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A-posteriori error analysis of hp-version discontinuous Galerkin finite element methods for second-order quasilinear elliptic problems

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We develop the a-posteriori error analysis of *hp*-version interior-penalty discontinuous Galerkin finite element methods for a class of second-order quasilinear elliptic partial differential equations. Computable upper and lower bounds on the error are derived in terms of a natural (mesh-dependent) energy norm. The bounds are explicit in the local mesh size and the local degree of the approximating polynomial. The performance of the proposed estimators within an automatic *hp*-adaptive refinement procedure is studied through numerical experiments.

Key words and phrases: *hp*-adaptivity, a-posteriori error estimation, discontinuous Galerkin methods, quasilinear elliptic PDEs.

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1 Introduction

In this article, we consider the a-posteriori error analysis, in a natural mesh-dependent energy norm, for a class of interior-penalty hp-version discontinuous Galerkin finite element methods for the numerical solution of the following quasilinear elliptic boundary value problem:

$$-\nabla \cdot (\mu(\mathbf{x}, |\nabla u|)\nabla u) = f \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \Gamma. \tag{1.2}$$

Here, Ω is a bounded polygonal domain in $\mathbb{R}^2$ with boundary Γ, and $f \in \mathrm{L}^2(\Omega)$. Additionally, we assume that the nonlinearity μ satisfies the following assumptions:

(A1) $\mu \in \mathrm{C}(\overline{\Omega} \times [0, \infty))$;

(A2) there exist positive constants m_μ and M_μ such that

$$m_\mu(t - s) \leq \mu(\mathbf{x}, t)t - \mu(\mathbf{x}, s)s \leq M_\mu(t - s), \qquad t \geq s \geq 0,\ \mathbf{x} \in \overline{\Omega}. \tag{1.3}$$

We remark that, if μ satisfies (1.3), there exist constants C_1 and C_2, $C_1 \geq C_2 > 0$, such that for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$, and all $\mathbf{x} \in \overline{\Omega}$,

$$|\mu(\mathbf{x}, |\mathbf{v}|)\mathbf{v} - \mu(\mathbf{x}, |\mathbf{w}|)\mathbf{w}| \leq C_1|\mathbf{v} - \mathbf{w}|, \tag{1.4}$$

$$C_2|\mathbf{v} - \mathbf{w}|^2 \leq (\mu(\mathbf{x}, |\mathbf{v}|)\mathbf{v} - \mu(\mathbf{x}, |\mathbf{w}|)\mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}); \tag{1.5}$$

see [34, Lemma 2.1].

Nonlinearities of this kind appear in numerous problems in continuum mechanics. In particular, they arise in mathematical models for non-Newtonian fluids, such as the following generalised power-law model: given $\mathbf{f} \in \mathrm{L}^2(\Omega)^2$, find $(\mathbf{u}, p) \in \mathrm{H}^1(\Omega)^2 \times \mathrm{L}^2(\Omega)/\mathbb{R}$ such that

$$\begin{aligned} -\nabla \cdot \{\mu(\mathbf{x}, |e(\mathbf{u})|)e(\mathbf{u})\} + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma, \end{aligned}$$

where $\mathbf{u} = (u_1, u_2)^{\mathrm{T}}$ is the velocity vector, p is the pressure, $\mathbf{f} = (f_1, f_2)^{\mathrm{T}}$ is the applied force, $e(\mathbf{u})$ is the symmetric 2×2 strain tensor defined by

$$e_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2,$$

and $|e(\mathbf{u})|$ is the Frobenius norm of $e(\mathbf{u})$.

For the sake of notational simplicity we shall suppress the dependence of μ on $\mathbf{x}$ and write $\mu(t)$ instead of $\mu(\mathbf{x}, t)$. Indeed, in many physical applications μ is in fact independent of $\mathbf{x}$; for example, in the Carreau law for a non-Newtonian fluid, we have $\mu(t) = \mu_\infty + (\mu_0 - \mu_\infty)(1 + \lambda t^2)^{\frac{r-2}{2}}$, where $\lambda > 0$, $1 < r \leq 2$ and $0 < \mu_\infty < \mu_0$.

In recent years there has been considerable interest in discontinuous Galerkin finite element methods for the numerical solution of a wide range of partial differential equations. We shall not attempt to give an extensive survey of this area of research: the reader is referred to [11] for a detailed review. Discontinuous Galerkin Finite Element Methods (DGFEMs) were introduced in the early 1970s for the numerical solution of first-order hyperbolic problems (see [12, 13, 15, 29, 33, 41]). Simultaneously, but quite independently, they were proposed as nonstandard schemes for the approximation of second-order elliptic equations [1, 38, 42]. The recent upsurge of interest in this class of methods has been stimulated by the computational convenience of DGFEMs due to a high degree of locality, the need to approximate advection-dominated diffusion problems without excessive numerical stabilisation, the necessity to accommodate high-order *hp*- and spectral element discretisations for first-order hyperbolic equations and advection-diffusion problems [17, 31], and the desire to handle nonlinear hyperbolic problems in a locally conservative manner and without auxiliary numerical stabilisation [9, 14]; see also [8, 10] for the error analysis of the local version of the DGFEM in the elliptic case, as well as [2] and [39].

In the recent article [20] a family of interior-penalty *hp*-DGFEMs was formulated for the numerical approximation of the scalar quasilinear boundary value problem (1.1)–(1.2), and a-priori bounds were derived on the error, measured in terms of a mesh-dependent energy norm. These error bounds are optimal with respect to the mesh size h and mildly suboptimal (by $p^{1/2}$) in the polynomial degree p; more precisely, for $u \in \mathrm{C}^1(\Omega) \cap \mathrm{H}^k(\Omega)$, $k \geq 2$, it was shown that, for any member of the family of methods considered, the error tends to zero at the rate $\mathcal{O}(h^{s-1}/p^{k-3/2})$, where $1 \leq s \leq \min\{p+1, k\}$, as h tends to zero and p tends to infinity. For related work on h-version local DGFEMs for quasilinear PDEs, we refer to the articles [7, 18], for example. Here, we extend this work by considering the a-posteriori error analysis of the interior-penalty *hp*-DGFEMs from [20]. In particular, we shall derive computable upper and lower bounds on the error, measured in terms of the underlying DG-energy norm, which are explicit in terms of their dependence on h and p. The upper bound is based on the general techniques developed in the articles [21, 22, 23, 24]. Indeed, here the proof crucially relies on the approximation of discontinuous finite element functions by conforming ones. Results of this type have been developed independently by a number of authors in the context of the h-version of the DGFEM; see, for example, [19, 30, 32]. The extension of this type of result to the *hp*-version of the DGFEM was recently undertaken in the article [23]. In contrast to [23], for example, here we avoid the need to introduce an auxiliary formulation of the underlying DGFEM through the use of lifting operators, while still only requiring minimal regularity assumptions on the unknown analytical solution. The proof of the lower (efficiency) bounds is based on the techniques presented in [36]. As in the case of the conforming *hp*-version finite element methods considered in [36], reliability and efficiency of our error bounds cannot be established uniformly with respect to the polynomial degree, since the proof of efficiency relies on employing inverse estimates which are suboptimal in the spectral order. Finally, numerical experiments highlighting the performance of the proposed estimator within an *hp*-adaptive mesh refinement algorithm will also be presented.

The outline of this article is as follows. In Section 2, we revisit the *hp*-DGFEM introduced in [20], for the numerical approximation of the boundary-value problem (1.1)–(1.2). In Section 3, our a-posteriori error bounds are presented and discussed; both upper and lower energy norm bounds will be derived. In Section 4, we present a series of numerical experiments to illustrate the performance of the proposed error estimator within an automatic *hp*-mesh refinement algorithm. Finally, in Section 5 we summarise the main results of this article and draw some conclusions.

Throughout the paper, we use the following standard function spaces. For a bounded Lipschitz domain $D \subset \mathbb{R}^d$, $d \geq 1$, we write $\mathrm{H}^t(D)$ to denote the usual (real) Sobolev space of order $t \geq 0$ with norm $\|\cdot\|_{t,D}$. In the case $t = 0$, we set $\mathrm{L}^2(D) = \mathrm{H}^0(D)$. We define $\mathrm{H}^1_0(D)$ to be the subspace of functions in $\mathrm{H}^1(D)$ with zero trace on ∂D. For a function space $\mathrm{X}(D)$, we write $\mathrm{X}(D)^d$ to denote the space of d-component vector fields whose components belong to $\mathrm{X}(D)$; this space is equipped with the usual product-norm which, for simplicity, is denoted in the same way as the norm in $\mathrm{X}(D)$.

2 *hp*-Version discontinuous Galerkin FEM

Let $\mathcal{T}_h$ be a subdivision of Ω into disjoint open element domains κ such that $\overline{\Omega} = \bigcup_{\kappa\in\mathcal{T}_h} \overline{\kappa}$. We assume that the family of subdivisions $\{\mathcal{T}_h\}_{h>0}$ is shape-regular (see, for example, [6, pp. 61, 113, and Remark 2.2, p. 114]) and each $\kappa \in \mathcal{T}_h$ is an affine image of a fixed master element $\widehat{\kappa}$; i.e., for each $\kappa \in \mathcal{T}_h$ there exists an affine mapping $F_\kappa : \widehat{\kappa} \to \kappa$ such that $\kappa = F_\kappa(\widehat{\kappa})$, where $\widehat{\kappa}$ is either the open unit triangle $\{(x, y) : -1 < x < 1, -1 < y < -x\}$ or the open unit square $(-1, 1)^2$ in $\mathbb{R}^2$. By h_κ we denote the element diameter of $\kappa \in \mathcal{T}_h$, $h = \max_{\kappa\in\mathcal{T}_h} h_\kappa$, and $\mathbf{n}_\kappa$ signifies the unit outward normal vector to κ. We allow the meshes $\mathcal{T}_h$ to be *1-irregular*, i.e., each edge of any one element $\kappa \in \mathcal{T}_h$ contains at most one hanging node (which, for simplicity, we assume to be the midpoint of the corresponding edge). Here, we suppose that $\mathcal{T}_h$ is *regularly reducible* (cf. [40, Section 7.1]), i.e., there exists a shape-regular conforming (regular) mesh $\widetilde{\mathcal{T}}_h$ (consisting of triangles and parallelograms) such that the closure of each element in $\mathcal{T}_h$ is a union of closures of elements of $\widetilde{\mathcal{T}}_h$, and that there exists a constant $C > 0$, independent of the element sizes, such that for any two elements $\kappa \in \mathcal{T}_h$ and $\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h$ with $\widetilde{\kappa} \subseteq \kappa$ we have $h_\kappa/h_{\widetilde{\kappa}} \leq C$. Note that these assumptions imply that the family $\{\mathcal{T}_h\}_{h>0}$ is of *bounded local variation*, i.e., there exists a constant $\rho_1 \geq 1$, independent of the element sizes, such that

$$\rho_1^{-1} \leq h_\kappa/h_{\kappa'} \leq \rho_1, \tag{2.1}$$

for any pair of elements $\kappa, \kappa' \in \mathcal{T}_h$ which share a common edge $e = \partial\kappa \cap \partial\kappa'$.

For a nonnegative integer k, we denote by $\mathcal{P}_k(\widehat{\kappa})$ the set of polynomials of total degree k on $\widehat{\kappa}$. When $\widehat{\kappa}$ is the unit square, we also consider $\mathcal{Q}_k(\widehat{\kappa})$, the set of all tensor-product polynomials on $\widehat{\kappa}$ of degree k in each co-ordinate direction. To each $\kappa \in \mathcal{T}_h$ we assign a polynomial degree p_κ (local approximation order).

We store the h_κ, p_κ and F_κ in the vectors $\mathbf{h} = \{h_\kappa : \kappa \in \mathcal{T}_h\}$, $\mathbf{p} = \{p_\kappa : \kappa \in \mathcal{T}_h\}$ and $\mathbf{F} = \{F_\kappa : \kappa \in \mathcal{T}_h\}$, respectively, and consider the finite element space

$$S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}) = \{v \in \mathrm{L}^2(\Omega) : v|_\kappa \circ F_\kappa \in \mathcal{R}_{p_\kappa}(\widehat{\kappa}) \quad \forall \kappa \in \mathcal{T}_h\} ,$$

where $\mathcal{R}$ is either $\mathcal{P}$ or $\mathcal{Q}$. We shall suppose that the polynomial degree vector $\mathbf{p}$, with $p_\kappa \geq 1$ for each $\kappa \in \mathcal{T}$, has *bounded local variation*, i.e., there exists a constant $\rho_2 \geq 1$ independent of $\mathbf{h}$ and $\mathbf{p}$, such that, for any pair of neighbouring elements $\kappa, \kappa' \in \mathcal{T}_h$,

$$\rho_2^{-1} \leq p_\kappa / p_{\kappa'} \leq \rho_2. \tag{2.2}$$

Let us consider the set $\mathcal{E}$ of all one-dimensional open edges, or, simply, *edges*, of all elements $\kappa \in \mathcal{T}_h$. Further, we denote by $\mathcal{E}_{\mathrm{int}}$ the set of all edges e in $\mathcal{E}$ that are contained in Ω (interior edges). Additionally, let $\Gamma_{\mathrm{int}} = \{\mathbf{x} \in \Omega : \mathbf{x} \in e \text{ for some } e \in \mathcal{E}_{\mathrm{int}}\}$, and introduce $\mathcal{E}_{\mathrm{B}}$ to be the set of boundary edges consisting of all $e \in \mathcal{E}$ that are contained in $\partial\Omega$. Moreover, let $\Gamma_{\mathrm{int,B}} = \Gamma_{\mathrm{int}} \cup \Gamma$.

Suppose that e is an edge of an element $\kappa \in \mathcal{T}_h$. Then, by h_e, we denote the length of e. Due to our assumptions on the subdivision $\mathcal{T}_h$ we have that, if $e \subset \partial\kappa$, then h_e is commensurate with h_κ, the diameter of κ.

Given that $e \in \mathcal{E}_{\mathrm{int}}$, there exist indices i and j such that $i > j$ and $\kappa_i, \kappa_j \in \mathcal{T}_h$ share the edge e; we define the (element-numbering-dependent) jump of an (element-wise smooth) function v across e and the mean-value of v on e by

$$[\![v]\!]_e = v|_{\partial\kappa_i \cap e} - v|_{\partial\kappa_j \cap e} \quad \text{and} \quad \langle\!\langle v \rangle\!\rangle_e = \frac{1}{2}\left(v|_{\partial\kappa_i \cap e} + v|_{\partial\kappa_j \cap e}\right),$$

respectively. For a boundary edge $e \subset \Gamma$, and thereby $e \subset \partial\kappa \cap \Gamma$ for some $\kappa \in \mathcal{T}$, we define

$$[\![v]\!]_e = \langle\!\langle v \rangle\!\rangle_e = v|_{\partial\kappa \cap e}.$$

When there is no danger of confusion, the subscript $\cdot_e$ will be suppressed. Additionally, we associate with each edge $e \subset \Gamma_{\mathrm{int}}$ the unit normal vector ν which points from κ_i to κ_j $(i > j)$; if $e \subset \Gamma$, then ν is defined as the outward unit normal vector on Γ.

With these notations and a parameter $\theta \in [-1, 1]$, we introduce the semilinear form

$$\begin{aligned} B(w, v) = &\int_\Omega \mu(|\nabla_h w|) \nabla_h w \cdot \nabla_h v \,\mathrm{d}\mathbf{x} \\ &- \int_{\Gamma_{\mathrm{int,B}}} \langle\!\langle \mu(|\nabla_h w|) \nabla_h w \cdot \nu \rangle\!\rangle [\![v]\!] \,\mathrm{d}s + \theta \int_{\Gamma_{\mathrm{int,B}}} \langle\!\langle \mu(h^{-1}|[\![w]\!]|) \nabla_h v \cdot \nu \rangle\!\rangle [\![w]\!] \,\mathrm{d}s \\ &+ \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![w]\!] \, [\![v]\!] \,\mathrm{d}s, \end{aligned} \tag{2.3}$$

and the linear functional

$$\ell(v) = \int_\Omega f v \,\mathrm{d}\mathbf{x}. \tag{2.4}$$

Here, ∇_h denotes the element-wise gradient operator defined, for $v \in \mathrm{H}^1(\Omega, \mathcal{T}_h)$, by $(\nabla_h v)|_\kappa = \nabla(v|_\kappa)$. For an edge $e \in \mathcal{E}$, the discontinuity penalisation parameter σ, featuring in $B(\cdot,\cdot)$ above, is defined by

$$\sigma|_e = \sigma_e = \gamma \frac{\langle\!\langle p^2 \rangle\!\rangle_e}{h_e}, \tag{2.5}$$

where $\gamma \geq 1$ is a (sufficiently large) constant, cf. Theorem 2 below.

The *hp*-DGFEM approximation of problem (1.1)–(1.2) reads as follows: find $u_{\mathrm{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$ such that

$$B(u_{\mathrm{DG}}, v) = \ell(v) \qquad \forall v \in S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F}). \tag{2.6}$$

Remark 1 In the case of an inhomogeneous Dirichlet boundary condition, $u = g$ on Γ, the third term in the semilinear form $B_{\mathrm{DG}}(\cdot,\cdot)$ must be replaced by

$$\theta \int_{\Gamma_{\mathrm{int}}} \langle\!\langle \mu(h^{-1}|[\![w]\!]|)\nabla_h v \cdot \nu \rangle\!\rangle [\![w]\!] \,\mathrm{d}s + \theta \int_\Gamma \mu(h^{-1}|w-g|)\nabla_h v \cdot \mathbf{n}\,(w-g)\,\mathrm{d}s,$$

while the linear functional $\ell(\cdot)$ defined in (2.4) must be substituted by

$$\ell(v) = \int_\Omega f v \,\mathrm{d}\mathbf{x} + \int_\Gamma \sigma g v \;\mathrm{d}s;$$

we refer to [20] for further details.

The existence and uniqueness of the DG solution u_{DG} satisfying (2.6) is guaranteed by the following result proved in [20, Theorem 2.5].

Theorem 2 *Suppose that γ in* (2.5) *is chosen sufficiently large. Then, there exists a unique element u_{DG} in $S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$ such that* (2.6) *holds.*

We conclude this section by equipping the DG space $S^{\mathbf{p}}(\Omega, \mathcal{T}, \mathbf{F})$ with the DG energy norm $\|\cdot\|_{\mathrm{DG}}$ defined by

$$\|v\|_{\mathrm{DG}} = \left(\sum_{\kappa \in \mathcal{T}} \int_\kappa |\nabla v|^2 \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{int,B}}} \sigma \,[\![v]\!]^2 \,\mathrm{d}s \right)^{1/2}$$

induced by the energy inner product

$$(v, w)_{\mathrm{DG}} = \sum_{\kappa \in \mathcal{T}} \int_\kappa \nabla v \cdot \nabla w \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{int,B}}} \sigma \,[\![v]\!]\,[\![w]\!] \,\mathrm{d}s.$$

The a-priori error analysis of the discontinuous Galerkin finite element method (2.6) has been developed in [20]; here, we shall be concerned with its a-posteriori error analysis.

3 hp-Version a-posteriori error analysis

Under the structural hypotheses (1.4)–(1.5) on the coefficient μ, the existence of a unique solution $u \in \mathrm{H}_0^1(\Omega)$ to (1.1)–(1.2) follows from the following result from the theory of monotone operators (see [37], Theorem 3.2.23), with $H = \mathrm{H}_0^1(\Omega)$, $\Lambda = C_1$ and $\lambda = C_2$. Henceforth, we shall therefore assume that $u \in \mathrm{H}_0^1(\Omega)$.

Proposition 3 *Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$, and let T be an operator from H into itself. Suppose that T is Lipschitz-continuous on H, i.e. there exists $\Lambda > 0$ such that*

$$\|T(\mathsf{w}_1) - T(\mathsf{w}_2)\|_H \leq \Lambda \|\mathsf{w}_1 - \mathsf{w}_2\|_H \qquad \forall \mathsf{w}_1, \mathsf{w}_2 \in H,$$

and strongly monotone on H, i.e. there exists $\lambda > 0$ such that

$$(T(\mathsf{w}_1) - T(\mathsf{w}_2), \mathsf{w}_1 - \mathsf{w}_2)_H \geq \lambda \|\mathsf{w}_1 - \mathsf{w}_2\|_H^2 \qquad \forall \mathsf{w}_1, \mathsf{w}_2 \in H.$$

Then, T is a bijection of H onto itself, and the inverse T^{-1} of T is Lipschitz-continuous on H:

$$\|T^{-1}\mathsf{f} - T^{-1}\mathsf{g}\|_H \leq (\Lambda/\lambda)\|\mathsf{f} - \mathsf{g}\|_H \qquad \forall \mathsf{f}, \mathsf{g} \in H.$$

3.1 Upper bound

In this section we will formulate the main result of this paper, Theorem 4. To this end, we first define, for an element $\kappa \in \mathcal{T}_h$ and an edge $e \in \mathcal{E}_{\mathrm{int}}$, the data-oscillation terms

$$\mathcal{O}_\kappa^{(1)} = h_\kappa^2 p_\kappa^{-2} \, \|(\mathbb{I} - \Pi_{\mathcal{T}_h})|_\kappa (f + \nabla \cdot (\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\kappa}^2, \tag{3.1}$$

and

$$\mathcal{O}_e^{(2)} = h_e \overline{p}_e^{-1} \|(\mathbb{I} - \Pi_{\mathcal{E}})|_e(\llbracket \mu(|\nabla u_{\mathrm{DG}}|\nabla u_{\mathrm{DG}}) \cdot \nu \rrbracket)\|_{0,e}^2, \tag{3.2}$$

which depend on the right-hand side f in (1.1) and the numerical solution u_{DG} from (2.6). Here, $\mathbb{I}$ is a generic identity operator and $\Pi_{\mathcal{T}_h}$ denotes the element-wise L^2-projector onto the space $S^{\mathbf{p}-1}(\Omega, \mathcal{T}_h, \mathbf{F})$, where $\mathbf{p}-1 = \{p_\kappa - 1\}_{\kappa \in \mathcal{T}_h}$. Furthermore, $\Pi_{\mathcal{E}}|_e$ is defined as the L^2-projector onto $\mathcal{P}_{\overline{p}_e - 1}(e)$; here, $\overline{p}_e = \max\{p_\kappa, p_{\kappa'}\}$, with $\kappa, \kappa' \in \mathcal{T}_h$, $e = \partial\kappa \cap \partial\kappa'$ (we note that, due to our assumptions on the polynomial degree vector $\mathbf{p}$, the quantities $\overline{p}_e$, p_κ and $p_{\kappa'}$ are all commensurate with one another).

Theorem 4 *Let the analytical solution u of* (1.1)–(1.2) *belong to* $\mathrm{H}_0^1(\Omega)$. *Furthermore, let $u_{\mathrm{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ be its discontinuous Galerkin approximation, i.e. the solution of* (2.6). *Then, the following hp-version a-posteriori error bound holds:*

$$\|u - u_{\mathrm{DG}}\|_{\mathrm{DG}} \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 + \mathcal{O}(f, u_{\mathrm{DG}}) \right)^{\frac{1}{2}}, \tag{3.3}$$

where the local error indicators η_κ, $\kappa \in \mathcal{T}_h$, are defined by

$$\begin{aligned}\eta_\kappa^2 = h_\kappa^2 p_\kappa^{-2} \|\Pi_{\mathcal{T}_h}(f + \nabla \cdot (\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\kappa}^2 \\ + h_\kappa p_\kappa^{-1} \|\Pi_{\mathcal{E}}([\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} \cdot \nu]\!])\|_{0,\partial\kappa\setminus\Gamma}^2 + \gamma^2 h_\kappa^{-1} p_\kappa^3 \|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial\kappa}^2,\end{aligned} \tag{3.4}$$

and

$$\mathcal{O}(f, u_{\mathrm{DG}}) = \sum_{\kappa \in \mathcal{T}_h} \mathcal{O}_\kappa^{(1)} + \sum_{e \in \mathcal{E}_{\mathrm{int}}} \mathcal{O}_e^{(2)}.$$

Here, $C > 0$ is a constant that is independent of $\mathbf{h}$, the polynomial degree vector $\mathbf{p}$, and the parameters γ and θ, and only depends on the shape-regularity of the mesh and on the constants ρ_1 and ρ_2 from (2.1) and (2.2), respectively.

Remark 5 We observe a slight suboptimality with respect to the polynomial degree in the last term of the local error estimator η_κ in (3.4). This results from the fact that, due to the possible presence of hanging nodes in $\mathcal{T}_h$, a nonconforming interpolant is used in the proof of the above Theorem 4; cf. Section 3.1.3. Indeed, for conforming (regular) meshes, i.e. meshes without any hanging nodes, a conforming (hp-version) Clément interpolant, as constructed in [35], can be employed; this then results in an a posteriori error bound of the form (3.3), with the term $\gamma^2 h_\kappa^{-1} p_\kappa^3 \|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial\kappa}^2$ in (3.4) replaced by the improved expression $\gamma h_\kappa^{-1} p_\kappa^2 \|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial\kappa}^2$; cf. [23].

Remark 6 In order to incorporate the inhomogeneous boundary condition $u = g$ on Γ, the error indicators η_κ are simply adjusted by modifying the jump indicators $\|\sigma^{\frac{1}{2}}[\![u_{\mathrm{DG}}]\!]\|_{0,\partial\kappa}^2$ on $\partial\kappa \cap \Gamma$, with the inclusion of an additional data-oscillation term; see [23] for details.

3.1.1 DG decompositions

The hp-version a-posteriori error analysis for the DGFEM (2.6) will be based on an approach similar to the one discussed in [23] (see also [21, 22, 24, 44], for related work). In contrast with the analysis presented in [23] though, here we shall also admit 1-irregular meshes containing hanging nodes. To this end, consider a given subdivision $\mathcal{T}_h$ of Ω that is regularly reducible, i.e., $\mathcal{T}_h$ can be refined to a shape-regular conforming mesh $\widetilde{\mathcal{T}}_h$ as described in Section 2. Furthermore, denote by $S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})$ the corresponding DG space, with a suitable affine element mapping vector $\widetilde{\mathbf{F}}$ and a polynomial degree vector $\widetilde{\mathbf{p}}$ that is defined by $p_{\widetilde{\kappa}} = p_\kappa$, for any $\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h$ with $\widetilde{\kappa} \subseteq \kappa$, for some $\kappa \in \mathcal{T}_h$. We note that $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}) \subseteq S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})$, and that, due to our assumptions in Section 2 (specifically, the commensurability of the local element sizes and of the local polynomial degrees in $\mathcal{T}_h$ and $\widetilde{\mathcal{T}}_h$, due to our bounded local variation assumptions), the DG energy norms $\|\cdot\|_{\mathrm{DG}}$ and $\|\cdot\|_{\widetilde{\mathrm{DG}}}$ corresponding to the DG spaces $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ and $S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})$, respectively, are equivalent on $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$; in particular, there exist positive constants N_1, N_2, independent of $\mathbf{h}$ and $\mathbf{p}$, such that

$$N_1 \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![v]\!]^2 \, \mathrm{d}s \leq \int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma} \, [\![v]\!]^2 \, \mathrm{d}s \leq N_2 \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![v]\!]^2 \, \mathrm{d}s \qquad \forall v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F}). \tag{3.5}$$

Here, $\widetilde{\Gamma}_{\mathrm{int,B}}$ denotes the union of all element edges of $\widetilde{\mathcal{T}}_h$, and $\widetilde{\sigma}$ is the discontinuity penalisation parameter on $S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})$ which is defined analogously as for $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ in (2.5); note that, for $v \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$, the jump $[\![v]\!]$ vanishes on $\widetilde{\Gamma}_{\mathrm{int,B}} \setminus \Gamma_{\mathrm{int,B}}$.

An important step in our analysis is the decomposition of the space $S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})$ into two orthogonal subspaces: a conforming part $[S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\parallel} = S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}}) \cap \mathrm{H}^1_0(\Omega)$, and a nonconforming part $[S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\perp}$ defined as the orthogonal complement of $[S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\parallel}$ in $S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})$ with respect to the DG energy inner product $(\cdot, \cdot)_{\widetilde{\mathrm{DG}}}$ (inducing the DG energy norm $\|\cdot\|_{\widetilde{\mathrm{DG}}}$), i.e.,

$$S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}}) = [S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\parallel} \oplus_{\|\cdot\|_{\widetilde{\mathrm{DG}}}} [S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\perp}.$$

Based on this setting, the DG-solution u_{DG} obtained by (2.6) may be split accordingly,

$$u_{\mathrm{DG}} = u_{\mathrm{DG}}^{\parallel} + u_{\mathrm{DG}}^{\perp}, \tag{3.6}$$

where $u_{\mathrm{DG}}^{\parallel} \in [S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\parallel}$ and $u_{\mathrm{DG}}^{\perp} \in [S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\perp}$. Furthermore, we define the error of the hp-DGFEM by

$$e_{\mathrm{DG}} = u - u_{\mathrm{DG}}, \tag{3.7}$$

and let

$$e_{\mathrm{DG}}^{\parallel} = u - u_{\mathrm{DG}}^{\parallel} \in \mathrm{H}^1_0(\Omega). \tag{3.8}$$

3.1.2 Auxiliary results

For the proof of the above Theorem 4, we shall require the following auxiliary results.

Proposition 7 *Under the assumptions in Section 2 on the (regularly reduced) subdivision $\widetilde{\mathcal{T}}_h$, the following norm-equivalence holds over the space $[S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\perp}$:*

$$\widetilde{C}_1 \|v\|^2_{\widetilde{\mathrm{DG}}} \leq \int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma} \, [\![v]\!]^2 \, \mathrm{d}s \leq \widetilde{C}_2 \|v\|^2_{\widetilde{\mathrm{DG}}} \qquad \forall v \in [S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\perp}, \tag{3.9}$$

where the constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ depend only on the shape-regularity of $\mathcal{T}_h$ and on the constants ρ_1 and ρ_2 in (2.1) *and* (2.2)*, respectively.*

Proof See [23, Proposition 4.5]. ■

Corollary 7A *With $u_{\mathrm{DG}}^{\perp}$ and $e_{\mathrm{DG}}^{\parallel}$ defined by* (3.6) *and* (3.8)*, respectively, the following bounds hold:*

$$\|u_{\mathrm{DG}}^{\perp}\|_{\widetilde{\mathrm{DG}}} \leq D_1 \left(\int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![u_{\mathrm{DG}}]\!]^2 \, \mathrm{d}s \right)^{\frac{1}{2}}, \qquad \|e_{\mathrm{DG}}^{\parallel}\|_{\mathrm{DG}} \leq D_2 \|e_{\mathrm{DG}}\|_{\mathrm{DG}},$$

where the constants $D_1, D_2 > 0$ are independent of γ, $\mathbf{h}$ and $\mathbf{p}$, and only depend on the shape-regularity of $\mathcal{T}_h$ and the constants ρ_1 and ρ_2 in (2.1) and (2.2), respectively.

Proof In order to prove the first of the above bounds, we recall that $u_{\mathrm{DG}}^{\|} \in \mathrm{H}_0^1(\Omega)$. This implies that $[\![u_{\mathrm{DG}}^{\|}]\!] = 0$ on $\widetilde{\Gamma}_{\mathrm{int,B}}$, and hence,

$$[\![u_{\mathrm{DG}}^{\perp}]\!] = [\![u_{\mathrm{DG}}^{\|}]\!] + [\![u_{\mathrm{DG}}^{\perp}]\!] = [\![u_{\mathrm{DG}}^{\|} + u_{\mathrm{DG}}^{\perp}]\!] = [\![u_{\mathrm{DG}}]\!].$$

Then, due to Proposition 7, we obtain,

$$\|u_{\mathrm{DG}}^{\perp}\|_{\widetilde{\mathrm{DG}}}^2 \leq C \int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma}\, [\![u_{\mathrm{DG}}^{\perp}]\!]^2 \,\mathrm{d}s = C \int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma}\, [\![u_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s. \tag{3.10}$$

Furthermore, since $u_{\mathrm{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$, and because of (3.5), we conclude that

$$\|u_{\mathrm{DG}}^{\perp}\|_{\widetilde{\mathrm{DG}}}^2 \leq C \int_{\Gamma_{\mathrm{int,B}}} \sigma\, [\![u_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s.$$

For the second bound, we use the triangle inequality, the bound (3.10), and the fact that, since the analytical solution u of (1.1)–(1.2) and $e_{\mathrm{DG}}^{\|}$ belong to $\mathrm{H}_0^1(\Omega)$, we have

$$[\![u]\!] = [\![e_{\mathrm{DG}}^{\|}]\!] = 0 \qquad \text{and} \qquad [\![e_{\mathrm{DG}}]\!] = [\![u]\!] - [\![u_{\mathrm{DG}}]\!] = -[\![u_{\mathrm{DG}}]\!] \tag{3.11}$$

on $\widetilde{\Gamma}_{\mathrm{int,B}}$ (and thereby also on $\Gamma_{\mathrm{int,B}}$). Thus,

$$\begin{aligned}\|e_{\mathrm{DG}}^{\|}\|_{\mathrm{DG}} = \|e_{\mathrm{DG}}^{\|}\|_{\widetilde{\mathrm{DG}}} &\leq \|e_{\mathrm{DG}}\|_{\widetilde{\mathrm{DG}}} + \|u_{\mathrm{DG}}^{\perp}\|_{\widetilde{\mathrm{DG}}} \leq \|e_{\mathrm{DG}}\|_{\widetilde{\mathrm{DG}}} + C\left(\int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma}\, [\![u_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq \|e_{\mathrm{DG}}\|_{\widetilde{\mathrm{DG}}} + C\left(\int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma}\, [\![e_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \leq C\|e_{\mathrm{DG}}\|_{\widetilde{\mathrm{DG}}}.\end{aligned} \tag{3.12}$$

In a similar way, we obtain

$$\|e_{\mathrm{DG}}\|_{\widetilde{\mathrm{DG}}}^2 = \sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \|\nabla e_{\mathrm{DG}}\|_{0,\widetilde{\kappa}}^2 + \int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma}\, [\![e_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s = \sum_{\kappa \in \mathcal{T}_h} \|\nabla e_{\mathrm{DG}}\|_{0,\kappa}^2 + \int_{\widetilde{\Gamma}_{\mathrm{int,B}}} \widetilde{\sigma}\, [\![u_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s.$$

Moreover, observing that $u_{\mathrm{DG}} \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$, and applying (3.5), leads to

$$\begin{aligned}\|e_{\mathrm{DG}}\|_{\widetilde{\mathrm{DG}}}^2 &\leq \sum_{\kappa \in \mathcal{T}_h} \|\nabla e_{\mathrm{DG}}\|_{0,\kappa}^2 + C \int_{\Gamma_{\mathrm{int,B}}} \sigma\, [\![u_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s \\ &= \sum_{\kappa \in \mathcal{T}_h} \|\nabla e_{\mathrm{DG}}\|_{0,\kappa}^2 + C \int_{\Gamma_{\mathrm{int,B}}} \sigma\, [\![e_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s \leq C\|e_{\mathrm{DG}}\|_{\mathrm{DG}}^2,\end{aligned}$$

which, referring to (3.12), yields the second bound. ∎

Next, we state the following approximation property.

Lemma 8 *For any $\varphi \in \mathrm{H}^1_0(\Omega)$, there exists a function $\varphi_{hp} \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ such that*

$$h_\kappa^{-2} p_\kappa^2 \|\varphi - \varphi_{hp}\|^2_{0,\kappa} + \|\nabla(\varphi - \varphi_{hp})\|^2_{0,\kappa} + h_\kappa^{-1} p_\kappa \|\varphi - \varphi_{hp}\|^2_{0,\partial\kappa} \leq C_I \|\nabla \varphi\|^2_{0,\kappa}, \quad (3.13)$$

for any $\kappa \in \mathcal{T}_h$, with an interpolation constant $C_I > 0$, which is independent of $\mathbf{h}$ *and* $\mathbf{p}$*, and only depends on the shape-regularity of the mesh and the constants ρ_1 and ρ_2 in* (2.1) *and* (2.2)*, respectively.*

Proof We first consider the proof of the upper bounds on the $\mathrm{L}^2(\kappa)$-norms of $\varphi - \varphi_{hp}$ and $\nabla(\varphi - \varphi_{hp})$. In this case, on quadrilateral elements, the above approximation property follows from the tensorisation of the corresponding one-dimensional approximation results for an H^1-projector; see [25], for details. For triangular elements, we employ a reflection technique. More precisely, writing $\widehat{\kappa}$ to denote the canonical triangle with vertices $(-1,-1)$, $(1,-1)$, and $(-1,1)$, we define $\widehat{\kappa}'$ to be triangle with vertices $(1,-1)$, $(1,1)$, and $(-1,1)$ obtained by reflecting $\widehat{\kappa}$ about its longest edge. Analogously, given $\widehat{v} \in \mathrm{H}^1(\widehat{\kappa})$, we write $\widehat{v}' \in \mathrm{H}^1(\widehat{\kappa}')$ to denote the reflection of v in the line $\xi_2 = -\xi_1$, where (ξ_1, ξ_2) denotes the local coordinate system for the reference element $\widehat{\kappa}$. With this notation we define the function $w \in \mathrm{H}^1(\widehat{S})$ by $w|_{\widehat{\kappa}} = \widehat{v}$ and $w|_{\widehat{\kappa}'} = \widehat{v}'$, where $\widehat{S}$ is the unit square $(-1,1)^2$. Due to symmetry, we deduce that there exists a positive constant C, such that

$$\sqrt{2}\, \|\widehat{v}\|_{0,\widehat{\kappa}} \leq \|\widehat{w}\|_{0,\widehat{S}} \leq C \|\widehat{v}\|_{0,\widehat{\kappa}} \qquad \text{and} \qquad \sqrt{2}\, \|\nabla \widehat{v}\|_{0,\widehat{\kappa}} \leq \|\nabla \widehat{w}\|_{0,\widehat{S}} \leq C \|\nabla \widehat{v}\|_{0,\widehat{\kappa}}.$$

Thereby, the approximation properties on the reference element $\widehat{\kappa}$ now follow from the corresponding results on the unit square $\widehat{S}$; the proof is then completed by employing a standard scaling argument.

The upper bound on the approximation error measured in terms of the $\mathrm{L}^2(\partial\kappa)$-norm now follows from the above results, together with the trace inequality

$$\|v\|^2_{0,\partial\kappa} \leq C \left(\|\nabla v\|_{0,\kappa} \|v\|_{0,\kappa} + h_\kappa^{-1} \|v\|^2_{0,\kappa} \right),$$

where $v \in \mathrm{H}^1(\kappa)$ and C is a positive constant which depends only on the shape-regularity of the element κ. ■

3.1.3 Proof of Theorem 4

We commence the proof of our main theorem by applying (1.5). This yields

$$\begin{aligned}
C_2 \|e_{\mathrm{DG}}\|^2_{\mathrm{DG}} &= C_2 \left(\sum_{\kappa \in \mathcal{T}_h} \int_\kappa |\nabla u - \nabla u_{\mathrm{DG}}|^2 \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![e_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s \right) \\
&= C_2 \left(\sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \int_{\widetilde{\kappa}} |\nabla u - \nabla u_{\mathrm{DG}}|^2 \,\mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![e_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s \right) \\
&\leq \sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \int_{\widetilde{\kappa}} (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}) \cdot \nabla e_{\mathrm{DG}} \,\mathrm{d}\mathbf{x} + C_2 \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![e_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s \\
&\equiv T_1 + T_2 + T_3,
\end{aligned} \quad (3.14)$$

where

$$T_1 = \sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \int_{\widetilde{\kappa}} (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}) \cdot \nabla e_{\mathrm{DG}}^{\|} \, \mathrm{d}\mathbf{x},$$

$$T_2 = -\sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \int_{\widetilde{\kappa}} (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}) \cdot \nabla u_{\mathrm{DG}}^{\perp} \, \mathrm{d}\mathbf{x},$$

$$T_3 = C_2 \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![e_{\mathrm{DG}}]\!]^2 \, \mathrm{d}s,$$

and $e_{\mathrm{DG}}^{\|} \in \mathrm{H}_0^1(\Omega)$ and $u_{\mathrm{DG}}^{\perp} \in [S^{\widetilde{\mathbf{p}}}(\Omega, \widetilde{\mathcal{T}}_h, \widetilde{\mathbf{F}})]^{\perp}$ are defined as in (3.6) and (3.8), respectively.

We will now analyse the three terms T_1, T_2 and T_3 separately.

Term T_1. We first note that

$$T_1 = \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} (\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}) \cdot \nabla e_{\mathrm{DG}}^{\|} \, \mathrm{d}\mathbf{x}.$$

Then, using integration by parts, we obtain

$$\begin{aligned} T_1 &= -\sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \nabla \cdot (\mu(|\nabla u|)\nabla u) e_{\mathrm{DG}}^{\|} \, \mathrm{d}\mathbf{x} - \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} \cdot \nabla e_{\mathrm{DG}}^{\|} \, \mathrm{d}\mathbf{x} \\ &= \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} f e_{\mathrm{DG}}^{\|} \, \mathrm{d}\mathbf{x} - \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} \cdot \nabla e_{\mathrm{DG}}^{\|} \, \mathrm{d}\mathbf{x}. \end{aligned}$$

We now let $\varphi_{hp} \in S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ be the element-wise projection of $e_{\mathrm{DG}}^{\|}$ satisfying Lemma 8. Then, by the definition of the hp-DGFEM (2.6), it follows that

$$\begin{aligned} T_1 &= \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} f(e_{\mathrm{DG}}^{\|} - \varphi_{hp}) \, \mathrm{d}\mathbf{x} - \sum_{\kappa \in \mathcal{T}_h} \int_{\kappa} \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} \cdot \nabla(e_{\mathrm{DG}}^{\|} - \varphi_{hp}) \, \mathrm{d}\mathbf{x} \\ &- \int_{\Gamma_{\mathrm{int,B}}} \langle\!\langle \mu(|\nabla_h u_{\mathrm{DG}}|)\nabla_h u_{\mathrm{DG}} \cdot \nu \rangle\!\rangle [\![\varphi_{hp}]\!] \, \mathrm{d}s + \theta \int_{\Gamma_{\mathrm{int,B}}} \langle\!\langle \mu(h^{-1}|[\![u_{\mathrm{DG}}]\!]|)\nabla_h \varphi_{hp} \cdot \nu \rangle\!\rangle [\![u_{\mathrm{DG}}]\!] \, \mathrm{d}s \\ &+ \int_{\Gamma_{\mathrm{int,B}}} \sigma \, [\![u_{\mathrm{DG}}]\!] \, [\![\varphi_{hp}]\!] \, \mathrm{d}s. \end{aligned}$$

Hence, integrating the second term on the right-hand side of the above equality by parts, leads to

$$
\begin{aligned}
T_1 &= \sum_{\kappa\in\mathcal{T}_h}\int_\kappa (f+\nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\,(e^{\|}_{\mathrm{DG}}-\varphi_{hp})\,\mathrm{d}\mathbf{x}\\
&-\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\mathbf{n}_\kappa)(e^{\|}_{\mathrm{DG}}-\varphi_{hp})\,\mathrm{d}s\\
&-\int_{\Gamma_{\mathrm{int,B}}}\langle\!\langle\mu(|\nabla_h u_{\mathrm{DG}}|)\nabla_h u_{\mathrm{DG}}\cdot\nu\rangle\!\rangle[\![\varphi_{hp}]\!]\,\mathrm{d}s+\theta\int_{\Gamma_{\mathrm{int,B}}}\langle\!\langle\mu(h^{-1}|[\![u_{\mathrm{DG}}]\!]|)\nabla_h\varphi_{hp}\cdot\nu\rangle\!\rangle[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s\\
&+\int_{\Gamma_{\mathrm{int,B}}}\sigma\,[\![u_{\mathrm{DG}}]\!]\,[\![\varphi_{hp}]\!]\,\mathrm{d}s.
\end{aligned}
$$

Using the fact that $[\![e^{\|}_{\mathrm{DG}}]\!]=0$ on $\Gamma_{\mathrm{int,B}}$, since $e^{\|}_{\mathrm{DG}}\in \mathrm{H}^1_0(\Omega)$, and a few elementary calculations, we have that

$$
\begin{aligned}
&-\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\mathbf{n}_\kappa)(e^{\|}_{\mathrm{DG}}-\varphi_{hp})\,\mathrm{d}s\\
&=-\int_{\Gamma_{\mathrm{int}}}[\![\mu(|\nabla_h u_{\mathrm{DG}}|)\nabla_h u_{\mathrm{DG}}\cdot\nu]\!]\langle\!\langle e^{\|}_{\mathrm{DG}}-\varphi_{hp}\rangle\!\rangle\,\mathrm{d}s+\int_{\Gamma_{\mathrm{int,B}}}\langle\!\langle\mu(|\nabla_h u_{\mathrm{DG}}|)\nabla_h u_{\mathrm{DG}}\cdot\nu\rangle\!\rangle[\![\varphi_{hp}]\!]\,\mathrm{d}s.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
T_1 = &\sum_{\kappa\in\mathcal{T}_h}\int_\kappa (f+\nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\,(e^{\|}_{\mathrm{DG}}-\varphi_{hp})\,\mathrm{d}\mathbf{x}\\
&-\int_{\Gamma_{\mathrm{int}}}[\![\mu(|\nabla_h u_{\mathrm{DG}}|)\nabla_h u_{\mathrm{DG}}\cdot\nu]\!]\langle\!\langle e^{\|}_{\mathrm{DG}}-\varphi_{hp}\rangle\!\rangle\,\mathrm{d}s\\
&+\theta\int_{\Gamma_{\mathrm{int,B}}}\langle\!\langle\mu(h^{-1}|[\![u_{\mathrm{DG}}]\!]|)\nabla_h\varphi_{hp}\cdot\nu\rangle\!\rangle[\![u_{\mathrm{DG}}]\!]\,\mathrm{d}s+\int_{\Gamma_{\mathrm{int,B}}}\sigma\,[\![u_{\mathrm{DG}}]\!]\,[\![\varphi_{hp}]\!]\,\mathrm{d}s,
\end{aligned}
$$

and thus,

$$
\begin{aligned}
T_1 \le &\sum_{\kappa\in\mathcal{T}_h}\int_\kappa |f+\nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}})||e^{\|}_{\mathrm{DG}}-\varphi_{hp}|\,\mathrm{d}\mathbf{x}\\
&+\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa\setminus\Gamma}|[\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\nu]\!]||\langle\!\langle e^{\|}_{\mathrm{DG}}-\varphi_{hp}\rangle\!\rangle|\,\mathrm{d}s\\
&+|\theta|\int_{\Gamma_{\mathrm{int,B}}}h\,\mu(h^{-1}|[\![u_{\mathrm{DG}}]\!]|)(h^{-1}|[\![u_{\mathrm{DG}}]\!]|)|\langle\!\langle\nabla_h\varphi_{hp}\cdot\nu\rangle\!\rangle|\,\mathrm{d}s+\int_{\Gamma_{\mathrm{int,B}}}\sigma|[\![u_{\mathrm{DG}}]\!]||[\![\varphi_{hp}]\!]|\,\mathrm{d}s\\
\le &\sum_{\kappa\in\mathcal{T}_h}\|f+\nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}})\|_{0,\kappa}\|e^{\|}_{\mathrm{DG}}-\varphi_{hp}\|_{0,\kappa}\\
&+C\sum_{\kappa\in\mathcal{T}_h}\|[\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\nu]\!]\|_{0,\partial\kappa\setminus\Gamma}\|e^{\|}_{\mathrm{DG}}-\varphi_{hp}\|_{0,\partial\kappa}\\
&+M_\mu|\theta|\int_{\Gamma_{\mathrm{int,B}}}|[\![u_{\mathrm{DG}}]\!]|\langle\!\langle|\nabla_h\varphi_{hp}|\rangle\!\rangle\,\mathrm{d}s+\int_{\Gamma_{\mathrm{int,B}}}\sigma|[\![u_{\mathrm{DG}}]\!]||[\![\varphi_{hp}]\!]|\,\mathrm{d}s,
\end{aligned}
$$

where we have applied (1.3) (with $s=0$ and $t=h^{-1}|[\![u_{\mathrm{DG}}]\!]|$) to bound the second-last of the above terms. Moreover, proceeding as in the proof of [20, Lemma 2.2] (cf., also [43, Lemma 3.5]) and recalling that $\gamma \geq 1$, we obtain

$$\begin{aligned}\int_{\Gamma_{\mathrm{int,B}}} |[\![u_{\mathrm{DG}}]\!]|\langle\!\langle|\nabla_h\varphi_{hp}|\rangle\!\rangle\,\mathrm{d}s &\leq \left(\int_{\Gamma_{\mathrm{int,B}}} \sigma\,[\![u_{\mathrm{DG}}]\!]^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{e\in\mathcal{E}}\int_e \frac{h}{\langle\!\langle p^2\rangle\!\rangle}\langle\!\langle|\nabla\varphi_{hp}|\rangle\!\rangle^2\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\leq C\left(\int_{\Gamma_{\mathrm{int,B}}} \sigma\,[\![u_{\mathrm{DG}}]\!]^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}\|\nabla\varphi_{hp}\|_{0,\kappa}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Furthermore, by (3.13), we have

$$\sum_{\kappa\in\mathcal{T}_h}\|\nabla\varphi_{hp}\|_{0,\kappa}^2 \leq C\sum_{\kappa\in\mathcal{T}_h}\|\nabla(e_{\mathrm{DG}}^{\|}-\varphi_{hp})\|_{0,\kappa}^2 + C\sum_{\kappa\in\mathcal{T}_h}\|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2 \leq C\sum_{\kappa\in\mathcal{T}_h}\|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2,$$

and hence,

$$\int_{\Gamma_{\mathrm{int,B}}} |[\![u_{\mathrm{DG}}]\!]|\langle\!\langle|\nabla_h\varphi_{hp}|\rangle\!\rangle\,\mathrm{d}s \leq C\left(\int_{\Gamma_{\mathrm{int,B}}} \sigma\,[\![u_{\mathrm{DG}}]\!]^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}\|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2\right)^{\frac{1}{2}}.$$

Moreover, using again the fact that $[\![e_{\mathrm{DG}}^{\|}]\!]=0$ on $\Gamma_{\mathrm{int,B}}$, and recalling (2.1)–(2.2), implies

$$\begin{aligned}\int_{\Gamma_{\mathrm{int,B}}} \sigma\,|[\![u_{\mathrm{DG}}]\!]||[\![\varphi_{hp}]\!]|\,\mathrm{d}s &= \int_{\Gamma_{\mathrm{int,B}}} \sigma\,|[\![u_{\mathrm{DG}}]\!]||[\![e_{\mathrm{DG}}^{\|}-\varphi_{hp}]\!]|\,\mathrm{d}s\\ &\leq C\left(\int_{\Gamma_{\mathrm{int,B}}} \sigma\langle\!\langle p\rangle\!\rangle\,[\![u_{\mathrm{DG}}]\!]^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa}\sigma\langle\!\langle p\rangle\!\rangle^{-1}|e_{\mathrm{DG}}^{\|}-\varphi_{hp}|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\\ &\leq C\gamma^{\frac{1}{2}}\left(\int_{\Gamma_{\mathrm{int,B}}} \sigma\langle\!\langle p\rangle\!\rangle\,[\![u_{\mathrm{DG}}]\!]^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^{-1}p_\kappa\|e_{\mathrm{DG}}^{\|}-\varphi_{hp}\|_{0,\partial\kappa}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, collecting the terms leads to

$$\begin{aligned}T_1 \leq &\sum_{\kappa\in\mathcal{T}_h}h_\kappa p_\kappa^{-1}\|f+\nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}})\|_{0,\kappa}h_\kappa^{-1}p_\kappa\|e_{\mathrm{DG}}^{\|}-\varphi_{hp}\|_{0,\kappa}\\ &+ C\sum_{\kappa\in\mathcal{T}_h}h_\kappa^{\frac{1}{2}}p_\kappa^{-\frac{1}{2}}\|[\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\nu]\!]\|_{0,\partial\kappa\setminus\Gamma}h_\kappa^{-\frac{1}{2}}p_\kappa^{\frac{1}{2}}\|e_{\mathrm{DG}}^{\|}-\varphi_{hp}\|_{0,\partial\kappa}\\ &+ C|\theta|\left(\int_{\Gamma_{\mathrm{int,B}}} \sigma\,[\![u_{\mathrm{DG}}]\!]^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}\|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2\right)^{\frac{1}{2}}\\ &+ C\gamma^{\frac{1}{2}}\left(\int_{\Gamma_{\mathrm{int,B}}} \sigma\langle\!\langle p\rangle\!\rangle\,[\![u_{\mathrm{DG}}]\!]^2\,\mathrm{d}s\right)^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h}h_\kappa^{-1}p_\kappa\|e_{\mathrm{DG}}^{\|}-\varphi_{hp}\|_{0,\partial\kappa}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Furthermore, applying again the approximation property (3.13), using that $\gamma \geq 1 \geq |\theta| \geq 0$, and incorporating (2.2), results in

$$
\begin{aligned}
T_1 \leq & \left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa^2 p_\kappa^{-2} \|f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}})\|_{0,\kappa}^2\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa^{-2} p_\kappa^2 \|e_{\mathrm{DG}}^{\|} - \varphi_{hp}\|_{0,\kappa}^2\right)^{\frac{1}{2}} \\
& + C\left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\nu]\!]\|_{0,\partial\kappa\setminus\Gamma}^2\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa^{-1} p_\kappa \|e_{\mathrm{DG}}^{\|} - \varphi_{hp}\|_{0,\partial\kappa}^2\right)^{\frac{1}{2}} \\
& + C|\theta|\left(\sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \sigma\, [\![u_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} \|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2\right)^{\frac{1}{2}} \\
& + C\gamma^{\frac{1}{2}}\left(\sum_{\kappa\in\mathcal{T}_h} \int_{\partial\kappa} \sigma \langle\!\langle p\rangle\!\rangle\, [\![u_{\mathrm{DG}}]\!]^2 \,\mathrm{d}s\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa^{-1} p_\kappa \|e_{\mathrm{DG}}^{\|} - \varphi_{hp}\|_{0,\partial\kappa}^2\right)^{\frac{1}{2}} \\
\leq & \, C\left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa^2 p_\kappa^{-2} \|f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}})\|_{0,\kappa}^2\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} \|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2\right)^{\frac{1}{2}} \\
& + C\left(\sum_{\kappa\in\mathcal{T}_h} h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\nu]\!]\|_{0,\partial\kappa\setminus\Gamma}^2\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} \|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2\right)^{\frac{1}{2}} \\
& + C\left(\gamma^2 \sum_{\kappa\in\mathcal{T}_h} h_\kappa^{-1} p_\kappa^3 \|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial\kappa}^2\right)^{\frac{1}{2}} \left(\sum_{\kappa\in\mathcal{T}_h} \|\nabla e_{\mathrm{DG}}^{\|}\|_{0,\kappa}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore,

$$
T_1 \leq C\left(\sum_{\kappa\in\mathcal{T}_h} \widetilde{\eta}_\kappa^2\right)^{\frac{1}{2}} \|e_{\mathrm{DG}}^{\|}\|_{\mathrm{DG}},
$$

which, by Corollary 7A, yields

$$
T_1 \leq C\left(\sum_{\kappa\in\mathcal{T}_h} \widetilde{\eta}_\kappa^2\right)^{\frac{1}{2}} \|e_{\mathrm{DG}}\|_{\mathrm{DG}}.
$$

Here, for $\kappa \in \mathcal{T}_h$, the term $\widetilde{\eta}_\kappa$ is defined by

$$
\begin{aligned}
\widetilde{\eta}_\kappa^2 = & \, h_\kappa^2 p_\kappa^{-2} \|f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}})\|_{0,\kappa}^2 \\
& + h_\kappa p_\kappa^{-1} \|[\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\cdot\nu]\!]\|_{0,\partial\kappa\setminus\Gamma}^2 + \gamma^2 h_\kappa^{-1} p_\kappa^3 \|[\![u_{\mathrm{DG}}]\!]\|_{0,\partial\kappa}^2.
\end{aligned}
$$

Noticing that

$$
\widetilde{\eta}_\kappa^2 \leq C\left(\eta_\kappa^2 + \mathcal{O}_\kappa^{(1)} + \sum_{\substack{e\in\mathcal{E}_{\mathrm{int}} \\ e\subset\partial\kappa}} \mathcal{O}_e^{(2)}\right),
$$

we obtain

$$T_1 \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \left(\eta_\kappa^2 + \mathcal{O}_\kappa^{(1)} + \sum_{\substack{e \in \mathcal{E}_{\text{int}} \\ e \subset \partial\kappa}} \mathcal{O}_e^{(2)} \right) \right)^{\frac{1}{2}} \|e_{\text{DG}}\|_{\text{DG}}. \tag{3.15}$$

Term T_2. In order to bound T_2 we recall (1.4). This yields

$$\begin{aligned} T_2 &\leq \sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \int_{\widetilde{\kappa}} |\mu(|\nabla u|)\nabla u - \mu(|\nabla u_{\text{DG}}|)\nabla u_{\text{DG}}||\nabla u_{\text{DG}}^{\perp}| \, \mathrm{d}\mathbf{x} \\ &\leq C_1 \sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \int_{\widetilde{\kappa}} |\nabla e_{\text{DG}}||\nabla u_{\text{DG}}^{\perp}| \, \mathrm{d}\mathbf{x} \leq C_1 \sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \|\nabla e_{\text{DG}}\|_{0,\widetilde{\kappa}} \|\nabla u_{\text{DG}}^{\perp}\|_{0,\widetilde{\kappa}} \\ &\leq C_1 \left(\sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \|\nabla e_{\text{DG}}\|_{0,\widetilde{\kappa}}^2 \right)^{\frac{1}{2}} \left(\sum_{\widetilde{\kappa} \in \widetilde{\mathcal{T}}_h} \|\nabla u_{\text{DG}}^{\perp}\|_{0,\widetilde{\kappa}}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, we have

$$T_2 \leq C_1 \|e_{\text{DG}}\|_{\text{DG}} \|u_{\text{DG}}^{\perp}\|_{\widetilde{\text{DG}}},$$

which, upon applying Corollary 7A, gives

$$T_2 \leq C\|e_{\text{DG}}\|_{\text{DG}} \left(\int_{\Gamma_{\text{int,B}}} \sigma \, [\![u_{\text{DG}}]\!]^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \leq C\|e_{\text{DG}}\|_{\text{DG}} \left(\gamma \sum_{\kappa \in \mathcal{T}_h} h_\kappa^{-1} p_\kappa^2 \|[\![u_{\text{DG}}]\!]\|_{0,\partial\kappa}^2 \right)^{\frac{1}{2}},$$

and thus, since $\gamma \geq 1$,

$$T_2 \leq C\|e_{\text{DG}}\|_{\text{DG}} \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 \right)^{\frac{1}{2}}. \tag{3.16}$$

Term T_3. A bound for T_3 is found by recalling (3.11). This gives

$$\begin{aligned} T_3 &\leq C_2 \int_{\Gamma_{\text{int,B}}} \sigma \, |[\![e_{\text{DG}}]\!]||[\![u_{\text{DG}}]\!]| \, \mathrm{d}s \leq C_2 \left(\int_{\Gamma_{\text{int,B}}} \sigma \, [\![e_{\text{DG}}]\!]^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \left(\int_{\Gamma_{\text{int,B}}} \sigma \, [\![u_{\text{DG}}]\!]^2 \, \mathrm{d}s \right)^{\frac{1}{2}} \\ &\leq C\|e_{\text{DG}}\|_{\text{DG}} \left(\gamma \sum_{\kappa \in \mathcal{T}_h} h_\kappa^{-1} p_\kappa^2 \|[\![u_{\text{DG}}]\!]\|_{0,\partial\kappa}^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thereby, we obtain

$$T_3 \leq C\|e_{\text{DG}}\|_{\text{DG}} \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 \right)^{\frac{1}{2}}. \tag{3.17}$$

Finally, combining the bounds (3.14) and (3.15)–(3.17) leads to

$$\|e_{\mathrm{DG}}\|_{\mathrm{DG}}^2 \leq C \left(\sum_{\kappa \in \mathcal{T}_h} \eta_\kappa^2 + \mathcal{O}(f, u_{\mathrm{DG}}) \right)^{\frac{1}{2}} \|e_{\mathrm{DG}}\|_{\mathrm{DG}}.$$

Dividing both sides of the above inequality by $\|e_{\mathrm{DG}}\|_{\mathrm{DG}}$ completes the proof of Theorem 4.

3.2 Local lower bounds

In this section we derive local lower bounds on the error measured in terms of the DG energy norm $\|\cdot\|_{\mathrm{DG}}$. As in the case of conforming *hp*-version finite element methods, estimators which are both optimally reliable and efficient in the polynomial degree are not currently available in the literature, cf. [36], for example. The key technical reason for this is that the proofs of the lower bounds exploit the use of inverse estimates which are suboptimal in the polynomial degree. To minimise the deterioration of the efficiency bounds with respect to the polynomial degree, weighted versions of the local a-posteriori error indicators η_κ may be employed. This idea was first used in the context of conforming finite element methods in [36]; subsequent extensions to DGFEMs have been undertaken in the article [23], for example. For simplicity of exposition, we only present lower bounds for our (unweighted) a posteriori error indicators η_κ; extensions to weighted versions of η_κ follow analogously, cf. [23]. We begin by quoting the following theorem under the assumption that the computational mesh $\mathcal{T}_h$ is conforming (regular). The extension of these bounds to nonconforming (irregular) meshes which are regularly reducible follows analogously; cf. Remark 10 below.

Theorem 9 *Let $\kappa, \kappa' \in \mathcal{T}_h$ be any two neighbouring elements, $e = \partial\kappa \cap \partial\kappa' \in \mathcal{E}_{\mathrm{int}}$, and $\omega_e = (\overline{\kappa} \cup \overline{\kappa}')^\circ$. Then, for all $\delta > 0$, the following local hp-version a-posteriori lower bounds on the error e_{DG} from (3.7) hold:*

a)

$$\|\Pi_{\mathcal{T}_h}(f + \nabla \cdot (\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\kappa} \leq C h_\kappa^{-1} p_\kappa^2 \left(\|\nabla e_{\mathrm{DG}}\|_{0,\kappa} + p_\kappa^{\delta - \frac{1}{2}} \sqrt{\mathcal{O}_\kappa^{(1)}} \right);$$

b)

$$\begin{aligned}
&\|\Pi_{\mathcal{E}}|_e((\llbracket \mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} \rrbracket \cdot \nu)|_e)\|_{0,e} \\
&\qquad \leq C h_\kappa^{-\frac{1}{2}} p_\kappa^{\delta + \frac{3}{2}} \left(\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e} + p_\kappa^{\delta - \frac{1}{2}} \sum_{\tau \in \{\kappa, \kappa'\}} \sqrt{\mathcal{O}_\tau^{(1)}} + p_\kappa^{-\frac{1}{2}} \sqrt{\mathcal{O}_e^{(2)}} \right);
\end{aligned}$$

c)

$$\|\llbracket u_{\mathrm{DG}} \rrbracket\|_{0,e} \leq C \gamma^{-\frac{1}{2}} h_\kappa^{\frac{1}{2}} p_\kappa^{-1} \|\sigma^{\frac{1}{2}} \llbracket e_{\mathrm{DG}} \rrbracket\|_{0,e}.$$

Here, the generic constant $C>0$ depends on δ, but is independent of $\mathbf{h}$ *and* $\mathbf{p}$.

Proof We proceed similarly as in [36]; see also [23]. To this end, we first introduce suitable cut-off functions as follows: on the reference element $\widehat{\kappa}$, we define a weight-function $\Phi_{\widehat{\kappa}}(\mathbf{x}) = \min_{\mathbf{y}\in\partial\widehat{\kappa}} |\mathbf{x}-\mathbf{y}|$. Then, for $\kappa \in \mathcal{T}_h$, we let $\Phi_\kappa = c_\kappa \Phi_{\widehat{\kappa}} \circ F_\kappa^{-1}$, where the factor c_κ is chosen so that $\int_\kappa (\Phi_\kappa - 1)\,\mathrm{d}\mathbf{x} = 0$. Furthermore, on the reference interval $\widehat{\iota} = (-1,1)$, we define the weight-function $\Phi_{\widehat{\iota}}(x) = 1 - x^2$. Then, for an interior edge $e \in \mathcal{E}_{\mathrm{int}}$, we let $\Phi_e = c_e \Phi_{\widehat{\iota}} \circ F_e^{-1}$, where F_e is the affine mapping from $\widehat{\iota}$ to e, and c_e is chosen so that $\int_e (\Phi_e - 1)\,\mathrm{d}s = 0$.

Proof of a): Let $\kappa \in \mathcal{T}_h$ and define $v_\kappa = \Phi_\kappa^\alpha \Pi_{\mathcal{T}_h}(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))$, where $\alpha \in (\frac{1}{2},1]$. Then, using (1.1), and integrating by parts, yields

$$
\begin{aligned}
\|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa}^2 &= \int_\kappa v_\kappa \Pi_{\mathcal{T}_h}(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\,\mathrm{d}\mathbf{x} \\
&= \int_\kappa v_\kappa\, \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} - \mu(|\nabla u|)\nabla u)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\kappa v_\kappa (\Pi_{\mathcal{T}_h} - \mathbb{I})(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\,\mathrm{d}\mathbf{x} \\
&= -\int_\kappa \nabla v_\kappa \cdot (\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} - \mu(|\nabla u|)\nabla u)\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\kappa v_\kappa (\Pi_{\mathcal{T}_h} - \mathbb{I})(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\,\mathrm{d}\mathbf{x} \\
&\le \int_\kappa |\nabla v_\kappa||\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}} - \mu(|\nabla u|)\nabla u|\,\mathrm{d}\mathbf{x} \\
&\quad + \int_\kappa |v_\kappa||(\Pi_{\mathcal{T}_h} - \mathbb{I})(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))|\,\mathrm{d}\mathbf{x}.
\end{aligned}
$$

Recalling (1.4), this can be transformed into

$$
\begin{aligned}
\|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa}^2 &\le C\int_\kappa |\nabla v_\kappa||\nabla e_{\mathrm{DG}}|\,\mathrm{d}\mathbf{x} + \int_\kappa |v_\kappa||(\Pi_{\mathcal{T}_h} - \mathbb{I})(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))|\,\mathrm{d}\mathbf{x} \\
&\le C\|\nabla v_\kappa\|_{0,\kappa}\|\nabla e_{\mathrm{DG}}\|_{0,\kappa} + \|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa}\|\Phi_\kappa^{\frac{\alpha}{2}}(\mathbb{I} - \Pi_{\mathcal{T}_h})(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\kappa} \\
&\le C\|\nabla v_\kappa\|_{0,\kappa}\|\nabla e_{\mathrm{DG}}\|_{0,\kappa} + h_\kappa^{-1} p_\kappa \|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa}\sqrt{\mathcal{O}_\kappa^{(1)}}.
\end{aligned}
$$

From the proof of [36, Lemma 3.4], we have

$$
\|\nabla v_\kappa\|_{0,\kappa} \le C h_\kappa^{-1} p_\kappa^{2-\alpha} \|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa};
$$

thereby,

$$
\|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa}^2 \le C h_\kappa^{-1} p_\kappa \|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa}\left(p_\kappa^{1-\alpha}\|\nabla e_{\mathrm{DG}}\|_{0,\kappa} + \sqrt{\mathcal{O}_\kappa^{(1)}}\right).
$$

Dividing both sides of the above inequality by $\|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa}$ and observing that (by applying the inverse inequality from [36, Theorem 2.5])

$$
\begin{aligned}
\|\Pi_{\mathcal{T}_h}(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\kappa} &\le C p_\kappa^\alpha \|\Phi_\kappa^{\frac{\alpha}{2}} \Pi_{\mathcal{T}_h}(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\kappa} \\
&= C p_\kappa^\alpha \|\Phi_\kappa^{-\frac{\alpha}{2}} v_\kappa\|_{0,\kappa},
\end{aligned}
$$

leads to

$$\|\Pi_{\mathcal{T}_h}(f+\nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\kappa} \le C h_\kappa^{-1} p_\kappa^{1+\alpha}\left(p_\kappa^{1-\alpha}\|\nabla e_{\mathrm{DG}}\|_{0,\kappa}+\sqrt{\mathcal{O}_\kappa^{(1)}}\right). \quad (3.18)$$

Choosing $\delta=\alpha-\frac{1}{2}$, completes the proof of *a)*.

Proof of b): Let $q_e=\Phi_e^\alpha \Pi_{\mathcal{E}}|_e\,((\llbracket\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\rrbracket\cdot\nu)|_e)$, where again $\alpha\in(\frac{1}{2},1]$. Then, referring to [36, Lemma 2.6 with $\varepsilon=p_\kappa^{-2}$], there exists $\chi_e\in \mathrm{H}_0^1(\omega_e)$ such that $\chi_e|_e=q_e$ and

$$\begin{aligned}\|\chi_e\|_{0,\omega_e} &\le C h_\kappa^{\frac{1}{2}} p_\kappa^{-1}\|\Phi_e^{-\frac{\alpha}{2}} q_e\|_{0,e},\\ \|\nabla\chi_e\|_{0,\omega_e} &\le C h_\kappa^{-\frac{1}{2}} p_\kappa\|\Phi_e^{-\frac{\alpha}{2}} q_e\|_{0,e}.\end{aligned} \quad (3.19)$$

Noting that $-\nabla\cdot(\mu(|\nabla u|)\nabla u)=f\in \mathrm{L}^2(\Omega)$, we conclude that $\llbracket\mu(|\nabla u|)\nabla u\rrbracket\cdot\nu=0$ on e. Hence, integrating by parts and assuming (without loss of generality) that the normal vector ν points from κ to κ', leads to

$$\begin{aligned}
&\|\Phi_e^{-\frac{\alpha}{2}} q_e\|_{0,e}^2\\
&=\int_e \Pi_{\mathcal{E}}(\llbracket\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\rrbracket\cdot\nu)\chi_e\,\mathrm{d}s\\
&=\int_e(\llbracket\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}-\mu(|\nabla u|)\nabla u\rrbracket\cdot\nu)\chi_e\,\mathrm{d}s+\int_e(\Pi_{\mathcal{E}}-\mathbb{I})(\llbracket\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\rrbracket\cdot\nu)\chi_e\,\mathrm{d}s\\
&=\int_{\partial\kappa}((\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}-\mu(|\nabla u|)\nabla u)\cdot\mathbf{n}_\kappa)\,\chi_e\,\mathrm{d}s\\
&\quad+\int_{\partial\kappa'}((\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}-\mu(|\nabla u|)\nabla u)\cdot\mathbf{n}_{\kappa'})\,\chi_e\,\mathrm{d}s\\
&\quad+\int_e(\Pi_{\mathcal{E}}-\mathbb{I})(\llbracket\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\rrbracket\cdot\nu)\chi_e\,\mathrm{d}s\\
&=\int_{\omega_e}(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}-\mu(|\nabla u|)\nabla u)\cdot\nabla\chi_e\,\mathrm{d}\mathbf{x}+\int_{\omega_e}(f+\nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\chi_e\,\mathrm{d}\mathbf{x}\\
&\quad+\int_e(\Pi_{\mathcal{E}}-\mathbb{I})(\llbracket\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\rrbracket\cdot\nu)\chi_e\,\mathrm{d}s\\
&\equiv R_1+R_2+R_3.
\end{aligned} \quad (3.20)$$

Employing (1.4) and (3.19), R_1 can be bounded as follows:

$$R_1\le C\int_{\omega_e}|\nabla e_{\mathrm{DG}}||\nabla\chi_e|\,\mathrm{d}\mathbf{x}\le C\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e}\|\nabla\chi_e\|_{0,\omega_e}\le C h_\kappa^{-\frac{1}{2}} p_\kappa\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e}\|\Phi_e^{-\frac{\alpha}{2}} q_e\|_{0,e}. \quad (3.21)$$

In order to obtain a bound for R_2, we use (3.18) and the definition of $\mathcal{O}_\kappa^{(1)}$ from (3.1); thereby,

$$\begin{aligned} R_2 &= \int_{\omega_e} \Pi_{\mathcal{T}_h}(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\chi_e \,\mathrm{d}\mathbf{x} \\ &\quad - \int_{\omega_e} (\Pi_{\mathcal{T}_h} - \mathbb{I})(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\chi_e \,\mathrm{d}\mathbf{x} \\ &\le \|\Pi_{\mathcal{T}_h}(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\omega_e}\|\chi_e\|_{0,\omega_e} \\ &\quad + \|(\Pi_{\mathcal{T}_h} - \mathbb{I})(f + \nabla\cdot(\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}))\|_{0,\omega_e}\|\chi_e\|_{0,\omega_e} \\ &\le Ch_\kappa^{-1}p_\kappa^{1+\alpha}\left(p_\kappa^{1-\alpha}\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e} + \sum_{\tau\in\{\kappa,\kappa'\}}\sqrt{\mathcal{O}_\tau^{(1)}}\right)\|\chi_e\|_{0,\omega_e}. \end{aligned} \tag{3.22}$$

Recalling (3.19), this gives

$$R_2 \le Ch_\kappa^{-\frac{1}{2}}p_\kappa^{\alpha}\left(p_\kappa^{1-\alpha}\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e} + \sum_{\tau\in\{\kappa,\kappa'\}}\sqrt{\mathcal{O}_\tau^{(1)}}\right)\|\Phi_e^{-\frac{\alpha}{2}}q_e\|_{0,e}.$$

A bound on R_3 is based on the definition of $\mathcal{O}_e^{(2)}$ from (3.2) and on the fact that $\chi_e = q_e$ on e:

$$R_3 \le \|\Phi_e^{\frac{\alpha}{2}}(\Pi_{\mathcal{E}} - \mathbb{I})([\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}]\!]\cdot\nu)\|_{0,e}\|\Phi_e^{-\frac{\alpha}{2}}\chi_e\|_{0,e} \le Ch_\kappa^{-\frac{1}{2}}p_\kappa^{\frac{1}{2}}\sqrt{\mathcal{O}_e^{(2)}}\|\Phi_e^{-\frac{\alpha}{2}}q_e\|_{0,e}. \tag{3.23}$$

Combining (3.20)–(3.23), gives

$$\|\Phi_e^{-\frac{\alpha}{2}}q_e\|_{0,e}^2 \le Ch_\kappa^{-\frac{1}{2}}p_\kappa\left(\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e} + p_\kappa^{\alpha-1}\sum_{\tau\in\{\kappa,\kappa'\}}\sqrt{\mathcal{O}_\tau^{(1)}} + p_\kappa^{-\frac{1}{2}}\sqrt{\mathcal{O}_e^{(2)}}\right)\|\Phi_e^{-\frac{\alpha}{2}}q_e\|_{0,e}.$$

As in the proof of *a)*, we divide the above inequality by $\|\Phi_e^{-\frac{\alpha}{2}}q_e\|_{0,e}$, and use the fact that $\Phi_e^{-\frac{\alpha}{2}}q_e = \Phi_e^{\frac{\alpha}{2}}\Pi_{\mathcal{E}}|_e\left(([\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}]\!]\cdot\nu)|_e\right)$. Then, applying the inverse inequality from [36, Lemma 2.4] (see also [4, 5]), we get

$$\|\Pi_{\mathcal{E}}([\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}]\!]\cdot\nu)\|_{0,e} \le Cp_\kappa^{\alpha}\|\Phi_e^{\frac{\alpha}{2}}\Pi_{\mathcal{E}}([\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}]\!]\cdot\nu)\|_{0,e} = Cp_\kappa^{\alpha}\|\Phi_e^{-\frac{\alpha}{2}}q_e\|_{0,e}.$$

Thereby,

$$\begin{aligned} &\|\Pi_{\mathcal{E}}([\![\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}]\!]\cdot\nu)\|_{0,e} \\ &\qquad \le Ch_\kappa^{-\frac{1}{2}}p_\kappa^{1+\alpha}\left(\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e} + p_\kappa^{\alpha-1}\sum_{\tau\in\{\kappa,\kappa'\}}\sqrt{\mathcal{O}_\tau^{(1)}} + p_\kappa^{-\frac{1}{2}}\sqrt{\mathcal{O}_e^{(2)}}\right). \end{aligned}$$

Again, selecting $\delta = \alpha - \frac{1}{2}$ leads to estimate *b)*.

Proof of c): This follows from (2.1), (2.2) and (3.11):

$$\|[\![u_{\mathrm{DG}}]\!]\|_{0,e} = \|[\![e_{\mathrm{DG}}]\!]\|_{0,e} \le C\gamma^{-\frac{1}{2}}h_\kappa^{\frac{1}{2}}p_\kappa^{-1}\|\sigma^{\frac{1}{2}}[\![e_{\mathrm{DG}}]\!]\|_{0,e}.$$

That completes the proof of the lower bounds. ■

Remark 10 For the case when the mesh $\mathcal{T}_h$ is 1-irregular (but assumed to be regularly reducible to a conforming mesh $\widetilde{\mathcal{T}}_h$, cf. Section 2), analogous bounds to the ones derived in Theorem 9 still hold. Indeed, bounds a) and c) follow directly; for the proof of b), employing the argument outlined in the proof of Theorem 9, we deduce that

$$\begin{aligned} &\|\Pi_{\mathcal{E}}|_e((\llbracket\mu(|\nabla u_{\mathrm{DG}}|)\nabla u_{\mathrm{DG}}\rrbracket\cdot\nu)|_e)\|_{0,e} \\ &\qquad\leq C h_{\widetilde{\kappa}}^{-\frac{1}{2}} p_{\widetilde{\kappa}}^{\delta+\frac{3}{2}} \left(\|\nabla e_{\mathrm{DG}}\|_{0,\widetilde{\omega}_e} + p_{\widetilde{\kappa}}^{\delta-\frac{1}{2}} \sum_{\tau\in\{\widetilde{\kappa},\widetilde{\kappa}'\}} \sqrt{\mathcal{O}_\tau^{(1)}} + p_{\widetilde{\kappa}}^{-\frac{1}{2}} \sqrt{\mathcal{O}_e^{(2)}} \right), \qquad (3.24) \end{aligned}$$

where $\widetilde{\omega}_e$ is defined so that the closure of $\widetilde{\omega}_e$ is the union of the closure of the two elements $\widetilde{\kappa},\widetilde{\kappa}'\in\widetilde{\mathcal{T}}_h$ which share the common edge e. The right-hand side of (3.24) may now be bounded from above by an similar expression involving quantities measured over the (nonmatching) elements κ and κ' which share the edge e; by this we mean that, in the estimate (3.24), the element size $h_{\widetilde{\kappa}}$ and polynomial degree $p_{\widetilde{\kappa}}$ are commensurate with h_κ and p_κ, respectively, and the error term $\|\nabla e_{\mathrm{DG}}\|_{0,\widetilde{\omega}_e}$ is bounded from above by $\|\nabla e_{\mathrm{DG}}\|_{0,\omega_e}$. We note that the data oscillation terms $\mathcal{O}_\tau^{(1)}$ appearing in (3.24) are, however, still measured over the elements $\widetilde{\kappa},\widetilde{\kappa}'\in\widetilde{\mathcal{T}}_h$ since they are in general not bounded by the corresponding oscillations on $\kappa,\kappa'\in\mathcal{T}_h$.

4 Numerical experiments

In this section we present a series of numerical examples to demonstrate the practical performance of the proposed a-posteriori error estimator derived in Theorem 4 within an automatic *hp*-adaptive refinement procedure which is based on 1-irregular quadrilateral elements. In each of the examples shown in this section the DG solution u_{DG} defined by (2.6) is computed with $\theta=0$, i.e., we employ an incomplete-interior-penalty-type discontinuous Galerkin method. Analogous results to those presented for $\theta=0$ are also observed with $\theta=-1$ and $\theta=1$; for brevity these results have been omitted. Additionally, we set the constant γ appearing in the definition of the interior-penalty parameter σ defined in (2.5) equal to 10. The resulting system of nonlinear equations is solved by employing a damped Newton method; within each inner (linear) iteration, we exploit a (left-) preconditioned GMRES algorithm using a block symmetric Gauss–Seidel preconditioner.

The *hp*-adaptive meshes are constructed by first marking the elements for refinement or derefinement according to the size of the local error indicators η_κ; this is achieved by employing the fixed fraction strategy, see [27], with refinement and derefinement fractions set to 25% and 10%, respectively. Once an element $\kappa\in\mathcal{T}_h$ has been flagged for refinement or derefinement, a decision must be made whether the local mesh size h_κ or the local degree p_κ of the approximating polynomial should be adjusted accordingly. The choice to perform either *h*-refinement/derefinement or *p*-refinement/derefinement is based on estimating the local smoothness of the (unknown) analytical solution. To this end, we employ the hp-adaptive strategy developed in [28], where the local regularity of the analytical solution is estimated from truncated local Legendre expansions of the computed numerical solution; see, also, [16, 26].

Here, the emphasis will be on investigating the asymptotic sharpness of the proposed a-posteriori error bound on a sequence of nonuniform hp-adaptively refined 1-irregular meshes. To this end, we shall compare the estimator derived in Theorem 4, which is slightly suboptimal (by a factor of $p^{1/2}$) in the spectral order p, with the corresponding optimal one (cf. Remark 5); we note that the derivation of the latter precludes the use of hanging nodes. Indeed, here we shall show that despite the loss of optimality in p, the former indicator performs extremely well on hp-refined meshes, in the sense that the *effectivity index*, which is defined as the ratio of the a-posteriori error bound and the energy norm of the actual error, is roughly constant on all of the meshes employed. Moreover, our numerical experiments indicate that both a-posteriori error indicators give rise to very similar quantitative results. For simplicity, as in [3], we set the constant C arising in Theorem 4 equal to one; in general, to ensure the reliability of the error estimator, this constant must be determined numerically for the underlying problem at hand. In all of our experiments, the data-approximation terms in the a-posteriori bound stated in Theorem 4 will be neglected.

4.1 Example 1

In this example, we let Ω be the unit square $(0,1)^2$ in $\mathbb{R}^2$. The nonlinear diffusion coefficient is defined as follows:

$$\mu(\mathbf{x}, |\nabla u|) = 2 + \frac{1}{1 + |\nabla u|};$$

further, we select f so that the analytical solution to (1.1)–(1.2) is given by

$$u(x,y) = x(1-x)y(1-y)(1-2y)\,\mathrm{e}^{-s(2x-1)^2},$$

where s is a positive constant, cf. [23, 36]; throughout this section we set $s = 20$.

In Figure 1(a) we present a comparison of the actual and estimated energy norm of the error versus the third root of the number of degrees of freedom in the finite element space $S^{\mathbf{p}}(\Omega, \mathcal{T}_h, \mathbf{F})$ on a linear-log scale, for the sequence of meshes generated by our hp-adaptive algorithm using the suboptimal indicator stated in Theorem 4 (denoted by p^3 in the figure) and the corresponding optimal one outlined in Remark 5 (denoted by p^2 in the figure). We note that for both indicators meshes employing hanging nodes are employed, despite the fact that the derivation of the latter, hp-optimal, error indicator necessitates the use of conforming (regular) meshes. The third root of the number of degrees of freedom is chosen on the basis of the a-priori error analysis performed in [43], for example. Here, we observe that the two error indicators perform in a very similar manner: in each case the error

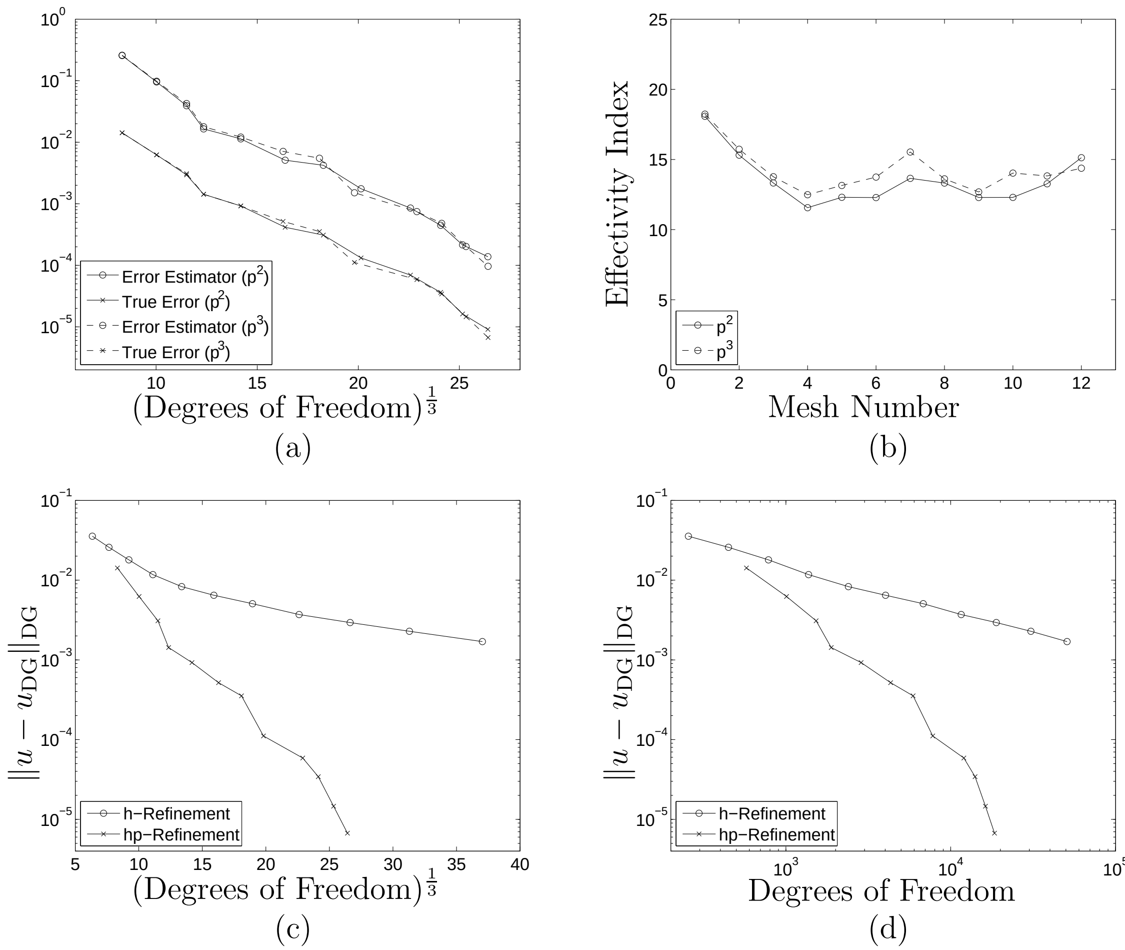
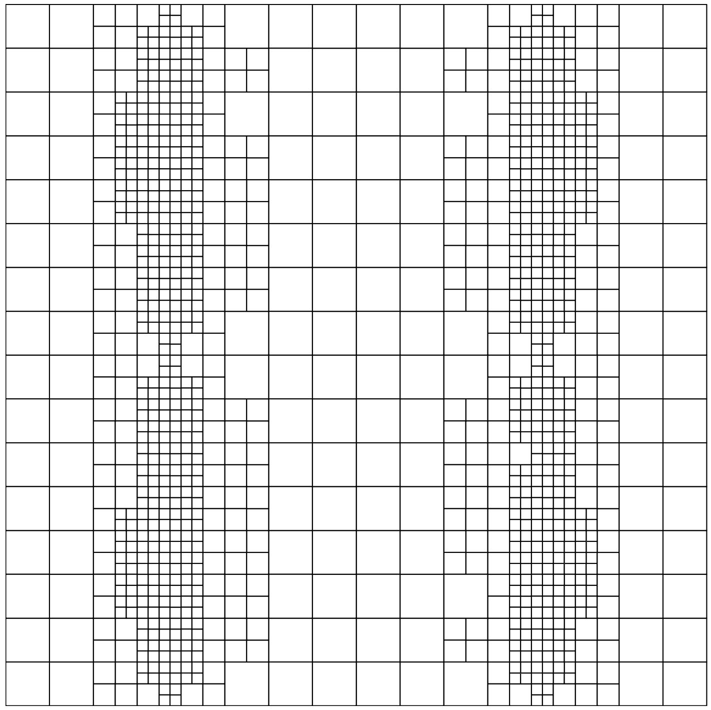
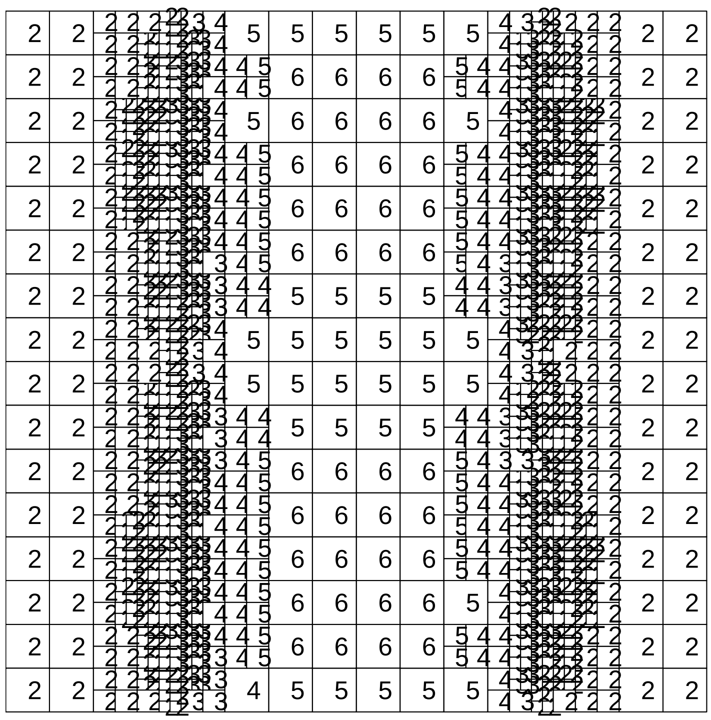


Figure 1: *Example 1.* (a) Comparison of the actual and estimated energy norm of the error with respect to the (third root of the) number of degrees of freedom with hp-adaptive mesh refinement; (b) Effectivity indices; (c) & (d) Comparison of the actual error with h- and hp-adaptive mesh refinement.

bound over-estimates the true error by a (reasonably) consistent factor; indeed, from Figure 1(b), we see that the computed effectivity indices oscillate around a value of approximately 13. Additionally, from Figure 1(a) we observe that the convergence lines using hp-refinement are (roughly) straight on a linear-log scale, which indicates that exponential convergence is attained for this smooth problem, as we would expect. In Figures 1(c) & (d), we present a comparison between the actual energy norm of the error employing both h- and hp-mesh refinement; here, the hp-refinement is based on employing the error indicator stated in Theorem 4. In the former case, the DG solution u_{DG} is computed using bilinear elements, i.e., $\mathbf{p} = 1$; here, the adaptive algorithm is again based on employing the fixed fraction strategy, with refinement and derefinement fractions set to 25% and 10%, respectively. From Figures 1(c) & (d), we clearly observe the superiority of employing a grid adaptation



(a)



(b)

Figure 2: *Example 1.* Finite element mesh after 11 adaptive refinements, with 1198 elements and 18443 degrees of freedom: (a) h-mesh alone; (b) hp-mesh.

strategy based on exploiting hp-adaptive refinement: on the final mesh, the energy norm of the error using hp-refinement is over two orders of magnitude smaller than the corresponding quantity computed when h-refinement is employed alone.

In Figure 2 we show the mesh generated using the proposed hp-version a-posteriori error indicator stated in Theorem 4 after 11 hp-adaptive refinement steps. For clarity, we show the h-mesh alone, as well as the corresponding polynomial degree distribution on this mesh. Here, we observe that some h-refinement of the mesh has been performed in the vicinity of the base of the exponential 'hills' situated in the left- and the right-hand sides of the domain, where the gradient/curvature of the analytical solution is relativity large. Once the h-mesh has adequately captured the structure of the solution, the hp-adaptive algorithm increased the degree of the approximating polynomial within the interior part of the domain containing these hills.

4.2 Example 2

In this section we let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$, and select

$$\mu(\mathbf{x}, |\nabla u|) = 1 + \mathrm{e}^{-|\nabla u|^2}.$$

Then, writing (r, φ) to denote the system of polar co-ordinates, we choose f and an appropriate inhomogeneous boundary condition for u so that

$$u = r^{2/3} \sin(2\varphi/3);$$

cf. [43], for example. We note that u is analytic in $\overline{\Omega} \setminus \{\mathbf{0}\}$, but ∇u is singular at the origin; indeed, here $u \notin \mathrm{H}^2(\Omega)$.

Figure 3(a) shows the history of the actual and estimated energy norm of the error on each of the meshes generated by our hp-adaptive algorithm using both the indicator stated in Theorem 4 (denoted by p^3 in the figure) and the corresponding one outlined in Remark 5 (denoted by p^2 in the figure). As in the previous example, we observe that the two error indicators perform in a very similar manner, though for this nonsmooth example the loss in optimality in the jump indicator in the estimator stated in Theorem 4 does lead to a slight increase in the effectivity indices in comparison with the latter indicator. However, from Figure 3(b) we observe that asymptotically both a-posteriori bounds over-estimate the true error by a consistent factor. Additionally, from Figure 3(a) we observe exponential convergence of the energy norm of the error using both estimators with hp-refinement; indeed, on a linear-log scale, the convergence lines are, on average, straight. Figures 3(c) & (d) highlight the superiority of employing hp-adaptive refinement in comparison with h-refinement: on the final mesh, the energy norm of the error using the hp-refinement indicator stated in Theorem 4 is over two orders of magnitude smaller than the corresponding quantity when h-refinement is employed alone, based on using bilinear elements.

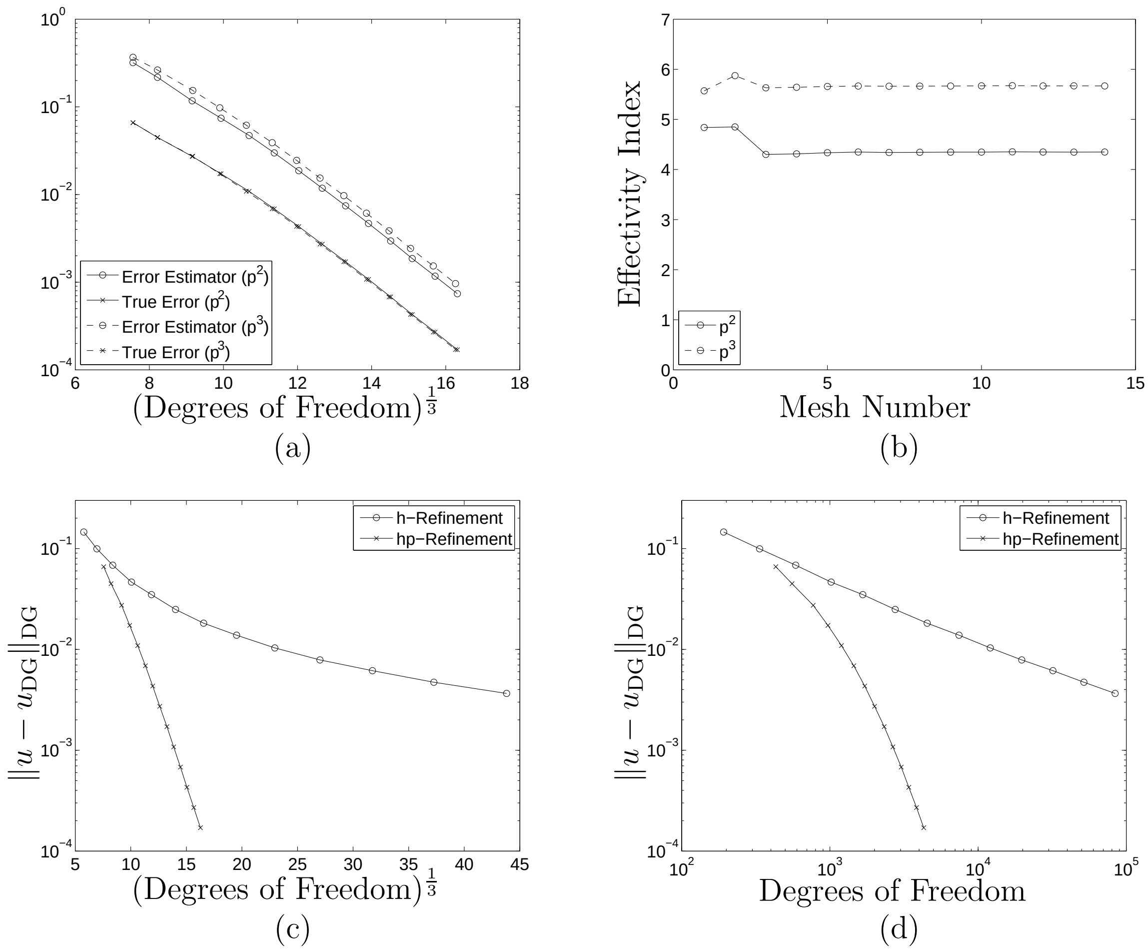


Figure 3: *Example 2.* (a) Comparison of the actual and estimated energy norm of the error with respect to the (third root of the) number of degrees of freedom with hp-adaptive mesh refinement; (b) Effectivity indices; (c) & (d) Comparison of the actual error with h- and hp-adaptive mesh refinement.

In Figure 4 we show the mesh generated using the local error indicators η_κ stated in Theorem 4 after 13 hp-adaptive refinement steps. Here, we see that the h-mesh has been refined in the vicinity of the re-entrant corner located at the origin; from the zoom, we see that h-refinement is more pronounced in the direction $y = x$. In the normal direction, $y = -x$, p-refinement is employed instead, as the solution is deemed to be smooth here. Additionally, we see that the polynomial degrees have been increased away from the re-entrant corner located at the origin, since the underlying analytical solution is smooth in this region.

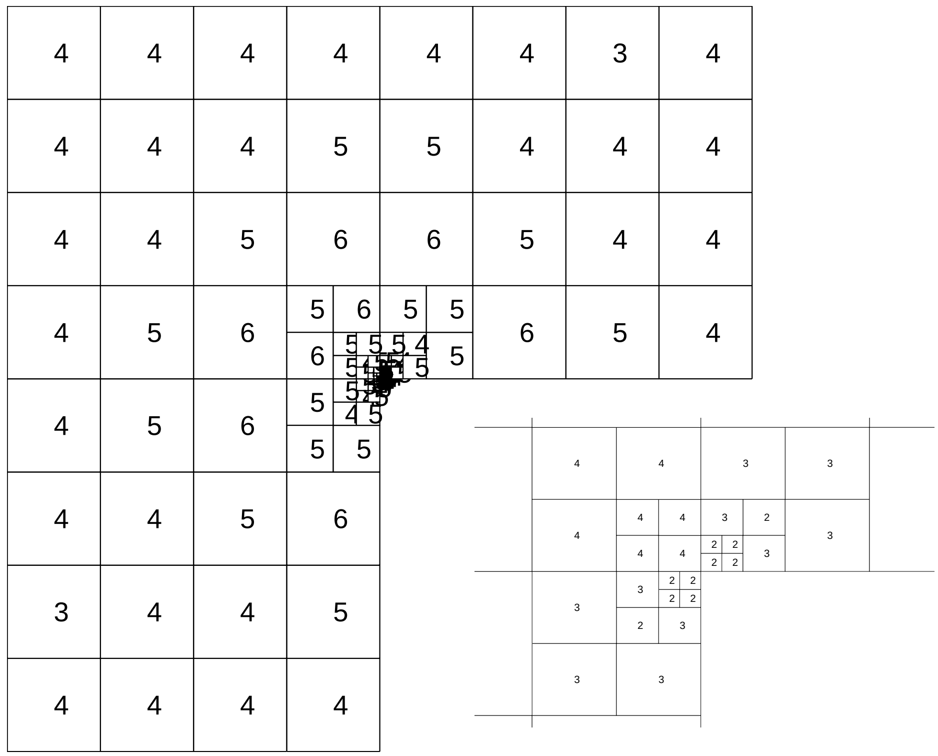


Figure 4: *Example 2. hp-mesh after* 13 *adaptive refinements, with* 162 *elements and* 4302 *degrees of freedom.*

5 Concluding remarks

In this paper, we derived global upper and local lower residual-based a-posteriori error bounds in the energy norm for the class of interior-penalty *hp*-DGFEMs developed in [20] for the numerical approximation of second-order quasilinear elliptic partial differential equations. The analysis is based on employing a suitable DG space decomposition, together with an *hp*-version projection operator. Numerical experiments presented in this article clearly demonstrate that the proposed a-posteriori estimator converges to zero at the same asymptotic rate as the energy norm of the actual error on sequences of *hp*-adaptively refined meshes. Future work will be devoted to the extension of our analysis to *hp*-adaptive discontinuous Galerkin approximations of quasi-Newtonian incompressible flow models.

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