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Citation for published version (APA):

Emre, E., & Hautus, M. L. J. (1980). A polynomial characterization of (A,B)-invariant and reachability subspaces. *SIAM Journal on Control and Optimization*, 18(4), 420-436. <https://doi.org/10.1137/0318031>

DOI:

[10.1137/0318031](https://doi.org/10.1137/0318031)

Document status and date:

Published: 01/01/1980

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
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A POLYNOMIAL CHARACTERIZATION OF (A, B)-INVARIANT AND REACHABILITY SUBSPACES*

E. EMRE† AND M. L. J. HAUTUS‡

Abstract. Based on the state space model of P. Fuhrmann, a link is laid between the geometric approach to linear system theory, as developed by W. M. Wonham and A. S. Morse, and the approach based on polynomial matrices. In particular polynomial characterizations of (A, B)-invariant and reachability subspaces are given. These characterizations are used to prove the equivalence of the disturbance decoupling problem and the exact model matching problem and also to connect the polynomial matrix and the geometric approach to the construction of observers.

Finally, constructive procedures and conditions are given for computing the supremal (A, B)-invariant subspace and reachability space and for checking the solvability of the exact model matching problem.

1. Introduction. The geometric approach to linear system theory has proved very successful in solving a variety of problems (see [17] for a detailed account of this theory). The principal concepts in this theory, which are instrumental in the description of many results, are (A, B)-invariant subspaces and reachability (controllability) subspaces. An alternative approach to linear system design has been developed in [13], [14], [16]. This theory depends to a large extent on polynomial matrix techniques. It is evident that a method for translating results of one theory to another is very desirable, because such a method would yield a better understanding of the relations between the two different approaches. This would be very useful, in particular since the geometric method may be viewed as exponent of the so-called “modern control theory” and the polynomial matrix method may be considered a generalization of the classical frequency domain methods.

A number of papers with the objective of translating the results of geometric control theory into polynomial matrix terms have appeared (e.g., [1], [3], [10], [12]).

It is the purpose of this paper to show that a very useful link between the two approaches can be based on the work of P. Fuhrmann ([7], [8], [9]). Specifically, it will be shown that using the state space model associated with a system matrix, introduced by Fuhrmann, one can give characterizations of the concepts of (A, B)-invariant subspaces and reachability subspaces in terms of polynomial matrices. This will be the subject of § 3 and 6. One application of the polynomial characterization of (A, B)-invariant subspaces will be given in § 4, where it will be shown that the disturbance decoupling problem (see [17, Chap. 4]) and the exact model matching problem (see [16], [14], [5]) are equivalent problems. Another application is given in § 5, where it is shown that the equivalence of the polynomial matrix and the geometric formulation of observers can be derived from the results of § 3. In § 7, the concept of row properness defined in [14], [16] is used to formulate a necessary and sufficient condition for the existence of a solution of the exact model matching problem and, hence, of the disturbance decoupling problem in terms of degrees of polynomial matrices. Also in § 7 a constructive characterization of the supremal (A, B)-invariant subspace and reachability space contained in $\ker C$ is given. Finally, in § 8, the results of § 3 are extended to the situation where the system is described by Rosenbrock’s system matrix.

The preliminary § 2 contains a short description of Fuhrmann’s state space model.

* Received by the editors October 18, 1978, and in revised form October 5, 1979.

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2. The state space model associated with a matrix fraction representation. Let K be a field. We denote by $K[s]$ the set of polynomials, and by $K(s)$ the set of rational functions over K . If \mathcal{S} is any set and $p, q \in \mathbb{N}$, we denote by \mathcal{S}^p the set of p -vectors with components in \mathcal{S} and by $\mathcal{S}^{p \times q}$ the set of $p \times q$ matrices with entries in \mathcal{S} . If A is a $p \times q$ matrix, we denote by $\{A\}$ the K -linear space generated by the columns of A . If $U(s) \in K^{q \times r}[s]$ and $\mathcal{L}: K^q[s] \rightarrow K^p[s]$ is a linear map, then $\mathcal{L}U(s)$ denotes the result obtained by applying \mathcal{L} to each of the columns of $U(s)$.

Let $x(s) \in K^p(s)$. We denote by $(x(s))_-$, the strictly proper part of $x(s)$ and by $(x(s))_{-1}$, the coefficient of s^{-1} in the expansion of $x(s)$ in powers of s^{-1} .

DEFINITION 2.1. Let $T(s) \in K^{p \times q}[s]$. Then X_T denotes the set of $x(s) \in K^p[s]$ for which there exists a strictly proper $u(s) \in K^q(s)$ such that $T(s)u(s) = x(s)$.

In what follows, X_T plays a fundamental role (compare the closely related concept of right rational annihilator [4]).

In particular, if $p = q$ and $T(s)$ is nonsingular, then

$$X_T = \{x(s) \in K^p[s] \mid T^{-1}(s)x(s) \text{ is strictly proper}\}.$$

In this particular situation we define the map

$$\pi_T: K^p[s] \rightarrow X_T: x(s) \mapsto T(s)(T^{-1}(s)x(s))_-.$$

(Compare [7] and [8] where further properties of this map are given.) In the following we consider X_T a K -linear space. We consider a linear system whose transfer matrix is given by the left matrix fraction representation

$$(2.2) \quad G(s) = T^{-1}(s)U(s).$$

We assume that $G(s)$ is strictly proper, $T(s) \in K^{q \times q}[s]$, $U(s) \in K^{q \times r}[s]$.

Define the linear maps

$$(2.3) \quad \begin{aligned} \mathcal{A}: X_T &\rightarrow X_T: x(s) \mapsto \pi_T(sx(s)), \\ \mathcal{B}: K^r &\rightarrow X_T: u \mapsto U(s)u, \\ \mathcal{C}: X_T &\rightarrow K^q: x(s) \mapsto (T^{-1}(s)x(s))_{-1}. \end{aligned}$$

By definition, for $x(s) \in X_T$ we have $\mathcal{A}x(s) = sx(s) - T(s)c(s)$ for some $c(s) \in K^q[s]$. Since $T^{-1}(s)x(s)$ and $T^{-1}\mathcal{A}x(s)$ are strictly proper it follows that $c(s)$ must be constant. Hence

$$(2.4) \quad \mathcal{A}x(s) = sx(s) - T(s)c$$

for some $c \in K^q$, depending on $x(s)$.

The following result is proved in [7].

THEOREM 2.5. The system $\Sigma := (\mathcal{C}, \mathcal{A}, \mathcal{B})$ with state space X_T is an observable realization of $G(s)$. The realization is reachable iff $T(s)$ and $U(s)$ are left coprime.

We will call this realization Σ the T -realization of $G(s)$.

Conversely, if we are given an observable system $\Sigma = (C, A, B)$, then we construct a left matrix fraction representation of the transfer matrix of Σ in the following way. Let

$$(2.6) \quad C(sI - A)^{-1} = T^{-1}(s)S(s),$$

where S and T are left coprime. Define

$$(2.7) \quad U(s) := S(s)B.$$

Then $G(s) = T^{-1}(s)U(s)$ is the required representation. We have the following result.

THEOREM 2.8. *The T -realization of $G(s) = T^{-1}(s)U(s)$, where T and U are defined by (2.6) and (2.7) is isomorphic to the system Σ .*

Proof. Using the dual of [11, Cor. 4.11], we see that $S(s)$ is a basis matrix of X_T . Hence the linear map

$$\mathcal{S}: K^n \rightarrow X_T : x \mapsto S(s)x$$

is an isomorphism. Using the equation

$$T(s)C = S(s)(sI - A),$$

which follows from (2.6), one derives easily the relations $\mathcal{A}\mathcal{S} = \mathcal{S}A$, $\mathcal{B} = \mathcal{S}B$, $\mathcal{C}\mathcal{S} = C$. In particular, $\mathcal{A}\mathcal{S}x = \mathcal{A}(S(s)x) = \pi_T(S(s)(sI - A)x) + \pi_T(S(s)Ax) = \pi_T(T(s)Cx) + \pi_T(S(s)Ax) = S(s)Ax = \mathcal{S}Ax$.

It follows that (C, A, B) and $(\mathcal{C}, \mathcal{A}, \mathcal{B})$ are isomorphic. \square

Using Theorem 2.8, we may transform results obtained for the particular realization $(\mathcal{C}, \mathcal{A}, \mathcal{B})$ to any observable system.

3. $(\mathcal{A}, \mathcal{B})$ -invariant subspaces. We give a characterization of the $(\mathcal{A}, \mathcal{B})$ -invariant subspaces of the T -realization of a transfer matrix $G(s) = T^{-1}(s)U(s)$, as defined in the previous section. For the definition of $(\mathcal{A}, \mathcal{B})$ -invariant subspaces we refer to [17].

THEOREM 3.1. *Let $\Psi(s)$ be a $q \times m$ polynomial matrix. Then $\{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace of X_T iff there exist $C_1 \in K^{q \times m}$, $F_1 \in K^{r \times m}$ and $A_1 \in K^{m \times m}$ such that*

$$(3.2) \quad T(s)C_1 + U(s)F_1 = \Psi(s)(sI - A_1).$$

Proof. Suppose that $\{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace, i.e.,

$$(3.3) \quad \mathcal{A}\{\Psi(s)\} \subseteq \{\Psi(s)\} + \text{Im } \mathcal{B}.$$

Applying (2.4) to each column of $\Psi(s)$, we find that $\mathcal{A}\Psi(s) = \Psi_1(s)$, where

$$(3.4) \quad \Psi_1(s) := s\Psi(s) - T(s)C_1$$

for some $C_1 \in K^{q \times m}$. On the other hand, (3.3) implies

$$(3.5) \quad \Psi_1(s) = \Psi(s)A_1 + U(s)F_1$$

for some $A_1 \in K^{m \times m}$ and $F_1 \in K^{r \times m}$. Combining (3.4) and (3.5) yields (3.2). Conversely, if we assume (3.2), then

$$(3.6) \quad T^{-1}(s)\Psi(s) = (C_1 + T^{-1}(s)U(s)F_1)(sI - A_1)^{-1}$$

is strictly proper and hence $\{\Psi(s)\} \subseteq X_T$. Furthermore, if we define $\Psi_1(s)$ by (3.4), then (3.5) follows from (3.2) and, hence, $\{\Psi_1(s)\} \subseteq X_T$. It follows that

$$\mathcal{A}(\Psi(s)) = \pi_T(s\Psi(s)) = \pi_T(\Psi_1(s) + T(s)C_1) = \Psi_1(s).$$

Thus, (3.5) implies (3.3). \square

If the matrix $\Psi(s)$ occurring in Theorem 3.1 has full column rank, it is possible to give an interpretation to the matrices A_1, F_1, C_1 . For in that case there exists a K -linear map $\mathcal{F}: X_T \rightarrow K^r$ satisfying

$$(3.7) \quad \mathcal{F}\Psi(s) = F_1.$$

Then (3.2) implies

$$(3.8) \quad (\mathcal{A} - \mathcal{B}\mathcal{F})\Psi(x) = \Psi(s)A_1.$$

It follows that $\{\Psi(s)\}$ is an $(\mathcal{A} - \mathcal{B}\mathcal{F})$ -invariant subspace, and that A_1 is the matrix of the restriction of $\mathcal{A} - \mathcal{B}\mathcal{F}$ to $\{\Psi(s)\}$ with respect to the basis matrix $\Psi(s)$. In addition, F_1 is the matrix (with respect to the basis matrix $\Psi(s)$ of $\{\Psi(s)\}$ and the natural basis in K^r) of \mathcal{F} . Finally, we have

$$(3.9) \quad \mathcal{C}\Psi(s) = C_1$$

so that C_1 is the matrix of the restriction of \mathcal{C} to $\{\Psi(s)\}$ with respect to the basis matrix $\Psi(s)$ of $\{\Psi(s)\}$ and the natural basis of K^q .

The last result gives a characterization of $(\mathcal{A}, \mathcal{B})$ -invariant subspaces contained in $\ker \mathcal{C}$.

COROLLARY 3.10. *Let $\Psi(s) \in K^{q \times m}[s]$. Then $\{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace contained in $\ker \mathcal{C}$ iff there exist matrices F_1, A_1 such that*

$$(3.11) \quad U(s)F_1 = \Psi(s)(sI - A_1).$$

Proof. According to (3.9), we must have $C_1 = 0$ in formula (3.2). □

COROLLARY 3.12. *X_U is the largest $(\mathcal{A}, \mathcal{B})$ -invariant subspace of X_T contained in $\ker \mathcal{C}$.*

Proof. According to (3.10), we have for an arbitrary $(\mathcal{A}, \mathcal{B})$ -invariant subspace $\{\Psi(s)\}$ contained in $\ker \mathcal{C}$:

$$\Psi(s) = U(s)F_1(sI - A_1)^{-1}.$$

Hence $\{\Psi(s)\} \subseteq X_U$ (see Definition 2.1). It remains to be shown that X_U itself is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace. If $\Phi(s)$ is a basis matrix of X_U then there exists a strictly proper matrix $Q(s)$ such that $U(s)Q(s) = \Phi(s)$. Let (F_1, A_1, B_1) be a reachable realization of $Q(s)$, so that

$$\Phi(s) = U(s)F_1(sI - A_1)^{-1}B_1.$$

It follows from Lemma 3.13 that

$$\Psi(s) := U(s)F_1(sI - A_1)^{-1}$$

is a polynomial matrix. By Corollary 3.10, $\{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace. Hence $\{\Psi(s)\} \subseteq \{\Phi(s)\}$. On the other hand, since $\Phi(s) = \Psi(s)B_1$, it follows that $\{\Phi(s)\} \subseteq \{\Psi(s)\}$. Consequently, $X_U = \{\Phi(s)\} = \{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace. □

LEMMA 3.13. *Let $Q(s) \in K^{l \times n}[s]$, $A \in K^{n \times n}$, $B \in K^{n \times r}$, (A, B) reachable. If $Q(s)(sI - A)^{-1}B$ is a polynomial matrix then $Q(s)(sI - A)^{-1}$ is a polynomial matrix.*

Proof. We decompose the rational matrix $Q(s)(sI - A)^{-1}$ into its polynomial and its strictly proper part:

$$Q(s)(sI - A)^{-1} = P(s) + R(s);$$

then

$$R_0 := R(s)(sI - A) = Q(s) - P(s)(sI - A)$$

is a polynomial of degree zero and hence constant. It follows that

$$R_0(sI - A)^{-1}B = Q(s)(sI - A)^{-1}B - P(s)B$$

is a strictly proper polynomial and hence zero. Since (A, B) is reachable, this implies $R_0 = 0$ and hence

$$Q(s)(sI - A)^{-1} = P(s). \quad \square$$

The foregoing implies that the set of $(\mathcal{A}, \mathcal{B})$ -invariant subspaces in $\ker \mathcal{C}$ is uniquely determined by the numerator polynomial matrix of the matrix fraction representation of the transfer function matrix:

COROLLARY 3.14. *Let $U(s) \in K^{q \times r}[s]$, $T_i(s) \in K^{q \times q}[s]$, $i = 1, 2$, such that*

$$G_i(s) := T_i^{-1}(s)U(s)$$

is strictly proper for $i = 1, 2$. Let $(\mathcal{C}_i, \mathcal{A}_i, \mathcal{B}_i)$ be the T_i -realization of $G_i(s)$ for $i = 1, 2$. Then $M \subseteq X_U$ is an $(\mathcal{A}_1, \mathcal{B}_1)$ -invariant subspace of X_{T_1} contained in $\ker \mathcal{C}_1$ iff M is an $(\mathcal{A}_2, \mathcal{B}_2)$ -invariant subspace of X_{T_2} contained in $\ker \mathcal{C}_2$.

REMARK 3.15. Theorem 3.1 may be specialized to the case $U(s) = 0$, that is, $\mathcal{B} = 0$. In this case we have a realization of $G(s) = 0$ with the same state space X_T and the same map \mathcal{C} as before. An $(\mathcal{A}, \mathcal{B})$ -invariant subspace of X_T then is just an \mathcal{A} -invariant subspace. Thus we obtain the following characterization of \mathcal{A} -invariant subspaces.

PROPOSITION. *Let $\Psi(s)$ be a $q \times m$ polynomial matrix. Then, $\{\Psi(s)\}$ is an \mathcal{A} -invariant subspace of X_T iff there exist $C_1 \in K^{q \times m}$, $A_1 \in K^{m \times m}$ such that*

$$T(s)C_1 = \Psi(s)(sI - A_1).$$

4. Exact model matching and disturbance decoupling. If we have an observable system (C, A, B) with state space X then we may consider the problem of characterizing the (A, B) -invariant subspaces contained in $\ker C$. Using the isomorphism given in Theorem 2.8, we transform the problem to the case of a suitable T -realization. For this case we may appeal to Corollary 3.10, by which a complete characterization is given. It is important that, as already noted in Corollary 3.14, this characterization depends only on the numerator polynomial $U(s)$. Consequently, we have the following result.

THEOREM. *Let $\bar{\Sigma} = (C, A, B)$ be a realization with state space X of a transfer matrix $G(s) = T^{-1}(s)U(s)$, and let $\Sigma = (\mathcal{C}, \mathcal{A}, \mathcal{B})$ be the T -realization of $G(s)$. If $\bar{\Sigma}$ and Σ are isomorphic by the isomorphism $\mathcal{L}: X \rightarrow X_T$, then $M \subseteq X$ is an (A, B) -invariant subspace contained in $\ker \mathcal{C}$ iff there exist constant matrices F_1, A_1 satisfying*

$$U(s)F_1 = \Psi(s)(sI - A_1),$$

where $\Psi(s)$ is a basis matrix of $\mathcal{L}(M)$.

Thus we see how characterizations for $(\mathcal{A}, \mathcal{B})$ -invariant subspaces of the particular state space model Σ can be generalized to arbitrary (observable) state space models.

In this section we use the theory developed thus far to show the equivalence of the exact model matching problem and the disturbance decoupling problem.

PROBLEM 4.1. (Disturbance decoupling problem (DDP)). *Given the system*

$$(4.2) \quad \dot{x}(t) = Ax(t) + Bu(t) + Eq(t), \quad y(t) = Cx(t),$$

where (C, A) is observable, determine a constant matrix F such that if

$$u(t) = Fx(t), \quad t \geq 0,$$

the output $y(t)$ does not depend on $q(t)$, $t \geq 0$.

The following result has been given in [17, Thm. 4.2] in a slightly different but equivalent formulation:

THEOREM 4.3. *Problem 4.1 has a solution iff there exists a subspace M of the state space such that*

$$AM \subseteq M + \{B\}, \quad \{E\} \subseteq M \subseteq \ker C. \quad \square$$

In this paper we will also consider a slightly modified problem (compare also [18]).

PROBLEM 4.4. (Modified disturbance decoupling problem (MDDP)). *Given system (4.2), determine constant matrices F and D such that if*

$$u(t) = Fx(t) + Dq(t),$$

the output does not depend on $q(t)$.

In the modified problem, one assumes that not only the state but also the disturbance is directly available for measurement. Similarly to (4.3) we have the following result.

THEOREM 4.5. *Problem 4.4 has a solution iff there exists a subspace M such that*

$$AM \subset M + \{B\}, \quad \{E\} \subset M + \{B\}, \quad M \subset \ker C. \quad \square$$

The exact model matching problem is defined as follows.

PROBLEM 4.6. *Given transfer function matrices $G_1(s)$ and $G_2(s)$ determine a (i) strictly proper or (ii) proper rational matrix $Q(s)$ such that*

$$G_1(s)Q(s) = G_2(s).$$

Problem 4.6(i) will be called the *exact model matching problem* (EMMP), and Problem 4.6(ii) will be called the *modified exact model matching problem* (MEMMP). It is the purpose of this section to show that the existence of a solution of Problem 4.1 is equivalent to the existence of a solution of Problem 4.6(i). Similarly: Problem 4.4 has a solution iff Problem 4.6(ii) has a solution. We will concentrate on the modified problems. The original problems can be dealt with similarly.

First we have to indicate which MEMMP corresponds to a given MDDP and vice versa. Let us start with system (4.2). The data $G_1(s)$ and $G_2(s)$ of MEMMP are then defined by

$$G_1(s) := C(sI - A)^{-1}B, \quad G_2(s) := C(sI - A)^{-1}E.$$

Conversely, if we are given $G_1(s)$ and $G_2(s)$ in MEMMP, we construct an observable realization $(C, A, [B, E])$ of the transfer matrix $[G_1(s), G_2(s)]$. Then C, A, B, E are the data for MDDP. Thus, we have a one to one correspondence between MEMMP's and MDDP's.

Following Theorem 2.8, we assume that

$$C(sI - A)^{-1} = T^{-1}(s)S(s)$$

with $T(s)$ and $S(s)$ relatively prime, and $U(s) = S(s)B$; and we consider the T -realization $(\mathcal{C}, \mathcal{A}, \mathcal{B})$ of $G_1(s) = T^{-1}(s)U(s)$. According to Theorem 2.8, the map $\mathcal{S}: x \mapsto S(s)x: K^n \rightarrow X_T$ is an isomorphism. Consequently, we introduce the polynomial matrix $R(s) := S(s)E$ as representative of E in X_T . Then we have $G_2(s) = T^{-1}(s)R(s)$ and we can state the following result.

THEOREM 4.7. *Let $\{\Psi(s)\}$ be an $(\mathcal{A}, \mathcal{B})$ -invariant subspace in $\ker \mathcal{C}$, so that there exist constant matrices F_1 and A_1 satisfying*

$$(4.8) \quad U(s)F_1 = \Psi(s)(sI - A_1).$$

In addition, assume that $\{R(s)\} \subseteq \{\Psi(s)\} + \{U(s)\}$, so that there exist matrices B_1 and D_1 such that

$$(4.9) \quad R(s) = \Psi(s)B_1 + U(s)D_1.$$

Then $Q(s) := F_1(sI - A_1)^{-1}B_1 + D_1$ is a solution of MEMMP. Conversely, let $Q(s)$ be a solution of MEMMP and let (F_1, A_1, B_1, D_1) be a reachable realization of $Q(s)$. Then there exists a polynomial matrix $\Psi(s)$ satisfying (4.8) and (4.9).

Proof. If $\Psi(s)$ satisfies (4.8) and (4.9) then

$$U(s)Q(s) = \Psi(s)B_1 + U(s)D_1 = R(s),$$

which implies $G_1(s)Q(s) = G_2(s)$. Conversely the latter equation implies $U(s)Q(s) = R(s)$. Hence,

$$(4.10) \quad U(s)F_1(sI - A_1)^{-1}B_1 = R(s) - U(s)D_1.$$

Since (A_1, B_1) is reachable it follows from Lemma 3.13 that

$$(4.11) \quad \Psi(s) := U(s)F_1(sI - A_1)^{-1}$$

is a polynomial. Now (4.10) and (4.11) imply (4.9) and (4.8). \square

COROLLARY 4.12. *MEMMP has a solution iff the corresponding MMDP has a solution.*

Similarly one proves

PROPOSITION 4.13. *EMMP has a solution iff the corresponding DDP has a solution.*

Thus, if we want to solve (M)EMMP, we may construct the data A, B, C, E of (M)DDP and solve the latter problem. Then we do not only obtain a solution $Q(s)$ of (M)EMMP but also a realization of this solution. In this respect, it is important to note that the solution of (M)EMMP only depends on the numerator polynomials $U(s)$ and $R(s)$. Consequently, by a suitable choice of $T(s)$ (not necessarily equal to the original denominator polynomial) we may try to obtain a simple (M)DDP; compare [3]. We will formulate this idea more explicitly in § 6. Also in § 6, we will give existence conditions for a solution of (M)EMMP and, hence, of (M)DDP in terms of $U(s)$ and $R(s)$.

The following result states that if disturbance decoupling is at all possible by a (dynamic) control depending causally upon $q(t)$, then it is possible by a feedback control of the form $u = Fx + D_1q$.

COROLLARY 4.14. *Let there exist a proper rational matrix $H(s)$ such that, if the control $u = H(s)q$ is used in (4.2), the output does not depend on q . Then MDDP has a solution. If there exists a strictly proper matrix $H(s)$ with this property, then DDP has a solution.*

Proof. If the control $u = H(s)q$ is used in (4.2), then the transfer function matrix from q to y is $G_1(s)H(s) + G(s)$. If y does not depend on q , then this transfer matrix must be zero, hence

$$G_1(s)H(s) = -G_2(s),$$

that is, $-H(s)$ is a solution of MEMMP. Consequently, by Corollary 4.12, MDDP has a solution. \square

5. Observers. We consider several formulations of the observer problem, which is a well-known problem in linear system theory. Further references on the subject can be found in [14], [15], [16], [19], [20], [2] and [6].

Thus far two types of formulation of this problem have appeared in the literature: the geometric formulation (see [19], [20], and [2]) and the polynomial matrix formulation (see [14], [15], and [16]).

Here, our purpose is (based on the results on the connections of the geometric theory of linear systems and polynomial matrix approaches developed in § 3) to show explicitly the algebraic equivalence of the geometric and the polynomial matrix formulations of this problem, including the case where some of the inputs may be unknown.

Let $\Sigma = (C, A, B)$ be a given system over \mathbb{R} . Let \mathbb{C}^- be a subset of \mathbb{C} satisfying $\mathbb{C}^- \cap \mathbb{R} \neq \emptyset$. We call a rational function $u(s)$ *stable* (with respect to \mathbb{C}^-) if $u(s)$ has no poles in $\mathbb{C} \setminus \mathbb{C}^-$. In the continuous time interpretation of Σ , one might choose $\mathbb{C}^- = \{s \in \mathbb{C} | \operatorname{Re} s < 0\}$ and in discrete time $\mathbb{C}^- = \{s \in \mathbb{C} | |s| < 1\}$, but also different choices of \mathbb{C}^- are possible.

We assume that $C \in \mathbb{R}^{q \times n}$, $B \in \mathbb{R}^{n \times r}$ and that in addition to Σ we are given a feedthrough matrix $D \in \mathbb{R}^{q \times r}$. In continuous time, the interpretation (Σ, D) reads:

$$(5.1) \quad \dot{x} = Ax + Bu, \quad y = Cx + Du,$$

and the transfer function of (Σ, D) is

$$G(s) = G_{\Sigma, D}(s) = C(sI - A)^{-1}B + D.$$

DEFINITION 5.2. Let $L \in \mathbb{R}^{p \times n}$. A system $(\bar{\Sigma}, \bar{D}) = (\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is an L -observer of (Σ, D) if for every initial value x_0 of Σ , \bar{x}_0 of $\bar{\Sigma}$ and every control function u , the output \bar{y} of

$$(5.3) \quad \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}y, \quad \bar{y} = \bar{C}\bar{x} + \bar{D}y$$

satisfies: $\bar{y} - Lx$ is stable (in particular rational).

The observer uses only the output of (Σ, D) . If one wants to consider the situation in which partial or total knowledge of the input of Σ is available, one can incorporate this in the problem by a suitable choice of D . In particular, if the input is completely known, one introduces new matrices \check{C}, \check{D} and a new output \check{y} of Σ according to

$$\check{y} = \begin{bmatrix} y \\ u \end{bmatrix}, \quad \check{D} = \begin{bmatrix} D \\ I \end{bmatrix}, \quad \check{C} = \begin{bmatrix} C \\ 0 \end{bmatrix}$$

so that $\check{y} = \check{C}x + \check{D}u$ represents the total data available for the estimation of Lx .

Let us use the following notation:

$$\begin{aligned} G(s) &:= G_{\Sigma, D}(s) = C(sI - A)^{-1}B + D, \\ \bar{G}(s) &:= G_{\bar{\Sigma}, \bar{D}}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D}, \\ G_L(s) &:= L(sI - A)^{-1}B. \end{aligned}$$

Then we have the following result.

THEOREM 5.4. Let the system Σ be reachable and let $\bar{\Sigma}$ be observable. Then the following statements are equivalent:

- (i) $(\bar{\Sigma}, \bar{D})$ is an L -observer of (Σ, D) .
- (ii) $G_L(s) = \bar{G}(s)G(s)$ and $\sigma(\bar{A}) \subseteq \mathbb{C}^-$.
- (iii) There exists a real matrix M such that

$$\begin{aligned} \bar{D}D &= 0, \\ MA &= \bar{A}M + \bar{B}C, \\ L &= \bar{C}M + \bar{D}C, \\ MB &= \bar{B}D \\ \text{and } \sigma(\bar{A}) &\subseteq \mathbb{C}^-. \end{aligned}$$

Proof. (i) \Rightarrow (ii). If $x_0 = 0$, $\bar{x}_0 = 0$, then we have

$$\hat{y} - L\hat{x} = H(s)\hat{u},$$

where $H(s) = \bar{G}(s)G(s) - G_L(s)$, and $\hat{y}, \hat{x}, \hat{u}$ are Laplace transforms. Choosing, in

particular, $u = t^n e^{\lambda t} u_0$ with $\lambda \in \mathbb{C}$, we find

$$\hat{y} - L\hat{x} = \gamma_n(s - \lambda)^{-n-1} H(s) u_0.$$

Since $\hat{y} - L\hat{x}$ has to be stable for all $n \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $u_0 \in \mathbb{R}^r$, it follows that $H(s) = 0$.

Furthermore, if $x_0 = 0$, $u = 0$, then

$$\hat{y} - L\hat{x} = \bar{C}(sI - \bar{A})^{-1} \bar{x}_0.$$

Since (\bar{C}, \bar{A}) is observable, the stability of $\hat{y} - L\hat{x}$ implies that $\sigma(\bar{A}) \subseteq \mathbb{C}^-$.

(ii) \Rightarrow (iii). We use the matrix fraction representation

$$G'_0(s) := B'(sI - A')^{-1} C' = T^{-1}(s) U(s)$$

defined by $B'(sI - A')^{-1} = T^{-1}(s) S(s)$, $U(s) = S(s) C'$, and we consider the T -realization $(\mathcal{C}, \mathcal{A}, \mathcal{B})$ of $G'_0(s)$ (see Theorem 2.8). Then the equation $\bar{G}(s)G(s) = G_L(s)$ may be rewritten as

$$(U(s) + T(s)D')\bar{G}'(s) = S(s)L'.$$

Hence, if we write

$$(5.5) \quad \Psi(s) := (U(s) + T(s)D')\bar{B}'(sI - \bar{A}')^{-1},$$

then

$$(5.6) \quad \Psi(s)\bar{C}' = SL' - (U + TD')\bar{D}'.$$

Since (\bar{C}, \bar{A}') is observable, it follows from Lemma 3.13 that $\Psi(s)$ is a polynomial matrix. Equation (5.5) implies

$$U(s)\bar{B}' + T(s)D'\bar{B}' = \Psi(s)(sI - \bar{A}').$$

Hence, $\{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace, and

$$(5.7) \quad \mathcal{A}\Psi(s) = s\Psi(s) - T(s)D'\bar{B}' = \Psi(s)\bar{A}' + U(s)\bar{B}'$$

(see (3.4)). We consider again the map \mathcal{S} defined in the proof of Theorem 2.8: $\mathcal{S}x = S(s)x$. Then we define $M' = \mathcal{S}^{-1}\Psi(s)$, so that $S(s)M' = \Psi(s)$. It follows from (5.7) that

$$\mathcal{S}A'M' = \mathcal{A}\mathcal{S}M' = \mathcal{A}\Psi(s) = \Psi(s)\bar{A}' + U(s)\bar{B}' = \mathcal{S}M'\bar{A}' + \mathcal{S}C'\bar{B}'.$$

Hence, $A'M' = M'\bar{A}' + C'\bar{B}'$.

Furthermore, (5.6) implies:

$$T^{-1}(s)\Psi(s)\bar{C}' - T^{-1}(s)S(s)L' + T^{-1}U\bar{D}' = -D'\bar{D}.$$

Since the left-hand side is strictly proper, it follows that $D'\bar{D} = 0$ and

$$\mathcal{S}M'\bar{C}' - \mathcal{S}L' + \mathcal{S}C'\bar{D}' = 0,$$

Hence,

$$M'\bar{C}' - L' + C'\bar{D}' = 0.$$

Finally, it follows from (5.7) that

$$T^{-1}(s)U(s)\bar{B}' + T^{-1}(s)\Psi(s)\bar{A}' = sT^{-1}\Psi(s) - D'\bar{B}'.$$

Hence,

$$B'M' = \mathcal{C}\Psi(s) = (T^{-1}(s)\Psi(s))_{-1} = D'\bar{B}'$$

since $T^{-1}(s)U(s)$ and $T^{-1}(s)\Psi(s)$ are strictly proper.

(iii) \Rightarrow (i). A short calculation yields

$$\begin{aligned} \bar{y} - Lx &= \bar{C}(\bar{x} - Mx), \\ \frac{d}{dt}(\bar{x} - Mx) &= \bar{A}(\bar{x} - Mx), \end{aligned}$$

and the result follows from $\sigma(\bar{A}) \subseteq \mathbb{C}^-$. □

The equivalence (ii) \Leftrightarrow (i) is given in [15], and the equivalence (i) \Leftrightarrow (iii), with the a priori assumption (in the proof of (i) \Rightarrow (ii)) that $\bar{x} - Mx$ is stable, is given in [2]. Notice that here the stability of $\bar{x} - Mx$ is a consequence, rather than an assumption (see the proof of (iii) \Rightarrow (i)). For the situation of availability of the whole input, this was also shown in [6].

REMARK 5.8. The results of this section can easily be extended to systems over an arbitrary field K , provided an appropriate definition of stable rational function has been defined. Such a definition can be given as follows: Let \mathcal{M} be a multiplicative subset of $K[s]$ (i.e., $p(s) \in \mathcal{M}, g(s) \in \mathcal{M} \Rightarrow p(s)g(s) \in \mathcal{M}; 1 \in \mathcal{M}$). Then we say that a rational function $r(s) \in K(s)$ is stable if $r(s)$ has the representation $r(s) = p(s)/q(s)$ with $p(s) \in K[s], q(s) \in \mathcal{M}$. Then the stable functions form a ring. In the situation described above we have

$$\mathcal{M} = \{p(s) \in K[s] \mid p(s) = 0 \Rightarrow s \in \mathbb{C}^-\}.$$

In the general situation Theorem 5.4 remains valid if one replaces the condition $\sigma(\bar{A}) \subseteq \mathbb{C}^-$ with “ $(sI - \bar{A})^{-1}$ is stable”.

A particular example, which is relevant for discrete time systems, over arbitrary fields, is

$$\mathcal{M} := \{s^n \mid n = 0, 1, \dots\}.$$

An observer constructed according this multiplicative set is called a deadbeat observer.

6. Reachability subspaces. Let $\Psi(s)$ be a full column rank basis matrix of an $(\mathcal{A}, \mathcal{B})$ -invariant subspace. Recall the interpretation of the matrices A_1, F_1, C_1 given in (3.7), (3.8) and (3.9). Let B_1 be any constant $m \times p$ matrix such that $\{\Psi(s)B_1\} \subseteq \{U(s)\}$, say

$$\Psi(s)B_1 = U(s)L_1.$$

Then B_1 is the matrix of the (codomain) restriction of $\mathcal{B}L_1$ to $\{\Psi(s)\}$. It follows that

$$(\mathcal{A} - \mathcal{B}\mathcal{F})^k \mathcal{B}L_1 v = \Psi(s)A_1^k B_1 v$$

for every $v \in K^p$. Consequently,

$$(6.1) \quad \langle \mathcal{A} - \mathcal{B}\mathcal{F} \mid \mathcal{B}L_1 \rangle = \{\Psi(s)[B_1, \dots, A_1^{m-1} B_1]\}.$$

This formula immediately implies the following result.

THEOREM 6.2. *Let $\Psi(s)$ be a (full column rank) basis matrix of an $(\mathcal{A}, \mathcal{B})$ -invariant subspace. Then*

- (i) $\{\Psi(s)\}$ is a reachability subspace iff there exists a constant matrix B_1 such that $\{\Psi(s)B_1\} \subseteq \{U(s)\}$, and (A_1, B_1) is reachable (here A_1 is given by (3.2)).
- (ii) If B_1 is a constant matrix such that

$$(6.3) \quad \{\Psi(s)B_1\} = \{U(s)\} \cap \{\Psi(s)\},$$

then $\{\Psi(s)[B_1, \dots, A_1^{m-1}B_1]\}$ is the supremal reachability subspace contained in $\{\Psi(s)\}$.

Let us now consider reachability subspaces contained in $\ker \mathcal{C}$. Let $\Psi(s)$ be a basis matrix of such a space. According to Corollary 3.10, there exist matrices F_1 and A_1 such that

$$(6.4) \quad \Psi(s) = U(s)F_1(sI - A_1)^{-1}.$$

It follows from Theorem 6.2 that there exists B_1 such that (A_1, B_1) is reachable and $\{\Psi(s)B_1\} \subseteq \{U(s)\}$, say $\Psi(s)B_1 = U(s)L_1$. Hence

$$(6.5) \quad U(s)Q(s) = U(s)L_1,$$

where $Q(s) := F_1(sI - A_1)^{-1}B_1$. Also, since $\Psi(s)$ has full column rank, (F_1, A_1) is observable, as follows from (6.4). Hence (F_1, A_1, B_1) is a minimal realization of $Q(s)$.

COROLLARY 6.6. *There exists a nontrivial reachability subspace contained in $\ker \mathcal{C}$ iff*

$$\{U(s)\} \cap X_U \neq \{0\}.$$

Proof. If $\Psi(s)$ is a basis matrix of the $(\mathcal{A}, \mathcal{B})$ -invariant subspace X_U and $\Psi(s) = U(s)F_1(sI - A_1)^{-1}$, then the supremal reachability subspace contained in X_U (or, equivalently, in $\ker \mathcal{C}$) is nontrivial iff $B_1 \neq 0$, where B_1 is a matrix satisfying (6.3). \square

According to (6.5), $Q(s) - L_1$ is a nontrivial right zero matrix of $U(s)$. Consequently, if the supremal reachability subspace contained in \mathcal{C} is nonzero, then $U(s)$ is not left invertible. The converse, however, is not true. For example, if $U(s) = [U_1(s), 0]$ where $U_1(s)$ is left invertible, then it is easily seen that $U(s)$ is not left invertible, and $\{U(s)\} \cap X_U = \{0\}$. In order to give a necessary and sufficient condition for the existence of a maximal reachability subspace contained in $\ker \mathcal{C}$, we consider the $K[s]$ -module

$$(6.7) \quad \Delta := \{v(s) \in K^r[s] \mid U(s)v(s) = 0\}.$$

This module is generated by the columns of a matrix $M(s)$ (see [5, Thm. 3.1]).

COROLLARY 6.8. *There exists a nontrivial reachability subspace contained in $\ker \mathcal{C}$ iff the module Δ defined in (6.7) is not generated by a constant matrix.*

Proof. Let $M(s)$ be a generator matrix of minimal degree, say $M(s) = M_0s^k + \dots + M_k$. Then $s^{-k}M(s) = Q(s) - L_1$, where $Q(s) = M_1s^{-1} + \dots + M_k s^{-k}$ and $L_1 = -M_0$. We have

$$U(s)Q(s) = U(s)L_1$$

and $U(s)L_1 \neq 0$, since, otherwise, $[M(s) - s^k M_0, M_0]$ would be a generator matrix of lower degree than k . It follows that $\{U(s)L_1\} \subseteq \{U(s)\} \cap X_U$, so that $\{U(s)\} \cap X_U \neq \{0\}$.

Conversely, suppose that Δ is generated by a constant matrix, say D , and that $v \in \{U(s)\} \cap X_U$, say $v = U(s)c = U(s)r(s)$, where c is a constant vector and $r(s)$ is a strictly proper rational vector. It follows that there exists a rational vector $q(s)$ such that $c - r(s) = Dq(s)$. Decomposing $q(s)$ into a polynomial and a strictly proper part $q(s) = q_1(s) + q_2(s)$, we conclude that $c = Dq_1(s)$, so that $v = U(s)c = 0$. Hence, $\{U(s)\} \cap X_U = \{0\}$. \square

Now we have a procedure for constructing reachability subspaces contained in $\ker \mathcal{C}$. Choosing any matrix L_1 such that $\{U(s)L_1\} \subseteq X_U$, we have $U(s)Q(s) = U(s)L_1$ for some strictly proper $Q(s)$. If (F_1, A_1, B_1) is a minimal realization of $Q(s)$, it follows that $\Psi(s) := U(s)F_1(sI - A_1)^{-1}$ is a basis matrix of a reachability subspace, provided the columns of $\Psi(s)$ are independent. In general, it seems difficult to formulate conditions

upon L_1 and $Q(s)$ that guarantee that $\Psi(s)$ has full column rank. A sufficient condition for this is that $Q(s)$ be a strictly proper rational matrix with minimal McMillan degree satisfying the equation $U(s)Q(s) = U(s)L_1$. Indeed, if in this case $\Psi(s)$ does not have full column rank, there exists $\Phi(s)$ with less columns than Ψ such that $\{\Phi(s)\} = \{\Psi(s)\}$. Since $\{\Phi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace, there exist F_2, A_2 such that $\Phi(s) = U(s)F_2(sI - A_2)^{-1}$. Also, there exists D_1 such that $\Psi(s) = \Phi(s)D_1$. Hence,

$$U(s)Q(s) = \Psi(s)B_1 = \Phi(s)D_1B_1 = U(s)Q_2(s) = U(s)L_1,$$

where $Q_2(s) := F_2(sI - A_2)^{-1}D_1B_1$ has lower McMillan degree than $Q(s)$.

THEOREM 6.9. *Let L_1 be a constant matrix such that $\{U(s)L_1\} = \{U(s)\} \cap X_U$. Let $Q(s)$ be a strictly proper rational matrix of minimal McMillan degree, satisfying the equation $U(s)Q(s) = U(s)L_1$. Let (F_1, A_1, B_1) be a minimal realization of $Q(s)$. Then $\Psi(s) := U(s)F_1(sI - A_1)^{-1}$ is a basis matrix of the supremal reachability space contained in $\ker \mathcal{C}$.*

Proof. The supremal reachability subspace contained in $\ker \mathcal{C}$ is the (unique) minimal $(\mathcal{A}, \mathcal{B})$ -invariant subspace \mathcal{V} satisfying $(\text{Im } \mathcal{B}) \cap \mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{W}$, where \mathcal{W} is the supremal $(\mathcal{A}, \mathcal{B})$ -invariant subspace contained in $\ker \mathcal{C}$. To see this, observe that an $(\mathcal{A}, \mathcal{B})$ -invariant subspace \mathcal{V} satisfying $(\text{Im } \mathcal{B}) \cap \mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{W}$ is $(\mathcal{A} - \mathcal{B}\mathcal{F})$ -invariant for every \mathcal{F} such that \mathcal{W} is $(\mathcal{A} - \mathcal{B}\mathcal{F})$ -invariant. Indeed, $(\mathcal{A} - \mathcal{B}\mathcal{F})\mathcal{V} \subseteq (\mathcal{A} - \mathcal{B}\mathcal{F})\mathcal{W} \subseteq \mathcal{W}$ and $(\mathcal{A} - \mathcal{B}\mathcal{F})\mathcal{V} \subseteq \mathcal{V} + \text{Im } \mathcal{B}$ imply

$$(\mathcal{A} - \mathcal{B}\mathcal{F})\mathcal{V} \subseteq \mathcal{W} \cap (\mathcal{V} + \text{Im } \mathcal{B}) = \mathcal{V} + \mathcal{W} \cap \text{Im } \mathcal{B} \subseteq \mathcal{V}.$$

Since $\{U(s)\} \cap X_U = \{U(s)L_1\} = \{\Psi(s)B_1\} \subseteq \{\Psi(s)\} \subseteq X_U$, and because of the minimal McMillan degree of $Q(s)$, the result follows. \square

In the next section, it will be shown how Theorem 6.9 can be used for the explicit construction of the supremal reachability subspace.

7. Constructive characterizations. Conditions for solvability and the characterization of solutions of various problems can be made explicit by the use of row and column proper matrices (see [16]). This will be the subject of this section.

If $R \in K^{p \times q}[s]$ has rows $r_1(s), \dots, r_p(s)$, then $\deg r_i(s)$ is called the *i th row degree* of $R(s)$. The coefficient vector of s^{ν_i} in $r_i(s)$, where $\nu_i = \deg r_i(s)$ is called the *i th leading coefficient row vector*, and is denoted by $[r_i]_r$. We denote by $[R]_r$ the matrix of leading coefficient row vectors, that is, the constant matrix with rows $[r_1]_r, \dots, [r_p]_r$. Similarly, $[R]_c$ denotes the matrix of leading coefficient column vectors, that is, $[R]_c = ([R']_r)'$. A matrix is called *row (column) proper* if $[R]_r([R]_c)$ is nonsingular. A row proper matrix is easily seen to be right invertible. Conversely, we have (see [16, Thm, 2.5.7])

LEMMA 7.1. *If $L(s) \in K^{p \times q}[s]$ is right invertible there exists a unimodular matrix $M(s) \in K^{p \times p}[s]$ such that $M(s)L(s)$ is row proper with row degrees ν_1, \dots, ν_p satisfying $\nu_1 \leq \dots \leq \nu_p$. If $L(s) \in K^{p \times q}[s]$ is not right invertible, there exists a unimodular matrix $M(s)$ such that*

$$M(s)L(s) = \begin{bmatrix} L_1(s) \\ 0 \end{bmatrix},$$

where $L_1(s)$ is row proper with row degrees $\nu_1 \leq \dots \leq \nu_l$. The number l of rows of $L_1(s)$ equals the rank of $L(s)$.

The row degrees ν_1 are independent of $M(s)$ (which is not unique) and will be called the *row indices* of $L(s)$.

The following result (see [14, Property 2.2]) states a simple criterion for the properness of a rational matrix $T^{-1}(s)U(s)$ if the denominator polynomial matrix is row proper.

LEMMA 7.2. *Let $T(s)$ be row proper with row degrees ν_1, \dots, ν_q . If the row degrees of $U(s)$ are $\lambda_1, \dots, \lambda_q$ then $T^{-1}(s)U(s)$ is proper iff $\lambda_1 \leq \nu_i$ ($i = 1, \dots, q$) and strictly proper iff $\lambda_i < \nu_i$ ($i = i, \dots, q$).*

Observe that if T is not row proper, there exists a unimodular matrix $M(s)$ such that $T_1(s) := M(s)T(s)$ is row proper. If we define $U_1(s) := M(s)U(s)$, we have $T^{-1}(s)U(s) = T_1^{-1}(s)U_1(s)$, and we may apply Lemma 7.2.

Let us now consider (M)EMMP as defined in Problem 4.6. Assume that we have a matrix fraction representation $T^{-1}(s)[U(s), R(s)]$ of $[G_1(s), G_2(s)]$. Then the equation for $Q(s)$ reads

$$(7.3) \quad U(s)Q(s) = R(s).$$

In order that this equation has a (not necessarily proper) rational solution, it is necessary and sufficient that $\text{rank } U(s) = \text{rank } [U(s), R(s)]$. For the existence of a proper solution additional conditions have to be imposed. Writing down the i th row of (7.3)

$$u_i(s)Q(s) = r_i(s),$$

we note that a necessary condition for the existence of a proper solution is $\deg u_i(s) \geq \deg r_i(s)$. The following result shows that this is also sufficient provided that $U(s)$ has the form

$$\begin{bmatrix} U_1(s) \\ 0 \end{bmatrix},$$

with $U_1(s)$ row proper. According to Lemma 7.1, this can always be obtained by premultiplying (7.3) with a suitable unimodular matrix $M(s)$.

THEOREM 7.4. *Let $M(s)$ be a unimodular matrix such that*

$$M(s)U(s) = \begin{bmatrix} U_1(s) \\ 0 \end{bmatrix}, \quad M(s)R(s) = \begin{bmatrix} R_1(s) \\ R_2(s) \end{bmatrix},$$

where $U_1(s)$ is row proper. Let the row degrees of $U_1(s)$ be ν_1, \dots, ν_l and let the row degrees of $R_1(s)$ be $\lambda_1, \dots, \lambda_l$. Then (7.3) has a proper solution iff $R_2(s) = 0$ and $\lambda_i \leq \nu_i$ ($i = 1, \dots, l$). Equation (7.3) has a strictly proper solution iff $R_2(s) = 0$ and $\lambda_i < \nu_i$ ($i = 1, \dots, l$).

Proof. The conditions are necessary according to the foregoing discussions. Now assume that the conditions hold. Then there exists $L \in K^{r \times l}$ such that $U_1(s)L$ is a row proper $l \times l$ matrix with row degrees ν_1, \dots, ν_l . Define

$$Q(s) := L(U_1(s)L)^{-1}R_1(s).$$

Then $Q(s)$ satisfies (7.3). It follows from (7.2) that $Q(s)$ is proper. The proof for the strictly proper solution is similar. \square

We can express the result of Theorem 7.4 in a way not involving explicitly the matrix $M(s)$:

COROLLARY 7.5. *Equation (7.3) has a proper solution iff $U(s)$ and $[U(s), R(s)]$ have the same rank and the same row indices.*

In [14], no explicit condition for the solvability is given. In [5], a condition is given in terms of the kernel of the matrix $[U(s), R(s)]$. The conditions given in Theorem 7.4 and Corollary 7.5 are directly expressed in terms of the matrices $U(s)$ and $R(s)$.

The set X_U is the largest $(\mathcal{A}, \mathcal{B})$ -invariant subspace contained in $\ker \mathcal{C}$. By definition $x(s) \in X_U$ iff the equation

$$U(s)v(s) = x(s)$$

has a strictly proper solution $v(s)$. Therefore, using Theorem 7.4, we can give a constructive characterization of X_U .

COROLLARY 7.6. *Let $M(s)$ be as in Theorem 7.4. Then $x(s) \in X_U$ iff $y(s) := M(s)x(s)$ satisfies the conditions*

$$\begin{aligned} \deg y_i(s) &< v_i & (i = 1, \dots, l), \\ y_i(s) &= 0 & (i = l + 1, \dots, q). \end{aligned}$$

Here $y_i(s)$ denotes the i th component of $y(s)$. In particular, if we introduce the row vector $w_k(s) := [s^{k-1}, \dots, 1]$, then $M^{-1}(s)W(s)$ is a basis matrix of X_U , where

$$W(s) := \begin{bmatrix} W_1(s) \\ 0 \end{bmatrix}$$

with $W_1(s) := \text{diag}(w_{v_1-1}(s), \dots, w_{v_l-1}(s))$.

One way of solving (7.3), already mentioned in § 4, is the reformulation of (7.3) as a (M)DDP. In doing so, it is not necessary to use the original denominator matrix $T(s)$. One might try to find a new denominator matrix $T_1(s)$ such that $T_1^{-1}(s)U(s)$ is strictly proper and $T_1(s)$ is as simple as possible. If we choose $T_1(s)$ row proper, then according to Lemma 7.2, it suffices for the strict properness of $T_1^{-1}U$, that the row degrees of T_1 are larger than the row degrees of U . If we denote the latter by $\lambda_1, \dots, \lambda_1$, the simplest choice of $T_1(s)$ is $T_1(s) = \text{diag}(s^{\lambda_1+1}, \dots, s^{\lambda_1+1})$.

For this computation, it is not necessary that $U(s)$ be in row proper form. But if we transform $U(s)$ such that it has the form given in Theorem 7.4, then the dimension of the state space will be minimal. These ideas are worked out in more detail in [3].

We conclude this section with a construction of the supremal reachability subspace contained in $\ker \mathcal{C}$. To this end, we consider the space

$$\Lambda := \{v(s) \in K^r(s) \mid U(s)v(s) = 0\},$$

and we choose a minimal basis for Λ (see [5]), that is, a basis for Δ (see (6.7)) which is column proper. We define $L_1 := [M]_c$. Furthermore we choose any $D(s) \in K^{l \times l}[s]$ which has the same column degrees $M(s)$ and such that $[D]_c = I$. Then we observe (by Lemma 7.2) that, if

$$N(s) := L_1 D(s) - M(s),$$

then $Q(s) := N(s)D^{-1}(s)$ is strictly proper. Now we have

- THEOREM 7.7.** (i) $\{U(s)L_1\} = X_U \cap \{U(s)\}$,
 (ii) $Q(s)$ is a strictly proper rational matrix of minimal McMillan degree satisfying

$$(7.8) \quad U(s)Q(s) = U(s)L_1.$$

Hence, if (F_1, A_1, B_1) is a minimal realization of $Q(s)$, then $\Psi(s) := U(s)F_1(sI - A_1)^{-1}$ is a basis of the supremal reachability subspace contained in $\ker \mathcal{C}$.

Proof. (i) Since $U(s)M(s) = 0$, it is easily seen that (7.8) is satisfied. This implies that $\{U(s)L_1\} \subseteq X_U \cap \{U(s)\}$. Suppose that there exists a matrix \bar{L}_1 of full column rank such that $\{U(s)L_1\} \subseteq \{U(s)\bar{L}_1\}$, and $U(s)\bar{L}_1 = U(s)\bar{Q}(s)$ for some strictly proper $\bar{Q}(s)$. Let \bar{N}, \bar{D} be right coprime polynomial matrices such that $\bar{Q}(s) = \bar{N}(s)\bar{D}^{-1}(s)$, and $\bar{D}(s)$ is column proper with $[\bar{D}]_c = I$. Then

$$U(s)(\bar{N}(s) - \bar{L}_1\bar{D}(s)) = 0.$$

Since $\bar{Q}(s)$ is strictly proper, the columns of $\bar{N}(s) - \bar{L}_1\bar{D}(s)$ are linearly independent

over $K(s)$. But then \bar{L}_1 cannot have more columns than L_1 . Consequently, $\{U(s)L_1\} = \{U(s)\bar{L}_1\}$.

(ii) Suppose that $\bar{Q}(s) = \bar{N}(s)\bar{D}^{-1}(s)$ has a lower McMillan degree than $Q(s)$ and that $N(s)$ and $D(s)$ are relatively prime and that $\bar{D}(s)$ is column proper with $[\bar{D}(s)]_c = I$. Then we have

$$U(s)(\bar{N}(s) - L_1\bar{D}(s)) = 0,$$

and hence, $\bar{N}(s) - L_1\bar{D}(s) = M(s)R(s)$. By the ‘‘predictable degree property’’ (see [5, § 3, Remark]), this implies that the sum of the column degrees of $\bar{D}(s)$, and hence $\deg \det \bar{D}(s)$ is not less than $\deg \det D(s)$, which contradicts our assumption. \square

8. Generalization to systems represented by Rosenbrock’s system matrix. In this section, we indicate how the result of § 3 can be generalized to the case where the system is represented by a system matrix

$$(8.1) \quad P(s) = \begin{bmatrix} T(s) & U(s) \\ -V(s) & W(s) \end{bmatrix},$$

where $T(s) \in K^{q \times q}[s]$ is nonsingular and $P(s) \in K^{(q+l) \times (q+r)}[s]$. We assume that the transfer function matrix

$$G(s) := V(s)T^{-1}(s)U(s) + W(s)$$

and the matrix $T^{-1}(s)U(s)$ are strictly proper. If the latter condition is not satisfied, we can obtain this by strict system equivalence (see [13, § 3.1]). Indeed, if we define

$$U_1(s) := \pi_T(U(s)),$$

then

$$Q(s) := T^{-1}(s)(U(s) - U_1(s))$$

is a polynomial matrix. Therefore,

$$P_1(s) := \begin{bmatrix} T(s) & U_1(s) \\ -V(s) & W(s) + V(s)Q(s) \end{bmatrix}$$

is a polynomial system matrix with the same transfer matrix $G(s)$.

In [9], it is shown that the maps

$$\mathcal{A} : X_T \rightarrow X_T : x(s) \mapsto \pi_T(sx(s)),$$

$$\mathcal{B} : K^r \rightarrow X_T : u \mapsto U(s)u,$$

$$\mathcal{C} : X_T \rightarrow K^l : x(s) \mapsto (V(s)T^{-1}(s)x(s))_{-1}$$

yield a realization $(\mathcal{C}, \mathcal{A}, \mathcal{B})$ of $G(s)$ which is reachable iff $T(s)$ and $U(s)$ are left coprime, and observable iff $T(s)$ and $V(s)$ are right coprime.

It is easily seen that Theorem 3.1 is equally valid in this situation. Instead of Corollary 3.10 we get

THEOREM 8.2. *Let $\Psi(s)$ be a $q \times m$ polynomial matrix. Then $\{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace in $\ker \mathcal{C}$ iff there exists $C_1 \in K^{q \times m}$, $F_1 \in K^{r \times m}$, $A_1 \in K^{m \times m}$ and an $l \times m$ polynomial matrix $\Phi(s)$ such that*

$$(8.3) \quad P(s) \begin{bmatrix} C_1 \\ F_1 \end{bmatrix} = \begin{bmatrix} \Psi(s) \\ \Phi(s) \end{bmatrix} (sI - A_1).$$

Proof. By Theorem 3.1, $\{\Psi(s)\}$ is an $(\mathcal{A}, \mathcal{B})$ -invariant subspace of X_T iff for some

C_1, F_1, A_1 we have (3.2) and hence (3.6). But then

$$\begin{aligned}\mathcal{C}\Psi(s) &= (V(s)T^{-1}(s)\Psi(s))_{-1} \\ &= ((V(s)C_1 + (G(s) - W(s)F_1)(sI - A_1)^{-1}))_{-1} \\ &= ((V(s)C_1 - W(s)F_1)(sI - A_1)^{-1})_{-1}\end{aligned}$$

since $G(s)$ and $(sI - A_1)^{-1}$ are both strictly proper. Now it follows from Lemma (8.5) that

$$(8.4) \quad \Phi(s) := (-V(s)C_1 + W(s)F_1)(sI - A_1)^{-1}$$

is a polynomial iff $\mathcal{C}\Psi(s) = 0$. Combining (3.2) and (8.4) yields the desired result. \square

LEMMA 8.5. Let $Q(s) \in K^{l \times n}[s]$, $A \in K^{n \times n}$. If

$$(Q(s)(sI - A^{-1}))_{-1} = 0,$$

then $Q(s)(sI - A)^{-1}$ is a polynomial matrix.

The proof is analogous to the proof of Lemma 3.13 and will be omitted.

The generalization of Corollary 3.12 can be expressed in terms of the map

$$\mathcal{P} : K^{q+l}[s] \rightarrow K^q[s] : \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} \mapsto x(s).$$

COROLLARY 8.6. The largest $(\mathcal{A}, \mathcal{B})$ -invariant subspace of X_T contained in $\ker \mathcal{C}$ is $\mathcal{P}(X_P)$.

The proof is similar to the proof of Corollary 3.12 and will be omitted.

Acknowledgment. One of the authors (E. Emre) would like to thank the Department of Mathematics of the Eindhoven University of Technology for financial support and hospitality while this research was being done.

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