)<+\infty$, then there exist $d \in \mathbf{R}$ and $R>0$ such that $\psi_1 \leqslant d \leqslant \psi_2$ q.e. on B_r. Therefore we can apply Theorem 2.2 with $w=d$. By Lemma I.7.2. the estimate of Theorem 2.2 implies that every $u \in \mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is continuous at x_0.

If $-\infty\leqslant\bar{\psi}_1(x_0)<\underline{\psi}_2(x_0)=+\infty$, then each $u \in \mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is locally bounded near x_0 by Theorem 2.2. Since $\sup_{r>0} \inf_{B_r} \psi_2=+\infty$, there exists $R>0$ such that $\sup_{B_R} u+1 \leqslant \inf_{B_R} \psi_2$, therefore u is a solution of the one-sided obstacle problem

$$\begin{cases} u \in H^1(B_R), \quad u \geqslant \psi_1 \text{ q.e. on } B_R \\ a_\Omega(u, v-u) \geqslant 0, \quad \Omega = B_R \\ \forall v \in H^1(B_R), \quad v \geqslant \psi_1 \text{ q.e. on } B_R, \quad v-u \in H_0^1(B_R) \end{cases}$$

to which we can apply the continuity results of Theorem 5.2 of [20]. Therefore u is continuous and finite at x_0.

The case $-\infty=\bar{\psi}_1(x_0)<\underline{\psi}_2(x_0)\leqslant+\infty$ can be treated in a symmetric way. □

Part III. Generalized solutions

In this part of the paper we study a notion of generalized solutions of the two-obstacle problem $\{\psi_1, \psi_2\}$ which extends the notion of variational solution to the case where there exists no function $u \in H^1$ such that $\psi_1 \leqslant u \leqslant \psi_2$ (see Definition II.1.1). We then extend to generalized solutions the Wiener criterion proved in Part II in the variational case.

III.1. Dominated generalized solutions

Let Ω be a bounded open subset of $\mathbf{R}^N$.

Definition 1.1. Let $\psi_1, \psi_2: \Omega \to \bar{\mathbf{R}}$ be two functions such that ψ_1, ψ_2 q.e. on Ω. We say that a function $u: \Omega \to \bar{\mathbf{R}}$ is a *(dominated) generalized solution* (in Ω) of the two-obstacle problem $\{\psi_1, \psi_2\}$ if there exist three sequences $\psi_{1,h}, \psi_{2,h}, u_h$ of functions from Ω into $\bar{\mathbf{R}}$ such that ψ_1, ψ_2, u are the H^1-dominated quasi uniform limits (in Ω) of $\psi_{1,h}, \psi_{2,h}, u_h$ respectively and for every $h \in \mathbf{N}$ the function u_h is a variational solution of the two-obstacle problem $\{\psi_{1,h}, \psi_{2,h}\}$ according to Definition II.1.1.

Remark 1.1. It follows immediately from the definition that, if u is a *generalized* solution of the obstacle problem $\{\psi_1, \psi_2\}$, then u is *quasi continuous* and $\psi_1 \leq u \leq \psi_2$ q.e. in Ω. Moreover for every set $\Omega' \Subset \Omega$ the function $u|_{\Omega'}$ is H^1*-dominated* in Ω'.

We prove now an existence result for generalized solutions of a two-obstacle problem.

THEOREM 1.1. *Let $\psi_1, \psi_2 \colon \Omega \to \bar{\mathbf{R}}$ be two functions such that $\psi_1 \leq \psi_2$ q.e. on Ω and let $g \colon \partial\Omega \to \bar{\mathbf{R}}$ be a quasi continuous function. Assume that there exists a H^1-dominated quasi continuous function $\psi \colon \Omega \to \bar{\mathbf{R}}$ such that $\psi_1 \leq \psi \leq \psi_2$ q.e. in Ω and $\psi = g$ q.e. on $\partial\Omega$. Then there exists one and only one quasi continuous function $u \colon \bar{\Omega} \to \bar{\mathbf{R}}$ such that u is a generalized solution of the obstacle problem $\{\psi_1, \psi_2\}$ in Ω and $u = g$ q.e. on $\partial\Omega$.*

To prove Theoren 1.1 we need the following lemmas.

LEMMA 1.1. *Let $\psi_1, \psi_2 \colon \Omega \to \bar{\mathbf{R}}$ be two functions such that $\psi_1 \leq \psi_2$ q.e. in Ω and let $u \in H^1(\Omega)$ be a variational solution of the obstacle problem $\{\psi_1, \psi_2\}$ in Ω. Let $w \in H^1(\Omega)$ be a non-negative supersolution of the operator L in Ω and let v be the unique variational solution of the obstacle problem $\{\psi_1 + w, \psi_2 + w\}$ in Ω such that $v - (u + w) \in H^1_0(\Omega)$ Then $v \leq u + w$ q.e. in Ω.*

Proof. The function $z_1 = v \wedge (u + w) = v - (v - u - w)^+$ satisfies the obstacle condition $\psi_1 + w \leq z_1 \leq \psi_2 + w$ q.e. in Ω, moreover $z_1 - v \in H^1_0(\Omega)$. Since v is a variational solution of the obstacle problem $\{\psi_1 + w, \psi_2 + w\}$ we have

$$a_\Omega(v, -(v - u - w)^+) \geq 0. \tag{1.1}$$

On the other hand the function $z_2 = (v - w) \vee u = u + (v - w - u)^+$ satisfies the obstacle condition $\psi_1 \leq z_2 \leq \psi_2$ q.e. in Ω, moreover $z_2 - u \in H^1_0(\Omega)$. Since u is a variational solution of the obstacle problem $\{\psi_1, \psi_2\}$ we have

$$a_\Omega(u, (v - w - u)^+) \geq 0. \tag{1.2}$$

Finally $(v - u - w)^+$ is not negative and belongs to $H^1_0(\Omega)$. Since w is a supersolution (relative to the operator L) we have

$$a_\Omega(w, (v - u - w)^+) \geq 0. \tag{1.3}$$

By adding (1.1), (1.2), and (1.3) we obtain $a(v - u - w, (v - u - w)^+) \leq 0$, which yields $(v - u - w)^+ = 0$, hence $v \leq u + w$ q.e. in Ω. □

LEMMA 1.2. *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 : \Omega \to \bar{\mathbf{R}}$ be four functions such that $\varphi_1 \leq \varphi_2$ and $\psi_1 \leq \psi_2$ q.e. in Ω, and let $w \in H^1(\Omega)$ be a non-negative supersolution of the operator L in Ω. Assume that $\psi_1 \leq \varphi_1 + w$ and $\psi_2 \leq \varphi_2 + w$ q.e. in Ω. Let u, v be two variational solutions of the obstacle problems $\{\varphi_1, \varphi_2\}$ and $\{\psi_1, \psi_2\}$ respectively, such that $v \leq u + w$ on $\partial\Omega$. Then $v \leq u + w$ q.e. in Ω.*

Proof. Let z be the unique variational solution of the obstacle problem $\{\varphi_1 + w, \varphi_2 + w\}$ in Ω such that $z - (u+w) \in H_0^1(\Omega)$. By Lemma 1.1 we have $z \leq u + w$ q.e. in Ω, and by an easy comparison argument (Lemma II.2.1) we have $v \leq z$ q.e. in Ω. □

Now we prove a lemma concerning the approximation of an *H^1-dominated quasi continuous function.*

LEMMA 1.3. *Let K be a compact subset of $\mathbf{R}^N$ and let $\psi : K \to \bar{\mathbf{R}}$ be a H^1-dominated quasi continuous function. Then ψ is the H^1-dominated quasi uniform limit in K of a decreasing sequence of functions w_h of $H^1(\mathbf{R}^N)$ such that $w_h \geq \psi$ q.e. in K.*

Proof. Since ψ is H^1-dominated, by adding a suitable function of $H^1(\mathbf{R}^N)$, we may assume that $\psi \geq 0$ q.e. in K. Since ψ is quasi continuous, there exists a decreasing sequence A_h of open sets such that $\psi|_{K-A_h}$ is continuous on $K - A_h$ and $\mathrm{Cap}(A_h) < 1/h$ for every $h \in \mathbf{N}$. Therefore for every $h \in \mathbf{N}$ there exists a function $\varphi_h \in C_0^\infty(\mathbf{R}^N)$ such that $\varphi_h \geq 0$ on $\mathbf{R}^N$ and $\psi \leq \varphi_h \leq \psi + 1/h$ q.e. in $K - A_h$. Since ψ is H^1-dominated, there exists $v \in H^1(\mathbf{R}^N)$ such that $\psi \leq v$ q.e. in K and $v \geq 0$ q.e. on $\mathbf{R}^N$. Let us define $v_h = v \wedge \varphi_h$. Then $v_h \in H^1(\mathbf{R}^N)$, $0 \leq v_h \leq v$ q.e. in $\mathbf{R}^N$ and $\psi \leq v_h \leq \psi + 1/h$ q.e. in $K - A_h$.

Let z_h be the solution of the minimum problem

$$\min\left\{ \int_{\mathbf{R}^N} (|\nabla z|^2 + z^2)\, dx : z \in H^1(\mathbf{R}^N),\, z \geq v \text{ q.e. in } A_h \right\}.$$

Since $v \in H^1(\mathbf{R}^N)$ and $\mathrm{Cap}(A_h) \to 0$ as $h \to +\infty$, the sequence z_h converges to 0 strongly in $H^1(\mathbf{R}^N)$. Since A_h is decreasing the sequence z_h is decreasing, therefore it converges to 0 quasi uniformly in $\mathbf{R}^N$ (in the capacity sense).

Let us define $w_h = v_h + z_h$. The inequalities $\psi \leq v_h \leq w_h$ q.e. in $K - A_h$ and $\psi \leq v \leq z_h \leq w_h$ q.e. in $K \cap A_h$ imply that

$$\psi \leq w_h \quad \text{q.e. in } K. \tag{1.4}$$

The inequality $v_h \leq \psi + 1/h$ q.e. in $K - A_h$ implies that $w_h \leq \psi + z_h + 1/h$ q.e. in $K - A_h$. On the other hand $w_h \leq v + z_h \leq \psi + 2z_h$ q.e. in $K \cap A_h$, hence

$$w_h \leq \psi + 2z_h + \frac{1}{h} \quad \text{q.e. in } K. \tag{1.5}$$

Since z_h converges to 0 quasi uniformly, (1.4) and (1.5) imply that w_h converges to ψ quasi uniformly. Since

$$|w_h - \psi| = w_h - \psi = (v \wedge \varphi_h) + z_h - \psi \leq v + z_1 \quad \text{q.e. in } K$$

for every $h \in \mathbf{N}$, the function ψ is the H^1-dominated quasi uniform limit of w_h. To obtain a decreasing sequence we take $w_1 \wedge w_2 \wedge \ldots \wedge w_h$ for every $h = 1, 2, \ldots$. □

Proof of Theorem 1.1. Let us prove the existence. By Lemma 1.3 the function ψ is the H^1-dominated quasi uniform limit in $\bar{\Omega}$ of a decreasing sequence w_h of functions of $H^1(\mathbf{R}^N)$. By Proposition I.6.1 there exists a decreasing sequence v_h in $H^1(\mathbf{R}^N)$ converging to 0 strongly in $H^1(\mathbf{R}^N)$ such that $\psi \leq w_h \leq \psi + v_h$ q.e. in $\bar{\Omega}$ for every $h \in \mathbf{N}$. Moreover we may assume that each function v_h is a supersolution of the operator L in Ω.

Since $\psi_1 \leq w_h \leq \psi_2 + v_h$ q.e. in Ω, for every $h \in \mathbf{N}$ there exists a variational solution (in $H_l(\Omega)$) of the obstacle problem $\{\psi_1, \psi_2 + v_h\}$ which satisfies the boundary condition $u_h - w_h \in H_0^1(\Omega)$. If we extend u_h to $\bar{\Omega}$ by setting $u_h = w_h$ q.e. on $\partial\Omega$, the extended function u_h is quasi continuous in $\bar{\Omega}$. Let us fix $h \leq k$. Since $\psi_2 + v_k \leq \psi_2 + v_h$ q.e. in Ω and $w_k \leq w_h$ on $\partial\Omega$, by an easy comparison argument (Lemma II.2.1) we have $u_k \leq u_h$ q.e. on Ω. Taking into account the inequalities $u_k = w_k \leq w_h = u_h$ q.e. on $\partial\Omega$, we obtain

$$u_k \leq u_h \quad \text{q.e. in } \bar{\Omega}. \tag{1.6}$$

Since $\psi_2 + v_h \leq (\psi_2 + v_k) + v_h$ q.e. in Ω and $w_h \leq w_k + v_h$ on $\partial\Omega$, by Lemma 1.2 we have $u_h \leq u_k + v_h$ q.e. in Ω. Taking into account the inequalities $u_h = w_h \leq w_k + v_h = u_k + v_h$ q.e. on $\partial\Omega$, we obtain

$$u_h \leq u_k + v_h \quad \text{q.e. in } \bar{\Omega}. \tag{1.7}$$

From (1.6) it follows that the sequence u_h is decreasing q.e. on $\bar{\Omega}$, so it converges pointwise q.e. to a function $u : \bar{\Omega} \to \bar{\mathbf{R}}$. By letting k tend to $+\infty$ in (1.6) and (1.7) we obtain

$$u \leq u_h \leq u + v_h \quad \text{q.e. in } \bar{\Omega},$$

for every $h \in \mathbf{N}$, hence u is the H^1-dominated quasi uniform limit of u_h in $\bar{\Omega}$. This implies that u is quasi continuous in $\bar{\Omega}$ and that $u = \psi = g$ q.e. on $\partial\Omega$ (recall that w_h converges to ψ quasi uniformly on $\bar{\Omega}$). Since ψ_2 is the H^1-dominated quasi uniform limit of $\{\psi_2 + v_h\}$, the function u is a generalized solution in Ω of the problem $\{\psi_1, \psi_2\}$. This concludes the proof of the existence.

Let us prove the uniqueness. Let u and $\hat{u}$ be two quasi continuous functions on $\bar{\Omega}$. Assume that u and $\hat{u}$ are generalized solutions of the obstacle problem $\{\psi_1, \psi_2\}$ in Ω and that $u=\hat{u}=g$ q.e. on $\partial\Omega$. By definition there exist six sequences $\psi_{1,h}, \psi_{2,h}, u_h, \hat{\psi}_{1,h}, \hat{\psi}_{2,h}, \hat{u}_h$ such that ψ_i and $\hat{\psi}_i$ and the H^1-dominated quasi uniform limits in Ω respectively of $\psi_{i,h}$ and $\hat{\psi}_{i,h}$ for $i=1,2$, u and $\hat{u}$ are the H^1-dominated quasi uniform limits in Ω of u_h and $\hat{u}_h$ respectively, and for every $h\in\mathbf{N}$, the functions u_h and $\hat{u}_h$ belong to $H^1(\Omega)$ and are the variational solutions of the obstacle problems $\{\psi_{1,h}, \psi_{2,h}\}$ and $\{\hat{\psi}_{1,h}, \hat{\psi}_{2,h}\}$ respectively.

By Proposition I.6.1 there exists a decreasing sequence v_h in $H^1(\mathbf{R}^N)$ converging to 0 strongly in $H^1(\mathbf{R}^N)$ such that v_h is a supersolution of the operator L in Ω and $|\psi_{1,h}-\psi_1|\leq v_h$, $|\psi_{2,h}-\psi_2|\leq v_h$, $|u_h-u|\leq v_h$, $|\hat{\psi}_{1,h}-\psi_1|\leq v_h$, $|\hat{\psi}_{2,h}-\psi_2|\leq v_h$, $|\hat{u}_h-\hat{u}|\leq v_h$ q.e. in Ω for every $h\in\mathbf{N}$.

Let w_h be the unique variational solution of the obstacle problem

$$\{\psi_{1,h}\vee\hat{\psi}_{1,h}, \psi_{2,h}\vee\hat{\psi}_{2,h}\}$$

in Ω such that $w_h-(u_h\vee\hat{u}_h)\in H_0^1(\Omega)$.

Since $u_h\vee\hat{u}_h\leq|u-\hat{u}|+u_h+2v_h$ q.e. in Ω and $|u-\hat{u}|=0$ q.e. on $\partial\Omega$, by Lemma I.2.1 we have

$$u_h \vee \hat{u}_h \leq u_h+2v_h \quad \text{on } \partial\Omega \text{ in the sense of } H^1(\Omega). \tag{1.8}$$

Since $\psi_{1,h}\vee\hat{\psi}_{1,h}\leq\psi_{1,h}+2v_h$ and $\psi_{2,h}\vee\hat{\psi}_{2,h}\leq\psi_{2,h}+2v_h$ q.e. in Ω, from (1.8) and Lemma 1.2 we obtain

$$w_h\leq u_h+2v_h \quad \text{q.e. in } \Omega. \tag{1.9}$$

Since $\hat{\psi}_{1,h}\leq\psi_{1,h}\vee\hat{\psi}_{1,h}$, $\hat{\psi}_{2,h}\leq\psi_{2,h}\vee\hat{\psi}_{2,h}$ q.e. in Ω, and $\hat{u}_h\leq u_h\vee\hat{u}_h$ on $\partial\Omega$, by an easy comparison argument (Lemma II.2.1) we have

$$\hat{u}_h\leq w_h \quad \text{q.e. in } \Omega. \tag{1.10}$$

From (1.9) and (1.10) we get $\hat{u}_h\leq u_h+2v_h$ q.e. in Ω. Since v_h converges to 0 quasi uniformly in Ω, we have $\hat{u}\leq u$ q.e. in Ω. The opposite inequality can be proved in the same way, so $\hat{u}=u$ q.e. in Ω and the uniqueness is proved. □

Remark 1.2. We could have defined a different notion of *generalized solution* by using *quasi uniform convergence* (in the capacity sense) in Definition 1.1 instead of H^1-

dominated quasi uniform convergence. However this new notion, called *Cap-generalized solution*, is not useful for our purposes because both existence and uniqueness results of Theorem 1.1 are lost, as the following examples show.

Example 1.1. Let $\Omega=B_1(0), L=-\Delta, \psi_1\equiv 0$, and $\psi_2\equiv+\infty$. Then for every $t\geq 0$ the functions $u(x)=t|x|^{2-N}-t$ are Cap-generalized solutions of the obstacle problem $\{\psi_1,\psi_2\}$ which satisfy the boundary condition $u=0$ q.e. on $\partial\Omega$. In fact, for every $h\in\mathbf{N}$ the function $u_h=(t|x|^{2-N}-t)\wedge h$ is a variational solution of the obstacle problem $\{\psi_{1,h},\psi_2\}$, where

$$\psi_{1,h}(x)=\begin{cases} h & \text{in } B_{r_h}(0), \\ 0 & \text{elsewhere,}\end{cases}$$

and $r_h^{N-2}=t/(t+h)$. Since $\mathrm{Cap}(B_{r_h}(0))\to 0$ as $h\to+\infty$, the sequence $\psi_{1,h}$ converges to $\psi_1=0$ quasi uniformly and the sequence u_h converges to u quasi uniformly.

Note that the *unique dominated generalized* solution u of $\{\psi_1,\psi_2\}$ with boundary condition $u=0$ q.e. on $\partial\Omega$ is the function $u=0$.

Example 1.2. Let $\Omega=B_1(0)$, $L=-\Delta$, $\psi_1(x)=|x|^{1-N}$, $\psi_2(x)=+\infty$ for every $x\in\Omega$, and let $g(x)=1$ for every $x\in\partial\Omega$. Then $\psi=\psi_1$ is a quasi continuous function on $\bar\Omega$ such that $\psi_1\leq\psi\leq\psi_2$ q.e. in Ω and $\psi=g$ q.e. on $\partial\Omega$, but there exists no quasi continuous function $u\colon\bar\Omega\to\bar{\mathbf{R}}$ such that $u|_\Omega$ is a Cap-generalized solution of the obstacle problem $\{\psi_1,\psi_2\}$ in Ω and $u=g$ q.e. on $\partial\Omega$.

We argue by contradiction. Suppose that such a function u exists. Then there exist three sequences $\psi_{1,h},\psi_{2,h}, u_h$ such that $\psi_{i,h}$ converges to ψ_i quasi uniformly for $i=1,2$, u_h converges to u quasi uniformly, and for every h the function u_h is a variational solution of the problem $\{\psi_{1,h},\psi_{2,h}\}$. Fix $t>1$, let v_h be the unique *variational* solution of the obstacle problem $\{\psi_{1,h}\wedge t,\psi_{2,h}\wedge t\}$ which satisfies the boundary condition $v_h=u_h\wedge t$ on $\partial\Omega$. By an easy comparison argument (Lemma II.2.1) we have

$$v_h\leq u_h \quad \text{q.e. in } \Omega. \tag{1.11}$$

Since the sequences $[(u\wedge t)-(u_h\wedge t)]^+$, $[(\psi_1\wedge t)-(\psi_{1,h}\wedge t)]^+$, and $[t-(\psi_{2,h}\wedge t)]^+$ are uniformly bounded and converge to 0 quasi uniformly in Ω, by Proposition I.6.1. there exists a decreasing sequence z_h in $H^1(\mathbf{R}^N)$ converging to 0 in $H^1(\mathbf{R}^N)$ such that each z_h is a supersolution of $-\Delta$ in Ω and

$$u\wedge t\leq(u_h\wedge t)+z_h \quad \text{q.e. in } \Omega, \tag{1.12}$$

(1.13) $$\psi_1\wedge t\leqslant(\psi_{1,h}\wedge t)+z_h \quad\text{and}\quad t\leqslant(\psi_{2,h}\wedge t)+z_h \quad \text{q.e. in } \Omega.$$

From (1.12) we obtain

(1.14) $$1\leqslant(u_h\wedge t)+z_h \quad \text{on } \partial\Omega \text{ in the sense of } H^1(\Omega).$$

Let w_t be the unique variational solution of the obstacle problem $\{\psi_1\wedge t, t\}$ which satisfies the boundary condition $w_t-1\in H_0^1(\Omega)$. By Lemma 1.2 and by (1.13) and (1.14) we have $w_t\leqslant v_h+z_h$ q.e. in Ω, so (1.11) implies $w_t\leqslant u_h+z_h$ q.e. in Ω. By taking the limit as $h\to+\infty$ we obtain

(1.15) $$w_t\leqslant u \quad \text{q.e. in } \Omega.$$

Since

$$w_t(x)=\left[\frac{t-1}{t^\alpha-1}(|x|^{2-N}-1)+1\right]\wedge t \quad \text{for every } x\in\Omega,$$

with $\alpha=(N-2)/(N-1)<1$, by taking the limit as $t\to+\infty$ in (1.15) we obtain $u(x)=+\infty$ for every $x\in\Omega$, which contradicts the assumption that u is quasi continuous in $\bar\Omega$ and $u=1$ q.e. on $\partial\Omega$.

Remark 1.3. Let $\psi_1,\psi_2\colon\Omega\to\bar{\mathbf{R}}$ be two functions such that $\psi_1\leqslant\psi_2$ q.e. in Ω and let $u\colon\Omega\to\bar{\mathbf{R}}$ be a *generalized* solution of the obstacle problem $\{\psi_1,\psi_2\}$. By Remark 1.1 the function u is *quasi continuous* and for every open set $\Omega'\Subset\Omega$ the function $u|_{\bar\Omega'}$ is H^1-dominated on $\bar\Omega'$, so we can apply the uniqueness result of Theorem 1.1 with $\Omega=\Omega'$, $\psi_1=\psi_1|_{\Omega'}$, $\psi_2=\psi_2|_{\Omega'}$, and $g=u|_{\partial\Omega'}$. Therefore from the proof of the existence in Theorem 1.1 it follows that there exist two decreasing sequences u_h and v_h in $H^1(\Omega')$ such that u_h converges to u *quasi uniformly* in Ω', v_h converges to 0 strongly in $H^1(\Omega')$, v_h is a *supersolution* of the operator L in Ω', u_h is a *variational* solution of the problem $\{\psi_1,\psi_2+v_h\}$ in Ω', and $u\leqslant u_h\leqslant u+v_h$ q.e. in Ω'.

THEOREM 1.2. *Let $\psi_1,\psi_2\colon\Omega\to\bar{\mathbf{R}}$ be two functions such that there exists $w\in H^1(\Omega)$ with $\psi_1\leqslant w\leqslant\psi_2$ q.e. in Ω. Let u be a generalized solution of the obstacle problem $\{\psi_1,\psi_2\}$ in Ω. Then for every open set $\Omega'\Subset\Omega$ the function $u|_{\Omega'}$ belongs to $H^1(\Omega')$ and is a variational solution of the obstacle problem $\{\psi_1,\psi_2\}$ in Ω'.*

Proof. Let Ω'' be an open set with $\Omega'\Subset\Omega''\Subset\Omega$. By Remark 1.3 there exist two decreasing sequences u_h and v_h in $H^1(\Omega'')$ such that u_h converges to u quasi uniformly in Ω'', v_h converges to 0 strongly in $H^1(\Omega'')$, u_h is a variational solution of the problem $\{\psi_1,\psi_2+v_h\}$ in Ω'', and $u\leqslant u_h\leqslant u+v_h$ q.e. in Ω''.

Let us prove that u_h is bounded in $H^1(\Omega')$. Let $\varphi \in C_0^\infty(\Omega'')$ with $0 \leqslant \varphi \leqslant 1$ in Ω'' and $\varphi=1$ in Ω'; then the function $\varphi^2 w + (1-\varphi^2)\,u_h$ satiesfies the inequality

$$\psi_1 \leqslant \varphi^2 w + (1-\varphi^2)\,u_h \leqslant \psi_2 + v_h \quad \text{q.e. in } \Omega''.$$

Moreover $\varphi^2 w + (1-\varphi^2)\,u_h = u_h$ on $\partial\Omega''$. Since u_h is a variational solution of $\{\psi_1, \psi_2+v_h\}$ in Ω'' we have $a_{\Omega''}(u_h, \varphi^2(w-u_h)) \geqslant 0$ hence

$$2\sum_{i,j=1}^{N}\int_{\Omega''} a_{ij}(u_h)_{x_j}\varphi_{x_j}\varphi(w-u_h)\,dx + \sum_{i,j=1}^{N}\int_{\Omega''} a_{ij}(u_h)_{x_j} w_{x_j}\varphi^2\,dx \geqslant \sum_{i,j=1}^{N}\int_{\Omega''} a_{ij}(u_h)_{x_j}(u_h)_{x_i}\varphi^2\,dx$$

so there exists a constant $c=c(\lambda, \Lambda, N)$ such that

$$\int_{\Omega''}|\nabla u_h|^2\varphi^2\,dx \leqslant c\left\{\int_{\Omega''}|\nabla u_h|\,|\nabla w|\,\varphi^2\,dx + \int_{\Omega''}|\nabla u_h|\,|\nabla\varphi|\,\varphi|w-u_h|\,dx\right\}$$

$$\leqslant \varepsilon\int_{\Omega''}|\nabla u_h|^2\varphi^2\,dx + \frac{c}{\varepsilon}\left\{\int_{\Omega''}|\nabla w|^2\varphi^2\,dx + \int_{\Omega''}|\nabla\varphi|^2|w-u_h|^2\,dx\right\}$$

for every $\varepsilon>0$. Taking $\varepsilon=1/2$ we obtain

$$\int_{\Omega'}|\nabla u_h|^2\,dx \leqslant \int_{\Omega''}|\nabla u_h|^2\varphi^2\,dx \leqslant 2c\left\{\int_{\Omega''}|\nabla w|^2\varphi^2\,dx + \int_{\Omega''}|\nabla\varphi|^2|w-u_h|^2\,dx\right\}. \tag{1.16}$$

Since $|u_h-u| \leqslant v_h$ q.e. in Ω'', the sequence u_h converges to u in $L^2(\Omega'')$, so (1.16) implies that u_h is bounded in $H^1(\Omega')$. Therefore $u \in H^1(\Omega')$ and u_h converges to u weakly in $H^1(\Omega')$.

It remains to prove that $u|_{\Omega'}$ is a variational solution of the obstacle problem $\{\psi_1, \psi_2\}$ in Ω'. Let $v \in H^1(\Omega')$ such that $\psi_1 \leqslant v \leqslant \psi_2$ q.e. in Ω' and $v-u \in H_0^1(\Omega')$. Then $u_h+v-u \in H^1(\Omega')$, $\psi_1 \leqslant u_h+v-u \leqslant v+v_h \leqslant \psi_2+v_h$ q.e. in Ω' and $u_h+v-u=u_h$ on $\partial\Omega'$. Since u_h is a variational solution of the problem $\{\psi_1, \psi_2+v_h\}$ in Ω', we have $a_{\Omega'}(u_h, v-u) \geqslant 0$. Since u_h converges to u weakly in $H^1(\Omega')$, we obtain $a_{\Omega'}(u, v-u) \geqslant 0$, and this proves that $u|_{\Omega'}$ is a variational solution of the obstacle problem $\{\psi_1, \psi_2\}$ in Ω. □

The following comparison theorem is useful in the proof of the Wiener criterion for generalized solutions of obstacle problems. It extends to generalized solutions the elementary result proved in Lemma II.2.1 for variational solutions.

THEOREM 1.3. *Let $\varphi_1, \varphi_2, \psi_1, \psi_2 \colon \Omega \to \bar{\mathbf{R}}$ be four functions such that $\varphi_1 \leqslant \varphi_2$ and $\psi_1 \leqslant \psi_2$ q.e. in Ω. Let $u, v \colon \bar{\Omega} \to \bar{\mathbf{R}}$ be quasi continuous H^1-dominated functions such that u and v are generalized solutions in Ω of the obstacle problems $\{\varphi_1, \varphi_2\}$ and*

$\{\psi_1, \psi_2\}$ *respectively. If* $\varphi_1 \leq \psi_1$ *q.e. in* Ω, $\varphi_2 \leq \psi_2$ *q.e. in* Ω, *and* $u \leq v$ *q.e. on* $\partial\Omega$, *then* $u \leq v$ *q.e. in* Ω.

Proof. From the proof of Theorem 1.1. it follows that there exist two increasing sequences $\varphi_{1,h}, u_h$ and two decreasing sequences $\psi_{2,h}, v_h$ such that φ_1, u, ψ_2, v are the H^1-dominated quasi uniform limits of $\varphi_{1,h}, u_h, \psi_{2,h}, v_h$ respectively, and u_h, v_h are variational solutions of the obstacle problems $\{\varphi_{1,h}, \varphi_2\}$ and $\{\psi_1, \psi_{2,h}\}$ respectively. Since $\varphi_{1,h} \leq \varphi_1 \leq \psi_1$ and $\varphi_2 \leq \psi_2 \leq \psi_{2,h}$ q.e. in Ω and $u_h \leq v_h$ on $\partial\Omega$ in the sense of $H^1(\Omega)$ (see Lemma I.2.1), by an easy comparison argument for variational solutions (see Lemma II.2.1) we have $u_h \leq v_h$ q.e. in Ω. By taking the limit as $h \to +\infty$ we obtain $u \leq v$ q.e. in Ω. □

III.2. Wiener criterion for generalized solutions

In this section we extend to generalized solutions of a two-obstacle problem the *Wiener criterion* and the Maz'ja estimates proved in Part II for variational solutions (Theorems II.1.1 and II.2.1).

Let $\psi_1, \psi_2 \colon \mathbf{R}^N \to \bar{\mathbf{R}}$ be two functions such that $\psi_1 \leq \psi_2$ q.e. in $\mathbf{R}^N$ and let x_0 be a point of $\mathbf{R}^N$. By $\mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ we denote, in this section, the set of all functions u which are *generalized solutions* of the obstacle problem $\{\psi_1, \psi_2\}$ in some open neighbourhood Ω of x_0 (depending on u).

Definition 2.1. We say that x_0 is a *regular point* of the *generalized* obstacle problem $\{\psi_1, \psi_2\}$ if the set $\mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is not empty and every solution $u \in \mathcal{U}_{\psi_1}^{\psi_2}(x_0)$ is finite and continuous at x_0.

Remark 2.1. If x_0 is a regular point for the variational problem $\{\psi_1, \psi_2\}$, according to Definition II.1.2, then x_0 is a regular point for the generalized obstacle problem $\{\psi_1, \psi_2\}$ according to Definition 2.1. In fact, in this case, Theorem 1.3 ensures that every generalized solution of $\{\psi_1, \psi_2\}$ is a variational solution in a neighbourhood of x_0.

Conversely, if x_0 is a regular point for the generalized obstacle problem $\{\psi_1, \psi_2\}$ and if there exists $w \in H^1(\mathbf{R}^N)$ such that $\psi_1 \leq w \leq \psi_2$ q.e. in a neighbourhood of x_0, then x_0 is a regular point for the variational obstacle problem $\{\psi_1, \psi_2\}$, as one can see by applying again Theorem 1.3.

The following theorem is the *Wiener criterion* for generalized two-obstacle problems.

THEOREM 2.1. *The point* x_0 *is regular for the generalized obstacle problem* $\{\psi_1, \psi_2\}$ *if and only if the following conditions* (2.1), (2.2) *and* (2.3) *are satisfied:*

(2.1) $\bar{\psi}_1(x_0) < +\infty$, $\underline{\psi}_2(x_0) > -\infty$, *and* $\bar{\psi}_1(x_0) \leq \underline{\psi}_2(x_0)$;

(2.2) *there exists an* H^1*-dominated quasi continuous function* $\psi: \mathbf{R}^N \to \bar{\mathbf{R}}$ *such that* $\psi_1 \leq \psi \leq \psi_2$ *q.e. in a neighbourhood of* x_0;

(2.3) x_0 *is a Wiener point of* $\{\psi_1, \psi_2\}$ *according to Definition* II.1.3.

To prove that conditions (2.1), (2.2), and (2.3) are sufficient for the regularity we use the following extension of Theorem II.2.1.

THEOREM 2.2. *Theorem* II.2.1 *and Propositions* II.2.1 *and* II.2.2 *continue to hold, with* $\mu = 0$, *for every generalized solution* u *of the obstacle problem* $\{\psi_1, \psi_2\}$ *on* $\Omega = B_R(x_0), R > 0$.

Proof. It is enough to prove Lemma II.2.2 and Proposition II.2.2 for generalized solutions. For every $r > 0$ we set $B_r = B_r(x_0)$. Let u be a generalized solution of the obstacle problem $\{\psi_1, \psi_2\}$ on a ball B_R and let $0 < R' < R$. By Remark 1.3 there exist two decreasing sequences u_h and $\psi_{2,h}$ converging to u and ψ_2 quasi uniformly in $B_{R'}$, such that u_h is a variational solution of the problem $\{\psi_1, \psi_{2,h}\}$ in $B_{R'}$.

Let us fix $0 < r < R'$ and $\varepsilon_1 > 0$. By the estimate (II.2.9) for variational solutions we have

$$(2.4) \qquad \inf_{B_r} u_h \geq \Psi_{1,h}(\varepsilon_1, R') - c\left[\inf_{B_{R'}} u_h - \Psi_{1,h}(\varepsilon_1, R')\right]^- \exp\left(-\beta \int_r^{R'} \delta_1^*(\varepsilon_1, \varrho) \frac{d\varrho}{\varrho}\right),$$

where

$$\Psi_{1,h}(\varepsilon_1, R') = \inf_{B_{R'}} \psi_{2,h} \wedge [\bar{\psi}_1(x_0) - \varepsilon_1].$$

Since u_h and $\psi_{2,h}$ are decreasing and converge to u and ψ_2 quasi uniformly in $B_{R'}$, we have

$$(2.5) \qquad \inf_{B_\varrho} u = \lim_{h \to +\infty} \inf_{B_\varrho} u_h \quad \text{and} \quad \inf_{B_\varrho} \psi_2 = \lim_{h \to +\infty} \inf_{B_\varrho} \psi_{2,h}$$

for every $0 < \varrho \leq R'$, hence

$$\Psi_1(\varepsilon_1, R') = \lim_{h \to +\infty} \Psi_{1,h}(\varepsilon_1, R').$$

Therefore, taking the limit in (2.4) first as $h \to +\infty$ and then as $R' \to R$ we obtain

$$\inf_{B_r} u \geq \Psi_1(\varepsilon_1, R) - c\left[\inf_{B_R} u - \Psi_1(\varepsilon_1, R)\right]^- \exp\left(-\beta \int_r^R \delta_1^*(\varepsilon_1, \varrho)\frac{d\varrho}{\varrho}\right),$$

which proves the inequality (II.2.9) of Lemma II.2.2.

The estimate (II.2.10) can be proved in a similar way, keeping now ψ_2 fixed and using an increasing approximation of u and ψ_1.

To prove Proposition II.2.2 for generalized solutions we define

$$t_1 = \sup_{B_R} \psi_1 \vee d_R \quad \text{and} \quad t_2 = \inf_{B_R} \psi_2 \wedge d_R$$

and we use the approximation from above of u and ψ_2 on $B_{R'}$, $0<R'<R$, considered in the first part of the proof. By Lemma II.2.3 the function $(u_h - t_2)^-$ is a non-negative subsolution of the operator L. Therefore Proposition I.5.1 implies that

$$\inf_{B_{sR'}} u_h \geq t_2 - \sup_{B_{sR'}} (u_h - t_2)^- \geq t_2 - c(R')^{-N/2} \|(u_h - d_R)^-\|_{L^2(B_{R'})}.$$

Since the sequence u_h is decreasing, by the monotone convergence theorem and by (2.5) we obtain

$$\inf_{B_{sR'}} u \geq t_2 - c(R')^{-N/2} \|(u - d_R)^-\|_{L^2(B_{R'})}.$$

Using an increasing approximation of u and ψ_1 we obtain also

$$\sup_{B_{sR'}} u \leq t_1 + c(R')^{-N/2} \|(u - d_R)^+\|_{L^2(B_{R'})}.$$

We now take the limit as $R' \to R$ in the last two inequalities and conclude the proof as in the variational case. □

Proof of Theorem 2.1. Let us prove the sufficiency. Assume that conditions (2.1), (2.2), and (2.3) are satisfied. If $\bar{\psi}_1(x_0) < \underline{\psi}_2(x_0)$ then there exists a constant $t \in \mathbf{R}$ such that $\psi_1 \leq t \leq \psi_2$ q.e. in a neighbourhood of x_0. Therefore x_0 is a regular point for the variational obstacle problem $\{\psi_1, \psi_2\}$ by Theorem II.1.1 and this implies that x_0 is a regular point for the generalized obstacle problem $\{\psi_1, \psi_2\}$ by Remark 2.1. If $\bar{\psi}_1(x_0) = \underline{\psi}_2(x_0)$, then every generalized solution of the obstacle problem $\{\psi_1, \psi_2\}$ is finite and continuous at x_0 by Theorem 2.2. Therefore in both cases x_0 is a regular point for the generalized obstacle problem $\{\psi_1, \psi_2\}$.

Let us prove the necessity. Conditions (2.1) and (2.2) are obvious. To prove (2.3) for every $\tau \in \mathbf{R}$ and for $r>0$ small enough we consider the unique function $u=u_{r,\tau}:\overline{B_{2r}(x_0)}\to\bar{\mathbf{R}}$ such that u is a generalized solution of the obstacle problem $\{\tilde{\psi}_1, \tilde{\psi}_2\}$ in B_{2r} and $u=\tau$ q.e. on $\partial B_{2r}(x_0)$ (see Theorem 1.1), where

$$\tilde{\psi}_1(x)=\begin{cases}\psi_1(x) & \text{if } x\in B_r,\\ -\infty & \text{if } x\notin B_r,\end{cases} \quad\text{and}\quad \tilde{\psi}_2(x)=\begin{cases}\psi_2(x) & \text{if } x\in B_r,\\ +\infty & \text{if } x\notin B_r.\end{cases}$$

Then we can prove Steps 1, 2 and 3 of the proof of the *Wiener criterion* (Theorem II.1.1) by using the comparison principle for generalized solutions provided by Theorem 1.3. □

We now apply the results of Sections II.2 and III.2 to the important case of obstacles defined on an arbitrary (possibly "thin") subset F of $\mathbf{R}^N$. In this case the estimates of the oscillation of the solution can be given in terms of the oscillation of the obstacles and of the Wiener modulus $W_F(r,R)$ of F introduced in (I.7.6).

Given two functions $h_1, h_2: F\to\bar{\mathbf{R}}$, we define

$$\psi_1=\begin{cases}h_1 & \text{on } F,\\ -\infty & \text{elsewhere},\end{cases}\qquad \psi_2=\begin{cases}h_2 & \text{on } F,\\ +\infty & \text{elsewhere}.\end{cases}$$

We fix a point $x_0\in\mathbf{R}^N$ and a radius $R>0$. We assume that

$$\sup_{B_R\cap F} h_1<+\infty \quad\text{and}\quad \inf_{B_R\cap F} h_2>-\infty,$$

where, for every $r>0$, we set $B_r=B_r(x_0)$.

COROLLARY 2.1. *Under the above hypotheses there exist two constants $c=c(\lambda,\Lambda,N)>0$ and $\beta=\beta(\lambda,\Lambda,N)>0$ such that for every (generalized) solution u of the obstacle problem $\{\psi_1,\psi_2\}$ on B_R we have*

$$\operatorname*{osc}_{B_r} u\le c\left[\operatorname*{osc}_{B_R\cap F} h_1+\operatorname*{osc}_{B_R\cap F} h_2+\left(\operatorname*{osc}_{B_R} u\right) W_F(r,R)^\beta\right] \tag{2.6}$$

for every $0<r\le R$.

Proof. We may assume that the right hand side of (2.6) is finite. Let us fix $0<r\le R$ and a solution u of the obstacle problem $\{\psi_1,\psi_2\}$ on B_R. Given $\varepsilon>0$, we choose $\sigma_1>0$ and $\sigma_2>0$ so that

$$\sigma_1 W_F(r,R) = \operatorname*{osc}_{B_R\cap F} h_1+\varepsilon \quad \text{and} \quad \sigma_2 W_F(r,R) = \operatorname*{osc}_{B_R\cap F} h_2+\varepsilon. \tag{2.7}$$

By Lemma I.7.3. we have

$$\omega_{1,\sigma_1}(r,R) \leqslant W_F(r,R) \quad \text{and} \quad \omega_{2,\sigma_2}(r,R) \leqslant W_F(r,R). \tag{2.8}$$

Suppose that

$$\underline{\psi}_2(x_0)-\bar{\psi}_1(x_0) \leqslant \operatorname*{osc}_{B_R\cap F} h_1+\operatorname*{osc}_{B_R\cap F} h_2. \tag{2.9}$$

By (2.7) and (2.8) we have

$$\begin{aligned}\Psi_{\sigma_1,\sigma_2}(r,R) &\leqslant \underline{\psi}_2(x_0)-\bar{\psi}_1(x_0)+\operatorname*{osc}_{B_R\cap F} h_1+\operatorname*{osc}_{B_R\cap F} h_2+2\varepsilon \\ &\leqslant 2\left[\operatorname*{osc}_{B_R\cap F} h_1+\operatorname*{osc}_{B_R\cap F} h_2+\varepsilon\right],\end{aligned}$$

thus Proposition II.2.1 for generalized solutions, together with (2.8), yields

$$\operatorname*{osc}_{B_r} u \leqslant 2\left[\operatorname*{osc}_{B_R\cap F} h_1+\operatorname*{osc}_{B_R\cap F} h_2+\varepsilon\right]+c\left(\operatorname*{osc}_{B_R} u\right) W_F(r,R)^\beta,$$

and as $\varepsilon\to 0$ we obtain (2.6).

If (2.9) is not satisfied, then there exists a constant d such that

$$\inf_{B_R} u \leqslant d \leqslant \sup_{B_R} u \tag{2.10}$$

and $\psi_1 \leqslant d \leqslant \psi_2$ q.e. on B_R. Taking (2.7) and (2.8) into account, Theorem II.2.2, applied with $w=d$, yields

$$\mathcal{V}(r) \leqslant c\left[R^{-N/2}\|u-d\|_{L^2(B_R)} W_F(r,R)^\beta+\operatorname*{osc}_{B_R\cap F} h_1+\operatorname*{osc}_{B_R\cap F} h_2+2\varepsilon\right]. \tag{2.11}$$

The estimate (2.6) follows now from (2.10) and (2.11), taking the limit as $\varepsilon\to 0$. □

III.3. Generalized Dirichlet problems

In this section we give an estimate for the modulus of continuity of solutions of Dirichlet problems with quasi continuous H^1-dominated boundary conditions.

Let D be a bounded open subset of $\mathbf{R}^N$ and let g: $\partial D \to \bar{\mathbf{R}}$ be an *H^1-dominated quasi continuous function.*

Definition 3.1. We say that a function $u:D\to\bar{\mathbf{R}}$ is a *dominated generalized solution* (in D) of the equation $Lu=0$ if u is the H^1-dominated quasi uniform limit (in D) of a sequence u_h of functions of $H^1(D)$ such that $Lu_h=0$ (in D) in the sense of Section I.4.

By Theorem 1.2, applied with $\Omega=D$, $\psi_1=-\infty$, and $\psi_2=+\infty$, every dominated generalized solution u of the equation $Lu=0$ in D belongs to $H^1_{\text{loc}}(D)$ and satisfies $Lu=0$ in the sense of distributions. The converse is false, as the following example shows.

Example 3.1. Let $D=B_1(0)\setminus\{0\}$ and let $L=-\Delta$. Then for every $t\in\mathbf{R}$ the functions $u_t(x)=t|x|^{2-N}-t$ belong to $H^1_{\text{loc}}(D)$ and satisfy $Lu_t=0$ on D in the sense of distributions, but u_t is a dominated generalized solution in the sense of Definition 3.1 only for $t=0$.

By Theorem 1.1, applied with $\Omega=D$, $\psi_1=-\infty$, and $\psi_2=+\infty$, there exists a unique *quasi continuous* function $u:\bar{D}\to\bar{\mathbf{R}}$ such that u is a dominated generalized solution of the equation $Lu=0$ in D and $u=g$ q.e. on ∂D. We shall refer to this function as the solution of the Dirichlet problem

$$Lu=0 \;\text{ in } D, \quad u=g \;\text{ on } \partial D. \tag{3.1}$$

It is easy to see that, if g is continuous, then u concides with the solution of the Dirichlet problem (3.1) in the sense of [22], Section 10.

We now show that the Maz'ja estimate at a boundary point (see [18] and [19]) for the solution of the Dirichlet problem (3.1) can be obtained from the estimates for generalized solutions of a two-obstacle problem given by Theorem 2.2.

Let $x_0\in\partial D$. For every $r>0$ we set $B_r=B_r(x_0)$.

THEOREM 3.1. *Let u be the solution of the Dirichlet problem* (3.1), *with g quasi continuous and essentially bounded on ∂D (in the capacity sense). Then there exist two constants $c=c(\lambda,\Lambda,N)>0$ and $\beta=\beta(\lambda,\Lambda,N)>0$ such that*

$$\operatorname*{osc}_{B_r\cap D} u \leq \operatorname*{osc}_{B_R\cap\partial D} g + c\Big(\operatorname*{osc}_{\partial D} g\Big)\exp\left(-\beta\int_r^R \frac{\operatorname{cap}(B_\varrho - D, B_{2\varrho})}{\operatorname{cap}(B_\varrho, B_{2\varrho})}\,\frac{d\varrho}{\varrho}\right)$$

for every $0<r<R$.

Proof. Let us fix $0<r<R'<R$. By adapting the proof of Tietze's extension theorem (see for instance [6]), we can extend $g|_{\bar{B}_{R'}\cap\partial D}$ to a quasi continuous function $\psi:\bar{B}_{R'}\to\mathbf{R}$

such that

$$\inf_{\bar{B}_{R'}} \psi = \inf_{\bar{B}_{R'} \cap \partial D} g \leq \sup_{\bar{B}_{R'} \cap \partial D} g = \sup_{\bar{B}_{R'}} \psi. \tag{3.2}$$

Then we can extend ψ to an essentially bounded quasi continuous function, still denoted by ψ, defined in $\mathbf{R}^N$ such that $\psi = g$ q.e. on ∂D.

Let $E = \mathbf{R}^N - D$, let $\psi_1, \psi_2 \colon \mathbf{R}^N \to \bar{\mathbf{R}}$ be the functions defined by

$$\psi_1 = \begin{cases} \psi & \text{in } E, \\ -\infty & \text{elsewhere}, \end{cases} \qquad \psi_2 = \begin{cases} \psi & \text{in } E, \\ +\infty & \text{elsewhere}, \end{cases} \tag{3.3}$$

and let Ω be a bounded open subset of $\mathbf{R}^N$ containing $\bar{D} \cup B_R$; then the obstacle problem $\{\psi_1, \psi_2\}$ has a unique generalized solution v in Ω (Theorem 1.3) and we have

$$v = \begin{cases} u & \text{q.e. in } D, \\ \psi & \text{q.e. in } \Omega - D. \end{cases} \tag{3.4}$$

By Theorem 2.2 the function v satisfies the estimate (II.2.4), therefore by Remark I.7.1 we have

$$\operatorname*{osc}_{B_r} v \leq \Psi(\varepsilon_1, \varepsilon_2, R') + c\left(\operatorname*{osc}_{B_{R'}} v\right) \sum_{i=1}^{2} \exp\left(-\beta \int_r^{R'} \delta_i^*(\varepsilon_i, \varrho) \frac{d\varrho}{\varrho}\right) \tag{3.5}$$

for every $\varepsilon_1 > 0, \varepsilon_2 > 0$. Given $\varepsilon > 0$, we set

$$\varepsilon_1 = \bar{\psi}(x_0) - \inf_{B_{R'}} \psi_2 + \varepsilon = \bar{\psi}_1(x_0) - \inf_{B_{R'} \cap E} \psi + \varepsilon$$

and

$$\varepsilon_2 = \sup_{B_{R'}} \psi_1 - \underline{\psi}_2(x_0) + \varepsilon = \sup_{B_{R'} \cap E} \psi - \underline{\psi}_2(x_0) + \varepsilon.$$

Then $E_1^*(\varepsilon_1, \varrho) \supseteq E \cap B_\varrho$ and $E_2^*(\varepsilon_2, \varrho) \supseteq E \cap B_\varrho$ for every $0 < \varrho \leq R'$, therefore (3.5) implies

$$\operatorname*{osc}_{B_r \cap D} u \leq \operatorname*{osc}_{B_r} v \leq \operatorname*{osc}_{B_{R'} \cap E} \psi + 2\varepsilon + c\left(\operatorname*{osc}_{B_{R'}} v\right) W_E(r, R')^\beta, \tag{3.6}$$

where, according to (I.7.6),

$$W_E(r, R') = \exp\left(-\int_r^{R'} \frac{\operatorname{cap}(E \cap B_\varrho, B_{2\varrho})}{\operatorname{cap}(B_\varrho, B_{2\varrho})} \frac{d\varrho}{\varrho}\right).$$

By (3.2) we have

$$\operatorname*{osc}_{B_{R'}\cap E} \psi \leq \operatorname*{osc}_{B_R\cap\partial D} g \tag{3.7}$$

and from the maximum principle we get

$$\begin{aligned}\operatorname*{osc}_{B_{R'}} v &\leq \left[\sup_{B_{R'}\cap D} u \vee \sup_{B_{R'}\cap E} \psi\right] - \left[\inf_{B_{R'}\cap D} u \wedge \inf_{B_{R'}\cap E} \psi\right] \\ &\leq \sup_{\partial D} g - \inf_{\partial D} g = \operatorname*{osc}_{\partial D} g.\end{aligned} \tag{3.8}$$

Since (3.6) holds for every $R'<R$ and for every $\varepsilon>0$, from (3.7) and (3.8) we obtain

$$\operatorname*{osc}_{B_r\cap D} u \leq \operatorname*{osc}_{B_R\cap\partial D} g + c\left(\operatorname*{osc}_{\partial D} g\right) W_E(r,R)^{\beta},$$

which is the Maz'ja estimate at the point $x_0\in\partial D$. □

More generally, given an *arbitrary* subset E of Ω, and an H^1-dominated quasi continuous function $\psi: \mathbf{R}^N \to \bar{\mathbf{R}}$ we consider the formal Dirichlet problem

$$\begin{cases} Lu = 0 & \text{in } \Omega - E \\ u = \psi & \text{in } E. \end{cases} \tag{3.9}$$

By a solution of (3.9) we mean any generalized solution in Ω of the obstacle problem $\{\psi_1, \psi_2\}$ where ψ_1 and ψ_2 are given by (3.3). By applying (II.2.4) to the case at hand, we obtain the estimate

$$\operatorname*{osc}_{B_r} u \leq \operatorname*{osc}_{B_R\cap E} \psi + c\left(\operatorname*{osc}_{B_R} u\right)\exp\left(-\beta\int_r^R \frac{\operatorname{cap}(E\cap B_\varrho, B_{2\varrho})}{\operatorname{cap}(B_\varrho, B_{2\varrho})}\frac{d\varrho}{\varrho}\right)$$

for every $x_0\in\Omega$ and for every $0<r<R$ with $B_R(x_0)\subseteq\Omega$, where $c=c(\lambda,\Lambda,N)$, $\beta=\beta(\lambda,\Lambda,N)$, and $B_\varrho=B_\varrho(x_0)$ for every $\varrho>0$.

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