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A POLYNOMIAL-TIME TEST FOR TOTAL DUAL INTEGRALITY IN FIXED DIMENSION

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In this note we show that, for any fixed number r, there exists a polynomial-time algorithm to test whether a given system of linear inequalities $Ax \leqslant b$ is totally dual integral, where A is an integer matrix of rank r.

Key words: Polynomial-Time, Algorithm, Total Dual Integrality, Hilbert Basis, Integer Linear Programming.

1. Introduction

Let A be a rational $m \times n$-matrix, and let b be a rational vector of length m. The system $Ax \leqslant b$ of linear inequalities is called *totally dual integral* if the minimum in the linear programming duality equation

$$\max\{cx \mid Ax \leqslant b\} = \min\{yb \mid y \geqslant 0, yA = c\} \tag{1}$$

has an integer optimal solution for each integer vector c for which the optima exist (cx and yb denoting inner products). Edmonds and Giles [3] motivated this concept by showing that if $Ax \leqslant b$ is totally dual integral and b is integer, then also the maximum in (1) has an integer optimal solution, for each vector c for which the optima exist. Equivalently, then the polyhedron

$$P = \{x \mid Ax \leqslant b\} \tag{2}$$

is the convex hull of the lattice points contained in it.

In this note we show that, for any fixed number r, there exists an algorithm which tests the total dual integrality of a given system $Ax \leqslant b$ of linear inequalities, where A is an integer matrix of rank r, in time bounded by a polynomial in the sizes of

A and b (in binary notation). The main tool here is Lenstra's polynomial-time algorithm for integer linear programming in fixed dimension [7].

Total dual integrality is closely connected to so-called 'Hilbert bases'. Following Giles and Pulleyblank [4], a set of integer vectors $a_1, \ldots, a_k$ is called a *Hilbert basis* if each integer vector in the convex cone spanned by $a_1, \ldots, a_k$ is a nonnegative integer combination of $a_1, \ldots, a_k$. Indeed, the system $Ax \leqslant b$ is totally dual integral if and only if for each minimal face F of P the rows of A which are active in F form a Hilbert basis (a row is *active in* F if the corresponding inequality in $Ax \leqslant b$ is satisfied with equality by each vector in F). In particular, all rows of A form a Hilbert basis if and only if the system $Ax \leqslant 0$ is totally dual integral.

It is easy to see that each pointed rational polyhedral cone C is spanned by a unique minimal Hilbert basis (minimal under inclusion)—take all nonzero integer vectors in C which cannot be expressed as a sum of other integer vectors in C (this is a finite set, as these vectors are all contained in the polytope $\{\lambda_1 b_1 + \cdots + \lambda_t b_t \mid 0 \leqslant \lambda_i \leqslant 1\ (1 \leqslant i \leqslant t)\}$, where $b_1, \ldots, b_t$ are integer vectors spanning C). This implies that each full-dimensional rational polyhedron is defined by a unique minimal totally dual integral system with integer left-hand sides (cf. [8]).

Since, if we fix the rank r of A, the number of minimal faces of P is polynomially bounded by the number of rows of A, and since they can be enumerated in polynomial time, testing total dual integrality for fixed rank in polynomial time can be reduced to the test of being a Hilbert basis for fixed rank in polynomial time.

Chandrasekharan [1] proved that there exists an algorithm which finds for any totally dual integral system $Ax \leqslant b$, with A integer, and for any integer vector c, an integer optimum solution y for the minimum in (1), in time polynomially bounded by the sizes of A, b and c (in binary notation).

The property of being a totally dual integral system, and that of being a Hilbert basis, is in the class co-NP, if the left-hand side coefficients (i.e. the entries of A) are integral. That is, there exists a polynomial-length proof for not being totally dual integral, or for not being a Hilbert basis. Indeed, if the integer vectors $a_1, \ldots, a_k$ do not form a Hilbert basis, there exists an integer vector c in the cone C spanned by $a_1, \ldots, a_k$ such that $c - a_i \notin C$ for all i for which a_i does not belong to the minimal face F of C, and such that c does not belong to F. Note that, given a vector c, these conditions can be checked with Khachiyan's method. Moreover, c can be chosen in the polytope $\{\lambda_1 a_1 + \cdots + \lambda_k a_k \mid 0 \leqslant \lambda_i \leqslant 1\ (1 \leqslant i \leqslant k)\}$, so that the size of c is polynomially bounded by the size of $a_1, \ldots, a_k$.

In fact it is the content of the theorem that if we fix the rank of $a_1, \ldots, a_k$, such a vector c can be found in polynomial time. (Parts of the proof of this theorem also occur in [2].)

Theorem. *For any fixed natural number r, there exists an algorithm which tests, for any integer matrix A of rank r and for any rational vector b, whether the system $Ax \leqslant b$ is totally dual integral, in time bounded by a polynomial in the sizes of A and b (in binary notation).*

Proof. I. Let A have rows $a_1, \ldots, a_m$. We first argue that we may assume that $b=0$, and that the convex cone C spanned by $a_1, \ldots, a_m$ is pointed and full-dimensional (equivalently, that the cone $\{x \mid Ax \leqslant 0\}$ is pointed and full-dimensional).

Indeed, we have to test whether in each minimal face F of $P=\{x \mid Ax \leqslant b\}$, the active rows of A form a Hilbert basis. Now we can enumerate all minimal faces of P as follows. Enumerate all collections of r linearly independent rows of A. If, say, $a_1, \ldots, a_r$ are linearly independent, we check whether $F=\{x \mid a_1x=b_1, \ldots, a_rx=b_r\}$ is contained in P, which can be done by checking whether $Ax_0 \leqslant b$ holds for some solution x_0 of $a_1x=b_1, \ldots, a_rx=b_r$. If $F \subseteq P$, then F is a minimal face of P. Let, say, $a_1x \leqslant b_1, \ldots, a_kx \leqslant b_k$ be those inequalities among $Ax \leqslant b$ which are active in F (i.e., which hold with equality for some vector, and hence for each vector, in F). We have to test whether $a_1, \ldots, a_k$ is a Hilbert basis. That is, if A' denotes the matrix with rows $a_1, \ldots, a_k$, we have to test whether $A'x \leqslant 0$ is totally dual integral. Therefore, since for fixed r the number of minimal faces is polynomially bounded by the size of A, we may assume that $b=0$.

Next we show that we may assume that the convex cone C spanned by $a_1, \ldots, a_m$ is pointed and full-dimensional. Let F be the unique minimal face of C, and let L be the linear hull of C. Let, say, $a_1, \ldots, a_t$ be the rows of A belonging to F (they can be determined in polynomial time as a row a_i of A belongs to F if and only if $-a_i$ belongs to C). Let $d := \dim F$ (and note that $r=\dim L$). Now there exists a unimodular matrix U such that $FU=\mathbb{R}^d \times 0^{n-d}$, and $LU=\mathbb{R}^r \times 0^{n-r}$. Such a matrix U can be found as follows. Choose d linearly independent vectors $v_1, \ldots, v_d$ from $a_1, \ldots, a_t$, and after that choose $r-d$ vectors $v_{d+1}, \ldots, v_r$ from $a_{t+1}, \ldots, a_m$ such that $v_1, \ldots, v_r$ are linearly independent. Let V be the matrix with rows $v_1, \ldots, v_r$, and determine, with the algorithm of Kannan and Bachem [5], a unimodular matrix U such that VU is in (lower-triangular) Hermite normal form. One easily checks that U has the required properties.

Now $Ax \leqslant 0$ is totally dual integral if and only if $AUx \leqslant 0$ is totally dual integral. So we may assume that $A=AU$, and that $F=\mathbb{R}^d \times 0^{n-d}$ and $L=\mathbb{R}^r \times 0^{n-r}$. Since now the last $n-r$ columns of A are zero, we may assume that $n=r$. It is easy to see that $Ax \leqslant 0$ is totally dual integral if and only if $a_1, \ldots, a_t$ form a Hilbert basis for F, and $A'x \leqslant 0$ is totally dual integral, where A' is the matrix consisting of the last $n-d$ columns of A.

Now the fact that F is a linear space implies that $a_1, \ldots, a_t$ is a Hilbert basis for F if and only if F is the convex cone generated by $a_1, \ldots, a_t$ (which is a given fact, by definition of $a_1, \ldots, a_t$), and $F \cap \mathbb{Z}^n$ is the lattice generated by $a_1, \ldots, a_t$ (which can be checked again with the algorithm of Kannan and Bachem). Indeed, necessity of these conditions is immediate. To see sufficiency, let z be an integer vector in F. Then $z=\nu_1a_1+\cdots+\nu_ta_t$ for certain integers $\nu_1, \ldots, \nu_t$. Moreover, for each $i=1, \ldots, t$, the vector $-a_i$ belongs to F, and can hence be written as a convex combination of $a_1, \ldots, a_t$. Therefore, $0=\mu_1a_1+\cdots+\mu_ta_t$ for positive rationals $\mu_1, \cdots, \mu_t$. By choosing M appropriately, $z=(\nu_1+M\mu_1)a_1+\cdots+(\nu_t+M\mu_t)a_t$ is a decomposition of z as a nonnegative integer combination of $a_1, \ldots, a_t$.

So it suffices to describe a test of total dual integrality for $A'x \leq 0$. That is, without loss of generality we may assume that the cone C generated by $a_1, \ldots, a_m$ is full-dimensional and pointed.

II. Here we describe an algorithm to test whether $a_1, \ldots, a_m$ is a Hilbert basis for the cone C generated by $a_1, \ldots, a_m$, where C is pointed and full-dimensional.

We first observe that $a_1, \ldots, a_m$ is a Hilbert basis for C if and only if the only integer vector in the set

$$C_0 = \{z \in C \mid z - a_i \notin C \text{ for } i = 1, \ldots, m\} \tag{3}$$

is the origin.

To prove necessity, let z be an integer vector in C_0. Then $z = \lambda_1 a_1 + \cdots + \lambda_m a_m$ for nonnegative integers $\lambda_1, \ldots, \lambda_m$. As $z - a_i \notin C$ we know that $\lambda_1 = \cdots = \lambda_m = 0$, i.e., that $z = 0$.

To prove sufficiency, let z be an integer vector in C. Let λ_1 be the highest rational number such that $z - \lambda_1 a_1$ belongs to C. Next, let λ_2 be the highest rational number such that $z - \lfloor \lambda_1 \rfloor a_1 - \lambda_2 a_2$ belongs to C. If $\lambda_1, \ldots, \lambda_j$ have been found, let λ_{j+1} be the highest rational number such that $z - \lfloor \lambda_1 \rfloor a_1 - \lfloor \lambda_2 \rfloor a_2 - \cdots - \lfloor \lambda_j \rfloor a_j - \lambda_{j+1} a_{j+1}$ belongs to C. When $\lambda_1, \ldots, \lambda_m$ have been found, the vector $z - \lfloor \lambda_1 \rfloor a_1 - \cdots - \lfloor \lambda_m \rfloor a_m$ is an integer vector in C_0, and hence is the origin. This expresses z as a nonnegative integer combination of $a_1, \ldots, a_m$.

So it suffices to check whether the only integer vector in C_0 is the origin. To this end let $b_1, \ldots, b_t$ be vectors such that

$$C = \{z \mid b_j z \geq 0 \text{ for } j = 1, \ldots, t\}. \tag{4}$$

Since the rank $r = n$ of $a_1, \ldots, a_m$ is fixed, such $b_1, \ldots, b_t$ can be found in polynomial time (as each facet of C is determined by r linearly independent vectors from $a_1, \ldots, a_m$).

Now it follows trivially from (3) and (4) that

$$C_0 = \{z \in C \mid \text{for all } i = 1, \ldots, m \text{ there exists } j = 1, \ldots, t \text{ with } b_j z < b_j a_i\}. \tag{5}$$

So if Φ denotes the collection of all functions $\phi : \{1, \ldots, m\} \to \{1, \ldots, t\}$, then

$$C_0 = \bigcup_{\phi \in \Phi} \{z \mid b_j z \geq 0 \text{ for } j = 1, \ldots, t, \text{ and } b_{\phi(i)} z < b_{\phi(i)} a_i \text{ for } i = 1, \ldots, m\}. \tag{6}$$

This expresses C_0 as a union of convex sets, and we have to test whether the only integer vector in each of these convex sets is the origin. Below we shall see that this can be done in polynomial time (for fixed dimension) with Lenstra's algorithm for integer linear programming [7]. (Note that Φ generally has exponential size, even for fixed rank r of $a_1, \ldots, a_m$.)

Let Z be the collection of vectors z determined by n linearly independent equations from:

$$\begin{aligned} b_j z &= 0 \qquad (j = 1, \ldots, t), \\ b_j z &= b_j a_i \quad (j = 1, \ldots, t;\ i = 1, \ldots, m). \end{aligned} \tag{7}$$

Since n is fixed, we can enumerate and store Z in polynomial time. Next let Σ be the collection of all subsets $\{z_1, \ldots, z_n\}$ of Z such that:

$$\begin{aligned}&\text{(i)}\ z_1, \ldots, z_n \text{ are linearly independent,}\\ &\text{(ii)}\ z_1, \ldots, z_n \text{ belong to } C,\\ &\text{(iii)}\ \forall i = 1, \ldots, m\ \exists j = 1, \ldots, t\ \forall k = 1, \ldots, n\colon b_j z_k \leqslant b_j a_i.\end{aligned} \tag{8}$$

Again, Σ can be enumerated and stored in polynomial time. Define

$$\sigma(z_1, \ldots, z_n) = \text{convex hull}\{0, z_1, \ldots, z_n\} \backslash \text{convex hull}\{z_1, \ldots, z_n\}, \tag{9}$$

for $\{z_1, \ldots, z_n\}$ in Σ. We finally show:

$$C_0 = \bigcup_{\{z_1, \ldots, z_n\} \in \Sigma} \sigma(z_1, \ldots, z_n). \tag{10}$$

we are finished as soon as we have proved (10): with Lenstra's algorithm we can test, for each $\{z_1, \ldots, z_n\}$ in Σ, whether $\sigma(z_1, \ldots, z_n) \backslash \{0\}$ contains integer vectors. Hence we can test whether C_0 contains integer vectors other than the origin.

To prove (10), first observe that C_0 is bounded. Indeed, $C_0 \subseteq \{\lambda_1 a_1 + \cdots + \lambda_m a_m \mid 0 \leqslant \lambda_i < 1 \text{ for } i = 1, \ldots, m\}$. Now let $w \in C_0$. Then by (6) there exists a function ϕ in Φ such that w belongs to the convex set

$$P = \{z \mid b_j z \geqslant 0 \text{ for } j = 1, \ldots, t, \text{ and } b_{\phi(i)} z < b_{\phi(i)} a_i \text{ for } i = 1, \ldots, m\}. \tag{11}$$

Since P is bounded and nonempty, it remains bounded if we replace in (11) the sign $<$ by $\leqslant$, thus obtaining the closure $\bar{P}$ of P. Since $w \in P$ there exists an $\varepsilon > 0$ such that $(1+\varepsilon)w$ belongs to P. As $0 \in \bar{P}$, $(1+\varepsilon)w$ is a convex combination of some linearly independent vertices $z_1, \ldots, z_n$ of $\bar{P}$. Then $\{z_1, \ldots, z_n\} \in \Sigma$ and $w \in \sigma(z_1, \ldots, z_n)$.

To prove the reverse inclusion for (10), let $\{z_1, \ldots, z_n\} \in \Sigma$ and $w \in \sigma(z_1, \ldots, z_n)$. There exists $\varepsilon > 0$ such that $(1+\varepsilon)w \in \sigma(z_1, \ldots, z_n)$. By (8) (ii) w belongs to C. Moreover, by (8) (iii), there exists a function ϕ in Φ such that $b_{\phi(i)} z_k \leqslant b_{\phi(i)} a_i$ for $i = 1, \ldots, m$ and $k = 1, \ldots, n$. Since $b_j z_k \geqslant 0$ for all $j = 1, \ldots, t$ and $k = 1, \ldots, n$, and since $z_1, \ldots, z_n$ are linearly independent, we know that $b_{\phi(i)} a_i > 0$ for all $i = 1, \ldots, m$. Therefore, $(1+\varepsilon) b_{\phi(i)} w \leqslant b_{\phi(i)} a_i$ for $i = 1, \ldots, m$, implies that $b_{\phi(i)} w < b_{\phi(i)} a_i$ for $i = 1, \ldots, m$, and hence that w belongs to C_0. □

The problem is still open whether testing total dual integrality is co-NP complete or polynomially solvable (or both).

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