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A Polynomial-Time Algorithm to Approximate the Mixed Volume within a Simply Exponential Factor

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Abstract Let $\mathbf{K} = (K_1, \ldots, K_n)$ be an n-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in $\mathbf{R}^n$. The Minkowski polynomial $V_{\mathbf{K}}$ is defined as $V_{\mathbf{K}}(\lambda_1, \ldots, \lambda_n) = V(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ and the mixed volume $V(K_1, \ldots, K_n)$ as

$$V(K_1, \ldots, K_n) = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} V_{\mathbf{K}}(\lambda_1, \ldots, \lambda_n).$$

Our main result is a poly-time algorithm which approximates $V(K_1, \ldots, K_n)$ with multiplicative error e^n and with better rates if the affine dimensions of most of the sets K_i are small. Our approach is based on a particular approximation of $\log(V(K_1, \ldots, K_n))$ by a solution of some convex minimization problem. We prove the mixed volume analogues of the Van der Waerden and Schrijver–Valiant conjectures on the permanent. These results, interesting on their own, allow us to justify the abovementioned approximation by a convex minimization, which is solved using the ellipsoid method and a randomized poly-time time algorithm for the approximation of the volume of a convex set.

Keywords Convex sets · Mixed volume · Convex optimization · Algorithm

1 Introduction

Let $\mathbf{K} = (K_1, \ldots, K_n)$ be an n-tuple of convex compact subsets in the Euclidean space $\mathbf{R}^n$, and let $V(\cdot)$ be the Euclidean volume in R^n. It is a well-known result of Hermann Minkowski (see, for instance, [5]) that the value of $V_{\mathbf{K}}(\lambda_1 K_1 + \cdots + \lambda_n K_n)$,

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where "+" denotes the Minkowski sum, and λK denotes the dilatation of K with coefficient λ, is a homogeneous polynomial of degree n in nonnegative variables $\lambda_1, \dots, \lambda_n$ (called the Minkowski polynomial). The coefficient $V(K_1, \dots, K_n)$ of $\lambda_1 \cdot \lambda_2 \cdots \lambda_n$ is called the *mixed volume* of $K_1, \dots, K_n$. Alternatively,

$$V(K_1, \dots, K_n) = \frac{\partial^n}{\partial \lambda_1 \cdots \partial \lambda_n} V_{\mathbf{K}}(\lambda_1, \dots, \lambda_n).$$

Mixed volume is known to be monotone [5], that is, $K_i \subseteq L_i$ for $i = 1, \dots, n$ implies $V(K_1, \dots, K_n) \le V(L_1, \dots, L_n)$. In particular, it is always nonnegative, and therefore all the coefficients of the Minkowski polynomial $V_{\mathbf{K}}$ are nonnegative real numbers.

The corresponding *Brunn–Minkowski theory*, which is the backbone of convex geometry and its numerous applications, is about various implications of the fact that the functional $(V_{\mathbf{K}}(\lambda_1, \dots, \lambda_n))^{\frac{1}{n}}$ is concave on the nonnegative orthant $R_+^n = \{(\lambda_1, \dots, \lambda_n) : \lambda_i \ge 0\}$. Its generalization, *Alexandrov–Fenchel theory*, is based on the fact that the functionals $(\frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} V_{\mathbf{K}}(0, \dots, 0, \lambda_{k+1}, \dots, \lambda_n))^{\frac{1}{n-k}}$ are concave on R_+^{n-k} for all $1 \le k \le n-1$.

The problem of computing the mixed volume of convex bodies is also important for combinatorics and algebraic geometry [7]. For instance, the number of toric solutions to a system of n polynomial equations on $\mathbf{C}^n$ is upper bounded by—and for a generic system, equal to—the mixed volume of the Newton polytopes of the corresponding polynomials. This remarkable result, called the *BKK Theorem*, is covered, for instance, in [23] and [5].

1.1 Previous Work

The BKK Theorem created an "industry" of computing (exactly) the mixed volume of integer polytopes and its various generalizations; most algorithms in the area are of exponential running time ([9, 11, 17], and many more). Most researchers in the "industry" do not bother to formally write down the complexity, rather they describe the actual amount of the computer time. Although there was a substantial algorithmic activity on the mixed volume of polytopes prior to [7], the paper [7] was the first, to our knowledge, systematic complexity-theoretic study in the area. It followed naturally upon the famous FPRAS algorithms [6] for volumes of convex bodies, solved several natural problems, and posed many important hard questions. The existence of FPRAS for the mixed volume even for polytopes or ellipsoids is still an open problem.

Efficient polynomial-time probabilistic algorithms that approximate the mixed volume extremely tightly (within a $(1+\epsilon)$ factor) were developed for some classes of well-presented convex bodies [7]. The algorithms in [7] are based on the multivariate polynomial interpolation and work if and only if the number k of distinct convex sets in the tuple $\mathbf{K}$ is "small", i.e., $k = O(\log(n))$.

The first efficient probabilistic algorithm that provides an $n^{O(n)}$-factor approximation for *arbitrary well-presented proper convex bodies* was obtained by Barvinok [2]. Barvinok's algorithms start by replacing convex bodies with ellipsoids. This first step already gives $n^{O(n)}$-factor in the worst case. After that the mixed volume of ellipsoids is approximated with a simply exponential factor c^n by two randomized algorithms; one of those deals with approximation of the mixed discriminant.

The question of existence of an efficient *deterministic* algorithm for approximating the mixed volume of arbitrary well-presented proper convex bodies with an error depending only on the dimension was posed in [7]. The authors quote a lower bound (*Bárány and Füredi bound*) [1] of $(\Omega(\frac{n}{\log n}))^{\frac{n}{2}}$ for the approximation factor of such an algorithm. (Notice that Barvinok's *randomized* algorithm [2] does not beat the *Bárány and Füredi bound*.)

Deterministic polynomial-time algorithms that approximate the mixed volume with a factor of $n^{O(n)}$ were given, for a fixed number of distinct proper convex bodies in $\mathbf{K} = (K_1, \dots, K_n)$, in [2, 7]. Finally, a deterministic polynomial-time algorithm that approximates the mixed volume with a factor of $n^{O(n)}$ in the general case of well-presented compact convex sets was given in [12, 13]. Let $\mathbf{A} = (A_1, \dots, A_n)$ be an n-tuple of $n \times n$ complex matrices; the corresponding determinantal polynomial is defined as $\mathrm{Det}_{\mathbf{A}}(\lambda_1, \dots, \lambda_n) = \det(\sum_{1 \le i \le n} \lambda_i A_i)$. The mixed discriminant is defined as $D(A_1, \dots, A_n) = \frac{\partial^n}{\partial\lambda_1 \cdots \partial\lambda_n} \mathrm{Det}_{\mathbf{A}}(\lambda_1, \dots, \lambda_n)$. Similarly to the randomized algorithm from [2], the algorithm in [12, 13] reduced the approximation of the mixed volume of well-presented compact convex sets to the approximation of the mixed volume of ellipsoids; this first step gives an $n^{O(n)}$ factor in the worst case, and the mixed volume of ellipsoids is approximated by $(D(A_1, \dots, A_n))^{\frac{1}{2}}$ of the corresponding positive semidefinite matrices $A_i \succeq 0$. This second step adds $\sqrt{3}^n$ to the multiplicative approximation error (see inequality (16) below).

The approximation of the mixed discriminant has also been relaxed to a convex optimization problem (geometric programming). In order to prove the accuracy of the convex relaxation, the author proved in [14] the mixed discriminant analogue of the Van der Waerden conjecture on permanents of doubly stochastic matrices [20], which was posed by Bapat [3].

To summarize, the interpolational approach from [7] is limited by the restriction that the number of distinct convex sets is $O(\log(n))$; previous approaches [2, 12, 13] cannot give the simply exponential approximation factor c^n because of the initial approximation of convex sets by ellipsoids.

1.2 Our Approach

Assume, modulo deterministic poly-time preprocessing [7], that the mixed volume $V(K_1, \dots, K_n) > 0$. We define the *capacity* of the volume polynomial $V_{\mathbf{K}}$ as

$$\mathrm{Cap}(V_{\mathbf{K}}) = \inf_{x_i > 0: 1 \le i \le n} \frac{V_{\mathbf{K}}(x_1, \dots, x_n)}{\prod_{1 \le i \le n} x_i}. \tag{1}$$

Since the coefficients of the volume polynomial $V_{\mathbf{K}}$ are nonnegative real numbers, we get the upper bound

$$V(K_1, \dots, K_n) \le \mathrm{Cap}(V_{\mathbf{K}}).$$

The trick is that $\log(\mathrm{Cap}(V_{\mathbf{K}}))$ is a solution of the following convex minimization problem:

$$\log\bigl(\mathrm{Cap}(V_{\mathbf{K}})\bigr) = \inf_{y_1+\cdots+y_n=0} \log\bigl(V_{\mathbf{K}}\bigl(e^{y_1}, \dots, e^{y_n}\bigr)\bigr). \tag{2}$$

Recall that the functional $\log(p(e^{y_1}, \dots, e^{y_n}))$ *is convex on* R^n *if* $p(x_1, \dots, x_n)$ *is any polynomial with nonnegative coefficients. More generally, a sum of log-convex functionals is log-convex.*

We view $\mathrm{Cap}(V_{\mathbf{K}})$ as an approximation for the mixed volume $V(K_1, \dots, K_n)$. To justify this we prove the lower bound

$$V(K_1, \dots, K_n) \geq \frac{n!}{n^n}\mathrm{Cap}(V_{\mathbf{K}}) \approx e^{-n}\mathrm{Cap}(V_{\mathbf{K}}),$$

which is the mixed volume analogue of the Van der Waerden conjecture on the permanent. We also present better upper bounds when "most" of the convex sets K_i have small affine dimension, which are analogues of the Schrijver–Valiant conjecture posed in [24] and proved in [25].

The idea of our approach is very similar to our treatment of *H-Stable* polynomials in [16]. Recall that a homogeneous polynomial $p(x_1, \dots, x_n)$ with nonnegative coefficients is called *H-Stable* if $p(z_1, \dots, z_n) \neq 0$, provided that the real parts $\mathrm{Re}(z_i) > 0$, $1 \leq i \leq n$.

Not all Minkowski polynomials $V_{\mathbf{K}}$ are H-Stable: any univariate polynomial with nonnegative coefficients $S(x) = \sum_{0 \leq i \leq n} \binom{n}{i} a_i x^i$ such that $a_i^2 \geq a_{i-1}a_{i-1}$, $1 \leq i \leq n-1$, can be presented as $S(x) = V(A + xB)$ for some convex compact subsets (simplexes) $A, B \subset R^n$ [26]. Fortunately, a modification of the inductive proof in [16] works for Minkowski polynomials and is presented in the next section.

After establishing the mixed volume analogues of the Van der Waerden and the Schrijver–Valiant (permanental) conjectures, we present a randomized poly-time algorithm to solve problem (2) based on the ellipsoid method and randomized poly-time algorithms for the volume approximation. Together with the mixed volume analogues of the Van der Waerden conjecture, this gives a randomized poly-time algorithm to approximate the mixed volume $V(K_1, \dots, K_n)$ within relative accuracy e^n. Notice that, in view of the *Bárány and Füredi bound*, this cannot be achieved by a deterministic poly-time oracle algorithm. We use the ellipsoid method because of its robustness: we deal essentially with a random oracle which computes $\log(V_{\mathbf{K}}(e^{y_1}, \dots, e^{y_n}))$ with an additive small error ϵ; we use this oracle to get an approximation of the gradient of $\log(V_{\mathbf{K}}(e^{y_1}, \dots, e^{y_n}))$.

2 Van der Waerden and Schrijver–Valiant Conjectures for the Mixed Volume

Consult Appendix A for the proofs of results in this section.

2.1 The Mixed Volume Analogue of the Van der Waerden–Falikman–Egorychev Inequality

Theorem 2.1

1. *Let* $\mathbf{K} = (K_1, \dots, K_n)$ *be an* n*-tuple of convex compact subsets in the Euclidean space* $\mathbf{R}^n$. *The mixed volume* $V(K_1, \dots, K_n)$ *satisfies the following lower bound*:

$$V(K_1, \dots, K_n) \geq \frac{n!}{n^n} \mathrm{Cap}(V_{\mathbf{K}}). \tag{3}$$

2. *The equality in* (3) *is attained if and only if either the mixed volume* $V(K_1, \dots, K_n) = 0$ *or* $K_i = a_i K_1 + b_i : a_i > 0,\ b_i \in R^n;\ 2 \leq i \leq n$.

2.2 The Mixed Volume Analogue of the Schrijver–Valiant Conjecture

Definition 2.2

1. Let $n \geq k \geq 1$ be two integers. We define the univariate polynomial

$$sv_{n,k}(x) = 1 + \sum_{1 \leq i \leq k} \left(\frac{x}{n}\right)^i \binom{n}{i}.$$

Note that $sv_{n,n}(x) = (1 + \frac{x}{n})^n$. Next we define the functions

$$\lambda(n,k) = \left(\min_{x>0} \left(\frac{sv_{n,k}(x)}{x}\right)\right)^{-1}. \tag{4}$$

Remark 2.3 It was observed in [15] that

$$\lambda(k,k) = g(k) =: \left(\frac{k-1}{k}\right)^{k-1}, \quad k \geq 1; \qquad \prod_{1 \leq k \leq n} g(k) = \frac{n!}{n^n}. \tag{5}$$

The following inequalities are easily verified:

$$\lambda(n,k) < \lambda(n,l) : n \geq k > l \geq 1; \qquad \lambda(m,k) > \lambda(n,k) : n > m \geq k. \tag{6}$$

It follows that

$$\lambda(\infty, k) =: \lim_{n \to \infty} \lambda(n,k) = \left(\min_{t>0} \frac{\sum_{0 \leq i \leq k} \frac{t^i}{i!}}{t}\right)^{-1}. \tag{7}$$

The equality $\lambda(n,2) = (1 + \sqrt{2}\sqrt{\frac{n-1}{n}}t)^{-1} \geq (1+\sqrt{2})^{-1}$ follows from basic calculus.

2. Let $n \geq m \geq 1$ be two integers. An univariate polynomial with nonnegative coefficients $R(t) = \sum_{0 \leq i \leq m} a_i t^i$ is called *n-Newton* if it satisfies the following inequalities:

$$NIs: \left(\frac{a_i}{\binom{n}{i}}\right)^2 \geq \frac{a_{i-1}}{\binom{n}{i-1}} \frac{a_{i+1}}{\binom{n}{i+1}} : 1 \leq i \leq m-1. \quad (8)$$

(The standard Newton's inequalities correspond to the case $n = m$ and are satisfied if all the roots of p are real.)

The main mathematical result in this paper is the following theorem:

Theorem 2.4 *Let* $\mathbf{K} = (K_1, \dots, K_n)$ *be an* n*-tuple of convex compact subsets in the Euclidean space* $\mathbf{R}^n$, *and let* $\mathit{aff}(i)$ *be the affine dimension of* K_i, $1 \leq i \leq n$. *Then the following inequality holds*:

$$\mathrm{Cap}(V_{\mathbf{K}}) \geq V(K_1, \dots, K_n) \geq \prod_{1 \leq i \leq n} \lambda\big(i, D(i)\big)\mathrm{Cap}(V_{\mathbf{K}});$$

$$D(i) = \min\big(i, \mathit{aff}(i)\big), \quad 1 \leq i \leq n. \quad (9)$$

Corollary 2.5 *Suppose that* $\mathit{aff}(i) \leq k : k+1 \leq i \leq n$. *Then*

$$V(K_1, \dots, K_n) \geq \frac{k!}{k^k} \lambda(n, k)^{n-k} \mathrm{Cap}(V_{\mathbf{K}}). \quad (10)$$

If $k = 2$, *we get the inequality*

$$V(K_1, \dots, K_n) \geq \frac{1}{2}(1 + \sqrt{2})^{2-n} \mathrm{Cap}(V_{\mathbf{K}}). \quad (11)$$

2.3 Comparison with Previous Results

Inequality (3) is an analogue of the famous Van der Waerden conjecture [20], proved in [10] and [8], on the permanent of doubly-stochastic matrices. Indeed, consider the "boxes" $K_i = \{(x_1, \dots, x_n) : 0 \leq x_j \leq A(i, j), 1 \leq j \leq n\}$. Then the mixed volume is equal to the permanent

$$V(\mathbf{K}) = V(K_1, \dots, K_n) = \mathrm{Perm}(A),$$

and if the $n \times n$ matrix A is doubly-stochastic, then $\mathrm{Cap}(V_{\mathbf{K}}) = 1$.

Though this mixed volume representation of the permanent has been known since the publication of [8] if not earlier, the author is not aware of any attempts prior to the present paper to generalize the Van der Waerden conjecture to the mixed volume. We think that our version, stated in terms of the *capacity*, is most natural and useful.

Inequality (10) is an analogue of Schrijver's lower bound [16, 25] on the number of perfect matchings in k-regular bipartite graphs: affine dimensions play role of the degrees of vertices.

The reader familiar with [16] can recognize the similarity between inequalities (9), (3), and (10) and the corresponding inequalities in [16] proved for *H-Stable* polynomials. The method of proof in the present paper is also similar to the one in [16] (in spite of the fact that not all Minkowski polynomials $V_{\mathbf{K}}$ are *H-Stable*). But we get worse constants: for instance, if $k = 2$ in the notation of (10), then in the *H-Stable* case we get the factor 2^{-n+1} instead of the $\frac{1}{2}(1+\sqrt{2})^{2-n}$ obtained in this paper. Whether the latter factor is asymptotically sharp is an open problem.

2.4 The Idea of our Proof

Our proof of Bapat's conjecture in [14], i.e., of Van der Waerden's conjecture for the mixed discriminant, is an adaptation of Egorychev's proof in [8]. In contrast, the proofs in [16] and in the present paper have practically nothing in common either with Egorychev's proof or with Falikman's proof in [10].

1. *How do we prove the lower bounds?* We follow the general approach introduced by the author in [16]. We associate with the Minkowski polynomial $V_{\mathbf{K}}$ the sequence of polynomials

$$q_n = V_{\mathbf{K}}, q_i(x_1, \dots, x_i) = \frac{\partial^{n-i}}{\partial x_{i+1} \cdots \partial x_n} q_n(x_1, \dots, x_i, 0, \dots, 0), \quad 1 \le i \le n-1.$$

Note that $q_1(x) = V(K_1, \dots, K_n)x$. Everything follows from the inequality

$$\mathrm{Cap}(q_i) \ge \lambda\big(i+1, \min\big(i+1, \mathit{aff}(i+1)\big)\big)\mathrm{Cap}(q_{i+1}), \quad 1 \le i \le n-1.$$

Not surprisingly, we do use the (still hard to prove) Alexandrov–Fenchel inequalities to prove this crucial inequality. (In contrast, the *H-Stable* case in [16] required just the elementary AG inequality).

2. *How do we prove the uniqueness?* The uniqueness proofs in [8] and [14] are critically based on the known characterization of the equality cases in the Alexandrov inequalities for the mixed discriminant. In the case of the Alexandrov–Fenchel inequalities for the mixed volume, such a characterization is not known. Luckily, the method of our proof of the lower bound (3) allows us to use the well-known characterization of the equality in the (much simpler) Brunn–Minkowski inequality. The uniqueness proof in the present paper is very similar to the uniqueness proof in [16]. The fundamental tool in [16] was Gårding's famous (and not hard to prove) result on the convexity of the hyperbolic cone.

3 Convex Optimization Relaxation of the Mixed Volume

Inequalities (9), (3), and (10) justify the following strategy for approximation of the mixed volume $V(K_1, \dots, K_n)$ within a simply exponential multiplicative factor: solve the convex optimization problem (2) with an additive $O(1)$ error. We follow here the approach from [12, 13] which dealt with the following problem:

$$\log\big(\mathrm{Cap}(\mathrm{Det}_{\mathbf{A}})\big) = \inf_{y_1+\cdots+y_n=0} \log\left(\det\left(\sum_{1\le i\le n} e^{y_i} A_i\right)\right) : A_i \succeq 0. \tag{12}$$

The main difference between the two problems is that the value and the gradient of determinantal polynomials can be exactly evaluated in deterministic polynomial time. The case of the Minkowski polynomials $V_{\mathbf{K}}$ requires some extra care. Yet, this is done using standard and well-known tools. This section of the paper is fairly routine and can be easily reproduced by any convex optimization professional.

We now give an overview of the main points.

3.1 A Brief Overview

1. *Representations of convex sets and a priori ball for the convex relaxation*: we deal in this paper with two types of representations. First, similar to [7], we consider *well-presented* convex compact sets; second, motivated by algebraic applications and the BKK theorem, we consider integer polytopes given as a list of extreme points. In both cases, we start with deterministic poly-time preprocessing which transforms the initial tuple $\mathbf{K} \in R^n$ into a collection of indecomposable tuples
$$\mathbf{K}^{(\mathbf{1})} \in R^{d_1}, \dots, \mathbf{K}^{(\mathbf{i})} \in R^{d_i}; \qquad \sum_{1 \le j \le i} d_j = n$$
such that the mixed volume $V(\mathbf{K}) = \prod_{1 \le j \le i} V(\mathbf{K}^{(\mathbf{i})})$. The tuple $\mathbf{K}$ is indecomposable if and only if the minimum in (2) is attained and unique. The preprocessing is essentially the same as in [13].

   After this preprocessing we deal only with the indecomposable case and get an a priori ball which is guaranteed to contain the unique minimizer of (2). The radius of this ball is expressed in terms of the complexity of the corresponding representation:
$r \le O(n^2(\log(n) + \langle \mathbf{K} \rangle))$, where $\langle \mathbf{K} \rangle$ is the complexity of the initial tuple $\mathbf{K}$.
This part is fairly similar to the analogous problem for (12) treated in [13].
2. *Lipschitz Property and Rounding*: In the course of our algorithm we need to evaluate the volumes $V(e^{y_1}K_1 + \cdots + e^{y_n}K_n)$ and the mixed volumes $V(K_i, B, \dots, B)$, $B = e^{y_1}K_1 + \cdots + e^{y_n}K_n$. This requires a well-presentation of the Minkowski sum $B = e^{y_1}K_1 + \cdots + e^{y_n}K_n$. Given the well-presentation of $\mathbf{K}$, one gets a well-presentation of $a_1K_1 + \cdots + a_nK_n$ if the sizes of positive rational numbers a_i are bounded by $\mathrm{poly}(n, \langle \mathbf{K} \rangle)$ [7]. Therefore we need a rounding procedure, which requires us to keep only a "small" number of fractional bits of y_i (integer bits are taken care of by an a priori ball). This can be done using the Lipschitz property (18) of $\log(V(e^{y_1}K_1 + \cdots + e^{y_n}K_n))$ (which is just Euler's identity for homogeneous functionals) and its partial derivatives, proved in Lemma 3.3.
3. *Complexity of our Algorithm*: We give an upper bound on the number of calls to oracles for Minkowski sums $a_1K_1 + \cdots + a_nK_n$. The number of calls to oracles for the initial tuple $\mathbf{K}$ will be larger but still polynomial (see the discussion in [4] in the context of surface area computation, which is, up to a constant, the mixed volume $V(\mathrm{Ball}_n(1), A, \dots, A)$).
4. *Ellipsoid Method with noisy first-order oracle*: let $g(.)$ be a differentiable convex functional defined on the closed ball $\mathrm{Ball}_n(r) = \{X \in R^n : \langle X, X \rangle \le r^2\}$ and $\mathrm{Var}(g) = \max_{X \in \mathrm{Ball}_n(r)} g(X) - \min_{X \in \mathrm{Ball}_n(r)} g(X)$. The standard version of the ellipsoid method requires exact values of the function and its gradient. Fortunately,

there exists a noisy version [21], which needs approximations of the value $\overline{g(X)}$ and of the gradient $\overline{(\bigtriangledown g)(X)}$ such that

$$\sup_{Y,X\in \mathrm{Ball}_n(r)} \left|\left(\bar{g}(X)+\left\langle\overline{(\bigtriangledown f)}(X),Y-X\right\rangle\right)-\left(g(X)+(\bigtriangledown g)(X),Y-X>\right)\right| \\ \leq \delta \mathrm{Var}(g).$$

In our case, $g(y_1,\dots,y_n)=\log(V_{\mathbf{K}}(e^{y_1},\dots,e^{y_n}))$. We get the additive approximation of $g(y_1,\dots,y_n)$ using an FPRAS for the volume approximation and the additive approximation of $(\bigtriangledown g)(X)$ using FPRAS from [7] for approximating the "simple" mixed volume (generalized surface area) $V(K_i,K,\dots,K)$, $K=\sum_{1\leq i\leq n} e^{y_i}K_i$.

3.2 Representations of Convex Compact Sets

Following [7], we consider the following well-presentation of convex compact set $K_i\subset R^n$, $1\leq i\leq n$: a weak membership oracle for K together with a rational $n\times n$ matrix A_i and a rational vector $Y_i\in R^n$ such that

$$Y_i+A_i\left(\mathrm{Ball}_n(1)\right)\subset K_i\subset Y_i+n\sqrt{n+1}A_i\left(\mathrm{Ball}_n(1)\right). \tag{13}$$

We define the size $\langle\mathbf{K}\rangle$ as the maximum of bit sizes of entries of matrices A_i, $1\leq i\leq n$. Since the mixed volume $V(K_1,\dots,K_n)=V(K_1+\{-Y_1\},\dots,K_n+\{-Y_n\}$, we will assume WLOG that $Y_i=0$, $1\leq i\leq n$. This assumption implies the following identity for affine dimensions:

$$\mathit{aff}\left(\sum_{i\in S}K_i\right)=\mathrm{Rank}\left(\sum_{i\in S}A_iA_i^T\right),\qquad S\subset\{1,\dots,n\}. \tag{14}$$

Definition 3.1 An n-tuple $\mathbf{K}=(K_1,\dots,K_n)$ of convex compact subsets in R^n is called indecomposable if $\mathit{aff}(\sum_{i\in S}K_i)>\mathrm{Card}(S):S\subset\{1,\dots,n\}$, $1\leq \mathrm{Card}(S)<n$.

We consider, similar to [13], $n(n-1)$ auxiliary n-tuples $\mathbf{K}^{ij}$, where $\mathbf{K}^{ij}$ is obtained from $\mathbf{K}$ by substituting K_i instead of K_j. Notice that

$$V(x_1K_1+\cdots+x_nK_n)=x_1x_2\cdots x_n\left(V(\mathbf{K})+\frac{1}{2}\sum_{1\leq i,j\leq n}\frac{x_i}{x_j}V\left(\mathbf{K}^{ij}\right)\right)+\cdots. \tag{15}$$

It follows from (14) that the n-tuple $\mathbf{K}=(K_1,\dots,K_n)$ of well-presented convex sets is indecomposable iff the n-tuple of positive semidefinite matrices $\mathbf{Q}=(Q_1\cdots Q_n):Q_i=A_iA_i^T$ is fully indecomposable as defined in [13], which implies that indecomposability of $\mathbf{K}$ is equivalent to the inequalities $V(\mathbf{K}^{ij})>0:1\leq i$, $j\leq n$. Here $V(\mathbf{K}^{ij})$ stands for the mixed volume of the n-tuple $\mathbf{K}^{ij}$.

It was proved in [13] *that an n-tuple of positive semidefinite matrices $\mathbf{Q}=(Q_1\cdots Q_n)$ is indecomposable if and only if there exists a unique minimum in the optimization problem*

$$\inf_{y_1+\cdots+y_n=0} \log\left(\det\left(\sum_{1\le i\le n} e^{y_i} Q_i\right)\right).$$

In the same way, an n-tuple $\mathbf{K} = (K_1, \dots, K_n)$ of convex compact subsets in R^n is indecomposable if and only if there exists a unique minimum in the optimization problem (2).

Applying the decomposition algorithm from Sect. 2 in [13] to n-tuple of positive semidefinite matrices $\mathbf{Q} = (Q_1 \cdots Q_n)$, we can, by deterministic poly-time preprocessing, determine whether or not the n-tuple of convex compact subsets $\mathbf{K}$ is indecomposable and if not, factor the mixed volume as $V(\mathbf{K}) = \prod_{1\le j\le m\le n} V(\mathbf{K_j})$. Here the $n(j)$-tuple $\mathbf{K_j} = (K_{j,1}, \dots, K_{j,n(j)}) \subset R^{n(j)}$ is well presented and indecomposable, $\sum_{1\le j\le m} n(j) = n$, and the sizes $\langle \mathbf{K_j}\rangle \le \langle \mathbf{K}\rangle + \mathrm{poly}(n)$.

Based on the above remarks, we will deal from now on only with indecomposable well-presented tuples of convex compact sets. Moreover, to simplify the exposition, we assume WLOG that the matrices A_i in (13) are integer.

Let $\mathcal{E}_A$ be the ellipsoid $A(\mathrm{Ball}_n(1))$ in R^n. The following inequality, proved in [2], connects the mixed volume of ellipsoids and the corresponding mixed discriminant:

$$\begin{aligned} &3^{-\frac{n+1}{2}} v_n D^{\frac{1}{2}}\left(A_1(A_1)^T, \dots, A_n(A_n)^T\right) \\ &\quad \le V(\mathcal{E}_{A_1} \cdots \mathcal{E}_{A_n}) \le v_n D^{\frac{1}{2}}\left(A_1(A_1)^T, \dots, A_n(A_n)^T\right). \end{aligned} \tag{16}$$

Here v_n is the volume of the unit ball in $\mathbf{R}^n$.

3.3 Properties of Volume Polynomials: Lipschitz, Bound on the Second Derivative, a Priori Ball

Proposition 3.2

1. *Lipschitz Property. Let $p(x_1, \dots, x_n)$ be a nonzero homogeneous polynomial of degree n with nonnegative coefficients, $x_i = e^{y_i}$. Then*

$$\frac{\partial}{\partial y_i} \log\left(p\left(e^{y_1}, \dots, e^{y_n}\right)\right) = \frac{\frac{\partial}{\partial x_i} p(x_1, \dots, x_n) e^{y_i}}{p(x_1, \dots, x_n)}. \tag{17}$$

It follows from Euler's identity that $\sum_{1\le i\le n} \frac{\partial}{\partial y_i} \log(p(e^{y_1}, \dots, e^{y_n})) = n$; therefore the functional $f(y_1, \dots, y_n) = \log(p(e^{y_1}, \dots, e^{y_n}))$ is Lipschitz on R^n:

$$\left| f(y_1 + \delta_1, \dots, y_n + \delta_n) - f(y_1, \dots, y_n)\right| \le n\|\Delta\|_\infty \le n\|\Delta\|_2. \tag{18}$$

2. *Upper bound on second derivatives. Let us fix real numbers $y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n$ and define univariate function $q(y_i) = \log(V_{\mathbf{K}}(e^{y_1}, \dots, e^{y_i}, \dots, e^{y_n}))$. Notice that $e^{q(y_i)} = \sum_{0\le j\le \mathit{aff}(K_i)} a_j e^{jy}, a_j \ge 0$.*

Proposition 3.3

$$0 \le q''(y) \le \mathit{aff}(K_i). \tag{19}$$

(*Lemma B.3 in Appendix* B *proves a more general inequality.*)

3. *A Priori Ball result from* [13]. *Let* $p \in \mathrm{Hom}_+(n,n)$, $p(x_1,\dots,x_n) = x_1 x_2 \cdots x_n \times (a + \frac{1}{2}\sum_{1\le i\neq j\le n} b^{i,j}\frac{x_i}{x_j}) + \cdots$. *Assume that* $\min_{1\le i\neq j\le n} b^{i,j} =: \mathit{Stf}(p) > 0$. *Then there exists a unique minimizer* $(z_1,\dots,z_n)$, $\sum_{1\le i\le n} z_i = 0$, *such that*

$$\log\bigl(p(z_1,\dots,z_n)\bigr) = \min_{\sum_{1\le i\le n} y_i = 0}\bigl(\log\bigl(p\bigl(e^{y_1},\dots,e^{y_n}\bigr)\bigr)\bigr).$$

Moreover,

$$|z_i - z_j| \le \log\left(\frac{2\mathrm{Cap}(p)}{\mathit{Stf}(p)}\right). \tag{20}$$

The next proposition directly adapts Lemma 4.1 from [13] to the mixed volume situation, using Barvinok's inequality (16).

Proposition 3.4 *Consider an indecomposable* n*-tuple of convex compact sets* $\mathbf{K} = (K_1,\dots,K_n)$ *with the well-presentation* $A_i(\mathrm{Ball}_n(1)) \subset K_i \subset y + n\sqrt{n+1} \times A_i(\mathrm{Ball}_n(1))$, $1 \le i \le n$, *with integer* $n\times n$ *matrices* A_i. *Then the minimum in the convex optimization problem* (2) *is attained and is unique. The unique minimizing vector* $(z_1,\dots,z_n)$, $\sum_{1\le i\le n} z_i = 0$ *satisfies the following inequalities*:

$$|z_i - z_j| \le O\bigl(n^{\frac{3}{2}}\bigl(\log(n) + \langle\mathbf{K}\rangle\bigr)\bigr); \qquad \|z_1,\dots,z_n)\|_2 \le O\bigl(n^2\bigl(\log(n) + \langle\mathbf{K}\rangle\bigr)\bigr). \tag{21}$$

In other words, the convex optimization problem (2) *can be solved on the following ball in* R^{n-1}:

$$\mathrm{Apr}(\mathbf{K}) = \left\{(z_1,\dots,z_n) : \|z_1,\dots,z_n)\|_2 \le O\bigl(n^2\bigl(\log(n) + \langle\mathbf{K}\rangle\bigr)\bigr), \sum_{1\le i\le n} z_i = 0\right\}. \tag{22}$$

The following inequality follows from the Lipschitz property (18):

$$\begin{aligned}&\bigl|\log\bigl(V_{\mathbf{K}}\bigl(e^{y_1},\dots,e^{y_n}\bigr) - \log\bigl(V_{\mathbf{K}}\bigl(e^{l_1},\dots,e^{l_n}\bigr)\bigr)\bigr)\bigr|\\ &\quad\le O\bigl(n^3\bigl(\log(n) + \langle\mathbf{K}\rangle\bigr)\bigr) : Y, L \in \mathrm{Apr}(\mathbf{K}).\end{aligned} \tag{23}$$

3.4 Ellipsoid Method with Noisy First-Order Oracles

We recall the following fundamental result [21]:

Let $f(Y)$ *be differentiable convex functional defined on the ball* $\mathrm{Ball}_n(r) = \{Y \in R^n : \langle Y, Y\rangle \le r^2\}$ *of radius* r. *Let* $\mathrm{Var}(f) = \max_{Y\in \mathrm{Ball}_n(r)} f(Y) - \min_{Y\in\mathrm{Ball}_n(r)} f(Y)$. *Assume that at each vector* $Y \in \mathrm{Ball}_n(r)$ *we have an oracle evaluating a value* $g(Y)$ *such that* $|g(Y) - f(Y)| \le 0.2\delta\mathrm{Var}(f)$ *and the vector* $gr(Y) \in R^n$ *such that* $\|gr(Y) - (\bigtriangledown f)(Y)\|_2 \le 0.2\delta r^{-1}\mathrm{Var}(f)$ *(here* $(\bigtriangledown f)(Y)$ *is the gradient of* f *evaluated at* Y*). Then the Ellipsoid method finds a vector* $Z \in \mathrm{Ball}_n(r)$ *such that* $f(Z) \le \min_{Y\in\mathrm{Ball}_n(r)} f(Y) + \epsilon\mathrm{Var}(f), \epsilon > \delta$. *The method requires* $O(n^2\log(\frac{1}{\epsilon-\delta}))$ *oracle calls plus* $O(n^2)$ *elementary operations to run the algorithm itself.*

3.5 Putting Things Together

Here we take advantage of randomized algorithms which can evaluate $\log(\mathrm{Vol}(K))$ for a well-presented convex set K, with an additive error ϵ and failure probability δ in $O(\epsilon^{-k}n^l\log(\frac{1}{\delta}))$ oracle calls. For instance, the best current algorithm [19] gives $k=2$, $l=4$. We will need below to evaluate volumes $V(\sum_{1\le i\le n}x_iK_i)$. In our case the functional $f=\log(V_{\mathbf{K}}(e^{y_1},\dots,e^{y_n})$ is defined on ball $\mathrm{Apr}(\mathbf{K})$ of radius $O(n^2(\log(n)+\langle\mathbf{K}\rangle))$ with the variance $\mathrm{Var}(f)\le O(n^3(\log(n)+\langle\mathbf{K}\rangle))$. Theorem 2.1 gives the bound:

$$\log\bigl(V(\mathbf{K})\bigr)\le\Bigl(\min_{Y\in\mathrm{Apr}(\mathbf{K})}f(Y)\Bigr)\le\log\bigl(V(\mathbf{K})\bigr)+\log\left(\frac{n^n}{n!}\right)\approx\log\bigl(V(\mathbf{K})\bigr)+n. \quad (24)$$

Therefore, to approximate the mixed volume $V(\mathbf{K})$ up to a multiplicative factor of e^n, it is sufficient to find $Z\in\mathrm{Apr}(\mathbf{K})$ such that $f(Z)\le\min_{Y\in\mathrm{Apr}(\mathbf{K})}f(Y)+O(1)$. In order to get that via the Ellipsoid method, we need to approximate $\log(V_{\mathbf{K}}(e^{y_1},\dots,e^{y_n}))$ with the additive error $O(\mathrm{Var}(f)^{-1})=O(n^{-3}(\log(n)+\langle\mathbf{K}\rangle)^{-1})$ and its gradient with the additive l_2 error $O(n^{-2}(\log(n)+\langle\mathbf{K}\rangle)^{-1})$.

1. *Approximation of* $\log(V_{\mathbf{K}}(e^{y_1},\dots,e^{y_n}))$ *with failure probability* δ. The complexity is $O(n^{10}(\log(n)+\langle\mathbf{K}\rangle)^2\log(\delta^{-1}))$.
2. *Approximation of the partial derivatives.* Let $x_i=e^{y_i}$ and recall that the partial derivatives are

$$\beta_i=\frac{\partial}{\partial y_i}\log\bigl(V_{\mathbf{K}}\bigl(e^{y_1},\dots,e^{y_n}\bigr)\bigr)=\frac{\frac{\partial}{\partial x_i}V_{\mathbf{K}}(x_1,\dots,x_n)e^{y_i}}{V_{\mathbf{K}}(x_1,\dots,x_n)}.$$

   Suppose that $0\le1-a\le\frac{\gamma_i}{\beta_i}\le1+a$. It follows from Euler's identity that $\sum_{1\le i\le n}|\gamma_i-\beta_i|\le a$. If $a=O(n^{-2}(\log(n)+\langle\mathbf{K}\rangle)^{-1})$, then the vector $(\gamma_1,\dots,\gamma_n)$ is the needed approximation of the gradient.
   Notice that $\Gamma_i=\frac{\partial}{\partial x_i}V_{\mathbf{K}}(x_1,\dots,x_n)=\frac{1}{(n-1)!}V(A,B,\dots,B)$, where the convex sets $A=K_i$ and $B=\sum_{1\le i\le n}e^{y_j}K_j$. The randomized algorithm from [7] approximates $V(A,B,\dots,B)$ with the complexity $O(n^{4+o(1)}\epsilon^{-(2+o(1)}\log(\delta)$. This gives the needed approximation of the gradient with the complexity $nO(n^{8+o(1)}(\log(n)+\langle\mathbf{K}\rangle)^{2+o(1)}\log(\delta^{-1}))$.
3. *Controlling the failure probability* δ. We need to approximate $O(n^2\log(\mathrm{Var}(f)))$ values and gradients. To achieve a probability of success $\frac{3}{4}$, we need that

$$\delta\approx\frac{1}{4}\bigl(n^2\bigl(n^{\frac{5}{2}}\bigl(\log(n)+\langle\mathbf{K}\rangle\bigr)\bigr)\bigr)^{-1}.$$

   This gives $\log((\delta)^{-1})\approx O(\log(n)+\log(\log(n)+\langle\mathbf{K}\rangle))$.

Remark 3.5 Let $g(y_1)=\log(V_{\mathbf{K}}(e^{y_1},\dots,e^{y_n}))$ and $\overline{g(y)}=g(y)+h(y)$, $|h(y)|\le a$. We present here an alternative elementary way to approximate the partial derivative $g'(y_1)$.

Recall that the function $g(.)$ is convex and $0 \le g''(x) \le n : x \in R$. It follows that

$$\left| \frac{\overline{g(y+\delta)} - \overline{g(y)}}{\delta} - g'(y) \right| \le \frac{n}{2}\delta + \frac{2a}{\delta}. \tag{25}$$

The optimal value in (25), $\delta_{\text{opt}} = 2\sqrt{\frac{a}{n}}$, gives the bound

$$\left| \frac{\overline{g(y+\delta_{\text{opt}})} - \overline{g(y)}}{\delta_{\text{opt}}} - g'(y) \right| \le 2\sqrt{na}. \tag{26}$$

The simple "estimator" (25) can be used instead of the interpolational algorithm from [7], but its worst-case complexity seems to be higher than that from [7].

Theorem 3.6 *Given an n-tuple $\mathbf{K}$ of well-presented convex compact sets in R^n, there is a poly-time algorithm which computes the number $AV(\mathbf{K})$ such that*

$$\text{Prob}\left\{ 1 \le \frac{AV(\mathbf{K})}{V(\mathbf{K})} \le 2 \prod_{1 \le i \le n} \lambda\big(i, \min\big(i, \mathit{aff}(i)\big)\big) \le 2\frac{n^n}{n!} \right\} \ge .75.$$

The complexity of the algorithm, neglecting the log *terms, is bounded by* $O(n^{12}(\log(n) + \langle \mathbf{K} \rangle)^2)$.

Next, we focus on the case of Newton polytopes, in other words, polytopes with integer vertices, i.e., we consider the mixed volumes $V(\mathbf{P}) = V(P_1, \dots, P_n)$, where

$$P_i = \text{Hull}\big(\{v_{i,j} : 1 \le j \le m(i), v_{i,j} \in Z_+^n\}\big).$$

We define

$$d(i) = \min\{k : P_i \subset k\text{Hull}(0, e_1, \dots, e_n)\},$$

i.e., $d(i)$ is the maximum coordinate sum attained on P_i. It follows from the monotonicity of the mixed volume that $V(P_1, \dots, P_n) \le \prod_{1 \le i \le k} d(i)$. Such polytopes are well presented if, for instance, they are given as a list of poly(n) vertices. This case corresponds to a system of sparse polynomial equations. Notice that the value $V(P_1, \dots, P_n)$ is either zero or an integer (BKK Theorem) and the *capacity* $\text{Cap}(V_{\mathbf{P}}) \le \frac{n^n}{n!} \prod_{1 \le i \le k} d(i)$ (inequality (3)).

The next theorem is proved in the same way as Theorem 3.6.

Theorem 3.7 *Given an n-tuple of $\mathbf{P} = (P_1, \dots, P_n)$ of well-presented integer polytopes in R^n, there is a poly-time algorithm which computes the number $AV(\mathbf{P})$ such that*

$$\text{Prob}\left\{ 1 \le \frac{AV(\mathbf{P})}{V(\mathbf{P})} \le 2 \prod_{1 \le i \le n} \lambda\big(i, \min\big(i, \mathit{aff}(P_i)\big)\big) \le 2\frac{n^n}{n!} \right\} \ge .75.$$

The complexity of the algorithm, neglecting the log *terms, is bounded by* $O(n^9(n + \log(\prod_{1 \le i \le n} d_i))^2)$.

4 Open Problems

1. Prove that, for “random” convex sets, $\frac{V(K_1,\ldots,K_n)}{\mathrm{Cap}(V_{\mathbf{K}})} \le \frac{n!}{n^n} O(1)$ with high probability. This is true for the permanents of random matrices with nonnegative entries.
2. The most important question is whether or not there exists an FPRAS algorithm for the mixed volume (or for the mixed discriminant). We conjecture that the answer is negative.
3. Another important open problem is whether or not our mixed volume generalization (10) of Schrijver’s lower bound on the number of perfect matchings in regular bipartite graphs [16, 25, 27] is asymptotically sharp.

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Appendix A: Proofs of Theorems 2.1 and 2.4

A.1 Useful (and well Known) Facts

Fact A.1 *Let $\pi \in S_n$ be a permutation and $\mathbf{K} = (K_1, \ldots, K_n)$ be an n-tuple of convex compact sets in R^n. Then the next identity holds*:

$$V(K_1, \ldots, K_n) = V(K_{\pi(1)}, \ldots, K_{\pi(n)}). \tag{27}$$

Fact A.2 *We recall here the fundamental Alexandrov–Fenchel inequalities for the mixed volume of n convex sets in R^n*:

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \ge V(K_1, K_1, K_3, \ldots, K_n)V(K_2, K_2, K_3, \ldots, K_n). \tag{28}$$

Fact A.3 *Let $(K_1, \ldots, K_i), i < n-1$; S, T be convex compact sets in R^n. Define*

$$a_0 = V(K_1, \ldots, K_i, S, S, \ldots, S), \qquad a_1 = V(K_1, \ldots, K_i, S, S, \ldots, S, T),$$
$$a_{n-i} = V(K_1, \ldots, K_i, T, T, \ldots, T).$$

Then the univariate polynomial U defined by $U(t) = V(K_1, \ldots, K_i, S + tT, S + tT, \ldots, S + tT)$ is expressed as

$$U(t) = \sum_{0 \le j \le n-i} \binom{n-i}{j} a_j t^j. \tag{29}$$

It follows from Fact A.1 and the Alexandrov–Fenchel inequalities that this univariate polynomial U is $(n-i)$-Newton.

Fact A.4 *Let* $\mathbf{K} = (K_1, \dots, K_n)$ *be an n-tuple of convex compact sets in* R^n. *For a nonnegative vector* $(x_1, \dots, x_n)$, *define the convex compact subset* $K = \sum_{1 \le i \le n} x_i K_i$. *Then the following identity holds*:

$$\frac{\partial^i}{\partial x_1 \cdots \partial x_i} V_{\mathbf{K}}(x_1, \dots, x_n) = \frac{1}{(n-i)!} V(K_1, \dots, K_i, K, \dots, K). \tag{30}$$

Fact A.5 *If* A *is an* $n \times n$ *real matrix and* $\lambda_i > 0$, $1 \le i \le n$, *then*

$$V\big(\lambda_1(AK_1), \dots, \lambda_n(AK_n)\big) = \big|\det(A)\big| \Bigg(\prod_{1 \le i \le n} \lambda_i\Bigg) V(K_1, \dots, K_n). \tag{31}$$

Fact A.6 *Let* $S, T; K_2, \dots, K_n$ *be convex compact sets in* R^n. *The next* (*additivity*) *identity holds*:

$$V(S+T, K_2, \dots, K_n) = V(S, K_2, \dots, K_n) + V(T, K_2, \dots, K_n). \tag{32}$$

Fact A.7 *Let* $\mathbf{K} = (K_1, \dots, K_n)$ *be an n-tuple of convex compact sets in* R^n. *Then the degree of the variable* x_i *in the polynomial* $V_{\mathbf{K}}$ *is* $\deg_{V_{\mathbf{K}}}(i) = \mathit{aff}(i)$, *where* $\mathit{aff}(i)$ *is the affine dimension of* K_i, $1 \le i \le n$.

A.2 Auxiliary Univariate Inequality

Lemma A.8 *Let* $R(t) = \sum_{0 \le i \le k} a_i t^i$ *be an n-Newton polynomial*, $n \ge k$, *and*

$$\lambda(n,k) = \left(\min_{x>0}\left(\frac{sv_{n,k}(x)}{x}\right)\right)^{-1},$$

where the polynomial $sv_{n,k}(x) = 1 + \sum_{1 \le i \le k} (\frac{x}{n})^i \binom{n}{i}$.

Then the following inequality holds:

$$a_1 = R'(0) \ge \lambda(n,k) \inf_{t>0} \frac{R(t)}{t}. \tag{33}$$

Equality in (33) *is attained if and only if* $R(t) = R(0)(1 + \sum_{1 \le i \le k} (\frac{at}{n})^i \binom{n}{i})$.

If $n = k$, *then*

$$\lambda(n,n) = \left(\min_{x>0}\left(\frac{(1+\frac{x}{n})^n}{x}\right)\right)^{-1} = \left(\frac{n-1}{n}\right)^{n-1} =: g(n). \tag{34}$$

Equality in (34) *is attained iff* $R(t) = R(0)(1 + \frac{at}{n})^n$.

Proof Note that $\lambda(n,k) \le 1$, $k \ge 1$. If $R(0) = 0$, then clearly

$$R'(0) = \inf_{t>0} \frac{R(t)}{t} \ge \lambda(n,k) \inf_{t>0} \frac{R(t)}{t}.$$

Therefore we need to consider only the case $R(0) > 0$. Assume WLOG that $R(0) = 1$. It follows directly from the Newton inequalities (8) that

$$a_i \le \binom{n}{i}\left(\frac{a_1}{n}\right)^i, \quad 1 \le i \le k.$$

Thus

$$R(x) = \sum_{0 \le i \le k} a_i x^i \le 1 + \sum_{1 \le i \le k} \left(\frac{a_1 x}{n}\right)^i \binom{n}{i} = sv(a_1 x), \quad x \ge 0.$$

It also follows that

$$\inf_{t>0} \frac{R(t)}{t} \le \min_{x>0}\left(\frac{sv_{n,k}(a_1 x)}{x}\right) = a_1 \min_{x>0}\left(\frac{sv_{n,k}(x)}{x}\right),$$

which gives that

$$R'(0) = a_1 \ge \lambda(n,k) \inf_{t>0} \frac{R(t)}{t}.$$

The remaining statements can be now easily verified. □

Corollary A.9 *Denote by* $\mathrm{Hom}_+(n,n)$ *the convex cone of homogeneous polynomials with nonnegative coefficients of degree* n *in* n *variables. Consider the polynomials*

$$p \in \mathrm{Hom}_+(n,n); \quad q \in \mathrm{Hom}_+(n-1,n-1),$$
$$q(x_1,\dots,x_{n-1}) = \frac{\partial}{\partial x_n} p(x_1,\dots,x_{n-1},0).$$

Suppose that for all positive vectors $X = (x_1,\dots,x_{n-1})$, *the univariate polynomials* $R_X(t) = p(x_1,\dots,x_{n-1},t)$ *are* n-Newton. *Then the following inequality holds*:

$$\mathrm{Cap}(q) \ge \lambda\big(n, \deg_p(n)\big)\mathrm{Cap}(p). \tag{35}$$

Proof Note that because the coefficients of p are nonnegative, the (univariate) degree $\deg(R_X) = \deg_p(n)$ for all positive vectors $X \in R^{n-1}_{++}$. It follows from the definition of the polynomial $q \in \mathrm{Hom}_+(n-1,n-1)$ that

$$q(x_1,\dots,x_{n-1}) = R'_X(0).$$

It follows from the definition (1) of the *capacity* that

$$R_X(t) = p(x_1,\dots,x_{n-1},t) \ge \mathrm{Cap}(p)\Bigg(\prod_{1\le i\le n-1} x_i\Bigg)t.$$

It now follows from inequality (33) that

$$q(x_1,\dots,x_{n-1}) = R'_X(0) \ge \lambda\big(n,\deg_p(n)\big)\mathrm{Cap}(p) \prod_{1\le i\le n-1} x_i,$$

which gives that $\mathrm{Cap}(q) \ge \lambda(n,\deg_p(n))\mathrm{Cap}(p)$. □

A.3 Proof of Theorem 2.4

Proof Let $\mathbf{K}=(K_1,\dots,K_n)$ be an n-tuple of convex compact sets in R^n.
We associate with the Minkowski polynomial

$$V_{\mathbf{K}}\in \mathrm{Hom}_+(n,n),\qquad V_{\mathbf{K}}(x_1,\dots,x_n)=V_n(x_1K_1+\cdots+x_nK_n)$$

the following sequence of polynomials:

$$q_n=V_{\mathbf{K}};\quad q_i\in \mathrm{Hom}_+(i,i),$$
$$q_i(x_1,\dots,x_i)=\frac{\partial^{n-i}}{\partial x_{i+1}\cdots\partial x_n}q_n(x_1,\dots,x_i,0,\dots,0),\quad 1\le i\le n-1.$$

Note that

$$q_1(x)=x\frac{\partial^n}{\partial x_1\cdots\partial x_n}V_{\mathbf{K}}(0,\dots,0)=V(K_1,\dots,K_n)x$$

and

$$q_i(x_1,\dots,x_i)=\frac{\partial}{\partial x_{i+1}}q_{i+1}(x_1,\dots,x_i,0),\quad 1\le i\le n-1.$$

Note also the obvious (but useful) inequality

$$\deg_{q_i}(i)\le \min\bigl(i,\deg_{q_n}(i)\bigr)=\min\bigl(i,\mathit{aff}(i)\bigr). \tag{36}$$

It follows from Fact A.4 and Fact A.3 that the polynomials q_i, $2\le i\le n$, satisfy the conditions of Corollary A.9. Therefore

$$\begin{aligned}\mathrm{Cap}(q_i)&\ge \lambda\bigl(i+1,\deg_{q_{i+1}}(i+1)\bigr)\mathrm{Cap}(q_{i+1})\\&\ge \lambda\bigl(i+1,\min\bigl(i+1,\mathit{aff}(K_{i+1})\bigr)\mathrm{Cap}(q_{i+1}).\end{aligned} \tag{37}$$

(We use here inequality (36) and the fact that $\lambda(n,k)$ is strictly decreasing in both variables.)

Multiplying inequalities (37), we get inequality (9). □

A.4 Proof of Theorem 2.1

Proof of Inequality (3) Since

$$\lambda(i,k)\ge\lambda(i,i)=g(i)=:\left(\frac{i-1}{i}\right)^{i-1},\quad 1\le k\le i$$

hence we get from (9) that

$$V(K_1,\dots,K_n)\ge \mathrm{Cap}(V_{\mathbf{K}})\prod_{2\le i\le n}g(i)=\frac{n!}{n^n}\mathrm{Cap}(V_{\mathbf{K}}).$$

□

Proof of the Uniqueness Part of Theorem 2.1 As remarked above, we follow in the present paper the proof of uniqueness in [16].

1. Assume that $\mathrm{Cap}(V_{\mathbf{K}}) > 0$. Suppose that $l = \mathit{aff}(n) < n$. As $g(l) > g(n)$ hence

$$V(K_1, \dots, K_n) \geq \mathrm{Cap}(V_{\mathbf{K}}) g(l) \prod_{2 \leq i \leq n-1} g(i) > \frac{n!}{n^n} \mathrm{Cap}(V_{\mathbf{K}}).$$

Therefore if

$$V(K_1, \dots, K_n) = \frac{n!}{n^n} \mathrm{Cap}(V_{\mathbf{K}})$$

then $\mathit{aff}(n) = n$. Using the permutation invariance (27), we get that

$$\mathit{aff}(i) = n, \quad 1 \leq i \leq n.$$

In other words the convex compact sets K_i all have nonempty interior. This fact together and the monotonicity of the mixed volume imply that all coefficients in the Minkowski polynomial $V_{\mathbf{K}}$ are strictly positive.

2. *Scaling*. All coefficients in the Minkowski polynomial $V_{\mathbf{K}}$ are strictly positive, hence there exists an unique positive vector $(a_1, \dots, a_n)$ such that the scaled polynomial $p = V_{\{a_1 K_1, \dots, a_n K_n\}}$ is doubly stochastic (see [16]):

$$\frac{\partial}{\partial x_i} p(1, 1, \dots, 1) = 1, \quad 1 \leq i \leq n.$$

We will deal, without loss of generality, only with this doubly stochastic case.

3. *Brunn–Minkowski*. Let $(z_1, \dots, z_{n-1})$ be the unique minimizer of the problem

$$\min_{x_i > 0, 1 \leq i \leq n-1; \prod_{1 \leq i \leq n-1} x_i = 1} q_{n-1}(x_1, \dots, x_{n-1}).$$

Such an unique minimizer exists since all the coefficients of q_{n-1} are positive. It follows from Lemma A.8 and the proof of Lemma A.9 that

$$V_{\mathbf{K}}(z_1, \dots, z_{n-1}, t) = V(S + tK_n) = (at + b)^n, \quad S = \sum_{1 \leq i \leq n-1} z_i K_i$$

for some positive numbers a, b. It follows from the equality case of the Brunn–Minkowski inequality [5] that

$$K_n = \alpha S + \{T_n\}, \quad \alpha > 0, \ T_n \in R^n.$$

In other words, $K_n = \sum_{1 \leq j \leq n-1} A(n, j) K_j + \{T_n\}$, where $A_{n,j} > 0$, $1 \leq j \leq n-1$, and $T_n \in R^n$.

In the same way, we get that there exists an $n \times n$ matrix A with the zero diagonal and positive off-diagonal part, and vectors $T_1, \dots, T_n \in R^n$ such that

$$K_i = \sum_{j \neq i} A(i, j) K_j + \{T_i\}.$$

It follows from the doubly-stochasticity of the polynomial $V_{\mathbf{K}}$ that all row sums of the matrix A are equal to one. Indeed, using identity (30), we get that

$$(n-1)! = (n-1)!\frac{\partial}{\partial x_i} V_{\mathbf{K}}(1,1,\dots,1) = V(SUM, SUM, \dots, SUM, K_i);$$
$$SUM = K_1 + \cdots + K_n,\ 1 \le i \le n.$$

As $K_i = \sum_{j\neq i} A(i,j)K_j + \{T_i\}$, we get, using Fact A.6 and Fact A.5, that

$$(n-1)! = V\Big(SUM, SUM, \dots, SUM, \sum_{j\neq i} A(i,j)K_j + \{T_i\}\Big)$$
$$= \sum_{j\neq i} A(i,j)V(SUM, SUM, \dots, SUM, K_j) = (n-1)!\sum_{j\neq i} A(i,j).$$

Therefore $\sum_{j\neq i} A(i,j) = 1,\ 1 \le i \le n$.

4. Associate with the convex compact set $K_i \subset R^n$ its support function

$$\gamma_i(X) = \max_{Y \in K_i} \langle X, Y\rangle, X \in R^n.$$

We get that

$$\gamma_i(X) = \sum_{j\neq i} A(i,j)\gamma_j(X) + \langle X, T_i\rangle, \quad X \in R^n.$$

Since the kernel

$$\mathrm{Ker}(I-A) = \big\{Y \in R^n : (I-A)Y = 0\big\} = \big\{c(1,1,\dots,1), c \in R\big\},$$

it follows finally that

$$\gamma_i(X) = \alpha(X) + \langle X, L_j\rangle, \quad X \in R^n$$

for some functional $\alpha(X)$ and vectors $L_1, \dots, L_n \in R^n$. This means, in the doubly-stochastic case, that $K_i = K_1 + \{L_i - L_1\}$, $2 \le i \le n$. □

Appendix B: Inequalities for Minkowski and Minkowski-like Polynomials

Let $f : R^n_{++} \to R_{++}$ be a differentiable positive-valued functional defined on the strictly positive orthant R^n_{++}. We assume that f is n-homogeneous, i.e., that $f(ax_1, \dots, ax_n) = a^n f(x_1, \dots, x_n)$ and the partial derivatives $\frac{\partial}{\partial x_i} f(x_1, \dots, x_n) > 0$, $(x_1, \dots, x_n) \in R^n_{++}$. We denote the set of such homogeneous functionals by PoH(n).

We define the *capacity* as

$$\mathrm{Cap}(f) = \inf_{x_i > 0} \frac{f(x_1, \dots, x_n)}{\prod_{1\le i\le n} x_i} = \inf_{x_i>0, \prod_{1\le i\le n} x_i = 1} f(x_1, \dots, x_n).$$

We define two subsets of $\mathrm{PoH}(n)$:
$\mathrm{Cav}(n)$, consisting of $f \in \mathrm{PoH}(n)$ such that $f^{\frac{1}{n}}$ is concave on all half-lines $\{X + tY : t \geq 0\} : X, Y \in R^n_{++}$; $\mathrm{Vex}(n)$, consisting of $f \in \mathrm{PoH}(n)$ such that $f^{\frac{1}{n}}$ is convex on all half-lines $\{X + tY : t \geq 0\} : X, Y \in R^n_{++}$.

Recall the Brunn–Minkowski theorem: the Minkowski polynomial $V_{\mathbf{K}}(x_1, \dots, x_n)$ belongs to $\mathrm{Cav}(n)$. Therefore the results in this Appendix apply to the Minkowski polynomials.

We also define the following Generalized Sinkhorn Scaling:

$$SH(x_1, \dots, x_n) = (y_1, \dots, y_n) : y_i = \frac{f(x_1, \dots, x_n)}{\frac{\partial}{\partial x_i} f(x_1, \dots, x_n)} = \frac{x_i}{\gamma_i},$$

$$\gamma_i = \frac{x_i \frac{\partial}{\partial x_i} f(x_1, \dots, x_n)}{f(x_1, \dots, x_n)}.$$

Theorem B.1 *If* $f \in \mathrm{Cav}(n)$, *then the following inequality holds*:

$$f\big(SH(x_1, \dots, x_n)\big) \leq f(x_1, \dots, x_n). \tag{38}$$

If $f \in \mathrm{Vex}(n)$, *then the reverse inequality holds*:

$$f\big(SH(x_1, \dots, x_n)\big) \geq f(x_1, \dots, x_n). \tag{39}$$

Proof Let $X = (x_1, \dots, x_n) \in R^n_{++}$ and $Y = SH(x_1, \dots, x_n)$. We can assume without loss of generality that $f(X) = 1$. If $f \in \mathrm{Cav}(n)$, then the univariate function $g(t) = (f(X + tY))^{\frac{1}{n}}$ is concave for $t \geq 0$. Therefore

$$g(t) \leq \left(g(0) + \frac{g'(0)}{n} t\right)^n = \left(1 + \frac{g'(0)}{n} t\right)^n.$$

We get, by elementary calculus, that

$$g'(0) = \sum_{1 \leq i \leq n} \left(\frac{\partial}{\partial x_i} f(x_1, \dots, x_n)\right) x_i = n.$$

The functional f is n-homogeneous, hence $g(t) = t^n f(Y + t^{-1}X) \leq (1 + t)^n$, and finally $f(Y + t^{-1}X) \leq (\frac{1+t}{t})^n$. Taking the limit $t \to \infty$, we get $f(SH(x_1, \dots, x_n)) \leq 1 = f(x_1, \dots, x_n)$.

The convex case is proven in the very same way. □

Theorem (B.1) suggests the following algorithm to approximate $\mathrm{Cap}(f)$:

$$X_{n+1} = \mathrm{Nor}(SH(X_n)) : \mathrm{Nor}(x_1, \dots, x_n) = \left(\frac{x_1}{a}, \dots, \frac{x_n}{a}\right), \quad a = \sqrt[n]{\left(\prod_{1 \leq i \leq n} x_i\right)}. \tag{40}$$

Corollary B.2 *Consider* $f \in \mathrm{Cav}(n)$. *Suppose that* $\mathrm{Cap}(f) > 0$,

$$\log\bigl(\mathrm{Cap}(f)\bigr) \le \log\bigl(f(x_1, \dots, x_n)\bigr) \le \log\bigl(\mathrm{Cap}(f)\bigr) + \epsilon, \quad 0 < \epsilon \le \frac{1}{10},$$

and $\prod_{1 \le i \le n} x_i = 1;\ x_i > 0,\ 1 \le i \le n$. *Then*

$$\sum_{1 \le i \le n} \left(1 - \frac{x_i \frac{\partial}{\partial x_i} f(x_1, \dots, x_n)}{f(x_1, \dots, x_n)}\right)^2 \le 10\epsilon. \tag{41}$$

Proof Let $\gamma_i = \frac{x_i \frac{\partial}{\partial x_i} f(x_1,\dots,x_n)}{f(x_1,\dots,x_n)}$. It follows from Euler's identity that $\sum_{1 \le i \le n} \gamma_i = n$, and thus $\log(\prod_{1 \le i \le n} \gamma_i) \le 0$.

Inequality (38) can be rewritten as

$$f\left(\frac{x_1}{\gamma_1}, \dots, \frac{x_n}{\gamma_n}\right) \le f(x_1, \dots, x_n).$$

Therefore $\log(\mathrm{Cap}(f)) \le \mathrm{Cap}(f) + \epsilon + \log(\prod_{1 \le i \le n} \gamma_i)$, which gives the inequality

$$-\epsilon \le \log\left(\prod_{1 \le i \le n} \gamma_i\right) \le 0.$$

Finally, using Lemma 3.10 in [18], we see that $\sum_{1 \le i \le n} (1 - \gamma_i)^2 \le 10\epsilon$. □

Corollary B.2 generalizes (with a much more transparent proof) the corresponding results from [13] and [22].

The following Lemma proves inequality (19).

Lemma B.3

1. *Let* $p(t) = \sum_{0 \le i \le n} a_i t^i,\ a_i \ge 0$, *be a polynomial with nonnegative coefficients. Assume that* $\log(p(t))$ *is concave on* R_{++} *and define* $q(x) = \log(p(e^x))$. *Then* $q(x)$ *is convex on* R, *and its second derivative satisfies the inequality*

$$0 \le q''(x) \le n. \tag{42}$$

2. *Let* $p(t) = \sum_{0 \le i \le n} a_i t^i,\ a_i \ge 0$, *be a polynomial with nonnegative coefficients. Assume that* $p(t)^{\frac{1}{m}},\ m \ge n$, *is concave on* R_+. *Then*

$$0 \le q''(x) \le f(n, m), \tag{43}$$

where $f(n, m) = n - \frac{n^2}{m}$ *if* $n \le \frac{m}{2}$ *and* $f(n, m) = \frac{m}{4}$ *otherwise. If* $n = m$, *then* $f(n, m) = \frac{n}{4}$, *and the upper bound* (43) *is attained on polynomials* $p(t) = (a + tb)^n;\ a, b > 0$.

Proof 1. The convexity of $q(x)$ is well known. The concavity of $\log(p(t))$ is equivalent to the inequality $(p'(t))^2 \geq p(t)p''(t) : t \geq 0$. Putting $y = e^x$, we get that

$$q''(x) = \frac{p''(y)y^2}{p(y)} + \frac{p'(y)y}{p(y)} - \left(\frac{p'(y)y}{p(y)}\right)^2.$$

The concavity of $\log(p(t))$ gives that $\frac{p''(y)y^2}{p(y)} - (\frac{p'(y)y}{p(y)})^2 \leq 0$.

The coefficients of the polynomial p are nonnegative, hence $\frac{p'(y)y}{p(y)} \leq n$. This last observation proves that

$$q''(x) \leq \frac{p'(y)y}{p(y)} \leq n.$$

2. Our proof of (43) is a direct adaptation of the above proof of (42). We use the following characterization of the concavity of $p(t)^{\frac{1}{m}}$, $m \geq n$:

$$\big(p'(t)\big)^2 \geq \frac{m}{m-1} p(t)p''(t), \quad t \geq 0.$$

□

We note that just the nonnegativity of the coefficients implies the quadratic bound

$$q''(x) \leq 0.25n^2.$$

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