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A POLYNOMIAL-TIME ALGORITHM FOR LINEAR OPTIMIZATION BASED ON A NEW KERNEL FUNCTION WITH TRIGONOMETRIC BARRIER TERM

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Abstract: In this paper, we propose a large-update interior-point algorithm for linear optimization based on a new kernel function. New search directions and proximity measure are defined based on this kernel function. We show that if a strictly feasible starting point is available, then the new algorithm has $O(n^{\frac{3}{4}}\log\frac{n}{\varepsilon})$ iteration complexity.

Keywords: Kernel function, Interior-point algorithm, Linear optimization, Polynomial complexity, Primal-dual method.

MSC: 90C05, 90C51.

1. INTRODUCTION

In this paper, we consider linear optimization (LO) problem in the standard form:

$$\begin{aligned} \min\ & c^T x \\ s.t.\ & Ax = b, \qquad\qquad (P) \\ & x \geq 0, \end{aligned}$$

where $A \in R^{m\times n}$ is a real $m \times n$ matrix of rank m, and $x, c \in R^n, b \in R^m$. The dual problem of (P) is given by

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$$\begin{aligned} &\max\ b^T y \\ &s.t.\ \ A^T y + s = c, \qquad\qquad (D) \\ &\qquad\qquad s \geq 0, \end{aligned}$$

with $y \in R^m$ and $s \in R^n$. In 1984, Karmarkar [12] proposed a polynomial-time algorithm, the so-called interior-point method (IPM) for linear optimization (LO). This method and its variants are frequently extended for solving wide classes of optimization problems, for example, quadratic optimization problem (QOP), semidefinite optimization (SDO) problem, second-order cone optimization (SOCO) problem, $P_*(\kappa)$ linear complementarity problems (LCPs), and convex optimization problem (CP). IPM is the most efficient method from computational point of view. Also, its promising performance in solving large-scale linear programming problems caused it to be used in practical issues. Usually, if parameter θ in this method is a constant, which is independent of the dimension of the problem, then the algorithm is called a large-update method. If it depends on the dimension, then the algorithm is said to be a small-update method. At present, the best known theoretical iteration bound for small-update IPM is better than the one for large-update IPM, but in practice large-update IPM is much more efficient than the small-update IPM [2, 3, 4, 5, 6]

Most of IPM algorithms for LO are based on the logarithmic barrier function [1, 10, 11]. In 2002, Peng et al. proposed new variants of IPM based on a specific self-regular barrier function. Such a function is strongly convex and smooth coercive on its domain, the positive real axis. They obtained the best known complexity results for large- and small-update methods, which have $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ complexity for large-update and $O(\sqrt{n}\log\frac{n}{\varepsilon})$ complexity for small-update methods [18, 19], and extended the results for LO, second order cone optimization (SOCO), semi-definite optimization (SDO) and NCPs. Later, Bai et al. [2] introduced eligible kernel functions and gave comprehensive complexity analysis. Cho [7] presented other barrier function which has $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$ complexity for large-update method and $O(\sqrt{n}\log\frac{n}{\varepsilon})$ complexity for small-update method. This kernel function does not belong to the family of self-regular functions. Kim et al. [16] defined new kernel functions that are both self-regular and eligible, and showed their properties. They also identified the relation between the classes of eligible and self-regular kernel functions. Kheirfam and Hasani [14] presented a large-update primal-dual interior-point algorithm for convex quadratic semi-definite optimization problems based on a new parametric kernel function. They investigate such a kernel function, and show that their algorithm has the best complexity bound, i.e., $O(\sqrt{n}\log n\log\frac{n}{\varepsilon})$.

In 2012, El Ghami et al. [9] proposed a new primal-dual IPM for LO problems based on a kernel function, which has a trigonometric barrier term, and Kheirfam defined another trigonometric barrier function and presented a new algorithm for semidefinite optimization [13]. They obtained $O(n^{\frac{3}{4}}\log\frac{n}{\varepsilon})$ iteration bound for large-update and $O(\sqrt{n}\log\frac{n}{\varepsilon})$ for small-update methods, respectively. El Ghami generalized the analysis presented in the above paper for $P_*(\kappa)$-LCPs [8]. Recently, Kheirfam [15] proposed a new kernel function with trigonometric barrier term which yields the complexity bound $O(\sqrt{\mathrm{n}}\log \mathrm{n}\log\frac{n}{\varepsilon})$ for large-update methods and is currently the best known bound for such methods. Some examples of kernel function, which have been analyzed in earlier papers can be seen in [8, 14, 16, 17]

In this paper, we define a new kernel function, which has a trigonometric barrier term, and propose a primal-dual interior-point algorithm for LO based on this function. We analyze the complexity for large-update method based on three conditions of kernel function. This algorithm has $O(n^{\frac{3}{4}}\log\frac{n}{\varepsilon})$ complexity bound for large-update method similar to complexity obtained in [9, 13].

The paper is organized as follows. In Section 2, we recall the generic path-following IPM. In Section 3, we define a new kernel function and give its properties, which are essential for the complexity analysis. In Section 4, we derive the complexity result for large-update method and obtain an upper bound to decrease the barrier function during an inner iteration. In the final section, we conclude with some remarks.

2. THE PRIMAL-DUAL ALGORITHM

In this section, we recall some basic concepts and the generic path-following IPM. Without loss of generality, we assume that a strictly feasible pair (x^0, s^0) exists, i.e., there exists (x^0, y^0, s^0) such that

$$Ax^0 = b, \qquad A^T y^0 + s^0 = c, \;\; x^0 > 0, \;\; s^0 > 0.$$

This assumption is called the interior-point condition (IPC) [20]. The IPC ensures the existence of an optimal primal-dual pair (x^*, s^*) with zero duality gap, i.e., $c^T x^* - b^T y^* = (x^*)^T s^* = 0$.

It is well known that finding an optimal solution of (P) and (D) is equivalent to solving the following system

$$\begin{aligned} Ax &= b, \;\; x \geq 0, \\ A^T y + s &= c, \;\; s \geq 0, \\ xs &= 0. \end{aligned} \tag{1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (1), the so-called complementarity condition for (P) and (D), by the parameterized equation $xs = \mu e$ with $\mu > 0$; where e denotes the all-one vector $(1,1,\ldots,1)^T$. Thus, we have the following parameterized system:

$$\begin{aligned} Ax &= b, \;\; x \geq 0, \\ A^T y + s &= c, \;\; s \geq 0, \\ xs &= \mu e. \end{aligned} \tag{2}$$

For each $\mu > 0$, the parameterized system (2) has a unique solution $(x(\mu), y(\mu), s(\mu))$ (see [20]), which is called the μ-center of (P) and (D). The set of μ-centers (with μ running through all positive real numbers) gives a parameterized curve, which is called the central path of (P) and (D). If $\mu \to 0$, then the limit of the central path exists and, since the limit point satisfies the complementarity condition, the limit yields optimal solutions for (P) and (D) [20].

A natural way to define a search direction is to follow the Newton approach and to linearize the third equation in (2) by replacing x, y and s with $x^+ = x + \Delta x, y^+ = y + \Delta y$ and $s^+ = s + \Delta s$ respectively. This leads to the following system:

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ x\Delta s + s\Delta x &= \mu e - xs. \end{aligned} \tag{3}$$

Since A has full row rank, the system (3) uniquely defines a search direction $(\Delta x, \Delta y, \Delta s)$ for any $x > 0$ and $s > 0$ [20]. We define the vector

$$v := \sqrt{\frac{xs}{\mu}}, \tag{4}$$

and its ith component as $\sqrt{\frac{x_i s_i}{\mu}}$. Introduce the scaled search directions as follows:

$$d_x := \frac{v\Delta x}{x}, \quad d_s := \frac{v\Delta s}{s}. \tag{5}$$

Using (5), we can rewrite the system (3) as follows:

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{6}$$

where

$$\bar{A} := \frac{1}{\mu}AV^{-1}X, \;\; V := \mathrm{d}iag(v), \;\; X := \mathrm{d}iag(x).$$

A crucial observation is that the right-hand side of the third equation in (6) is the negative gradient of the classical logarithmic barrier function $\Psi_c(v)$, that is,

$$d_x + d_s = -\nabla\Psi_c(v), \tag{7}$$

where

$$\Psi_c(v) := \sum_{i=1}^{n} \psi_c(v_i), \psi_c(t) = \frac{t^2 - 1}{2} - \log(t).$$

One may easily verify that $\psi_c(t)$ satisfies

$$\begin{aligned} &\psi_c'(1) = \psi_c(1) = 0, \\ &\psi_c''(t) > 0, t > 0, \\ &\lim_{t\to 0^+}\psi_c(t) = \lim_{t\to +\infty}\psi_c(t) = +\infty. \end{aligned} \tag{8}$$

This shows that $\Psi_c(v)$ is strictly convex, and attains its minimal value at $v = e$ with $\Psi_c(e) = 0$. Thus,

$$\Psi_c(e) = 0 \Leftrightarrow \nabla\Psi_c(e) = 0 \Leftrightarrow v = e \Leftrightarrow xs = \mu e.$$

In this paper, we replace the right-hand side of the third equation in (6) by $-\nabla\Psi(v)$, where Ψ is a barrier function induced by a new kernel function $\psi(t)$ as defined in (12). Thus, system (6) can be reformulated as follows:

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T\Delta y + d_s &= 0, \\ d_x + d_s &= -\nabla\Psi(v). \end{aligned} \tag{9}$$

The new search direction $(d_x, \Delta y, d_s)$ is obtained by solving (9) so that $(\Delta x, \Delta y, \Delta s)$ is computed via (5). By taking a step along the search direction determined by (9), with a step size α defined by some line search rules, a new triple (x_+, y_+, s_+) is constructed according to

$$x_+ = x + \alpha\Delta x,\ y_+ = y + \alpha\Delta y, s_+ = s + \alpha\Delta s. \tag{10}$$

Since d_x and d_s are orthogonal, we have

$$\Psi(v) = 0 \Leftrightarrow v = e \Leftrightarrow \nabla\Psi(v) = 0 \Leftrightarrow d_x = d_s = 0 \Leftrightarrow xs = \mu e.$$

We use $\Psi(v)$ as the proximity function to measure the distance between the current iterate and the μ-center for given $\mu > 0$. We also define the norm-based proximity measure, $\delta(v)$, as follows:

$$\delta(v) := \frac{1}{2} \parallel \nabla\Psi(v) \parallel = \frac{1}{2}\sqrt{\sum_{i=1}^{n} (\psi'(v_i))^2},\ \ v \in R^n_{++}. \tag{11}$$

We assume that (P) and (D) are strictly feasible, and the starting point (x^0, y^0, s^0) is strictly feasible for (P) and (D). Choose τ and $v^0 = \sqrt{\frac{x^0 s^0}{\mu^0}}$ initial strictly feasible point such that $\Psi(v^0) \leq \tau$, where τ is threshold value. We then decrease μ to $\mu := (1-\theta)\mu$, for some $\theta \in (0,1)$. In general, this will increase the value of $\Psi(v)$ above τ. To get this value smaller again, and coming closer to the current μ-center, we solve the scaled search directions from (9), and unscaled these directions by using (5). By choosing an appropriate step size α, we move along the search direction, and construct a new pair (x_+, y_+, s_+) given by (10). If necessary, we repeat the procedure until we find iterates such that $\Psi(v)$ no longer exceeds the threshold value τ, which means that the iterates are in a small enough neighborhood of $(x(\mu), y(\mu), s(\mu))$. Then μ is again reduced by the factor $1-\theta$ and we apply the same procedure targeting at the new μ-centers. This process is repeated until μ is small enough, say $n\mu \leq \varepsilon$ for a certain accuracy parameter ε, at this stage we have found an ε-approximate solution of (P) and (D). The generic IPM outlined above is summarized in Algorithm 1.

Algorithm1 : Primal-Dual Algorithm for LO

Input:

Accuracy parameter $\epsilon > 0$;

barrier update parameter $\theta, 0 < \theta < 1$;

threshold parameter $\tau \geq 1$;

$x^0 > 0, s^0 > 0$, and $\mu^0 = 1$ such that $\Psi(x^0, s^0, \mu^0) \leq \tau$.

begin

$x := x^0; s := s^0; \mu = \mu^0;$

while $n\mu \geq \epsilon$ **do**

begin

μ-update:

$\mu := (1-\theta)\mu;$

while $\Psi(x, y, s) > \tau$ **do**

begin

Solve the system (9) and use (5) for $\Delta x, \Delta y, \Delta s$;

Determine a step size α;

$x := x + \alpha\Delta x;$

$y := y + \alpha\Delta y;$

$s := s + \alpha\Delta s;$

end

end

end

A crucial question is how to choose the parameters τ, θ , and the step size α that minimizes the iteration complexity of the algorithm.

3 THE NEW KERNEL FUNCTION

In this section, we define a new kernel function and give its properties needed for the complexity analysis. Now, we define a new kernel function $\psi(t)$ as follows:

$$\psi(t) = t^2 - 2t + \frac{1}{\sin(u(t))}, \qquad t > 0,$$

where $u(t) = \frac{\pi t}{1+t}$. Then, we have the first three derivatives of $\psi(t)$ as follows:

$$\psi'(t) = 2t - 2 - \frac{u'(t)\cos(u(t))}{\sin^2(u(t))}, \tag{13}$$

$$\psi''(t) = 2 + \frac{u'(t)^2\sin^2(u(t)) - u''(t)\sin(u(t))\cos(u(t)) + 2u'(t)^2\cos^2(u(t))}{\sin^3(u(t))}, \tag{14}$$

$$\psi'''(t) =$$
$$\frac{1}{\sin^4(u(t))}(3u'(t)u''(t)\sin^3(u(t)) - 5u'(t)^3\sin^2(u(t))\cos(u(t))$$
$$-u'''(t)u(t)\sin^2(u(t))\cos(u(t)) + 6u'(t)u''(t)\sin(u(t))\cos^2(u(t))$$
$$-6u'\left(t\right)^3\cos^3\left(u(t)\right)\Big). \tag{15}$$

Lemma 1 *For the function* $u(t)$ *defined in (12), one has*

1. $-\pi\cos(u(t)) < (1+t)\sin(u(t)) < \pi, t > 0.$
2. $\pi t\cos(u(t)) < (1+t)\sin(u(t)) < \pi t, t > 0.$
3. $(1+t)\sin(u(t)) < 2\pi t\cos(u(t)), 0 < t \leq \frac{1}{2}.$

Proof. *For* $x > 0$, *we define*

$$f(x) = \frac{\pi x}{1+x} - \sin\left(\frac{\pi}{1+x}\right), \quad g(x) = \frac{\pi x}{1+x} + \tan\left(\frac{\pi x}{1+x}\right).$$

We have

$$f'(x) = \frac{\pi x}{(1+x)^2}\left(1+\cos\left(\frac{\pi}{1+x}\right)\right) > 0, \quad g'(x) = -\frac{\pi}{(1+x)^2}\tan^2\left(\frac{\pi}{1+x}\right) < 0.$$

Thus, $f(x)$ is strictly increasing and $g(x)$ is strictly decreasing for $x > 0$. Therefore $f(x) > 0$ and $g(x) < 0$, which implys that $\sin(\frac{\pi}{1+x}) < \frac{\pi x}{1+x}$ and $\frac{\pi x}{1+x} < tan(\frac{\pi}{1+x})$ respectively. Letting $x = \frac{1}{t}$ follows the first part. To prove the second part, we define

$$f(t) = \frac{\pi t}{1+t} - \sin\left(\frac{\pi t}{1+t}\right), \qquad g(t) = \frac{\pi t}{1+t} - \tan\left(\frac{\pi t}{1+t}\right).$$

It can be easily seen that $f(t)$ is strictly increasing and $g(t)$ is strictly decreasing and $f(0) = g(0) = 0$. Therefore $f(t) > 0$ and $g(t) < 0$, which implys the desired inequalities. Now, for $0 < t \leq \frac{1}{2}$. , we define

$$f(t) = 2\pi t\cos\left(\frac{\pi t}{1+t}\right) - (1+t)\sin\left(\frac{\pi t}{1+t}\right).$$

We have

$$f''(t) = \frac{-2\pi^2}{(1+t)^2}\sin\left(\frac{\pi t}{1+t}\right) - \frac{2\pi^3 t}{(1+t)^4}\cos(\frac{\pi t}{1+t}) - \frac{\pi^2(1-2t)}{(1+t)^3}\sin(\frac{\pi t}{1+t}) < 0,$$

the inequality follows from $\sin(\frac{\pi t}{1+t}) > 0$ and $\cos(\frac{\pi t}{1+t}) > 0$, for $0 < t \leq \frac{1}{2}$. This implies that $f(t)$ for $0 < t \leq \frac{1}{2}$ is strictly concave, since $f(0) = 0$ and $f(\frac{1}{2}) > 0$, therefore $f(t) > 0$. This completes the proof of lemma. ■

The next lemma is fundamental in the analysis of algorithm based on the kernel function (12).

Lemma 2 *For* $\psi(t)$ *defined in (12), we have*

$$\psi''(t) > 2, \tag{16}$$

$$t\psi''(t)+\psi'(t)>0,\ \ t<1 \tag{17}$$

$$t\psi''(t)-\psi'(t)>0, \tag{18}$$

$$\psi'''(t)<0. \tag{19}$$

Proof. *From (14), by* $u'(t)=\frac{\pi}{(1+t)^2}$ *and* $u''(t)=-\frac{2\pi}{(1+t)^3}$*, we get*

$$\psi''(t)=\frac{\pi^2}{(1+t)^4\sin^3(u(t))}(1+\frac{2+2t}{\pi}\sin(u(t))\cos(u(t))+\cos^2(u(t)))+2.$$

Case 1 Assume that $0<t\le 1$.
In this case $\sin(u(t))>0$, $\cos(u(t))>0$ and the proof is obvious.
Case 2 Assume that $t>1$.
Using the first part of Lemma 1, we obtain

$$\begin{aligned}\psi''(t)&\ge\frac{\pi^2}{(1+t)\pi^3}(1+\frac{2+2t}{\pi}\sin(u(t))\cos(u(t))+\cos^2(u(t)))+2\\&\ge\frac{1}{\pi(1+t)}\left(1+2\cos\big(u(t)\big)+\cos^2\big(u(t)\big)\right)+2\\&=\frac{1}{\pi(1+t)}(1+\cos(u(t)))^2+2>2.\end{aligned}$$

This proves (16). By using (13) and (14), we have

$$\begin{aligned}t\psi''(t)+\psi'(t)&=2t+\frac{\pi t}{(1+t)^4\sin^3(u(t))}(\pi\sin^2\big(u(t)\big)\\&\quad+2(1+t)\sin(u(t))\cos(u(t))+2\pi^2\cos^2(u(t)))\\&\quad-\frac{\pi\cos(u(t))}{(1+t)^2\sin^2(u(t))}+2t-2\\&=\frac{1}{(1+t)^4\sin^3(u(t))}(\pi^2t\sin^2(u(t))+2\pi t(1+t)\sin(u(t))\cos(u(t))\\&\quad+2\pi^2t\cos^2(u(t))-\pi(1+t)^2\sin(u(t))\cos(u(t))\\&\quad+4t(1+t)^4\sin^3(u(t))-2(1+t)^4\sin^3(u(t)))\\&:=\frac{1}{(1+t)^4\sin^3(u(t))}h(t).\end{aligned} \tag{20}$$

Case 1 Assume that $0<t\le\frac{1}{2}$.
Using the second part of Lemma 1, we obtain

$$\begin{aligned}h(t)&\ge\pi(1+t)\sin^3(u(t))+2\pi^2t^2\cos^2(u(t))+2\pi^2t\cos^2(u(t))\\&\quad-\pi(1+t)^2\sin(u(t))\cos(u(t))+(4t-2)(1+t)^4\sin^3(u(t))\\&=(1+t)\sin^3(u(t))(\pi+(4t-2)(1+t)^3)\\&\quad+\pi(1+t)(2\pi t\cos(u(t))-(1+t)\sin(u(t)))\cos(u(t))>0,\end{aligned}$$

the last inequality follows from $\sin(u(t))>0$ for $0<t\le\frac{1}{2}$, the third part of Lemma 1 and

$$\pi + (4t-2)(1+t)^3 = \pi + 4t^4 + 10t^3 + 6t^2 - 2t - 2$$

$$> (\pi - 3) + (t-1)^2 > 0.$$

Case 2 Assume that $\frac{1}{2} < t \leq 1$.
In this case, by the second part of Lemma 1, we have

$$h(t) = \pi^2 t + 2\pi t(1+t)\sin(u(t))\cos(u(t)) + \pi^2 t\cos^2(u(t))$$

$$-\pi(1+t)^2\sin(u(t))\cos(u(t)) + (4t-2)(1+t)^4\sin^3(u(t))$$

$$\geq \pi(1+t)\sin(u(t)) + \pi(1+t)(2t-(1+t))\sin(u(t))\cos(u(t))$$

$$+(4t-2)(1+t)^4\sin^3(u(t))$$

$$\geq \pi(1+t)\sin(u(t))\Big(1+(t-1)\cos(u(t))\Big)$$

$$+(4t-2)(1+t)^4\sin^3(u(t)) > 0,$$

the last inequality is true by $\sin(u(t)) > 0$ and $1 + (t-1)\cos(u(t)) > 0$, for $\frac{1}{2} < t \leq 1$. The two cases together prove (17). To prove (18), considering the first two derivatives of $\psi(t)$, we have

$$t\psi''(t) - \psi'(t) = \frac{1}{(1+t)^4\sin^3(u(t))}(\pi^2 t\sin^2(u(t))$$

$$+2\pi t(1+t)\sin(u(t))\cos(u(t))$$

$$+2\pi^2 t\cos^2(u(t)) + \pi(1+t)^2\sin(u(t))\cos(u(t))$$

$$+2(1+t)^4\sin^3(u(t))).$$

Case 1: Assume that $0 < t \leq 1$.

In this case $0 < u(t) \leq \frac{\pi}{2}$, so $\sin(u(t)) > 0$ and $\cos(u(t)) \geq 0$. Therefore $t\psi''(t) - \psi'(t) > 0$.

Case 2: Assume that $t > 1$. In this case, we have

$$t\psi''(t) - \psi'(t) > \frac{1}{(1+t)^4\sin^3(u(t))}(\pi^2 t\sin^2(u(t)) - 2\pi^2 t\cos^2(u(t))$$
$$+ 2\pi^2 t\cos^2(u(t))$$
$$-(1+t)^3\sin^2(u(t)) + 2(1+t)^4\sin^3(u(t)))$$
$$= \frac{1}{(1+t)^4\sin^3(u(t))}(\pi^2 t\sin^2(u(t))$$

$$+\left((2+2t)\sin\left(u(t)\right)-1\right)\left(1+t)^3\sin^2\left(u(t)\right)\right)$$
$$> \frac{1}{(1+t)^4\sin^3(u(t))}\left(\pi^2 t\sin^2\left(u(t)\right) + 3(1+\mathrm{t})^3\sin^2\left(u(t)\right)\right)$$
$$> 0.$$

The two cases together prove (18). To prove (19), using the first three derivatives of $u(t)$ and substituting into (15), we obtain

$$\psi'''(t) = \frac{-\pi}{(1+t)^6\sin^4(u(t))}h(t),$$

where

$$h(t) = 6\pi(1+t)\sin^3\left(u(t)\right) + 5\pi^2\sin^2\left(u(t)\right)\cos\left(u(t)\right)$$
$$+6(1+t)^2\sin^2(u(t))\cos(u(t))$$
$$+12\pi(1+t)\sin(u(t))\cos^2(u(t)) + 6\pi^2\cos^3(u(t)).$$

Case 1: Assume that $0 < t \leq 1$.

In this case $0 < u(t) \leq \frac{\pi}{2}$, so $\sin(u(t)) > 0$ and $\cos(u(t)) \geq 0$, and implies that $h(t) > 0$.

Case 2: Assume that $t > 1$.

In this case, $\cos(u(t)) < 0$ and $\sin(u(t)) > 0$. Therefore,

$$h(t) > 6\pi\sin^2\left(u(t)\right)\left((1+t)\sin\left(u(t)\right) + \pi\cos\left(u(t)\right)\right)$$
$$+6(1+t)\sin(u(t))\cos(u(t))((1+t)\sin(u(t)) + \pi\cos(u(t)))$$
$$+6\pi\cos^2(u(t))((1+t)\sin(u(t)) + \pi\cos(u(t)))$$
$$= 6(\pi\sin^2(u(t)) + (1+t)\sin(u(t))\cos(u(t)) + \pi\cos^2(u(t)))$$
$$\times ((1+t)\sin(u(t)) + \pi\cos(u(t)))$$
$$\geq 6\pi\left(1+\cos\left(u(t)\right)\right)\left((1+t)\sin\left(u(t)\right) + \pi\cos\left(u(t)\right)\right) > 0.$$

The second and the last inequalities follow by Lemma 1. ■

Lemma 3 *For* $t \geq 1$, *one has*

1. $\cos(u(t)) \geq 1 - t$.

2. $\frac{1}{\sin(u(t))} \leq t^2 - 2t + 2$.

Proof. For $t \geq 1$, $\frac{\pi}{2} \leq u(t) \leq \pi$ and $\sin(u(t)) \leq 1$. Define $f(t) := \cos(u(t)) + t - 1$. Then, we have

$$f'(t) = -\frac{\pi\sin(u(t))}{(1+t)^2} + 1 = \frac{-\pi\sin(u(t)) + (1+t)^2}{(1+t)^2} \geq \frac{-\pi+4}{(1+t)^2} > 0.$$

Thus, $f(t)$ is increasing for $t \geq 1$, and hence $f(t) \geq f(1) = 0$. This implies the first inequality. For the second part, defining

$$g(t) = t^2 - 2t + 2 - \frac{1}{\sin(u(t))},$$

we have

$$g'(t) = 2t - 2 + \frac{\pi\cos(u(t))}{(1+t)^2\sin^2(u(t))} \geq 2(t-1) - \frac{\pi(t-1)}{(1+t)^2\sin^2(u(t))}$$

$$= (2(1+t)^2\sin^2(u(t)) - \pi)\frac{t-1}{(1+t)^2\sin^2(u(t))}$$

$$\geq (8-\pi)\frac{t-1}{(1+t)^2\sin^2(u(t))} \geq 0.$$

The two inequalities are followed by $(1+t)\sin(u(t)) > 2$ for $t \geq 1$, and the first part, respectively. Therefore, $g(t)$ is increasing and hence $g(t) \geq g(1) = 0$. This completes the proof. ■

Note that $\psi'(1) = \psi(1) = 0$, and $\psi''(t) > 0$ imply that $\psi(t)$ is a nonnegative strictly convex such that $\psi(t)$ achieves its minimum at $t = 1$, i.e., $\psi(1) = 0$. This implies that, since $\psi(t)$ is twice differentiable, it is completely determined by its second derivative:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi. \tag{21}$$

The next lemma is very useful in the analysis of interior-point algorithms based on the kernel functions (see for example [2, 17]).

Lemma 4 *(Lemma 2.1.2 in [19]) Let $\psi(t)$ be a twice differentiable function for $t > 0$. Then the following three properties are equivalent:*

1. $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$ for $t_1, t_2 > 0$.
2. $\psi'(t) + t\psi''(t) \geq 0, t > 0$.
3. 3. $\psi(e^\xi)$ is convex.

Following [19], the property described in Lemma 4 is called exponential convexity, or shortly e-convexity. Therefore, Lemma 4 and (17) show that our new kernel function (12) is e-convex for $t > 0$.

Lemma 5 *(Lemmas 7 and 8 in [13]) For $\psi(t)$ defined as (12), one has*

1. $\psi(t) < \frac{1}{2}\psi''(t)(t-1)^2, t > 1.$
2. $\psi(t) \leq t\psi'(t), t \geq 1.$

The next theorem gives a lower bound on $\delta(v)$ in terms of $\Psi(v)$. This is due to the fact that $\psi(t)$ satisfies (19).

Theorem 6. *(Theorem 4.9 in [2]) Let* $\rho: [0,\infty) \to [1,\infty)$ *be the inverse function of* $\psi(t)$ *for* $t \geq 1$*. One has*

$$\delta(v) \geq \frac{1}{2}\psi'(\rho(\Psi(v))).$$

Lemma 7. *If* $\Psi(v) \geq 1$*, then*

$$\delta(v) \geq \frac{1}{4}\sqrt{\Psi(v)}.$$

Proof. Using (21) and (16), we have

$$s = \psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \geq \int_1^t \int_1^\xi 2(\zeta) d\zeta d\xi = (t-1)^2,$$

which implies $t = \rho(s) \leq 1 + \sqrt{s}$. Using Theorem 6 and Lemma 5, we have

$$\delta(v) \geq \frac{\psi\left(\rho(\Psi(v))\right)}{2\rho(\Psi(v))} = \frac{\Psi(v)}{2\rho(\Psi(v))} \geq \frac{\Psi(v)}{2\left(1+\sqrt{\Psi(v)}\right)}$$

$$\geq \frac{\Psi(v)}{4\sqrt{\Psi(v)}} \geq \frac{1}{4}\sqrt{\Psi(v)}.$$

The proof of the statement is completed. ■

At the start of each outer iteration, just before the update of μ with the factor $1-\theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ , the vector v defined by (4) is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which leads to an increase of the value of $\Psi(v)$. Then, during the inner iterations, $\Psi(v)$ decreases until it passes the threshold τ again. Hence, during the course of the algorithm, the largest values of $\Psi(v)$ occur just after the updates of μ. In the rest of this section, we derive an estimate for the effect of μ-update on the value of $\Psi(v)$. We start with an important theorem. This is due to the fact that $\psi(t)$ satisfies (16) and (18).

Theorem 8. *(Theorem 3.2 in [2]) Let* ρ *be the inverse function of* $\psi(t)$ *for* $t \geq 1$*. Then for any positive vector* v*, and any* $\beta \geq 1$

$$\Psi(\beta v) \leq n\psi(\beta\rho(\frac{\Psi(v)}{n})).$$

Corollary 9. *Let* $0 \leq \theta < 1$ *and* $v_+ = \frac{v}{\sqrt{1-\theta}}$*. If* $\Psi(v) \leq \tau$*, then*

$$\Psi(v_+) \leq n\psi\left(\frac{\rho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right).$$

Suppose that the barrier update parameter θ and threshold value τ are given. According to the algorithm, at the start of each outer iteration we have $\Psi(v) \leq \tau$. Define

$$L := L(n, \theta, \tau) := n\psi\left(\frac{\rho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right). \tag{22}$$

According to Corollary 9, L is an upper bound of $\Psi(v_+)$, the value of $\Psi(v)$ after the μ-update.

4. ANALYSIS OF THE ALGORITHM

In this section, we determine a default step size and obtain an upper bound to the decrease of the barrier function $\Psi(v)$ during an inner iteration.

4.1. Decrease the value of $\psi(v)$ and choose a default step size α

In each iteration, the search directions $\Delta x, \Delta y$ and Δs are obtained by solving the system (9) and via (5). After a step with size α and due to (5), the new iterate is obtained by

$$x_+ = x + \alpha\Delta x = \frac{x}{v}(v + \alpha d_x), \;\; s_+ = s + \alpha\Delta s = \frac{s}{v}(v + \alpha d_s).$$

Thus we have

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

Since the proximity after one step is defined by $\Psi(v_+)$, it follows that

$$\Psi(v_+) = \Psi(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}).$$

Hence, by Lemma 4

$$\Psi(v_+) \leq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

Let us denote the difference between the proximity before and after one step by a function of the step size, that is

$$f(\alpha) := \Psi(v_+) - \Psi(v).$$

Then $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)) - \Psi(v).$$

Obviously

$$f(0) = f_1(0) = 0.$$

Taking the derivative with respect to α, we obtain

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n} (\psi'(v_i + \alpha d_{xi})d_{xi} + \psi'(v_i + \alpha d_{si})d_{si}), \tag{23}$$

and

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n} (\psi''(v_i + \alpha d_{xi})d_{xi}^2 + \psi''(v_i + \alpha d_{si})d_{si}^2), \tag{24}$$

where d_{xi} and d_{si} denote the ith components of the vectors d_x and d_s, respectively. From (23), using (11) and the third equation of (9), we obtain

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x + d_s)$$

$$= -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{25}$$

In what follows, we use the short notation $\delta := \delta(v)$, and state four important lemmas without proofs. These are due to the fact that $\psi''(t)$ is monotonically decreasing.

Lemma 10. *(Lemma 4.1 in [2]) One has*

$$f_1''(\alpha) \leq 2\delta^2\psi''(v_{\min} - 2\alpha\delta).$$

Lemma 11. *(Lemma 4.2 in [2]) If the step size α satisfies*

$$-\psi'(v_{\min} - 2\alpha\delta) + \psi'(v_{\min}) \leq 2\delta, \tag{26}$$

then $f_1'(\alpha) \leq 0$.

Lemma 12. *(Lemma 4.3 in [2]) Let $\rho: [0,\infty) \to (0,1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ on the interval $(0,1]$, then the largest possible value of the step size of α satisfying (26) is given by*

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)).$$

Lemma 13. *(Lemma 4.4 in [2]) Let ρ and $\bar{\alpha}$ be the same as defined in Lemma 11. Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

For the purpose of finding an upper bound for $f(\alpha)$, we need a default step size $\tilde{\alpha}$ that is the lower bound of the $\bar{\alpha}$, and consists of δ.

Lemma 14. *Let $\rho: [0,\infty) \to (0,1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ on the interval $(0,1]$ and $\Psi(v) \geq \tau \geq 1$. Then*

$$\frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{(16 + 24\sqrt{6}\pi^2)\delta^{\frac{3}{2}}}.$$

Proof. To obtain the inverse function $t = \rho(s)$ of $-\frac{1}{2}\psi'(t)$ for $0 < t \leq 1$, we need to solve the equation

$$-\psi'(t) = -2t + 2 + \frac{\pi\cos(u(t))}{(1+t)^2\sin^2(u(t))} = 2s.$$

The above equation implies

$$2 + \frac{\pi\cos(u(t))}{(1+t)^2\sin^2(u(t))} = 2s + 2t \leq 2s + 2. \tag{27}$$

By setting $t = \rho(2\delta)$, we have

$$-\psi'(t) = 4\delta.$$

Hence, we have

$$\frac{1}{\psi''(t)} = \frac{1}{2 + \frac{\pi^2}{(1+t)^4\sin^3(u(t))}(1 + \frac{2(1+t)}{\pi}\sin(u(t))\cos(u(t)) + \cos^2(u(t)))}.$$

Using the second part of Lemma 1, $(1+t)\sin(u(t)) \leq \pi t \leq \pi$ for $0 < t \leq 1$, we have

$$\frac{1}{\psi''(t)} \geq \frac{1}{2 + \frac{\pi^2}{(1+t)^4\sin^3(u(t))}(1 + 2\cos(u(t)) + \cos^2(u(t)))}$$

$$\geq \frac{1}{2 + \frac{\pi^2}{(1+t)^3\sin^3(u(t))}(1 + \cos(u(t)))^2}$$

$$= \frac{1}{2 + \frac{(\pi\cos(u(t)) + 2(1+t)^2\sin^2(u(t)))^{\frac{3}{2}}\pi^2(1+\cos(u(t)))^2}{(1+t)^3\sin^3(u(t))\pi^2(1+\cos(u(t)))^2}}$$

$$\geq \frac{1}{2 + (\frac{\pi\cos(u(t)) + 2(1+t)^2\sin^2(u(t))}{(1+t)^2\sin^2(u(t))})^{\frac{3}{2}} \frac{\pi^2(1+\cos(u(t)))^2}{(2\cos^2(u(t)) + 2\sin^2(u(t)))^{\frac{3}{2}}}}$$

$$\geq \frac{1}{2 + (\frac{\pi\cos(u(t))}{(1+t)^2\sin^2(u(t))} + 2)^{\frac{3}{2}} \frac{4\pi^2}{2\sqrt{2}}}.$$

Using (27) and Lemma 7, we get

$$\frac{1}{\psi''(t)} \geq \frac{1}{2 + (4\delta + 2)^{\frac{3}{2}}\sqrt{2}\pi^2} \geq \frac{1}{(16 + 24\sqrt{6}\pi^2)\delta^{\frac{3}{2}}}.$$

This completes the proof. ■

In the sequel, we use the notation

$$\tilde{\alpha} = \frac{1}{(16 + 24\sqrt{6}\pi^2)\delta^{\frac{3}{2}}}, \tag{28}$$

as the default step size. By Lemma 13, $\bar{\alpha} \geq \tilde{\alpha}$.
Lemma 15. *(Lemma 4.5 in [2]) If the step size α is such that $\alpha \leq \bar{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Theorem 16. *If $\tilde{\alpha}$ is the default step size as given by (28), then*

$$f(\tilde{\alpha}) \leq \frac{-\Psi^{\frac{1}{4}}}{16 + 24\sqrt{6}\pi^2}.$$

Proof. Using Lemma 15 with $\alpha = \tilde{\alpha}$ and (28), we have

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 \leq \frac{-\delta^2}{(16 + 24\sqrt{6}\pi^2)\delta^{\frac{3}{2}}} = \frac{-\delta^{\frac{1}{2}}}{16 + 24\sqrt{6}\pi^2}.$$

Using Lemma 7, we obtain

$$f(\tilde{\alpha}) \leq -\frac{\Psi(v)^{\frac{1}{4}}}{32 + 48\sqrt{6}\pi^2}.$$

This proves the theorem. ∎

4.2. Iteration Bound

We need to count how many inner iterations are required to return to the situation where $\Psi(v) \leq \tau$ after a μ-update. We define the value of $\Psi(v)$ after μ-update as Ψ_0 , and the subsequent values in the same outer iteration are denoted as $\Psi_k, k = 1,2, \dots, K,$ where K denotes the total number of inner iterations in the outer iteration. According to decrease of $f(\tilde{\alpha})$, for $k = 1,2, \dots, K - 1$, we obtain

$$\Psi_{k+1}(v) \leq \Psi_k(v) - \frac{\Psi(v)^{\frac{1}{4}}}{32+48\sqrt{6}\pi^2}. \tag{29}$$

Lemma 17 *(Lemma 14 in [18]) Suppose $t_0, t_1, \dots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\xi}, k = 0,1, \dots, K - 1,$$

where $\beta > 0$ and $0 < \xi \leq 1$. Then $K \leq \left\lceil \frac{t_0^{\xi}}{\beta\xi} \right\rceil$
Letting $t_k = \Psi_k(v), \xi = \frac{3}{4}$ and $\beta = \frac{1}{32+48\sqrt{6}\pi^2}$, we can get the following theorem from Lemma 17.
Theorem 18. *Let K be the total number of inner iterations in the outer iteration. Then we have*

$$K \leq \frac{4}{3}(32 + 48\sqrt{6}\pi^2)\Psi_0^{\frac{3}{4}},$$

where, Ψ_0 is the value of $\Psi(v)$ after the μ-update in outer iteration.
According to Lemma 3, it is clear that $\psi(t) \leq 2(t-1)^2$ for $t \geq 1$. Applying Corollary 9 and the proof of Lemma 7, we obtain

$$\Psi_0 \leq n\psi(\frac{\rho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}) \leq n\psi(\frac{1+\sqrt{\frac{\tau}{n}}}{\sqrt{1-\theta}}) \leq 2n(\frac{1+\sqrt{\frac{\tau}{n}}}{\sqrt{1-\theta}}-1)^2$$
$$\leq \frac{2n}{1-\theta}(\theta+\sqrt{\frac{\tau}{n}})^2,$$

where the last inequality holds from $1-\sqrt{1-\theta} \leq \theta,$ for $0 < \theta \leq 1.$ The number of outer iterations is bounded above by $\frac{1}{\theta}\log(\frac{n}{\varepsilon})$ (Lemma $II.17$ in [20]). By multiplying the number of outer iterations and the number of inner iterations, we get an upper bound for the total number of iterations, namely

$$\frac{4(32+48\sqrt{6}\pi^2)}{3\theta}(\frac{2n}{1-\theta}(\theta+\sqrt{\frac{\tau}{n}})^2)^{\frac{3}{4}}\log\frac{n}{\varepsilon}.$$

Large-update methods use $\theta = \Theta(1)$ and $\tau = O(n)$, so the iteration bound becomes

$$O(n^{\frac{3}{4}}\log\frac{n}{\varepsilon}).$$

5. CONCLUSION

In this paper, we have proposed a new barrier function and a large-update version of primal-dual interior-point algorithm. The kernel function (12) has a new trigonometric barrier term with some properties described in Section 3. We have showed that the large-update IPM based on this kernel function has $O(n^{\frac{3}{4}}\log\frac{n}{\varepsilon})$ iteration bound, which improves the classical iteration complexity for large-update methods, i.e., $O(n\log\frac{n}{\varepsilon})$.

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