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A PORO-THERMOELASTIC PROBLEM WITH DISSIPATIVE HEAT CONDUCTION

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Abstract: In this work, we study from the mathematical and numerical points of view a poro-thermoelastic problem. A long-term memory is assumed on the heat equation. Under some assumptions on the constitutive tensors, the resulting linear system is composed of hyperbolic partial differential equations with a dissipative mechanism in the temperature equation. An existence and uniqueness result is proved using the theory of contractive semigroups. Then, a fully discrete approximation is introduced applying the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives. A discrete stability property is obtained. A priori error estimates are also shown, from which the linear convergence of the approximation is derived under suitable additional regularity conditions. Finally, one- and two-numerical simulations are presented to demonstrate the accuracy of the algorithm and the behavior of the solution.

Keywords: Thermoelasticity, porosity, dissipative mechanism, finite elements, discrete stability, a priori error estimates.

1. INTRODUCTION

We believe that the porous thermoelasticity is the easiest extension of the classical theory of thermoelasticity. Apart of the usual elastic deformations and temperature, it incorporates the volume fraction, which determines the porous structure of the material. The linear isothermal theory was proposed by Cowin and Nunziato [1, 2], but it is also worth recalling the contribution of Ieşan [3], where thermal considerations are incorporated to the linear case. Porous thermoelasticity is a theory widely accepted, and the quantity of contributions involving it is huge (see, among others, [4, 5]), because it describes the evolution of thermoelastic solids with small distributed pores. Among the physical applications of this theory, we can point out rocks, soils, ceramics or woods, as well as biological materials as bones or dentures. In recent years, a big deal has been developed to understand the time decay of perturbations [6, 7, 8, 9, 10, 11, 12, 13, 14].

Heat conduction theory based on classical Fourier law is very often used in engineering studies; however, this proposition violates the “causality principle”. Indeed, it is well known that inside the Fourier theory the thermal waves propagate instantaneously. This fact is not well accepted and many scientists have proposed alternative theories to overcome this paradox. The most

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known theory is the one based on the Cattaneo-Maxwell law, which introduces a relaxation parameter in the Fourier law and which brings to a hyperbolic heat equation. Other theories have been proposed [15, 16, 17]. We here want to emphasize the one considered by Gurtin [18]. In this case, the author imposes a thermoelastic theory where the entropy is invariant under time reversal. It is worth noting that, in comparison with other non-classical heat conduction theories, the one suggested by Gurtin has deserved few attention. Although it can be extended to other thermomechanical situations, the number of contributions that we can find in the literature is low [19, 20].

In this paper, we start with the general equations in the porous thermoelasticity which can be obtained from the papers of Gurtin [18] and Ciarletta and Scalia [21]. Then, we propose a particular sub-class for the constitutive relaxation functions in such a way that we can recover the recent proposition called as Moore-Gibson-Thompson thermoelasticity. We believe that this fact is remarkable as well as it is worth recalling that this theory has deserved a big interest in the last two years (see, for instance, [22, 23, 24]).

Here, we consider the poro-thermoelastic theory based on the heat conduction of the Moore-Gibson-Thompson type. First, we recall the basic equations for the theory. An existence and uniqueness result combined with the stability of the solutions is obtained. Later, using the finite element method to approximate the spatial variable and the implicit Euler scheme to discretize the time derivatives, fully discrete approximations are introduced. A discrete stability property and a priori error estimates are shown, from which the linear convergence of the algorithm is deduced under suitable additional regularity conditions. Finally, some one- and two-dimensional numerical simulations are provided to demonstrate the accuracy of the approximation and the behavior of the solution.

2. BASIC EQUATIONS

In this section we describe briefly the poro-thermomechanical problem (see [18, 21] for details).

Let $B \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the domain which occupies the thermoelastic material and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body, denoted by ∂B , is assumed to be Lipschitz. Moreover, let $\boldsymbol{x} \in B$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x} = (x_i)_{j=1}^d$ and t , and a subscript after a comma, under a variable, represents its spatial derivative with respect to the prescribed variable, that is $f_{i,j} = \frac{\partial f_i}{\partial x_j}$. The time derivatives are represented as a dot for the first order and two dots for the second order, over each variable. Finally, as usual the repeated index notation is used for the summation.

We recall that the evolutions equations are the following:

$$\begin{aligned}\rho \ddot{\boldsymbol{u}}_i &= t_{ij,j}, \\ J \ddot{\phi} &= h_{i,i} + g, \\ T_0 \dot{\eta} &= q_{i,i},\end{aligned}$$

where $\boldsymbol{u} = (u_i)_{i=1}^d$, t_{ij} , ϕ and η denote the displacement vector, the stress tensor, the volume fraction and the entropy, respectively, and ρ , J , h_i , g , T_0 and q are the mass density, the equilibrated inertia, the equilibrated stress, the equilibrated body force, the reference temperature (that we assume equal to one to simplify the calculations) and the heat flux vector, respectively.

Following the thermoelastic theory proposed by Gurtin and restricting our attention to materials with a center of symmetry, the constitutive equations are:

$$\begin{aligned} t_{ij} &= \int_{-\infty}^t \left[G_{ijmn}(t-s) \dot{e}_{mn}(s) + B_{ij}(t-s) \dot{\phi}(s) - l_{ij}(t-s) \dot{\theta}(s) \right] ds, \\ h_i &= \int_{-\infty}^t \left[A_{ij}(t-s) \dot{\phi}_{,j}(s) + M_{ij}(t-s) \theta_{,j}(s) \right] ds, \\ g &= \int_{-\infty}^t \left[-B_{ij}(t-s) \dot{e}_{ij}(s) - b(t-s) \dot{\phi}(s) + m(t-s) \dot{\theta}(s) \right] ds, \\ \eta &= \int_{-\infty}^t \left[l_{ij}(t-s) \dot{e}_{ij}(s) + a(t-s) \dot{\theta}(s) + m(t-s) \dot{\phi}(s) \right] ds, \\ q_i &= \int_{-\infty}^t \left[k_{ij}(t-s) \theta_{,j}(s) + M_{ji}(t-s) \dot{\phi}_{,j}(s) \right] ds, \end{aligned}$$

where θ represents the temperature field and G_{ijmn} , B_{ij} , l_{ij} , A_{ij} , M_{ij} and k_{ij} are constitutive tensors, and b , m and a are constitutive functions.

In this theory it can be proved that

$$G_{ijmn} = G_{mni}, \quad l_{ij} = l_{ji}, \quad A_{ij} = A_{ji}, \quad k_{ij} = k_{ji},$$

In this paper we restrict our attention to the case that the constitutive tensors and functions are given in the following form:

$$\begin{aligned} G_{ijmn}(\mathbf{x}, s) &= G_{ijmn}(\mathbf{x}), \quad B_{ij}(\mathbf{x}, s) = B_{ij}(\mathbf{x}), \quad l_{ij}(\mathbf{x}, s) = l_{ij}(\mathbf{x}), \quad A_{ij}(\mathbf{x}, s) = A_{ij}(\mathbf{x}), \\ M_{ij}(\mathbf{x}, s) &= M_{ij}(\mathbf{x}), \quad m(\mathbf{x}, s) = m(\mathbf{x}), \quad a(\mathbf{x}, s) = a(\mathbf{x}), \quad b(\mathbf{x}, s) = b(\mathbf{x}), \\ k_{ij}(\mathbf{x}, s) &= k_{ij}^*(\mathbf{x}) + (\tau^{-1} k_{ij}(\mathbf{x}) - k_{ij}^*(\mathbf{x})) e^{-\tau^{-1}s}, \end{aligned}$$

where τ is a positive constant.

Hence, we obtain the system:

$$(2.1) \quad \begin{cases} \rho \ddot{u}_i = \left(G_{ijmn} u_{m,n} + B_{ij} \phi - l_{ij}(\theta + \tau \dot{\theta}) \right)_{,j}, \\ J \ddot{\phi} = (A_{ij} \phi_{,j} + M_{ij}(\alpha_{,j} + \tau \theta_{,j}))_{,i} - B_{ij} u_{i,j} - b \phi + m(\theta + \tau \dot{\theta}), \\ \tau a \ddot{\theta} + a \dot{\theta} = -l_{ij} \dot{u}_{i,j} - m \dot{\phi} + \left(M_{ji} \phi_{,j} + k_{ij}^* \alpha_{,j} + k_{ij} \theta_{,j} \right)_{,i}, \end{cases}$$

where α is the thermal displacement given by

$$(2.2) \quad \alpha(\mathbf{x}, t) = \alpha^0(\mathbf{x}) + \int_0^t \theta(\mathbf{x}, s) ds$$

for a given function α^0 .

This is a system of hyperbolic partial differential equations with a dissipative mechanism at the temperature. In the following section, we will obtain existence and uniqueness of solutions under suitable assumptions.

Remark 2.1. If we assume that the material is isotropic and homogeneous, then system (2.1) becomes

$$(2.3) \quad \begin{cases} \rho \ddot{u}_i = \mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + B \phi_{,i} - l(\theta_{,i} + \tau \dot{\theta}_{,i}), \\ J \ddot{\phi} = A \phi_{,jj} + M(\alpha_{,ii} + \tau \theta_{,jj}) - B u_{i,i} - b \phi + m(\theta + \tau \dot{\theta}), \\ \tau a \ddot{\theta} + a \dot{\theta} = -l \dot{u}_{i,i} - m \dot{\phi} + M \phi_{,ii} + K^* \alpha_{,ii} + K \theta_{,jj}, \end{cases}$$

where the thermal displacement is obtained from equation (2.2).

In the one-dimensional case, the previous system reads:

$$(2.4) \quad \begin{cases} \rho\ddot{u} = \mu u_{xx} + B\phi_x - l(\theta + \tau\dot{\theta})_x, \\ J\ddot{\phi} = A\phi_{xx} + M(\alpha_{xx} + \tau\theta_{xx}) - Bu_x - b\phi + m(\theta + \tau\dot{\theta}), \\ \tau a\ddot{\theta} + a\dot{\theta} = -l\dot{u}_x - m\dot{\phi} + M\phi_{xx} + K^*\alpha_{xx} + K\theta_{xx}. \end{cases}$$

To determine a problem we need to impose initial and boundary conditions. Thus, we assume, for a.e. $\mathbf{x} \in B$,

$$(2.5) \quad \begin{cases} u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), & \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), & \phi(\mathbf{x}, 0) = \phi^0(\mathbf{x}), & \dot{\phi}(\mathbf{x}, 0) = \psi^0(\mathbf{x}), \\ \alpha(\mathbf{x}, 0) = \alpha^0(\mathbf{x}), & \dot{\alpha}(\mathbf{x}, 0) = \theta^0(\mathbf{x}), & \ddot{\alpha}(\mathbf{x}, 0) = \xi^0(\mathbf{x}), \end{cases}$$

where functions $\mathbf{u}^0 = (u_i^0)_{i=1}^d$, $\mathbf{v}^0 = (v_i^0)_{i=1}^d$, ϕ^0 , ψ^0 , θ^0 and ξ^0 are prescribed functions, and

$$(2.6) \quad u_i(\mathbf{x}, t) = \phi(\mathbf{x}, t) = \alpha(\mathbf{x}, t) = 0 \quad \text{for a.e. } \mathbf{x} \in \partial B.$$

We state now the assumptions on the constitutive coefficients. First, we impose that all of them are bounded and also that

- (i) $\rho(\mathbf{x}) \geq \rho_0 > 0$, $J(\mathbf{x}) \geq J_0 > 0$, $a(\mathbf{x}) \geq a_0 > 0$, $b(\mathbf{x}) \geq b_0 > 0$.
- (ii) There exists a positive constant C_1 such that

$$\int_B (G_{ijmn}u_{i,j}u_{m,n} + 2B_{ij}u_{i,j}\phi + b\phi^2) d\mathbf{x} \geq C_1 \int_B (u_{i,j}u_{i,j} + \phi^2) d\mathbf{x},$$

for every vector u_i vanishing on the boundary of B and every function ϕ .

- (iii) There exists a positive constant C_2 such that

$$A_{ij}\phi_i\phi_j + 2M_{ij}\phi_i\xi_j + k_{ij}^*\xi_i\xi_j \geq C_2(\phi_i\phi_i + \xi_i\xi_i),$$

for every vectors (ϕ_i) , (ξ_i) .

- (iv) There exists a positive constant C_3 such that

$$\bar{k}_{ij}\xi_i\xi_j \geq C_3\xi_i\xi_i,$$

for every vector ξ_i where $\bar{k}_{ij} = k_{ij} - \tau k_{ij}^*$.

We note that the previous conditions are natural. The physical meaning of condition (i) is obvious. Condition (ii) can be interpreted with the help of the *elastic stability* meanwhile condition (iii) is also needed to guarantee that the internal energy of the system is positive. Condition (iv) is usual in the context of the problems related with viscoelasticity (see [22]).

3. EXISTENCE AND UNIQUENESS OF SOLUTION

In this section we assume that the assumptions (i)-(iv) hold. Our aim is to show an existence and uniqueness theorem for the solutions to the problem determined by system (2.1) with initial conditions (2.5) and boundary conditions (2.6).

We study our problem in the Hilbert space:

$$\mathcal{H} = [H_0^1(B)]^d \times [L^2(B)]^d \times H_0^1(B) \times L^2(B) \times H_0^1(B) \times H_0^1(B) \times L^2(B).$$

We define the inner product:

$$\begin{aligned} <(\mathbf{u}, \mathbf{v}, \phi, \psi, \alpha, \theta, \xi), (\mathbf{u}^*, \mathbf{v}^*, \phi^*, \psi^*, \alpha^*, \theta^*, \xi^*)> = & \frac{1}{2} \int_B \left(\rho v_i \overline{v_i^*} + G_{ijmn} u_{i,j} \overline{u_{m,n}^*} + b\phi \overline{\phi^*} \right. \\ & B_{ij}(u_{i,j} \overline{\phi^*} + \overline{u_{i,j}^*} \phi) + M_{ij}((\alpha_{,i} + \tau \theta_{,i}) \overline{\phi_{,j}} + \overline{(\alpha_{,i}^* + \tau \theta_{,i}^*)} \phi_{,j}) + A_{ij} \phi_{,i} \overline{\phi_{,j}} + \tau \bar{k}_{ij} \theta_{,i} \overline{\theta_{,j}} \\ & \left. + J \psi \overline{\psi^*} + a(\tau \xi + \theta)(\tau \xi^* + \theta^*) + k_{ij}^*(\alpha_{,i} + \tau \theta_{,i})(\alpha_{,j}^* + \tau \theta_{,j}^*) \right) d\mathbf{x}. \end{aligned}$$

We note that this inner product is equivalent to the usual one in the Hilbert space \mathcal{H} .

We define the operators:

$$\begin{aligned} A_i \mathbf{u} &= \frac{1}{\rho} (G_{ijmn} u_{m,n})_{,j}, \quad B_i \phi = \frac{1}{\rho} (B_{ij} \phi)_{,j}, \\ C_i \theta &= \frac{-1}{\rho} (l_{ij} \theta)_{,j}, \quad D_i \xi = \frac{-1}{\rho} (\tau l_{ij} \xi)_{,j}, \\ E \phi &= \frac{1}{J} [(A_{ij} \phi_{,j})_{,i} - b\phi], \quad F \alpha = \frac{-1}{J} (M_{ij} \alpha_{,j})_{,i}, \\ G \theta &= \frac{1}{J} [(\tau M_{ij} \theta_{,j})_{,i} + m\theta], \quad H \mathbf{u} = \frac{-1}{J} B_{ij} u_{i,j}, \\ K \psi &= \frac{1}{J} m \tau \psi, \quad L \mathbf{v} = \frac{-1}{\tau a} l_{ij} v_{i,j}, \\ M^* \psi &= \frac{-m}{\tau a} \psi, \quad N \phi = \frac{1}{\tau a} (M_{ji} \phi_{,j})_{,i}, \\ P \alpha &= \frac{1}{\tau a} (k_{ij}^* \alpha_{,j})_{,i}, \quad Q \theta = \frac{1}{\tau a} (k_{ij} \theta_{,j})_{,i}, \\ R \xi &= \tau^{-1} \xi, \quad \mathbf{A} = (A_i), \quad \mathbf{B} = (B_i), \quad \mathbf{C} = (C_i), \quad \mathbf{D} = (D_i), \end{aligned}$$

and the matrix operator:

$$\mathcal{A} = \begin{pmatrix} 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 \\ \mathbf{A} & 0 & \mathbf{B} & 0 & 0 & \mathbf{C} & \mathbf{D} \\ 0 & 0 & 0 & \mathbf{I} & 0 & 0 & 0 \\ H & 0 & E & 0 & F & G & K \\ 0 & 0 & 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{I} \\ 0 & L & N & M^* & P & Q & R \end{pmatrix}.$$

So, we can write our problem as

$$\frac{d\mathbf{U}}{dt} = \mathcal{A}\mathbf{U}, \quad \mathbf{U}(0) = \mathbf{U}^0,$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \phi, \psi, \alpha, \theta, \xi)$ and $\mathbf{U}^0 = (\mathbf{u}^0, \mathbf{v}^0, \phi^0, \psi^0, \alpha^0, \theta^0, \xi^0)$.

We note that the domain of the operator \mathcal{A} is the subset of the Hilbert space \mathcal{H} such that

$$\begin{aligned} \mathbf{v} \in [H_0^1(B)]^d, \quad \psi, \theta, \xi \in H_0^1(B), \quad \mathbf{u} \in [H^2(B)]^d, \\ E\phi + F\alpha + G\theta \in L^2(B), \quad N\phi + P\alpha + Q\theta \in L^2(B). \end{aligned}$$

It is clear that it is a dense subspace of \mathcal{H} .

Theorem 3.1. *The operator \mathcal{A} generates a contractive semigroup.*

Proof. We have seen that the domain of \mathcal{A} is dense. We also have

$$Re < \mathcal{A}\mathbf{U}, \mathbf{U} > = -\frac{1}{2} \int_B \bar{k}_{ij} \theta_{,i} \theta_{,j} d\mathbf{x} \leq 0.$$

Therefore, in order to prove the theorem, it will be sufficient to see that zero belongs to the resolvent of the operator. We consider $(\mathbf{f}_1, \mathbf{f}_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}$. So, we just need to prove that the system

$$\begin{aligned} \mathbf{v} &= \mathbf{f}_1, & \psi &= f_3, & \theta &= f_5, & \xi &= f_6, \\ \mathbf{A}\mathbf{u} + \mathbf{B}\phi + \mathbf{C}\theta + \mathbf{D}\xi &= \mathbf{f}_2, \\ H\mathbf{u} + E\phi + F\alpha + G\theta + K\xi &= f_4, \\ L\mathbf{v} + N\phi + M^*\psi + P\alpha + Q\theta + R\xi &= f_7, \end{aligned}$$

has a solution in the domain of the operator \mathcal{A} . It is clear that we obtain a solution for \mathbf{v}, ψ, θ and ξ . Therefore, it follows that

$$\begin{aligned} \mathbf{A}\mathbf{u} + \mathbf{B}\phi &= \mathbf{f}_2 - \mathbf{C}f_5 - \mathbf{D}f_6 = \mathbf{F}_1, \\ H\mathbf{u} + E\phi + F\alpha &= f_4 - Gf_5 - Kf_6 = F_2, \\ N\phi + P\alpha &= f_7 - L\mathbf{f}_1 - M^*f_3 - Qf_5 - Kf_6 = F_3. \end{aligned}$$

We see that $(\mathbf{F}_1, F_2, F_3) \in [H^{-1}(B)]^d \times H^{-1}(B) \times H^{-1}(B)$.

On the other side, if we define the bilinear form

$$\mathcal{B}[(\mathbf{u}, \phi, \alpha), (\mathbf{u}^*, \phi^*, \alpha^*)] = <\mathbf{A}\mathbf{u} + \mathbf{B}\phi, \mathbf{u}^*> + < H\mathbf{u} + E\phi + F\alpha, \phi^*> + < N\phi + P\alpha, \alpha^*>,$$

it is clear that it is bounded. On the other side, it follows that

$$\mathcal{B}[(\mathbf{u}, \phi, \alpha), (\mathbf{u}, \phi, \alpha)] = \int_B [G_{ijmn}u_{i,j}u_{m,n} + 2B_{ij}u_{i,j}\phi + b\phi^2 + A_{ij}\phi_{,i}\phi_{,j} + 2M_{ij}\phi_{,i}\alpha_{,j} + k_{ij}^*\alpha_{,i}\alpha_{,j}] dx,$$

and therefore, we see that this form \mathcal{B} is coercive. In view of the Lax-Milgram lemma we conclude the proof. \square

We note that the above theorem leads to the following existence and uniqueness result.

Theorem 3.2. *If the constitutive functions are bounded and assumptions (i)-(iv) hold, for each $(\mathbf{u}^0, \mathbf{v}^0, \phi^0, \psi^0, \alpha^0, \theta^0, \xi^0)$ in the domain of \mathcal{A} there exists a unique solution to problem (2.1), (2.5) and (2.6).*

4. FULLY DISCRETE APPROXIMATIONS: AN A PRIORI ERROR ANALYSIS

In order to introduce a finite element approximation of the problem determined by system (2.1) with initial conditions (2.5) and boundary conditions (2.6), we obtain first its variational formulation.

Let $Y = L^2(B)$, $H = [L^2(B)]^d$ and $Q = [L^2(B)]^{d \times d}$ and denote by $(\cdot, \cdot)_Y$, $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational spaces V and E as follows,

$$\begin{aligned} V &= \{\mathbf{w} \in [H^1(B)]^d; \mathbf{w} = \mathbf{0} \text{ on } \partial B\}, \\ E &= \{z \in H^1(B); z = 0 \text{ on } \partial B\}, \end{aligned}$$

with respective scalar products $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_E$, and norms $\|\cdot\|_V$ and $\|\cdot\|_E$.

By using Green's formula and boundary conditions (2.6), we write the variational formulation of problem (2.1), (2.5) and (2.6) in terms of the variables $\mathbf{v} = \dot{\mathbf{u}}$, $\psi = \dot{\phi}$ and $\xi = \dot{\theta}$.

Find the velocity field $\mathbf{v} : [0, T] \rightarrow V$, the volume fraction speed $\psi : [0, T] \rightarrow E$ and the temperature speed $\xi : [0, T] \rightarrow E$ such that $\mathbf{v}(0) = \mathbf{v}^0$, $\psi(0) = \psi^0$, $\xi(0) = \xi^0$, and, for a.e. $t \in (0, T)$ and for all $\mathbf{w} \in V$, $r, z \in E$,

$$(4.1) \quad (\rho \dot{\mathbf{v}}(t), \mathbf{w})_H + G(\mathbf{u}(t), \mathbf{w}) + B(\phi(t), \mathbf{w}) = L(\theta(t) + \tau \xi(t), \mathbf{w}),$$

$$(4.2) \quad \begin{aligned} (J\dot{\psi}(t), r)_Y + A(\phi(t), r) + M(\alpha(t) + \tau \theta(t), r) + (b\phi(t), r)_Y + B(r, \mathbf{u}(t)) \\ = m(\theta(t) + \tau \xi(t), r)_Y, \end{aligned}$$

$$(4.3) \quad \begin{aligned} (\tau a\dot{\xi}(t) + a\xi(t), z)_Y + K(\theta(t), z) + K^*(\alpha(t), z) + M(\phi(t), z) \\ = -L(z, \mathbf{v}(t)) - m(\psi(t), z), \end{aligned}$$

where the displacement field \mathbf{u} , the volume fraction ϕ , the temperature θ and the thermal displacement α are then recovered from the relations:

$$(4.4) \quad \begin{aligned} \mathbf{u}(t) &= \int_0^t \mathbf{v}(s) ds + \mathbf{u}^0, \quad \phi(t) = \int_0^t \psi(s) ds + \phi^0, \\ \theta(t) &= \int_0^t \xi(s) ds + \theta^0, \quad \alpha(t) = \int_0^t \theta(s) ds + \alpha^0, \end{aligned}$$

and we have used the following bilinear operators:

$$\begin{aligned} G(\mathbf{u}, \mathbf{w}) &= (G_{ijmn} u_{m,n}, w_{i,j})_Y \quad \forall \mathbf{u}, \mathbf{w} \in V, \\ B(r, \mathbf{w}) &= (B_{ij} w_{i,j}, r)_Y \quad \forall \mathbf{w} \in V, r \in E, \\ L(r, \mathbf{w}) &= (L_{ij} w_{i,j}, r)_Y \quad \forall r \in E, \mathbf{w} \in V, \\ A(r, z) &= (A_{ij} r_{,j}, z_{,i})_Y \quad \forall r, z \in E, \\ M(r, z) &= (M_{ij} r_{,j}, z_{,i})_Y \quad \forall r, z \in E, \\ K(r, z) &= (k_{ij} r_{,j}, z_{,i})_Y \quad \forall r, z \in E, \\ K^*(r, z) &= (k_{ij}^* r_{,j}, z_{,i})_Y \quad \forall r, z \in E. \end{aligned}$$

Now, we consider a fully discrete approximation of problem (4.1)-(4.4). This is done in two steps. First, we assume that the domain \overline{B} is polyhedral and we denote by \mathcal{T}^h a regular triangulation in the sense of [27]. Thus, we construct the finite dimensional spaces $V^h \subset V$ and $E^h \subset E$ given by

$$(4.5) \quad V^h = \{\mathbf{w}^h \in [C(\overline{B})]^d ; \mathbf{w}_{Tr}^h \in [P_1(Tr)]^d \quad \forall Tr \in \mathcal{T}^h, \quad \mathbf{w}^h = \mathbf{0} \quad \text{on} \quad \partial B\},$$

$$(4.6) \quad E^h = \{r^h \in C(\overline{B}) ; r_{|Tr}^h \in P_1(Tr) \quad \forall Tr \in \mathcal{T}^h, \quad r^h = 0 \quad \text{on} \quad \partial B\},$$

where $P_1(Tr)$ represents the space of polynomials of degree less or equal to one in the element Tr , i.e. the finite element spaces V^h and E^h are composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by \mathbf{u}^{0h} , \mathbf{v}^{0h} , ϕ^{0h} , ψ^{0h} , α^{0h} , θ^{0h} and ξ^{0h} , are given by

$$(4.7) \quad \begin{aligned} \mathbf{u}^{0h} &= \mathcal{P}_1^h \mathbf{u}^0, \quad \mathbf{v}^{0h} = \mathcal{P}_1^h \mathbf{v}^0, \quad \phi^{0h} = \mathcal{P}_2^h \phi^0, \quad \psi^{0h} = \mathcal{P}_2^h \psi^0, \\ \alpha^{0h} &= \mathcal{P}_2^h \alpha^0, \quad \theta^{0h} = \mathcal{P}_2^h \theta^0, \quad \xi^{0h} = \mathcal{P}_2^h \xi^0, \end{aligned}$$

where \mathcal{P}_1^h and \mathcal{P}_2^h are the classical finite element interpolation operators over V^h and E^h , respectively (see, e.g., [27]).

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \dots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = nk$ for $n = 0, 1, \dots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Find the discrete velocity field $\mathbf{v}^{hk} = \{\mathbf{v}_n^{hk}\}_{n=0}^N \subset V^h$, the discrete volume fraction speed $\psi^{hk} = \{\psi_n^{hk}\}_{n=0}^N \subset E^h$ and the discrete temperature speed $\xi^{hk} = \{\xi_n^{hk}\}_{n=0}^N \subset E^h$ such that $\mathbf{v}_0^{0h} = \mathbf{v}^{0h}$, $\psi_0^{0h} = \psi^{0h}$, $\xi_0^{0h} = \xi^{0h}$, and, for $n = 1, \dots, N$ and for all $\mathbf{w}^h \in V^h$, $r^h, z^h \in E^h$,

$$(4.8) \quad (\rho\delta\mathbf{v}_n^{hk}, \mathbf{w}^h)_H + G(\mathbf{u}_n^{hk}, \mathbf{w}^h) + B(\phi_n^{hk}, \mathbf{w}^h) = L(\theta_n^{hk} + \tau\xi_n^{hk}, \mathbf{w}^h),$$

$$(4.9) \quad \begin{aligned} & (J\delta\psi_n^{hk}, r^h)_Y + A(\phi_n^{hk}, r^h) + M(\alpha_n^{hk} + \tau\theta_n^{hk}, r^h) + (b\phi_n^{hk}, r^h)_Y + B(r^h, \mathbf{u}_n^{hk}) \\ & = m(\theta_n^{hk} + \tau\xi_n^{hk}, r^h)_Y, \end{aligned}$$

$$(4.10) \quad \begin{aligned} & (\tau a\delta\xi_n^{hk} + a\xi_n^{hk}, z^h)_Y + K(\theta_n^{hk}, z^h) + K^*(\alpha_n^{hk}, z^h) + M(\phi_n^{hk}, z^h) \\ & = -L(z^h, \mathbf{v}_n^{hk}) - m(\psi_n^{hk}, z^h), \end{aligned}$$

where the discrete displacement, the discrete volume fraction, the discrete temperature and the discrete thermal displacement field are then recovered from the relations:

$$(4.11) \quad \begin{aligned} \mathbf{u}_n^{hk} &= k \sum_{j=1}^n \mathbf{v}_j^{hk} + \mathbf{u}^{0h}, \quad \phi_n^{hk} = k \sum_{j=1}^n \psi_j^{hk} + \phi^{0h}, \\ \theta_n^{hk} &= k \sum_{j=1}^n \xi_j^{hk} + \theta^{0h}, \quad \alpha_n^{hk} = k \sum_{j=1}^n \theta_j^{hk} + \alpha^{0h}. \end{aligned}$$

We note that the existence of a unique discrete solution to problem (4.8)-(4.11) is obtained in a straightforward way using the classical Lax-Milgram lemma and assumptions (i)-(iv).

Remark 4.1. It is worth noting that assumptions (iii) and (iv) imply that

$$(4.12) \quad A_{ij}\phi_i\phi_j + 2\tau M_{ij}\phi_i\xi_j + \tau k_{ij}\xi_i\xi_j \geq C_4(\phi_i\phi_i + \xi_i\xi_i),$$

where C_4 is a positive constant.

To prove this fact, using (iv) we first note that

$$\tau k_{ij}\xi_i\xi_j \geq \tau^2 k_{ij}^* \xi_i\xi_j.$$

Now, keeping in mind that

$$\begin{aligned} A_{ij}\phi_i\phi_j + 2\tau M_{ij}\phi_i\xi_j + \tau k_{ij}\xi_i\xi_j &\geq A_{ij}\phi_i\phi_j + 2\tau M_{ij}\phi_i\xi_j + \tau^2 k_{ij}^* \xi_i\xi_j \\ &= A_{ij}\phi_i\phi_j + 2M_{ij}\phi_i(\tau\xi_j) + k_{ij}^*(\tau\xi_i)(\tau\xi_j) \\ &\geq C_2(\phi_i\phi_i + (\tau\xi_i)(\tau\xi_i)) \\ &\geq C_2 \min\{1, \tau^2\}(\phi_i\phi_i + \xi_i\xi_i), \end{aligned}$$

we obtain estimates (4.12).

In the rest of the section, in order to prove a discrete stability property and a priori error estimates, we need to impose the following restriction on tensor l_{ij} :

$$(4.13) \quad l_{ij}(\mathbf{x}) = l\delta_{ij}, \quad i, j = 1, \dots, d,$$

where δ_{ij} represents the kronecker symbol and l is a constant.

First, we have the following stability property.

Lemma 4.2. *Let the assumptions of Theorem 3.2 hold. Under additional assumption (4.13), it follows that the sequences $\{\mathbf{u}^{hk}, \mathbf{v}^{hk}, \phi^{hk}, \psi^{hk}, \alpha^{hk}, \theta^{hk}, \xi^{hk}\}$ generated by problem (4.8)-(4.11) satisfy the stability estimate:*

$$\|\mathbf{v}_n^{hk}\|_H^2 + \|\nabla \mathbf{u}_n^{hk}\|_Q^2 + \|\psi_n^{hk}\|_Y^2 + \|\nabla \phi_n^{hk}\|_H^2 + \|\phi_n^{hk}\|_Y^2 + \|\nabla \alpha_n^{hk}\|_H^2 + \|\nabla \theta_n^{hk}\|_H^2 + \|\xi_n^{hk}\|_Y^2 \leq C,$$

where C is a positive constant which is independent of the discretization parameters h and k .

Proof. In order to simplify the calculations, we assume that $\tau = 1$.

Taking \mathbf{v}_n^{hk} as a test function in variational equation (4.8) we find that

$$(\rho\delta\mathbf{v}_n^{hk}, \mathbf{v}_n^{hk})_H + G(\mathbf{u}_n^{hk}, \mathbf{v}_n^{hk}) + B(\phi_n^{hk}, \mathbf{v}_n^{hk}) = L(\theta_n^{hk} + \xi_n^{hk}, \mathbf{v}_n^{hk}).$$

Keeping in mind that

$$\begin{aligned} (\rho\delta\mathbf{v}_n^{hk}, \mathbf{v}_n^{hk})_H &\geq \frac{\rho_0}{2k} \left\{ \|\mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1}^{hk}\|_H^2 \right\}, \\ G(\mathbf{u}_n^{hk}, \mathbf{v}_n^{hk}) &= \frac{1}{2k} \left\{ G(\mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1}^{hk}) + G(\mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}) \right\}, \\ L(\theta_n^{hk}, \mathbf{v}_n^{hk}) &= \int_B l_{ij} v_{ni,j}^{hk} \theta_n^{hk} d\mathbf{x} = - \int_B l_{ij} v_{ni,j}^{hk} \theta_{n,j}^{hk} d\mathbf{x} \leq C \left(\|\nabla \theta_n^{hk}\|_H^2 + \|\mathbf{v}_n^{hk}\|_H^2 \right), \end{aligned}$$

we have

$$\begin{aligned} \frac{\rho_0}{2k} \left\{ \|\mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1}^{hk}\|_H^2 \right\} + \frac{1}{2k} \left\{ G(\mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1}^{hk}) + G(\mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}) \right\} \\ (4.14) \quad + B(\phi_n^{hk}, \mathbf{v}_n^{hk}) \leq C(\|\nabla \theta_n^{hk}\|_H^2 + \|\mathbf{v}_n^{hk}\|_H^2) + L(\xi_n^{hk}, \mathbf{v}_n^{hk}). \end{aligned}$$

Taking now $r^h = \psi_n^{hk}$ as a test function in (4.9), it follows that

$$\begin{aligned} (J\delta\psi_n^{hk}, \psi_n^{hk})_Y + A(\phi_n^{hk}, \psi_n^{hk}) + M(\alpha_n^{hk} + \theta_n^{hk}, \psi_n^{hk}) + (b\phi_n^{hk}, \psi_n^{hk})_Y \\ + B(\psi_n^{hk}, \mathbf{u}_n^{hk}) = m(\theta_n^{hk} + \xi_n^{hk}, \psi_n^{hk}). \end{aligned}$$

Using the following estimates:

$$\begin{aligned} (J\delta\psi_n^{hk}, \psi_n^{hk})_Y &\geq \frac{J_0}{2k} \left\{ \|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2 \right\}, \\ A(\phi_n^{hk}, \psi_n^{hk}) &= \frac{1}{2k} \left\{ A(\phi_n^{hk}, \phi_n^{hk}) - A(\phi_{n-1}^{hk}, \phi_{n-1}^{hk}) + A(\phi_n^{hk} - \phi_{n-1}^{hk}, \phi_n^{hk} - \phi_{n-1}^{hk}) \right\}, \\ (b\phi_n^{hk}, \psi_n^{hk})_Y &= \frac{1}{2k} \left\{ (b\phi_n^{hk}, \phi_n^{hk})_Y - (b\phi_{n-1}^{hk}, \phi_{n-1}^{hk})_Y + (b(\phi_n^{hk} - \phi_{n-1}^{hk}), \phi_n^{hk} - \phi_{n-1}^{hk})_Y \right\}, \end{aligned}$$

it follows that

$$\begin{aligned} \frac{J_0}{2k} \left\{ \|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (b\phi_n^{hk}, \phi_n^{hk})_Y - (b\phi_{n-1}^{hk}, \phi_{n-1}^{hk})_Y + (b(\phi_n^{hk} - \phi_{n-1}^{hk}), \phi_n^{hk} - \phi_{n-1}^{hk})_Y \right\} \\ + \frac{1}{2k} \left\{ A(\phi_n^{hk}, \phi_n^{hk}) - A(\phi_{n-1}^{hk}, \phi_{n-1}^{hk}) + A(\phi_n^{hk} - \phi_{n-1}^{hk}, \phi_n^{hk} - \phi_{n-1}^{hk}) \right\} \\ + B(\psi_n^{hk}, \mathbf{u}_n^{hk}) + M(\theta_n^{hk}, \psi_n^{hk}) + M(\alpha_n^{hk}, \psi_n^{hk}) \\ (4.15) \quad \leq C(\|\theta_n^{hk}\|_Y^2 + \|\xi_n^{hk}\|_Y^2 + \|\mathbf{v}_n^{hk}\|_H^2) + m(\psi_n^{hk}, \xi_n^{hk}). \end{aligned}$$

Finally, taking ξ_n^{hk} as a test function in variational equation (4.10) we find that

$$\begin{aligned} (a\delta\xi_n^{hk} + a\xi_n^{hk}, \xi_n^{hk})_Y + K(\theta_n^{hk}, \xi_n^{hk}) + K^*(\alpha_n^{hk}, \xi_n^{hk}) + M(\phi_n^{hk}, \xi_n^{hk}) \\ = -L(\xi_n^{hk}, \mathbf{v}_n^{hk}) - m(\psi_n^{hk}, \xi_n^{hk}). \end{aligned}$$

Taking into account that

$$\begin{aligned} (a\delta\xi_n^{hk}, \xi_n^{hk})_Y &\geq \frac{a_0}{2k} \left\{ \|\xi_n^{hk}\|_Y^2 - \|\xi_{n-1}^{hk}\|_Y^2 \right\}, \\ K(\theta_n^{hk}, \xi_n^{hk}) &= \frac{1}{2k} \left\{ K(\theta_n^{hk}, \theta_n^{hk}) - K(\theta_{n-1}^{hk}, \theta_{n-1}^{hk}) + K(\theta_n^{hk} - \theta_{n-1}^{hk}, \theta_n^{hk} - \theta_{n-1}^{hk}) \right\}, \\ K^*(\alpha_n^{hk}, \xi_n^{hk}) &= \frac{1}{k} \left\{ K^*(\alpha_n^{hk}, \theta_n^{hk}) - K^*(\alpha_{n-1}^{hk}, \theta_{n-1}^{hk}) \right\} - K^*(\theta_n^{hk}, \theta_{n-1}^{hk}), \end{aligned}$$

we find that

$$\begin{aligned} &\frac{a_0}{2k} \left\{ \|\xi_n^{hk}\|_Y^2 - \|\xi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ K(\theta_n^{hk}, \theta_n^{hk}) - K(\theta_{n-1}^{hk}, \theta_{n-1}^{hk}) + K(\theta_n^{hk} - \theta_{n-1}^{hk}, \theta_n^{hk} - \theta_{n-1}^{hk}) \right\} \\ &+ \frac{1}{k} \left\{ K^*(\alpha_n^{hk}, \theta_n^{hk}) - K^*(\alpha_{n-1}^{hk}, \theta_{n-1}^{hk}) \right\} + M(\phi_n^{hk}, \xi_n^{hk}) \\ (4.16) \quad &\leq C(\|\nabla\theta_n^{hk}\|_H^2 + \|\nabla\theta_{n-1}^{hk}\|_H^2 + \|\mathbf{v}_n^{hk}\|_H^2) - m(\psi_n^{hk}, \xi_n^{hk}) - L(\xi_n^{hk}, \mathbf{v}_n^{hk}). \end{aligned}$$

Combining estimates (4.14), (4.15) and (4.16) we have

$$\begin{aligned} &\frac{\rho_0}{2k} \left\{ \|\mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1}^{hk}\|_H^2 \right\} + \frac{1}{2k} \left\{ G(\mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1}^{hk}) + G(\mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}) \right\} \\ &+ \frac{J_0}{2k} \left\{ \|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (b\phi_n^{hk}, \phi_n^{hk})_Y - (b\phi_{n-1}^{hk}, \phi_{n-1}^{hk})_Y + (b(\phi_n^{hk} - \phi_{n-1}^{hk}), \phi_n^{hk} - \phi_{n-1}^{hk})_Y \right\} \\ &+ \frac{1}{2k} \left\{ A(\phi_n^{hk}, \phi_n^{hk}) - A(\phi_{n-1}^{hk}, \phi_{n-1}^{hk}) + A(\phi_n^{hk} - \phi_{n-1}^{hk}, \phi_n^{hk} - \phi_{n-1}^{hk}) \right\} \\ &+ B(\psi_n^{hk}, \mathbf{u}_n^{hk}) + M(\theta_n^{hk}, \psi_n^{hk}) + M(\alpha_n^{hk}, \psi_n^{hk}) + \frac{a_0}{2k} \left\{ \|\xi_n^{hk}\|_Y^2 - \|\xi_{n-1}^{hk}\|_Y^2 \right\} \\ &+ \frac{1}{2k} \left\{ K(\theta_n^{hk}, \theta_n^{hk}) - K(\theta_{n-1}^{hk}, \theta_{n-1}^{hk}) + K(\theta_n^{hk} - \theta_{n-1}^{hk}, \theta_n^{hk} - \theta_{n-1}^{hk}) \right\} \\ &+ \frac{1}{k} \left\{ K^*(\alpha_n^{hk}, \theta_n^{hk}) - K^*(\alpha_{n-1}^{hk}, \theta_{n-1}^{hk}) \right\} + M(\phi_n^{hk}, \xi_n^{hk}) + B(\phi_n^{hk}, \mathbf{v}_n^{hk}) \\ &\leq C(\|\nabla\theta_n^{hk}\|_H^2 + \|\mathbf{v}_n^{hk}\|_H^2 + \|\theta_n^{hk}\|_Y^2 + \|\xi_n^{hk}\|_Y^2 + \|\nabla\theta_{n-1}^{hk}\|_H^2 + \|\nabla\theta_{n-1}^{hk}\|_H^2). \end{aligned}$$

Observing that

$$\begin{aligned} B(\phi_n^{hk}, \mathbf{v}_n^{hk}) + B(\psi_n^{hk}, \mathbf{u}_n^{hk}) &= \frac{1}{k} \left\{ B(\phi_n^{hk}, \mathbf{u}_n^{hk}) - B(\phi_{n-1}^{hk}, \mathbf{u}_{n-1}^{hk}) + B(\phi_n^{hk} - \phi_{n-1}^{hk}, \mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}) \right\}, \\ M(\theta_n^{hk}, \psi_n^{hk}) + M(\xi_n^{hk}, \phi_n^{hk}) &= \frac{1}{k} \left\{ M(\theta_n^{hk}, \phi_n^{hk}) - M(\theta_{n-1}^{hk}, \phi_{n-1}^{hk}) + M(\theta_n^{hk} - \theta_{n-1}^{hk}, \phi_n^{hk} - \phi_{n-1}^{hk}) \right\}, \end{aligned}$$

and that using assumptions (ii), (iv) and (4.12) we find that

$$\begin{aligned} G(\mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}) + 2B(\phi_n^{hk} - \phi_{n-1}^{hk}, \mathbf{u}_n^{hk} - \mathbf{u}_{n-1}^{hk}) + (b(\phi_n^{hk} - \phi_{n-1}^{hk}), \phi_n^{hk} - \phi_{n-1}^{hk})_Y &\geq 0, \\ A(\phi_n^{hk} - \phi_{n-1}^{hk}, \phi_n^{hk} - \phi_{n-1}^{hk}) + 2M(\theta_n^{hk} - \theta_{n-1}^{hk}, \phi_n^{hk} - \phi_{n-1}^{hk}) + K(\theta_n^{hk} - \theta_{n-1}^{hk}, \theta_n^{hk} - \theta_{n-1}^{hk}) &\geq 0, \end{aligned}$$

it follows that

$$\begin{aligned}
& \frac{\rho_0}{2k} \left\{ \|v_n^{hk}\|_H^2 - \|v_{n-1}^{hk}\|_H^2 \right\} + \frac{1}{2k} \left\{ G(\mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1}^{hk}) \right\} \\
& + \frac{J_0}{2k} \left\{ \|\psi_n^{hk}\|_Y^2 - \|\psi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ (\phi_n^{hk}, \phi_n^{hk})_Y - (\phi_{n-1}^{hk}, \phi_{n-1}^{hk})_Y \right\} + M(\alpha_n^{hk}, \psi_n^{hk}) \\
& + \frac{1}{2k} \left\{ A(\phi_n^{hk}, \phi_n^{hk}) - A(\phi_{n-1}^{hk}, \phi_{n-1}^{hk}) \right\} + \frac{1}{k} \left\{ B(\phi_n^{hk}, \mathbf{u}_n^{hk}) - B(\phi_{n-1}^{hk}, \mathbf{u}_{n-1}^{hk}) \right\} \\
& + \frac{1}{2k} \left\{ K(\theta_n^{hk}, \theta_n^{hk}) - K(\theta_{n-1}^{hk}, \theta_{n-1}^{hk}) \right\} + \frac{1}{k} \left\{ M(\theta_n^{hk}, \phi_n^{hk}) - M(\theta_{n-1}^{hk}, \phi_{n-1}^{hk}) \right\} \\
& + \frac{1}{k} \left\{ K^*(\alpha_n^{hk}, \theta_n^{hk}) - K^*(\alpha_{n-1}^{hk}, \theta_{n-1}^{hk}) \right\} + \frac{a_0}{2k} \left\{ \|\xi_n^{hk}\|_Y^2 - \|\xi_{n-1}^{hk}\|_Y^2 \right\} \\
& \leq C(\|\nabla \theta_n^{hk}\|_H^2 + \|v_n^{hk}\|_H^2 + \|\theta_n^{hk}\|_Y^2 + \|\xi_n^{hk}\|_Y^2 + \|\nabla \theta_{n-1}^{hk}\|_H^2).
\end{aligned}$$

Multiplying the previous estimates by k and summing up to n we get

$$\begin{aligned}
& \rho_0 \|v_n^{hk}\|_H^2 + G(\mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) + J_0 \|\psi_n^{hk}\|_Y^2 + (b\phi_n^{hk}, \phi_n^{hk})_Y + A(\phi_n^{hk}, \phi_n^{hk}) + 2B(\phi_n^{hk}, \mathbf{u}_n^{hk}) \\
& + K(\theta_n^{hk}, \theta_n^{hk}) + 2M(\theta_n^{hk}, \phi_n^{hk}) + 2K^*(\alpha_n^{hk}, \theta_n^{hk}) + a_0 \|\xi_n^{hk}\|_Y^2 + Ck \sum_{j=1}^n M(\alpha_j^{hk}, \psi_j^{hk}) \\
& \leq k \sum_{j=1}^n \left(\|\nabla \theta_j^{hk}\|_H^2 + \|v_j^{hk}\|_H^2 + \|\theta_j^{hk}\|_Y^2 + \|\xi_j^{hk}\|_Y^2 \right) + C \left(\|\mathbf{v}^{0h}\|_H^2 + \|\mathbf{u}^{0h}\|_V^2 + \|\psi^{0h}\|_Y^2 \right. \\
& \quad \left. + \|\phi^{0h}\|_E^2 + \|\alpha^{0h}\|_2 + \|\theta^{0h}\|_E^2 + \|\xi^{0h}\|_Y^2 \right).
\end{aligned}$$

Finally, taking into account again assumptions (ii), (iv) and (4.12) we find that

$$\begin{aligned}
& G(\mathbf{u}_n^{hk}, \mathbf{u}_n^{hk}) + 2B(\phi_n^{hk}, \mathbf{u}_n^{hk}) + (b\phi_n^{hk}, \phi_n^{hk})_Y \geq C_1 (\|\nabla \mathbf{u}_n^{hk}\|_Q^2 + \|\phi_n^{hk}\|_Y^2), \\
& A(\phi_n^{hk}, \phi_n^{hk}) + 2M(\theta_n^{hk}, \phi_n^{hk}) + K(\theta_n^{hk}, \theta_n^{hk}) \geq C_2 (\|\nabla \phi_n^{hk}\|_H^2 + \|\nabla \theta_n^{hk}\|_H^2),
\end{aligned}$$

and using the following straightforward estimates:

$$\begin{aligned}
& k \sum_{j=1}^n M(\alpha_j^{hk}, \psi_j^{hk}) = \sum_{j=1}^n M(\alpha_j^{hk}, \phi_j^{hk} - \phi_{j-1}^{hk}) \\
& = M(\alpha_n^{hk}, \phi_n^{hk}) + \sum_{j=1}^{n-1} M(\alpha_j^{hk} - \alpha_{j+1}^{hk}, \phi_j^{hk}) + M(\alpha_1^{hk}, \phi_0^{hk}), \\
& \sum_{j=1}^{n-1} M(\alpha_{j-1}^{hk} - \alpha_j^{hk}, \phi_j^{hk}) \leq Ck \sum_{j=1}^{n-1} \|\nabla \phi_j^{hk}\|_H^2 + \frac{C}{k} \sum_{j=1}^{n-1} \|\nabla(\alpha_j^{hk} - \alpha_{j+1}^{hk})\|_H^2 \\
& \leq Ck \sum_{j=1}^n (\|\nabla \phi_j^{hk}\|_H^2 + \|\nabla \theta_j^{hk}\|_H^2),
\end{aligned}$$

and a discrete version of Gronwall's inequality, we deduce the desired stability property. \square

Now, we derive some a priori error estimates on the numerical errors $\mathbf{u}_n - \mathbf{u}_n^{hk}$, $v_n - v_n^{hk}$, $\phi_n - \phi_n^{hk}$, $\psi_n - \psi_n^{hk}$, $\alpha_n - \alpha_n^{hk}$, $\theta_n - \theta_n^{hk}$ and $\xi_n - \xi_n^{hk}$. We have the following.

Theorem 4.3. *Let the assumptions of Lemma 4.2 hold. If we denote by (\mathbf{v}, ψ, ξ) the solution to problem (4.1)-(4.4) and $(\mathbf{v}^{hk}, \psi^{hk}, \xi^{hk})$ the solution to discrete problem (4.8)-(4.11), then we*

have the following error estimates for all $\mathbf{w}^h \in V^h$, $r^h, z^h \in E^h$:

$$\begin{aligned} & \max_{0 \leq n \leq N} \left\{ \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 + \|\mathbf{u}_n - \mathbf{u}_n^{hk}\|_V^2 + \|\psi_n - \psi_n^{hk}\|_Y^2 + \|\phi_n - \phi_n^{hk}\|_E^2 \right. \\ & \quad \left. + \|\xi_n - \xi_n^{hk}\|_Y^2 + \|\theta_n - \theta_n^{hk}\|_E^2 + \|\alpha_n - \alpha_n^{hk}\|_E^2 \right\} \\ & \leq Ck \sum_{j=1}^N \left(\|\dot{\mathbf{v}}_j - \delta \mathbf{v}_j\|_H^2 + \|\nabla(\dot{\mathbf{v}}_j - \delta \mathbf{u}_j)\|_Q^2 + \|\mathbf{v}_j - \mathbf{w}_j^h\|_V^2 + \|\dot{\psi}_j - \delta \psi_j\|_Y^2 + \|\dot{\phi}_j - \delta \phi_j\|_E^2 \right. \\ & \quad \left. + \|\psi_j - r_j^h\|_E^2 + \|\dot{\xi}_j - \delta \xi_j\|_Y^2 + \|\dot{\theta}_j - \delta \theta_j\|_E^2 + \|\xi_j - z_j^h\|_E^2 + \|\dot{\alpha}_j - \delta \alpha_j\|_E^2 \right) \\ & \quad + \frac{C}{k} \sum_{j=1}^{N-1} \left\{ \|\mathbf{v}_j - \mathbf{w}_j^h - (\mathbf{v}_{j+1} - \mathbf{w}_{j+1}^h)\|_H^2 + \|\psi_j - r_j^h - (\psi_{j+1} - r_{j+1}^h)\|_Y^2 \right. \\ & \quad \left. + \|\xi_j - z_j^h - (\xi_{j+1} - z_{j+1}^h)\|_Y^2 \right\} + C \max_{0 \leq n \leq N} \left(\|\mathbf{v}_n - \mathbf{w}_n^h\|_H^2 + \|\psi_n - r_n^h\|_Y^2 + \|\xi_n - z_n^h\|_Y^2 \right) \\ & \quad + C \left(\|\mathbf{v}^0 - \mathbf{v}^{0h}\|_H^2 + \|\mathbf{u}^0 - \mathbf{u}^{0h}\|_V^2 + \|\psi^0 - \psi^{0h}\|_Y^2 + \|\phi^0 - \phi^{0h}\|_E^2 + \|\theta^0 - \theta^{0h}\|_E^2 \right. \\ & \quad \left. + \|\alpha^0 - \alpha^{0h}\|_E^2 + \|\xi^0 - \xi^{0h}\|_Y^2 \right). \end{aligned}$$

Proof. Again, we assume that parameter $\tau = 1$ to simplify the calculations.

First, we obtain the estimates for the discrete velocity \mathbf{v}_n^{hk} . Then, we subtract variational equation (4.1) at time $t = t_n$ for a test function $\mathbf{w} = \mathbf{w}^h \in V^h \subset V$ and discrete variational equation (4.8) to obtain, for all $\mathbf{w}^h \in V^h$,

$$(\rho(\dot{\mathbf{v}}_n - \delta \mathbf{v}_n^{hk}), \mathbf{w}^h)_H + G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{w}^h) + B(\phi_n - \phi_n^{hk}, \mathbf{w}^h) - L(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, \mathbf{w}^h) = 0,$$

and so,

$$\begin{aligned} & (\rho(\dot{\mathbf{v}}_n - \delta \mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{v}_n^{hk})_H + G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) + B(\phi_n - \phi_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) \\ & \quad - L(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) \\ & = (\rho(\dot{\mathbf{v}}_n - \delta \mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{w}^h)_H + G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{v}_n - \mathbf{w}^h) + B(\phi_n - \phi_n^{hk}, \mathbf{v}_n - \mathbf{w}^h) \\ & \quad - L(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, \mathbf{v}_n - \mathbf{w}^h) \quad \forall \mathbf{w}^h \in V^h. \end{aligned}$$

Taking into account that

$$\begin{aligned} & (\rho(\dot{\mathbf{v}}_n - \delta \mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{v}_n^{hk})_H = (\rho(\dot{\mathbf{v}}_n - \delta \mathbf{v}_n), \mathbf{v}_n - \mathbf{v}_n^{hk})_H + (\rho(\delta \mathbf{v}_n - \delta \mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{v}_n^{hk})_H \\ & \quad + (\rho(\dot{\mathbf{v}}_n - \delta \mathbf{v}_n), \mathbf{v}_n - \mathbf{v}_n^{hk})_H + \frac{\rho_0}{2k} \left\{ \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1} - \mathbf{v}_{n-1}^{hk}\|_H^2 \right\}, \\ & G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) = G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \dot{\mathbf{u}}_n - \delta \mathbf{u}_n) + G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \delta \mathbf{u}_n - \delta \mathbf{u}_n^{hk}) \\ & = G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \dot{\mathbf{u}}_n - \delta \mathbf{u}_n) + \frac{1}{2k} \left\{ G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}) \right. \\ & \quad \left. + G(\mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}), \mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk})) \right\}, \\ & B(\phi_n - \phi_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) = B(\phi_n - \phi_n^{hk}, \dot{\mathbf{u}}_n - \delta \mathbf{u}_n) + B(\phi_n - \phi_n^{hk}, \delta \mathbf{u}_n - \delta \mathbf{u}_n^{hk}), \\ & |L(\theta_n - \theta_n^{hk}, \mathbf{w})| \leq C(\|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\mathbf{w}\|_H^2), \\ & |L(\xi_n - \xi_n^{hk}, \mathbf{v}_n - \mathbf{w}^h)| \leq C(\|\xi_n - \xi_n^{hk}\|_Y^2 + \|\nabla(\mathbf{v}_n - \mathbf{w}^h)\|_Q^2), \end{aligned}$$

it follows that

$$\begin{aligned}
& \frac{\rho_0}{2k} \left\{ \| \mathbf{v}_n - \mathbf{v}_n^{hk} \|_H^2 - \| \mathbf{v}_{n-1} - \mathbf{v}_{n-1}^{hk} \|_H^2 \right\} + B(\phi_n - \phi_n^{hk}, \delta \mathbf{u}_n - \delta \mathbf{u}_n^{hk}) \\
& + \frac{1}{2k} \left\{ G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}) \right. \\
& \quad \left. + G(\mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}), \mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk})) \right\} - L(\xi_n - \xi_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) \\
& \leq C \left(\| \dot{\mathbf{v}}_n - \delta \mathbf{v}_n \|_H^2 + \| \nabla(\dot{\mathbf{u}}_n - \delta \mathbf{u}_n) \|_Q^2 + \| \mathbf{v}_n - \mathbf{w}^h \|_V^2 + \| \nabla(\phi_n - \phi_n^{hk}) \|_H^2 + \| \xi_n - \xi_n^{hk} \|_Y^2 \right. \\
& \quad \left. + \| \nabla(\theta_n - \theta_n^{hk}) \|_H^2 + \| \mathbf{v}_n - \mathbf{v}_n^{hk} \|_H^2 + (\rho(\delta \mathbf{v}_n - \delta \mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{w}^h)_H \right).
\end{aligned}$$

Now, we obtain the estimates on the discrete volume fraction speed. Therefore, subtracting variational equation (4.2) at time $t = t_n$ for a test function $r = r^h \in E^h \subset E$ and discrete variational equation (4.9), we obtain, for all $r^h \in E^h$,

$$\begin{aligned}
& (J(\dot{\psi}_n - \delta \psi_n^{hk}), r^h)_Y + A(\phi_n - \phi_n^{hk}, r^h) + M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, r^h) + (b(\phi_n - \phi_n^{hk}), r^h)_Y \\
& + B(r^h, \mathbf{u}_n - \mathbf{u}_n^{hk}) - m(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, r^h)_Y = 0,
\end{aligned}$$

and so, we have, for all $r^h \in E^h$,

$$\begin{aligned}
& (J(\dot{\psi}_n - \delta \psi_n^{hk}), \psi_n - \psi_n^{hk})_Y + A(\phi_n - \phi_n^{hk}, \psi_n - \psi_n^{hk}) + M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, \psi_n - \psi_n^{hk}) \\
& + (b(\phi_n - \phi_n^{hk}), \psi_n - \psi_n^{hk})_Y + B(\psi_n - \psi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) - m(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, \psi_n - \psi_n^{hk})_Y \\
& = (J(\dot{\psi}_n - \delta \psi_n^{hk}), \psi_n - r^h)_Y + A(\phi_n - \phi_n^{hk}, \psi_n - r^h) + M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, \psi_n - r^h) \\
& + (b(\phi_n - \phi_n^{hk}), \psi_n - r^h)_Y + B(\psi_n - r^h, \mathbf{u}_n - \mathbf{u}_n^{hk}) - m(\theta_n - \theta_n^{hk} + \xi_n - \xi_n^{hk}, \psi_n - r^h)_Y.
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
& (J(\dot{\psi}_n - \delta \psi_n^{hk}), \psi_n - \psi_n^{hk})_Y = (J(\dot{\psi}_n - \delta \psi_n), \psi_n - \psi_n^{hk})_Y + (J(\delta \psi_n - \delta \psi_n^{hk}), \psi_n - \psi_n^{hk})_Y \\
& \geq (J(\dot{\psi}_n - \delta \psi_n), \psi_n - \psi_n^{hk})_Y + \frac{J_0}{2k} \left\{ \| \psi_n - \psi_n^{hk} \|_Y^2 - \| \psi_{n-1} - \psi_{n-1}^{hk} \|_Y^2 \right\}, \\
& A(\phi_n - \phi_n^{hk}, \psi_n - \psi_n^{hk}) = A(\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta \phi_n) + A(\phi_n - \phi_n^{hk}, \delta \phi_n - \delta \phi_n^{hk}) \\
& = A(\phi_n - \phi_n^{hk}, \dot{\phi}_n - \delta \phi_n) + \frac{1}{2k} \left\{ A(\phi_n - \phi_n^{hk}, \phi_n - \phi_n^{hk}) - A(\phi_{n-1} - \phi_{n-1}^{hk}, \phi_{n-1} - \phi_{n-1}^{hk}) \right. \\
& \quad \left. + A(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \right\}, \\
& (b(\phi_n - \phi_n^{hk}), \psi_n - \psi_n^{hk}) = (b(\phi_n - \phi_n^{hk}), \dot{\phi}_n - \delta \phi_n) + (b(\phi_n - \phi_n^{hk}), \delta \phi_n - \delta \phi_n^{hk}) \\
& = (b(\phi_n - \phi_n^{hk}), \dot{\phi}_n - \delta \phi_n) + \frac{1}{2k} \left\{ (b(\phi_n - \phi_n^{hk}), \phi_n - \phi_n^{hk})_Y - (b(\phi_{n-1} - \phi_{n-1}^{hk}), \phi_{n-1} - \phi_{n-1}^{hk})_Y \right. \\
& \quad \left. + (b(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}))_Y \right\}, \\
& B(\psi_n - \psi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) = B(\dot{\phi}_n - \delta \phi_n, \mathbf{u}_n - \mathbf{u}_n^{hk}) + B(\delta \phi_n - \delta \phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}), \\
& M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, \psi_n - \psi_n^{hk}) = M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, \dot{\phi}_n - \delta \phi_n) \\
& \quad + M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, \delta \phi_n - \delta \phi_n^{hk}),
\end{aligned}$$

we obtain

$$\begin{aligned}
& \frac{J_0}{2k} \left\{ \|\psi_n - \psi_n^{hk}\|_Y^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ A(\phi_n - \phi_n^{hk}, \phi_n - \phi_n^{hk}) \right. \\
& \quad \left. - A(\phi_{n-1} - \phi_{n-1}^{hk}, \phi_{n-1} - \phi_{n-1}^{hk}) + A(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \right\}, \\
& \quad + \frac{1}{2k} \left\{ (b(\phi_n - \phi_n^{hk}), \phi_n - \phi_n^{hk})_Y - (b(\phi_{n-1} - \phi_{n-1}^{hk}), \phi_{n-1} - \phi_{n-1}^{hk})_Y \right. \\
& \quad \left. + (b(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}))_Y \right\} \\
& \quad + M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, \delta\phi_n - \delta\phi_n^{hk}) + B(\delta\phi_n - \delta\phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) \\
& \leq C \left(\|\dot{\psi}_n - \delta\psi_n\|_Y^2 + \|\psi_n - \psi_n^{hk}\|_Y^2 + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\dot{\phi}_n - \delta\phi_n\|_E^2 + \|\nabla(\alpha_n - \alpha_n^{hk})\|_H^2 \right. \\
& \quad \left. + \|\psi_n - r^h\|_E^2 + \|\phi_n - \phi_n^{hk}\|_Y^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\theta_n - \theta_n^{hk}\|_Y^2 + (J(\delta\phi_n - \delta\phi_n^{hk}), \psi_n - r^h)_Y \right).
\end{aligned}$$

Finally, we obtain the estimates on the temperature speed. Thus, if we subtract variational equation (4.3) at time $t = t_n$ for a test function $z = z^h \in E^h \subset E$ and discrete variational equation (4.10) it follows that, for all $z^h \in E^h$,

$$\begin{aligned}
& (a(\dot{\xi} - \delta\xi_n^{hk} + \xi_n - \xi_n^{hk}), z^h)_Y + K(\theta_n - \theta_n^{hk}, z^h) + K^*(\alpha_n - \alpha_n^{hk}, z^h) + M(\phi_n - \phi_n^{hk}, z^h) \\
& \quad + L(z^h, \mathbf{v}_n - \mathbf{v}_n^{hk}) + m(\psi_n - \psi_n^{hk}, z^h) = 0.
\end{aligned}$$

Therefore, we have, for all $z^h \in E^h$,

$$\begin{aligned}
& (a(\dot{\xi} - \delta\xi_n^{hk} + \xi_n - \xi_n^{hk}), \xi_n - \xi_n^{hk})_Y + K(\theta_n - \theta_n^{hk}, \xi_n - \xi_n^{hk}) + K^*(\alpha_n - \alpha_n^{hk}, \xi_n - \xi_n^{hk}) \\
& \quad + M(\phi_n - \phi_n^{hk}, \xi_n - \xi_n^{hk}) + L(\xi_n - \xi_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) + m(\psi_n - \psi_n^{hk}, \xi_n - \xi_n^{hk}) \\
& = (a(\dot{\xi} - \delta\xi_n^{hk} + \xi_n - \xi_n^{hk}), \xi_n - z^h)_Y + K(\theta_n - \theta_n^{hk}, \xi_n - z^h) + K^*(\alpha_n - \alpha_n^{hk}, \xi_n - z^h) \\
& \quad + M(\phi_n - \phi_n^{hk}, \xi_n - z^h) + L(\xi_n - z^h, \mathbf{v}_n - \mathbf{v}_n^{hk}) + m(\psi_n - \psi_n^{hk}, \xi_n - z^h).
\end{aligned}$$

Now, taking into account that

$$\begin{aligned}
(a(\dot{\xi}_n - \delta\xi_n^{hk}), \xi_n - \xi_n^{hk})_Y &= (a(\dot{\xi}_n - \delta\xi_n), \xi_n - \xi_n^{hk})_Y + (a(\delta\xi_n - \delta\xi_n^{hk}), \xi_n - \xi_n^{hk})_Y \\
&\geq (a(\dot{\xi}_n - \delta\xi_n), \xi_n - \xi_n^{hk})_Y + \frac{a_0}{2k} \left\{ \|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2 \right\}, \\
K(\theta_n - \theta_n^{hk}, \xi_n - \xi_n^{hk}) &= K(\theta_n - \theta_n^{hk}, \theta_n - \delta\theta_n) + K(\theta_n - \theta_n^{hk}, \delta\theta_n - \delta\theta_n^{hk}) \\
&= K(\theta_n - \theta_n^{hk}, \theta_n - \delta\theta_n) + \frac{1}{2k} \left\{ K(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk}) - K(\theta_{n-1} - \theta_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) \right. \\
& \quad \left. + K(\theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk}), \theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk})) \right\}, \\
\frac{1}{k} \left\{ K^*(\alpha_n - \alpha_n^{hk}, \theta_n - \theta_n^{hk}) - K^*(\alpha_{n-1} - \alpha_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) \right\} &= K^*(\alpha_n - \alpha_n^{hk}, \delta\theta_n - \dot{\theta}_n) + K^*(\alpha_n - \alpha_n^{hk}, \xi_n - \xi_n^{hk}) \\
& \quad + K^*(\delta\alpha_n - \dot{\alpha}_n, \theta_{n-1} - \theta_{n-1}^{hk}) + K^*(\theta_n - \theta_n^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}), \\
M(\phi_n - \phi_n^{hk}, \xi_n - \xi_n^{hk}) &= M(\phi_n - \phi_n^{hk}, \dot{\theta}_n - \delta\theta_n) + M(\phi_n - \phi_n^{hk}, \delta\theta_n - \delta\theta_n^{hk}),
\end{aligned}$$

$$|L(\xi_n - z^h, \mathbf{v}_n - \mathbf{v}_n^{hk})| \leq C(\|\xi_n - z^h\|_E^2 + \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2),$$

we obtain, for all $z^h \in E^h$,

$$\begin{aligned} & \frac{a_0}{2k} \left\{ \|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ K(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk}) - K(\theta_{n-1} - \theta_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) \right. \\ & \quad \left. + K(\theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk}), \theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk})) \right\} \\ & \quad + \frac{1}{k} \left\{ K^*(\alpha_n - \alpha_n^{hk}, \theta_n - \theta_n^{hk}) - K^*(\alpha_{n-1} - \alpha_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) \right\} \\ & \quad + M(\phi_n - \phi_n^{hk}, \xi_n - \xi_n^{hk}) + L(\xi_n - \xi_n^{hk}, \mathbf{v}_n - \mathbf{v}_n^{hk}) \\ & \leq C \left(\|\dot{\xi}_n - \delta\xi_n\|_Y^2 + \|\dot{\theta}_n - \delta\theta_n\|_E^2 + \|\xi_n - \xi_n^{hk}\|_Y^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\xi_n - z^h\|_E^2 + \|\psi_n - \psi_n^{hk}\|_Y^2 \right. \\ & \quad \left. + \|\dot{\alpha}_n - \delta\alpha_n\|_E^2 + \|\nabla(\alpha_n - \alpha_n^{hk})\|_H^2 + \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 + (a(\delta\xi_n - \delta\xi_n^{hk}), \xi_n - z^h)_Y \right). \end{aligned}$$

Combining now the estimates for the velocities, the volume fraction speed and the temperature speed, we obtain

$$\begin{aligned} & \frac{\rho_0}{2k} \left\{ \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1} - \mathbf{v}_{n-1}^{hk}\|_H^2 \right\} + B(\phi_n - \phi_n^{hk}, \delta\mathbf{u}_n - \delta\mathbf{u}_n^{hk}) \\ & \quad + \frac{1}{2k} \left\{ G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}) \right. \\ & \quad \left. + G(\mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}), \mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk})) \right\} \\ & \quad + \frac{J_0}{2k} \left\{ \|\psi_n - \psi_n^{hk}\|_Y^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ A(\phi_n - \phi_n^{hk}, \phi_n - \phi_n^{hk}) \right. \\ & \quad \left. - A(\phi_{n-1} - \phi_{n-1}^{hk}, \phi_{n-1} - \phi_{n-1}^{hk}) + A(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \right\}, \\ & \quad + \frac{1}{2k} \left\{ (b(\phi_n - \phi_n^{hk}), \phi_n - \phi_n^{hk})_Y - (b(\phi_{n-1} - \phi_{n-1}^{hk}), \phi_{n-1} - \phi_{n-1}^{hk})_Y \right. \\ & \quad \left. + (b(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}))_Y \right\} \\ & \quad + M(\alpha_n - \alpha_n^{hk} + \theta_n - \theta_n^{hk}, \delta\phi_n - \delta\phi_n^{hk}) + B(\delta\phi_n - \delta\phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) \\ & \quad + \frac{a_0}{2k} \left\{ \|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ K(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk}) \right. \\ & \quad \left. - K(\theta_{n-1} - \theta_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) + K(\theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk}), \theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk})) \right\} \\ & \quad + \frac{1}{k} \left\{ K^*(\alpha_n - \alpha_n^{hk}, \theta_n - \theta_n^{hk}) - K^*(\alpha_{n-1} - \alpha_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) \right\} + M(\phi_n - \phi_n^{hk}, \xi_n - \xi_n^{hk}) \\ & \leq C \left(\|\dot{\mathbf{v}}_n - \delta\mathbf{v}_n\|_H^2 + \|\nabla(\dot{\mathbf{u}}_n - \delta\mathbf{u}_n)\|_Q^2 + \|\mathbf{v}_n - \mathbf{w}^h\|_V^2 + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\xi_n - \xi_n^{hk}\|_Y^2 \right. \\ & \quad \left. + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 + (\rho(\delta\mathbf{v}_n - \delta\mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{w}^h)_H + \|\dot{\psi}_n - \delta\psi_n\|_Y^2 \right. \\ & \quad \left. + \|\psi_n - \psi_n^{hk}\|_Y^2 + \|\dot{\phi}_n - \delta\phi_n\|_E^2 + \|\psi_n - r^h\|_E^2 + \|\nabla(\alpha_n - \alpha_n^{hk})\|_H^2 + \|\phi_n - \phi_n^{hk}\|_Y^2 \right. \\ & \quad \left. + \|\theta_n - \theta_n^{hk}\|_Y^2 + (J(\delta\phi_n - \delta\phi_n^{hk}), \psi_n - r^h)_Y + \|\dot{\xi}_n - \delta\xi_n\|_Y^2 + \|\theta_n - \delta\theta_n\|_E^2 \right. \\ & \quad \left. + \|\xi_n - z^h\|_E^2 + \|\dot{\alpha}_n - \delta\alpha_n\|_E^2 + (a(\delta\xi_n - \delta\xi_n^{hk}), \xi_n - z^h)_Y \right). \end{aligned}$$

Observing that

$$\begin{aligned} & B(\phi_n - \phi_n^{hk}, \delta\mathbf{u}_n - \delta\mathbf{u}_n^{hk}) + B(\delta\phi_n - \delta\phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) = \frac{1}{k} \left\{ B(\phi_n - \phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) \right. \\ & \quad \left. - B(\phi_{n-1} - \phi_{n-1}^{hk}, \mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}) + B(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}), \mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk})) \right\}, \\ & M(\theta_n - \theta_n^{hk}, \delta\phi_n - \delta\phi_n^{hk}) + M(\delta\theta_n - \delta\theta_n^{hk}, \phi_n - \phi_n^{hk}) = \frac{1}{k} \left\{ M(\theta_n - \theta_n^{hk}, \phi_n - \phi_n^{hk}) \right. \\ & \quad \left. - M(\theta_{n-1} - \theta_{n-1}^{hk}, \phi_{n-1} - \phi_{n-1}^{hk}) + M(\theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk}), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \right\}, \end{aligned}$$

and that using assumptions (ii), (iv) and (4.12) we have

$$\begin{aligned} & G(\mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}), \mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk})) \\ & + 2B(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}), \mathbf{u}_n - \mathbf{u}_n^{hk} - (\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk})) \\ & + (b(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}))_Y \geq 0, \\ & A(\phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk}), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \\ & + 2M(\theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk}), \phi_n - \phi_n^{hk} - (\phi_{n-1} - \phi_{n-1}^{hk})) \\ & + K(\theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk}), \theta_n - \theta_n^{hk} - (\theta_{n-1} - \theta_{n-1}^{hk})) \geq 0, \end{aligned}$$

we find that

$$\begin{aligned} & \frac{\rho_0}{2k} \left\{ \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 - \|\mathbf{v}_{n-1} - \mathbf{v}_{n-1}^{hk}\|_H^2 \right\} + B(\phi_n - \phi_n^{hk}, \delta\mathbf{u}_n - \delta\mathbf{u}_n^{hk}) \\ & + \frac{1}{2k} \left\{ G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) - G(\mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}, \mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}) \right\} \\ & + \frac{J_0}{2k} \left\{ \|\psi_n - \psi_n^{hk}\|_Y^2 - \|\psi_{n-1} - \psi_{n-1}^{hk}\|_Y^2 \right\} \\ & + \frac{1}{2k} \left\{ A(\phi_n - \phi_n^{hk}, \phi_n - \phi_n^{hk}) - A(\phi_{n-1} - \phi_{n-1}^{hk}, \phi_{n-1} - \phi_{n-1}^{hk}) \right\} \\ & + \frac{1}{2k} \left\{ (b(\phi_n - \phi_n^{hk}), \phi_n - \phi_n^{hk})_Y - (b(\phi_{n-1} - \phi_{n-1}^{hk}), \phi_{n-1} - \phi_{n-1}^{hk})_Y \right\} \\ & + M(\alpha_n - \alpha_n^{hk}, \delta\phi_n - \delta\phi_n^{hk}) + \frac{1}{k} \left\{ B(\phi_n - \phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) - B(\phi_{n-1} - \phi_{n-1}^{hk}, \mathbf{u}_{n-1} - \mathbf{u}_{n-1}^{hk}) \right\} \\ & + \frac{a_0}{2k} \left\{ \|\xi_n - \xi_n^{hk}\|_Y^2 - \|\xi_{n-1} - \xi_{n-1}^{hk}\|_Y^2 \right\} + \frac{1}{2k} \left\{ K(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk}) - K(\theta_{n-1} - \theta_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) \right\} \\ & + \frac{1}{k} \left\{ K^*(\alpha_n - \alpha_n^{hk}, \theta_n - \theta_n^{hk}) - K^*(\alpha_{n-1} - \alpha_{n-1}^{hk}, \theta_{n-1} - \theta_{n-1}^{hk}) \right\} \\ & + \frac{1}{k} \left\{ M(\theta_n - \theta_n^{hk}, \phi_n - \phi_n^{hk}) - M(\theta_{n-1} - \theta_{n-1}^{hk}, \phi_{n-1} - \phi_{n-1}^{hk}) \right\} \\ & \leq C \left(\|\dot{\mathbf{v}}_n - \delta\mathbf{v}_n\|_H^2 + \|\nabla(\dot{\mathbf{u}}_n - \delta\mathbf{u}_n)\|_Q^2 + \|\mathbf{v}_n - \mathbf{w}^h\|_V^2 + \|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\xi_n - \xi_n^{hk}\|_Y^2 \right. \\ & \quad \left. + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2 + \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 + (\rho(\delta\mathbf{v}_n - \delta\mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{w}^h)_H + \|\dot{\psi}_n - \delta\psi_n\|_Y^2 \right. \\ & \quad \left. + \|\psi_n - \psi_n^{hk}\|_Y^2 + \|\dot{\phi}_n - \delta\phi_n\|_E^2 + \|\psi_n - r^h\|_E^2 + \|\nabla(\alpha_n - \alpha_n^{hk})\|_H^2 + \|\phi_n - \phi_n^{hk}\|_Y^2 \right. \\ & \quad \left. + \|\theta_n - \theta_n^{hk}\|_Y^2 + (J(\delta\phi_n - \delta\phi_n^{hk}), \psi_n - r^h)_Y + \|\dot{\xi}_n - \delta\xi_n\|_Y^2 + \|\theta_n - \delta\theta_n\|_E^2 \right. \\ & \quad \left. + \|\xi_n - z^h\|_E^2 + \|\dot{\alpha}_n - \delta\alpha_n\|_E^2 + (a(\delta\xi_n - \delta\xi_n^{hk}), \xi_n - z^h)_Y \right). \end{aligned}$$

Multiplying the previous estimates by k and summing up to n we get

$$\begin{aligned} & \|\mathbf{v}_n - \mathbf{v}_n^{hk}\|_H^2 + B(\phi_n - \phi_n^{hk}, \delta\mathbf{u}_n - \delta\mathbf{u}_n^{hk}) + G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) + \|\psi_n - \psi_n^{hk}\|_Y^2 \\ & + A(\phi_n - \phi_n^{hk}, \phi_n - \phi_n^{hk}) + (b(\phi_n - \phi_n^{hk}), \phi_n - \phi_n^{hk})_Y + 2 \\ & + 2B(\phi_n - \phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) + \|\xi_n - \xi_n^{hk}\|_Y^2 + K(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk})_n \\ & + K^*(\alpha_n - \alpha_n^{hk}, \theta_n - \theta_n^{hk}) + M(\theta_n - \theta_n^{hk}, \phi_n - \phi_n^{hk}) + \sum_{j=1}^n M(\alpha_j - \alpha_j^{hk}, \delta\phi_j - \delta\phi_j^{hk}) \\ & \leq Ck \sum_{j=1}^n \left(\|\dot{\mathbf{v}}_j - \delta\mathbf{v}_j\|_H^2 + \|\nabla(\dot{\mathbf{u}}_j - \delta\mathbf{u}_j)\|_Q^2 + \|\mathbf{v}_j - \mathbf{w}_j^h\|_V^2 + \|\nabla(\phi_j - \phi_j^{hk})\|_H^2 + \|\xi_j - \xi_j^{hk}\|_Y^2 \right. \\ & \quad \left. + \|\nabla(\theta_j - \theta_j^{hk})\|_H^2 + \|\mathbf{v}_j - \mathbf{v}_j^{hk}\|_H^2 + (\rho(\delta\mathbf{v}_{j,n} - \delta\mathbf{v}_j^{hk}), \mathbf{v}_j - \mathbf{w}_j^h)_H + \|\dot{\psi}_j - \delta\psi_j\|_Y^2 \right. \\ & \quad \left. + \|\psi_j - \psi_j^{hk}\|_Y^2 + \|\dot{\phi}_j - \delta\phi_j\|_E^2 + \|\psi_j - r_j^h\|_E^2 + \|\nabla(\alpha_j - \alpha_j^{hk})\|_H^2 + \|\phi_j - \phi_j^{hk}\|_Y^2 \right. \\ & \quad \left. + \|\theta_j - \theta_j^{hk}\|_Y^2 + (J(\delta\phi_j - \delta\phi_j^{hk}), \psi_j - r_j^h)_Y + \|\dot{\xi}_j - \delta\xi_j\|_Y^2 + \|\theta_j - \delta\theta_j\|_E^2 + \|\xi_j - z_j^h\|_E^2 \right. \\ & \quad \left. + \|\dot{\alpha}_j - \delta\alpha_j\|_E^2 + (a(\delta\xi_j - \delta\xi_j^{hk}), \xi_j - z_j^h)_Y \right) + C \left(\|\mathbf{v}^0 - \mathbf{v}^{0h}\|_H^2 + \|\mathbf{u}^0 - \mathbf{u}^{0h}\|_V^2 \right. \\ & \quad \left. + \|\psi^0 - \psi^{0h}\|_Y^2 + \|\phi^0 - \phi^{0h}\|_E^2 + \|\theta^0 - \theta^{0h}\|_E^2 + \|\alpha^0 - \alpha^{0h}\|_E^2 + \|\xi^0 - \xi^{0h}\|_Y^2 \right). \end{aligned}$$

Now, using again assumptions (ii), (iv) and (4.12) we find that

$$\begin{aligned} G(\mathbf{u}_n - \mathbf{u}_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) + 2B(\phi_n - \phi_n^{hk}, \mathbf{u}_n - \mathbf{u}_n^{hk}) + (b(\phi_n - \phi_n^{hk}), \phi_n - \phi_n^{hk})_Y \\ \geq C_1(\|\nabla(\mathbf{u}_n - \mathbf{u}_n^{hk})\|_Q^2 + \|\phi_n - \phi_n^{hk}\|_Y^2), \\ A(\phi_n - \phi_n^{hk}, \phi_n - \phi_n^{hk}) + 2M(\theta_n - \theta_n^{hk}, \phi_n - \phi_n^{hk}) + K(\theta_n - \theta_n^{hk}, \theta_n - \theta_n^{hk}) \\ \geq C_2(\|\nabla(\phi_n - \phi_n^{hk})\|_H^2 + \|\nabla(\theta_n - \theta_n^{hk})\|_H^2). \end{aligned}$$

Keeping in mind the following straightforward estimates:

$$\begin{aligned} k \sum_{j=1}^n M(\alpha_j - \alpha_j^{hk}, \delta\phi_j - \psi_j^{hk}) &= \sum_{j=1}^n M(\alpha_j - \alpha_j^{hk}, \phi_j - \phi_j^{hk} - (\phi_{j-1} - \phi_{j-1}^{hk})) \\ &= M(\alpha_n - \alpha_n, \phi_n - \phi_n^{hk}) + \sum_{j=1}^{n-1} M(\alpha_j - \alpha_j^{hk} - (\alpha_{j+1} - \alpha_{j+1}^{hk}), \phi_j - \phi_j^{hk}) + M(\alpha^0 - \alpha^{0h}, \phi_1 - \phi_1^{hk}), \\ \sum_{j=1}^{n-1} M(\alpha_j - \alpha_j^{hk} - (\alpha_{j+1} - \alpha_{j+1}^{hk}), \phi_j - \phi_j^{hk}) &\leq Ck \sum_{j=1}^{n-1} \|\nabla(\phi_j - \phi_j^{hk})\|_H^2 + \frac{C}{k} \sum_{j=1}^{n-1} \|\nabla(\alpha_j - \alpha_j^{hk} - (\alpha_{j+1} - \alpha_{j+1}^{hk}))\|_H^2 \\ &\leq Ck \sum_{j=1}^{n-1} (\|\nabla(\phi_j - \phi_j^{hk})\|_H^2 + \|\nabla(\theta_j - \theta_j^{hk})\|_H^2 + \|\nabla(\dot{\alpha}_j - \delta\alpha_j)\|_H^2), \\ k \sum_{j=1}^n (\rho(\delta\mathbf{v}_j - \delta\mathbf{v}_j^{hk}), \mathbf{v}_j - \mathbf{w}_j^h)_H &= (\rho(\mathbf{v}_n - \mathbf{v}_n^{hk}), \mathbf{v}_n - \mathbf{w}_n^h)_H + (\rho(\mathbf{v}^0 - \mathbf{v}^{0h}), \mathbf{v}_1 - \mathbf{w}_1^h)_H \\ &\quad + \sum_{j=1}^{n-1} (\rho(\mathbf{v}_j - \mathbf{v}_j^{hk}), \mathbf{v}_j - \mathbf{w}_j^h - (\mathbf{v}_{j+1} - \mathbf{w}_{j+1}^h))_H, \\ k \sum_{j=1}^n (J(\delta\phi_j - \delta\phi_j^{hk}), \phi_j - r_j^h)_Y &= (J(\phi_n - \phi_n^{hk}), \phi_n - r_n^h)_Y + (J(\phi^0 - \phi^{0h}), \phi_1 - r_1^h)_Y \\ &\quad + \sum_{j=1}^{n-1} (J(\phi_j - \phi_j^{hk}), \phi_j - r_j^h - (\phi_{j+1} - r_{j+1}^h))_Y, \\ k \sum_{j=1}^n (a(\delta\theta_j - \delta\theta_j^{hk}), \theta_j - z_j^h)_Y &= (a(\theta_n - \theta_n^{hk}), \theta_n - z_n^h)_Y + (a(\theta^0 - \theta^{0h}), \theta_1 - z_1^h)_Y \\ &\quad + \sum_{j=1}^{n-1} (a(\theta_j - \theta_j^{hk}), \theta_j - z_j^h - (\theta_{j+1} - z_{j+1}^h))_Y, \end{aligned}$$

using the above estimates and a discrete version of Gronwall's inequality (see [26]) the proof is done.

□

The estimates provided in Theorem 4.3 can be used to obtain the convergence order of the approximations given by problem (4.8)-(4.11). As an example, if we assume the additional regularity on the solution to problem (4.1)-(4.4):

$$\begin{aligned} (4.17) \quad \mathbf{u} &\in H^3(0, T; H) \cap W^{1,\infty}(0, T; [H^2(\Omega)]^d) \cap H^2(0, T; [H^1(\Omega)]^d), \\ \phi &\in H^3(0, T; Y) \cap W^{1,\infty}(0, T; H^2(\Omega)) \cap H^2(0, T; H^1(\Omega)), \\ \alpha &\in H^4(0, T; Y) \cap W^{2,\infty}(0, T; H^2(\Omega)) \cap H^3(0, T; H^1(\Omega)), \end{aligned}$$

then we have the following result which states the linear convergence of the approximations given by problem (4.8)-(4.11).

Corollary 4.4. *Let the assumptions of Theorem 4.3 hold. If we assume the additional regularity (4.17) then there exists a positive constant C , independent of the discretization parameters h and k , such that*

$$\begin{aligned} \max_{0 \leq n \leq N} & \left\{ \| \mathbf{v}_n - \mathbf{v}_n^{hk} \|_H + \| \mathbf{u}_n - \mathbf{u}_n^{hk} \|_V + \| \psi_n - \psi_n^{hk} \|_Y + \| \phi_n - \phi_n^{hk} \|_E \right. \\ & \left. + \| \xi_n - \xi_n^{hk} \|_Y + \| \theta_n - \theta_n^{hk} \|_E + \| \alpha_n - \alpha_n^{hk} \|_E \right\} \leq C(h+k). \end{aligned}$$

Remark 4.5. We note that, proceeding as in this section, we could also analyze, from the numerical point of view, similar problems where the dissipative mechanism is included into the displacement field or the volume fraction. Hence, we could consider the following problems:

$$\begin{aligned} \rho \ddot{u}_i &= \left(G_{ijmn} u_{m,n} + B_{ij}(\phi + \tau \dot{\phi}) - l_{ij} \theta \right)_{,j}, \\ \tau J \ddot{\phi} + J \ddot{\phi} &= -B_{ij} u_{i,j} - b\phi - b\tau \dot{\phi} + m\theta + \left(M_{ij} \alpha_{,j} + A_{ij}^* \phi_{,j} + A_{ij} \dot{\phi}_{,j} \right)_{,i}, \\ a \dot{\theta} &= -l_{ij} \dot{u}_{i,j} - m(\dot{\phi} + \tau \ddot{\phi}) + \left(M_{ji} \phi_{,j} + M_{ij} \tau \dot{\phi}_{,j} + k_{ij} \alpha_{,j} \right)_{,i}, \end{aligned}$$

when the dissipative mechanism is assumed on the volume fraction (here, A_{ij} is a new constitutive tensor), and

$$\begin{aligned} \tau \rho \ddot{u}_i + \rho \ddot{u}_i &= \left(G_{ijmn}^* u_{m,n} + G_{ijmn} \dot{u}_{m,n} + B_{ij} \phi - l_{ij} \theta \right)_{,j}, \\ J \ddot{\phi} &= -B_{ij}(u_{i,j} + \tau \dot{u}_{i,j}) - b\phi + m\theta + (M_{ij} \alpha_{,j} + A_{ij} \phi_{,j})_{,i}, \\ a \dot{\theta} &= -l_{ij}(\dot{u}_{i,j} + \tau \ddot{u}_{i,j}) - m\dot{\phi} + (M_{ji} \phi_{,j} + k_{ij} \alpha_{,j})_{,i}, \end{aligned}$$

when the dissipative mechanism is assumed on displacement field (G_{ijmn}^* is another new constitutive tensor).

5. NUMERICAL RESULTS

In this section, we describe the numerical scheme implemented in MATLAB for solving problem (4.8)-(4.11), and we present some numerical examples to demonstrate the accuracy of the approximation and the dependence of the solution with respect to a constitutive parameter. Therefore, we solve the following linear problem (we restrict ourselves to the one-dimensional case for the sake of simplicity), for all for all $w^h \in V^h$, and $r^h, z^h \in E^h$.

$$\begin{aligned} (\rho v_n^{hk}, w^h)_Y + \mu k^2((v_n^{hk})_x, w_x^h)_Y &= \rho(v_{n-1}^{hk}, w^h)_Y - \mu k((u_n^{hk})_x, w_x^h)_Y - \ell k((\theta_n^{hk} + \tau \xi_n^{hk})_x, w^h)_Y \\ &\quad + Bk((\phi_n^{hk})_x, w^h)_Y, \\ (J \psi_n^{hk}, r^h)_Y + Ak^2((\psi_n^{hk})_x, r_x^h)_Y + bk^2(\psi_n^{hk}, r^h)_Y &= (J \psi_{n-1}^{hk}, r^h)_Y - Ak((\phi_{n-1}^{hk})_x, r_x^h)_Y \\ &\quad - bk(\phi_{n-1}^{hk}, r^h)_Y - Mk((\alpha_n^{hk} + \tau \theta_n^{hk})_x, r_x^h)_Y - Bk((u_n^{hk})_x, r^h)_Y + mk(\theta_n^{hk} + \tau \xi_n^{hk}, r^h)_Y, \\ (\tau a \xi_n^{hk} + ak \xi_n^{hk}, z^h)_Y + Kk^2((\xi_n^{hk})_x, z_x^h)_Y + K^* k^3((\xi_n^{hk})_x, z_x^h)_Y &= (\tau a \xi_{n-1}^{hk}, z^h)_Y - \ell k((v_n^{hk})_x, z^h)_Y \\ &\quad - mk((\psi_n^{hk})_x, z^h)_Y - Kk((\theta_{n-1}^{hk})_x, z_x^h)_Y - K^* k((\alpha_{n-1}^{hk} + k \theta_{n-1}^{hk})_x, z_x^h)_Y - Mk((\phi_n^{hk})_x, z_x^h)_Y, \end{aligned}$$

where the discrete displacement, the discrete volume fraction, the discrete temperature and the discrete thermal displacement field are then recovered from the relations:

$$u_n^{hk} = k v_n^{hk} + u_{n-1}^h, \quad \phi_n^{hk} = k \psi_n^{hk} + \phi_{n-1}^h, \quad \theta_n^{hk} = k \xi_n^{hk} + \theta_{n-1}^h, \quad \alpha_n^{hk} = k \theta_n^{hk} + \alpha_{n-1}^h.$$

This numerical scheme was implemented on a 3.2 Ghz PC using MATLAB, and a typical run ($h = k = 0.001$) took about 0.65 seconds of CPU time.

First example: numerical convergence

As an academical example, in order to show the accuracy of the approximations the following simpler case is considered. We solve the one-dimensional problem (2.4), (2.5) and (2.6) with the following data:

$$\begin{aligned} \rho &= 1, & \mu &= 1, & B &= 1, & \ell &= 1, & \tau &= 1, & J &= 1, & A &= 1, \\ b &= 1, & m &= 1, & M &= 1, & a &= 1, & K^* &= 1, & K &= 1. \end{aligned}$$

In this case, if we consider homogeneous Dirichlet boundary conditions at boundaries $x = 0, 1$ and the initial conditions:

$$u^0(x) = v^0(x) = \phi^0(x) = \psi^0(x) = \alpha^0(x) = \theta^0(x) = \xi^0(x) = x(x - 1) \quad \text{for all } x \in (0, 1),$$

and we add the following artificial supply terms, for $(x, t) \in (0, 1) \times (0, 1)$,

$$F_1(x, t) = e^t(2x + x(x - 1) - 3), \quad F_2(x, t) = e^t(2x - 7), \quad F_3(x, t) = e^t(2x + 3x(x - 1) - 7),$$

then the exact solution to problem (2.4), (2.5) and (2.6) can be easily calculated and it has the following form, for $(x, t) \in [0, 1] \times [0, 1]$:

$$u(x, t) = \phi(x, t) = \alpha(x, t) = e^t x(x - 1).$$

Thus, the approximation errors estimated by

$$\begin{aligned} \max_{0 \leq n \leq N} \Big\{ &\|v_n - v_n^{hk}\|_Y + \|(u_n - u_n^{hk})_x\|_Y + \|\psi_n - \psi_n^{hk}\|_Y + \|(\phi_n - \phi_n^{hk})_x\|_Y \\ &+ \|(\alpha_n - \alpha_n^{hk})_x\|_Y + \|(\theta_n - \theta_n^{hk})_x\|_Y + \|\xi_n - \xi_n^{hk}\|_Y \Big\} \end{aligned}$$

are presented in TABLE 1 for several values of the discretization parameters h and k . Moreover, the evolution of the error depending on the parameter $h + k$ is plotted in FIGURE 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 4.4, is achieved.

If we assume now that there are not supply terms, and we use the final time $T = 200$, the following data:

$$\begin{aligned} \rho &= 2, & \mu &= 10, & B &= 1, & \ell &= 2, & \tau &= 1, & J &= 2, & A &= 6, \\ b &= 1, & m &= 1, & M &= 1, & a &= 3, & K^* &= 1, & K &= 3, \end{aligned}$$

and the initial conditions:

$$u^0 = v^0 = \alpha^0 = \theta^0 = \xi^0 = 0, \quad \phi^0(x) = \psi^0(x) = \frac{1}{4}x(x - 1) \quad \text{for } x \in (0, 1),$$

taking the discretization parameters $h = 10^{-3}$ and $k = 10^{-3}$, the evolution in time of the discrete energy E_n^{hk} given by

$$\begin{aligned} E_n^{hk} = \frac{1}{2} \Big\{ &\rho \|v_n^{hk}\|_Y^2 + \mu \|(u_n^{hk})_x\|_Y^2 + b \|\phi_n^{hk}\|_Y^2 + 2B(\phi_n^{hk}, (u_n^{hk})_x)_Y + 2M((\alpha_n^{hk} + \tau\theta_n^{hk})_x, (\phi_n^{hk})_x)_Y \\ &+ A \|(\phi_n^{hk})_x\|_Y^2 + \tau K \|(\theta_n^{hk})_x\|_Y^2 + J \|\psi_n^{hk}\|_Y^2 + a \|(\tau\xi_n^{hk} + \theta_n^{hk})\|_Y^2 + K^* \|(\alpha_n^{hk} + \tau\theta_n^{hk})_x\|_Y^2 \Big\} \end{aligned}$$

is plotted in FIGURE 2 (in both natural and semi-log scales). As can be seen, it converges to zero and an exponential decay seems to be achieved.

Second example: dependence on the parameter τ in a two-dimensional case

As a second example, we analyze the dependence of the solution with respect to parameter τ in a two-dimensional setting. Therefore, the domain Ω occupies the square $[0, 1] \times [0, 1]$, and it is assumed to be clamped on its vertical boundaries $\{0, 1\} \times [0, 1]$ and traction-free on the rest of the boundary. So, we consider now the two-dimensional system (2.3) and we use the following data:

$$\begin{aligned} T &= 1, & \rho &= 2, & \mu = \lambda &= 1, & B &= 1, & l &= 2, & J &= 2, & A &= 6, \\ b &= 1, & m &= 1, & M &= 1, & a &= 3, & K^* &= 1, & K &= 3, \end{aligned}$$

and the initial conditions:

$$\begin{aligned} \mathbf{u}^0 &= \mathbf{v}^0 = \mathbf{0}, & \alpha^0 &= \theta^0 = \xi^0 = 0, \\ \phi^0(x, y) &= \psi^0(x, y) = x(x - 1)y(y - 1) & \text{for all } (x, y) &\in (0, 1) \times (0, 1). \end{aligned}$$

Taking the discretization parameter $k = 0.01$, we assume that parameter τ takes values 4 and 1. Therefore, in FIGURE 3 we plot the volume fraction at final time for these two values of parameter τ . As can be observed, there are important differences between the solutions, increasing the porosity of the material. Finally, in FIGURES 4 and 5 we show the temperature and the temperature speed at final time. Again, the solutions are rather different and, even for the temperature speed, it seems that some oscillations are produced for value $\tau = 4$. We also point out that we have omitted the results regarding the displacement fields because they are similar for the two values of the parameter.

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$h \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0002	0.0001
$1/2^3$	0.854946	0.822368	0.810040	0.808537	0.808143	0.808010	0.807983
$1/2^4$	0.478785	0.431473	0.407077	0.400576	0.398678	0.398255	0.398193
$1/2^5$	0.311233	0.248407	0.213827	0.204072	0.199943	0.198000	0.197709
$1/2^6$	0.242864	0.168093	0.122649	0.108689	0.102640	0.099654	0.098856
$1/2^7$	0.217701	0.135860	0.082576	0.064227	0.055561	0.051112	0.049927
$1/2^8$	0.209600	0.124048	0.066415	0.045099	0.033978	0.027597	0.025789
$1/2^9$	0.207320	0.120193	0.060355	0.037507	0.024891	0.016821	0.014207
$1/2^{10}$	0.206728	0.119089	0.058252	0.034666	0.021334	0.012287	0.009004
$1/2^{11}$	0.206578	0.118799	0.057600	0.033659	0.020001	0.010509	0.006860
$1/2^{12}$	0.206540	0.118725	0.057421	0.033337	0.019517	0.009836	0.006028
$1/2^{13}$	0.206531	0.118707	0.057375	0.033246	0.019356	0.009582	0.005712

TABLE 1. Example 1: Numerical errors for some h and k .

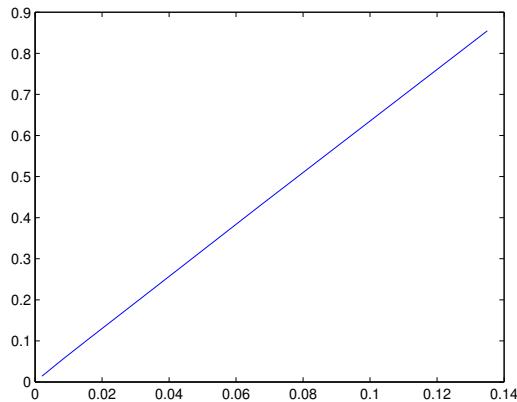


FIGURE 1. Example 1: Asymptotic constant error.

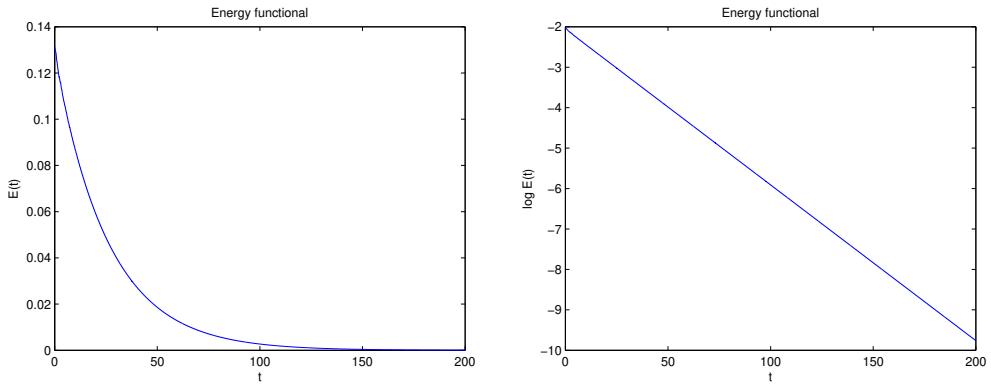


FIGURE 2. Example 1: Evolution in time of the discrete energy (natural and semi-log scales).

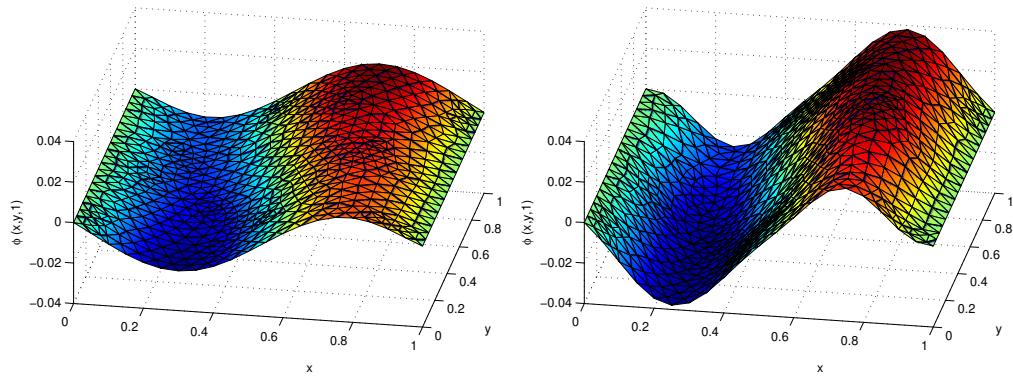


FIGURE 3. Example 2: Volume fraction at final time for $\tau = 1$ (left) and $\tau = 4$ (right).

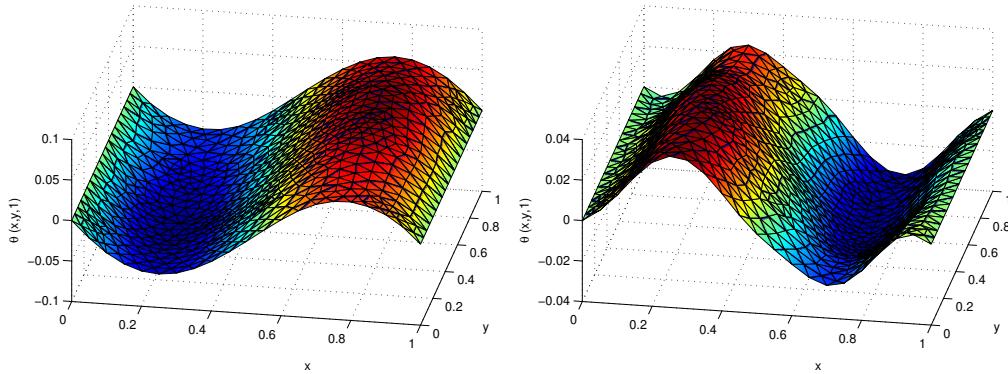


FIGURE 4. Example 2: Temperature at final time for $\tau = 1$ (left) and $\tau = 4$ (right).

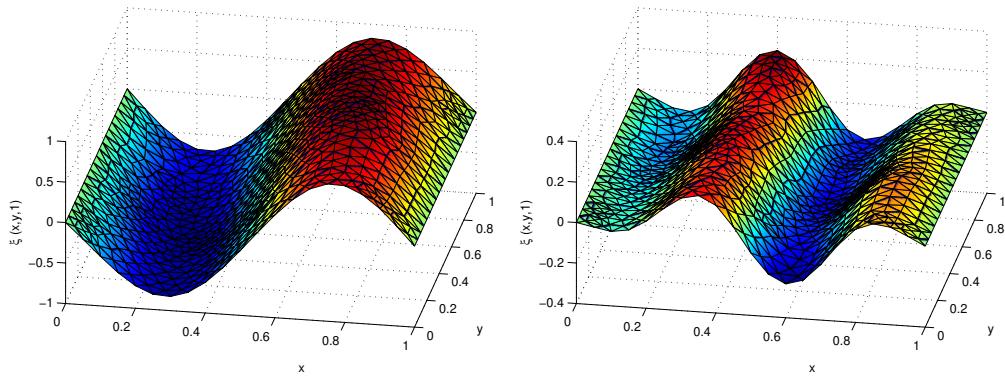


FIGURE 5. Example 2: Temperature speed at final time for $\tau = 1$ (left) and $\tau = 4$ (right).