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A Posteriori Error Estimates Based on Polynomial Preserving Recovery

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Abstract Superconvergence of order $O(h^{1+\rho})$, for some $\rho > 0$, is established for gradients recovered using Polynomial Preserving Recovery technique when the mesh is mildly structured. Consequently this technique can be used in building a posteriori error estimator that is asymptotically exact.

Key Words. finite element method, least-squares fitting, SPR, PPR, superconvergence

AMS Subject Classification. 65N30, 65N15, 65N12, 65D10, 74S05, 41A10, 41A25

1 Introduction

Adaptive control based on a posteriori error estimates have become standard in finite element methods. Generally speaking, error estimators can be classified under two categories. The first one is the residual type estimators, as in [4], and the second one is the recovery type estimators, as in [12]. In recovery type estimators, a recovery operation uses the finite element solution or its gradient to build another solution, as in [7] and [9], or another gradient, as in [13]. The recovered quantities are then used in building a posteriori error estimators (see [1] and [3] for some general discussion and literature).

As it is known, if the recovered gradient is superconvergent to the exact gradient, then the a posteriori error estimator based on this recovered gradient is exact in asymptotic sense. A good example of such estimators is Zienkiewicz-Zhu error estimator based on Superconvergence Patch Recovery (SPR), as in [14]. The Polynomial Preserving Recovery (PPR) is a new recovery technique, introduced in [11], which has good properties that enable it to be used in constructing a posteriori error estimator.

To fix the ideas, let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Consider

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the model elliptic boundary value problem of finding the solution u of

$$-\nabla(\mathcal{D}\nabla u + \boldsymbol{b}u) + cu = f \text{ in } \Omega \tag{1.1}$$

subject to the boundary conditions

$$\boldsymbol{n}\cdot(\mathcal{D}\nabla u + \boldsymbol{b}u) = g \text{ on } \Gamma_N \tag{1.2}$$

and

$$u = 0 \text{ on } \Gamma_D. \tag{1.3}$$

Here $\mathcal{D}$ is a 2×2 symmetric positive definite matrix with smooth entries. The rest of the data are assumed to be smooth, c is a non-negative function, $\boldsymbol{n}$ is the unit outward normal vector to $\partial\Omega$, and the boundary segments are assumed to be disjoint with $\overline{\Gamma}_D \cup \overline{\Gamma}_D = \partial\Omega$. As usual, $W_p^m(\Omega)$ and $H^m(\Omega)$ are the classical Sobolev spaces equipped with the norms $\| \quad \|_{m,p,\Omega}$, and $\| \quad \|_{m,\Omega}$, respectively.

The variational form of this problem is to find $u \in V$ such that

$$B(u,v) = L(v) \text{ for all } v \in V, \tag{1.4}$$

where

$$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\},$$

$$B(u,v) = \int_\Omega [(\mathcal{D}\nabla u + \boldsymbol{b}u)\nabla v + cuv]dxdy,$$

and

$$L(v) = \int_\Omega fvdx + \int_{\Gamma_N} gvds.$$

Let $\mathcal{T}_h$ be a triangular partition of Ω, and let $\Omega_h = \bigcup_{T\in\mathcal{T}_h} T$. Consider the C^0 linear finite element space $S_h \subset V$ associated with $\mathcal{T}_h$ and defined by

$$S_h = \{v \in V : v \in P_1(T) \text{ for every triangle } T \in \mathcal{T}_h\},$$

where $P_r(\mathcal{A})$ denotes the set of all polynomials defined on $\mathcal{A} \subseteq \mathbb{R}^2$ of total degree $\leq r$. The finite element solution of this problem is to find u_h such that

$$B(u_h, v) = L(v) \text{ for all } v \in S_h^0, \tag{1.5}$$

where

$$S_h^0 = \{v \in S_h : v|_{\Gamma_D} = 0\}.$$

Unfortunately, ∇u_h is piecewise continuous, and so a smoothing operation is needed to get a continuous gradient of the solution. Recovery techniques like SPR and PPR can be used for this purpose. Both of them recover the gradient at mesh nodes. This is enough to uniquely define $R_h u_h \in S_h \times S_h$, where R_h denotes the gradient recovery operator associated with SPR or PPR.

To recover the gradient using SPR or PPR at a mesh node $z = (x_z, y_z)$, a patch ω_z of triangles is selected as shown in Fig. 1(a). To recover the $x-$derivative at z using SPR, we find a polynomial $p_x \in P_1(\omega_z)$ that best fits $\partial_x v_h$, in least-squares sense, at the triangles centers in ω_z. The recovered $x-$derivative at z is defined to be $p_x(x_z, y_z)$. Similarly, we can define the recovered $y-$derivative at z. To get the required polynomials, ω_z must have at least three triangles whose centers are not lying on one straight line. This requirement is guaranteed for $z \in \Omega$, but not for $z \in \partial\Omega$. For $z \in \partial\Omega$, the gradient is computed in every patch corresponding to an adjacent node in Ω by evaluating the obtained polynomials at z, and then taking the average. If there are no adjacent nodes in Ω, then the gradient is defined to be $\nabla v_h(z)$ (see [13] for more details).

To recover the gradient at z using PPR, we find a polynomial $p \in P_2(\omega_z)$ that best fits v_h, in least-squares sense, at the mesh nodes in ω_z. The recovered gradient is defined to be $\nabla p(x_z, y_z)$. To get p, ω_z must contain at least 6 mesh nodes that are not on a conic section, as we shall see later. This might not be achieved if ω_z contains only the triangles attached to z. As it was proposed in [11], ω_z is extended by adding the triangles sharing an edge with ω_z, as shown in Fig. 1(b). Nodes on $\partial\Omega$ are handled in the same way, although they need extra care in constructing their patches. Unlike SPR, PPR recovers the exact gradient if $v_h \in P_2(\omega_z)$ without any restrictions on $\mathcal{T}_h$.

After the previous description of PPR and SPR recovery techniques, we have the following important remarks.

Remark 1.1 It is easy to see that PPR preserves the first derivatives of polynomials in $P_2(\omega_z)$. This is not true for SPR, except for some special cases. Basically, PPR can be viewed as a dynamic way to generate difference schemes for first derivatives that can recover the exact derivatives of polynomials in $P_2(\omega_z)$. In [3], a technique was proposed to generate such kind of difference schemes a priori, as in example 4.8*.4, where the derivative at a mesh node z is expressed as a weighted sum of the function values at the mesh nodes directly attached to z. The weights are determined such that

1. the first derivative derivatives of the basis of $P_2(\omega)$ are preserved, and

2. the sum of the squares of the weights is minimum.

It can be shown that PPR generates weights satisfying the above conditions without worrying about the mesh structure, or doing that a priori as it is used in real time mode.

Remark 1.2 The idea of best fitting the function values by a quadratic polynomial was used before, as in [7] and [9], but it is used in a way that differs with the way it is used in PPR in many aspects.

1. It is mainly used for recovering the functions values, and not the derivatives.
2. The patches used in PPR are constructed for nodes, while in function recovery technique patches are constructed for triangles.
3. In PPR, the best fit quadratic polynomial is used to compute the gradient at the node for which the patch is constructed, while in function recovery technique the best fit polynomial is used to approximate the function on the triangle for which the patch was constructed.

As it was shown in [3], a posteriori error estimators based on function recovery techniques are inferior to many estimators, especially the one based on SPR.

Let $I_h u \in S_h$ be the Lagrange interpolation of u, and let $\omega_T = \bigcup\{\omega_z : z \text{ is a vertex of } T\}$ be a patch corresponding to $T \in \mathcal{T}_h$. If

$$\|\nabla(I_h u - u_h)\|_{L^2(\omega_T)} \le Ch^{1+\rho} \tag{1.6}$$

for some $\rho > 0$, and

$$\|R_h v\|_{L^2(T)} \le C|v|_{1,\omega_T} \tag{1.7}$$

for all $T \in \mathcal{T}_h$, and for all $v \in S_h$, then it is possible to show that

$$\|\nabla u - R_h u_h\|_{L^2(\Omega)} \le Ch^{1+\rho}, \tag{1.8}$$

which means that the recovered gradient is superconvergent to the exact gradient. From that it is straight forward to prove that the a posteriori error estimator η_h defined by

$$\eta_h = \|R_h u_h - \nabla u_h\|_{L^2(\Omega)} \tag{1.9}$$

is asymptotically exact. As it is usual in finite element analysis, C is a generic constant that may depend on u, Ω, or mesh parameters other than h.

The assumption in (1.6) is satisfied if $\mathcal{T}_h$ is mildly structured in the following sense. Let $\mathcal{T}_h = \mathcal{T}_{h,1} \cup \mathcal{T}_{h,2}$ and $\Omega_h = \Omega_{h,1} \cup \Omega_{h,2}$, where $\Omega_{h,i} = \bigcup_{T \in \mathcal{T}_{h,i}} T$ for $i = 1, 2$.

Definition 1.3 *The triangulation $\mathcal{T}_h$ is said to satisfy the* <u>*condition* (α, σ)</u>*, if there exist positive constants α and σ such that: Every two adjacent triangles in $\mathcal{T}_{h,1}$ form an $O(h^{1+\alpha})$ parallelogram, and $|\Omega_{h,2}| = O(h^\sigma)$.*

An $O(h^{\alpha+1})$ parallelogram is a quadrilateral in which the difference between the lengths of any two opposite sides is $O(h^{\alpha+1})$. When $\alpha = \infty$, then every pair of adjacent triangles in $\mathcal{T}_{h,1}$ form a parallelogram. The case in which $\alpha = \sigma = \infty$ corresponds to $\mathcal{T}_h$ that is uniformly generated by lines parallel to three fixed directions. This case was handled in [6], where the error was expanded at mesh nodes, and the case in which $\alpha = 1$ was handled in [5]. The general case was treated in [10], where the following theorem was established.

Theorem 1.4 *Let u be the solution of* (1.4)*, let $u_h \in S_h$ be the finite element solution of* (1.5)*, and let $I_h u \in S_h$ be the linear interpolation of u. If the triangulation $\mathcal{T}_h$ satisfies the condition (α, σ) and $u \in H^3(\Omega) \cap W^2_\infty(\Omega)$, then*

$$\|u_h - I_h u\|_{1,\Omega} \leq h^{1+\rho}(\|u\|_{3,\Omega} + |u|_{2,\infty,\Omega}),$$

where $\rho = \min(\alpha, \frac{1}{2}, \frac{\sigma}{2})$.

Remark 1.5 The condition(α, σ) is sufficient to guarantee the superconvergence result in (1.6), but it is not necessary, as we shall see later. But, this condition is satisfied for meshes generated by many automatic mesh generators, as described in [10],i.e. it covers a wide range of meshes.

The assumption in (1.7) requires the operator associated with the recovery technique to be bounded. This is somewhat easy to establish when the recovery technique works directly on ∇v, which is the case in many recovery techniques like weighted average and SPR. In PPR the situation is much harder as it works on v. It is even not clear how to relate $R_h v$ to ∇v.

From now on, $\mathrm{G^h} : S_h \to S_h \times S_h$ denotes the operator associated with PPR. The main target in this paper is to show that $\mathrm{G^h}$ satisfies the assumption in (1.7). Having this in hand paves the way to prove the superconvergence property in (1.8) when $R_h u_h = \mathrm{G^h} u_h$. Finally, some numerical results are provided as a support for the theoretical results, where attention is paid to the regions near $\partial\Omega$. Also, the behavior of PPR is compared with that of SPR, as the later is widely used in practice.

Before going through the details, we should mention that PPR can be generalized to higher order elements, although it needs more care in selecting the sampling points. This will be handled in a future work.

2 Definition and existence of $\mathrm{G^h}$

Let $v \in S_h$. To define $\mathrm{G^h} v \in S_h \times S_h$, it is enough to define it at each mesh node. Consider a mesh node $z = (x_0, y_0) \in \overline{\Omega}$, and its corresponding patch ω_z with triangles $T_1, T_2, \ldots, T_m$, and the nodes $z_0 = z, z_1 = (x_1, y_1), \ldots, z_n = (x_n, y_n)$. Without loss of generality, z is considered as the origin of a local coordinate system for ω_z, i.e. $z = (0,0)$; otherwise, we replace node $z_i \in \omega_z$ by $z_i - z$, for $i = 0, \ldots, n$. Set $h_z = \max_{1 \le i \le n} \|z_i - z\|$. We may assume $h_z = 1$, i.e. the whole patch is inside the unite circle centered at z; otherwise we replace node $z_i \in \omega_z$ by z_i / h_z for $i = 1, \ldots, n$. Let $v_{z,i} = v(z_i)$ for $i = 0, 1, \ldots, n$. Let $p_z \in P_2(\omega_z)$ be the quadratic polynomial that best fits v, in least-squares sense, on ω_z. We write $p_z(x,y) = \boldsymbol{x}^T \boldsymbol{c_z}$, where $(x,y) \in \omega_z$, $\boldsymbol{c}_z = \begin{bmatrix} c_{z,1} & c_{z,2} & c_{z,3} & c_{z,4} & c_{z,5} & c_{z,6} \end{bmatrix}^T$, and $\boldsymbol{x}^T = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}$. By definition, $\boldsymbol{c}_z$ is determined by the linear system $A^T A_z \boldsymbol{c}_z = A^T \boldsymbol{v}_z$, where $\boldsymbol{v}_z = \begin{bmatrix} v_{z,0} & v_{z,1} & \ldots & v_{z,n} \end{bmatrix}^T$, and

$$A_z = \begin{bmatrix} 1 & x_0 & y_0 & \cdots & y_0^2 \\ 1 & x_1 & y_1 & \cdots & y_1^2 \\ 1 & x_2 & y_2 & \cdots & y_2^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & y_n & \cdots & y_n^2 \end{bmatrix}. \tag{2.1}$$

Let $B_z = A_z^T A_z$, then, assuming the existence of B_z^{-1}, $\boldsymbol{c}_z = B_z^{-1} A_z^T \boldsymbol{v}_z$. By definition,

$$\mathrm{G^h} v(z) = \frac{\nabla p_z(0,0)}{h_z} = \frac{1}{h_z} \begin{bmatrix} c_{z,2} & c_{z,3} \end{bmatrix}^T = \frac{1}{h_z} \begin{bmatrix} \boldsymbol{v}_z^T A_z B_z^{-1} e_2 & \boldsymbol{v}_z^T A_z B_z^{-1} e_3 \end{bmatrix}^T, \tag{2.2}$$

where e_2 and e_3 are the second and the third columns of the identity matrix $I_{6\times 6}$.

It is clear that computing p_z requires at least 6 nodes, i.e., $n \ge 5$. Basically, the patch ω_z contains the triangles attached to z. If $n < 5$, then ω_z is extended by attaching the triangles sharing an edge with it. For $z \in \Omega$, this extension is enough to get $n \ge 5$, but for $z \in \partial\Omega$ we may need to iterate this process more than once. Unfortunately having $n \ge 5$ is not enough to recover the gradient and other conditions have to imposed especially for nodes on the boundary. Before we continue, note that all quantities defined for z, or ω_z, will be subscripted with z.

From the above discussion, it is important to answer the following questions, especially the second one:

1. Assuming existence of p_z, does it depend on the orientation of the local coordinate system at z? Also, does this orientation affect the accuracy of the numerical computations of p_z,i.e. the condition number of A_z?

2. Are there any sufficient conditions that guarantee the existence of B_z^{-1}?

The following lemma addresses the answer of the first question.

Lemma 2.1 *If p_z exits, then it is invariant under the rotation of the local coordinate system at z.*

Proof. Rotate the local coordinate system at z by an angle θ in counterclockwise direction. The superscript $\tilde{}$ will be used for quantities expressed in the rotated coordinate system. If (x, y) refers to a point in the original coordinate system and $(\tilde{x}, \tilde{y})$ refers to the same point in the rotated coordinate system, then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}$$

With this in mind, it is easy to verify that $\tilde{\boldsymbol{x}}^T = R_\theta \boldsymbol{x}^T$ and $\tilde{A}_z = A_z R_\theta$, where

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 & 0 & 0 \\ 0 & \sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos^2\theta & -\cos\theta\sin\theta & \sin^2\theta \\ 0 & 0 & 0 & 2\cos\theta\sin\theta & \cos^2\theta - \sin^2\theta & -2\cos\theta\sin\theta \\ 0 & 0 & 0 & \sin^2\theta & \cos\theta\sin\theta & \cos^2\theta \end{bmatrix}.$$

Also, it is easy to verify the following properties for R_θ:

1. $\mathrm{Det}(R_\theta) = 1$, i.e., R_θ is invertible. Moreover, $R_\theta^{-1} = R_{-\theta}$.

2. The singular values of R_θ are $s_\theta^{-1}, 1, 1, 1, 1$, and s_θ, where $s_\theta = [9 - \cos 4\theta + ((1-\cos 4\theta)(17 - \cos 4\theta))^{1/2}]^{1/2}/8$, and $1 \le s_\theta \le \sqrt{2}$. Note that $s_{-\theta} = s_\theta$, and so the singular values of $R_{-\theta}$ are the same as those of R_θ. Hence, $\|R_\theta\| = \|R_{-\theta}\| = s_\theta$.

Let $\tilde{p}_z(\tilde{x}, \tilde{y}) = \tilde{\boldsymbol{x}}^T \tilde{\boldsymbol{c}}_z \in P_2(\omega_z)$ be the least-squares best fit of v on the same patch ω_z, but with respect to rotated coordinate system at z. As before, $\tilde{\boldsymbol{c}}_z$ is determined by solving the linear system $\tilde{A}_z^T \tilde{A}_z \tilde{\boldsymbol{c}}_z = \tilde{A}_z^T \boldsymbol{v}_z$. Since $\tilde{B}_z = \tilde{A}_z^T \tilde{A}_z = R_\theta^T B_z R_\theta$, and B_z^{-1} is assumed to exist, then $\tilde{B}_z^{-1} = R_\theta^{-1} B_z^{-1} R_\theta^{-T}$. Hence, $\tilde{\boldsymbol{c}}_z = \tilde{B}_z^{-1} \tilde{A}_z^T \boldsymbol{v}_z = R_\theta^{-1} \boldsymbol{c}_z$, i.e., $\tilde{p}_z = \tilde{\boldsymbol{x}}^T R_\theta^{-1} \boldsymbol{c}_z = \boldsymbol{x}\boldsymbol{c}_z = p_z$, and the proof is complete. □

For any Matrix H of order $k_1 \times k_2$, let $\sigma_1(H)$, and $\sigma_{\min(k1,k2)}(H)$ denote the largest and the smallest singular values of H, respectively. As we know, $\sigma_l^2(H) = \sigma_l(H^T H)$, for $l = 1, 2, \ldots, \min(k1, k2)$.

Remark 2.2 From the proof of Lemma 2.1, we know that $\tilde{B}_z = R_\theta^T B_z R_\theta$. Using this relation, the properties of singular values, and the properties of R_θ, it is easy to verify that

$$\frac{\sigma_l(A_z)}{\sqrt{2}} \le s_\theta^{-1} \sigma_l(A_z) \le \sigma_l(\tilde{A}_z) \le s_\theta \sigma_l(A_z) \le \sqrt{2}\sigma_l(A_z) \text{ for } l = 1, 6.$$

This shows that the patch orientation has almost no effect on the condition number of A_z.

We turn our attention now to the answer of the second question. We start with the following theorem from [8], after adopting our notation.

Theorem 2.3 *Pointwise interpolation in $P_2(\mathcal{A})$, $\mathcal{A} \subseteq \mathbb{R}^2$, at six distinct points $(x_i, y_i) \in \mathcal{A}, i = 1, 2, \ldots, 6$, has the finite interpolation property if and only if there is no conic section passing through all of six points.*

The above theorem simply says that the interpolation by a quadratic polynomial at six nodes exists and is unique as long as the six points are not on a conic section.

Definition 2.4 *The patch ω_z is said to satisfy the angle condition if the sum of any two adjacent angles inside ω_z is at most π, and is said to satisfy the line condition if its nodes are not lying on two lines.*

We write $n = n_1 + n_2$, where n_1 denotes the number of nodes that are directly attached to z. If $z \in \Omega$, then $n_1 \geq 3$. Practically, a good mesh generator can detect any node z for which $n_1 = 3$ and removes it. So, we may assume that $n_1 \geq 4$. It is obvious that for $z \in \Omega$ with $n_1 > 4$, ω_z satisfies the line condition, unless one of its triangles is degenerate. If $n_1 = 4$, ω_z may violate this condition as shown in Fig. 3(a). The following elementary lemma is needed in the proof of Theorem 2.6.

Lemma 2.5 *Any tangent to a branch of a hyperbola can not intersect with the other branch.*

Theorem 2.6 *Let $z \in \Omega$ be a mesh node. If the patch ω_z corresponding to z satisfies the angle and the line conditions, then B_z is invertible.*

Proof. As we know, if $\operatorname{Rank} A_z = 6$, then $\operatorname{Rank} B_z = 6$, and B_z is invertible. By Theorem 2.3, it is enough to show that ω_z has six distinct nodes that are not on a conic section. Having $z \in \Omega$ implies that sum of the angles at z_0 is 2π. Hence, the nodes in ω_z can not lie on a circle, a parabola, an ellipse, or on one branch of a hyperbola. Since ω_z satisfies the line condition, the nodes can not be on two lines. The remaining possibility is to have the nodes distributed on two branches of a hyperbola. Depending on n_1, we can have one of the following two cases.

Case 1: $n_1 > 4$. Proceed by contradiction, and assume that the nodes in ω_z are distributed on two branches of a hyperbola. Without loss of generality, one may assume that the real axis of the hyperbola is horizontal, and z_0 lies on the right branch. The left branch must have at least one node of ω_z. If it has three nodes, as in Fig. 2(a), then the angle condition is violated as the measure of the angle $z_1 z_2 z_3 > \pi$. Using the same argument, the left branch can not have more than two nodes. If it has two nodes, as shown in Fig. 2(b), then ω_z must have nodes z_3 and z_4 connected to z_1 and z_2, respectively. To satisfy the angle condition, ω_z can not have any nodes, like z_5, along the hyperbola to the right of z_3. This is because the line $z_1 z_3$ can not be tangent to the hyperbola at z_3 by Lemma 2.5. Using the same argument, we can not have any nodes, like z_6, along the hyperbola to the right of z_4. Thus, to comply with the angle condition, ω_z must have $n_1 = 4$, and this is a contradiction as $n_1 > 4$. Finally, if the left branch has just one node of ω_z, as in Fig. 2(c), then ω_z must have nodes z_2 and z_3 connected to z_1. Again, to satisfy the angle condition, $n_1 = 3$, and this is a contradiction.

Case 2: $n_1 = 4$. In this case the triangles attached to z_0 forms a quadrilateral. Let $z_1, z_2, z_3,$, and z_4 be the vertices of this quadrilateral taken in a counterclockwise direction, as shown in Fig. 3(a). Since ω_z satisfies the angle condition, z_0 must be the intersection point of the quadrilateral diagonals. The nodes in ω_z can not be distributed on a hyperbola; otherwise each diagonal intersects with the hyperbola at three points, which is impossible, and this completes the proof. □

The situation for nodes on $\partial\Omega$ is more delicate. Angle and line conditions are not sufficient any more, as shown in Fig. 3(b). So, in constructing ω_z for $z \in \partial\Omega$ we must impose another condition that ensures the invertibility of B_z. From what has been established, this condition is obvious and is a direct corollary of Theorem 2.6.

Corollary 2.7 *Consider a mesh node $z \in \partial\Omega$, and let ω_z be its corresponding patch. Suppose that ω_z contains another patch $\omega_{\hat{z}}$ corresponding to a node $\hat{z} \in \Omega \cap \omega_z$ and $\omega_{\hat{z}}$ satisfies the angle and line conditions. Then, B_z is invertible.*

Hence in constructing ω_z for $z \in \partial\Omega$, ω_z is extended, as described before, till it contains a patch corresponding to a node in Ω. The time-cost for this extension might be expensive. A cheaper procedure is to construct ω_z such that it contains all the nodes up to the nearest nodes in Ω and their corresponding patches. In this case the number of nodes is larger, and the computational cost might get higher.

3 Boundedness of $\mathrm{G^h}$

Before we start investigating the boundedness of $\mathrm{G^h}$, we go over some basic facts. Consider a mesh triangle $T \subset \overline{\Omega}$ with vertices $(x_1, y_1), (x_2, y_2)$, and (x_3, y_3) taken in counterclockwise direction. If the nodal value of $v \in S_h$ at (x_j, y_j) is v_j, and the basis function associated with this node is θ_j, then, for $(x, y) \in T$, $v(x, y) = \sum_{j=1}^{3} v_j \theta_j(x, y)$. Consider the reference triangle $\hat{T}$ with the vertices $(\xi_1, \eta_1) = (0,0), (\xi_2, \eta_2) = (1,0)$, and $(\xi_3, \eta_3) = (1,0)$. The basis functions on $\hat{T}$ are $\lambda_1(\xi,\eta) = 1-\xi-\eta, \lambda_2(\xi,\eta) = \xi$, and $\lambda_3(\xi,\eta) = \eta$, where λ_j is associated with the vertex (ξ_j, η_j), $j = 1,2,3$. Let $F : \hat{T} \to T$ be the transformation defined by

$$F(\xi,\eta) = \begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix},$$

where

$$x(\xi,\eta) = \sum_{j=1}^{3} x_j\lambda_j = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta,$$

and

$$y(\xi,\eta) = \sum_{j=1}^{3} y_j\lambda_j = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta.$$

The jacobian of this transformation is

$$J = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}.$$

As we know, $\lambda_j = \theta_j \circ F$ for $j = 1,2,3$. If $\hat{v} = v \circ F$, then

$$\hat{v}(\xi,\eta) = v(x(\xi,\eta), y(\xi,\eta)) = \sum_{j=1}^{3} v_j\lambda_j(\xi,\eta).$$

Writing the gradient as a column vector, we get

$$\nabla v = \frac{1}{h_z} J^{-T}\nabla\hat{v}.$$

Since

$$\nabla\hat{v} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix},$$

then

$$\nabla v = \frac{1}{2|T|} \begin{bmatrix} y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Setting

$$a_j = \frac{1}{2|T|}(y_{j+1} - y_{j+2}), \text{ and } b_j = \frac{1}{2|T|}(x_{j+2} - x_{j+1}), \tag{3.1}$$

where the addition in indices is mod 3, we have, for $(x,y) \in T$,

$$\partial_x v(x,y) = \sum_{j=1}^{3} a_j v_j \text{ and } \partial_y v(x,y) = \sum_{j=1}^{3} b_j v_j. \tag{3.2}$$

Let ω_z be the patch corresponding to $z \in \overline{\Omega}$, and consider a triangle $T_k \subset \omega_z$ for some $1 \le k \le m$. If the vertices of T_k, taken in counterclockwise direction, are $(x_{k,1}, y_{k,1}), (x_{k,2}, y_{k,2})$, and $(x_{k,3}, y_{k,3})$, then for $(x, y) \in T_k$, and using (3.1) and (3.2),

$$\partial_x v(x,y) = \frac{1}{h_z}\sum_{j=1}^{3} a_{k,j}v_{k,j}, \text{ and } \partial_y v(x,y) = \frac{1}{h_z}\sum_{j=1}^{3} b_{k,j}v_{k,j},$$

where

$$a_{k,j} = \frac{1}{2|T_k|}(y_{k,j+1} - y_{k,j+2}), \text{ and } b_{k,j} = \frac{1}{2|T_k|}(x_{k,j+2} - x_{k,j+1}),$$

and $v_{k,j} = v(x_{k,j}, y_{k,j})$. Hence,

$$\partial_x v(x,y) = \frac{1}{h_z}\boldsymbol{v}_k^T \boldsymbol{a}_k, \text{ and } \partial_y v(x,y) = \frac{1}{h_z}\boldsymbol{v}_k^T \boldsymbol{b}_k,$$

where $\boldsymbol{a}_k = \begin{bmatrix} a_{k,1} & a_{k,2} & a_{k,3} \end{bmatrix}^T$, $\boldsymbol{b}_k = \begin{bmatrix} b_{k,1} & b_{k,2} & b_{k,3} \end{bmatrix}^T$, and $\boldsymbol{v}_k = \begin{bmatrix} v_{k,1} & v_{k,2} & v_{k,3} \end{bmatrix}^T$. Let E_k be an $(n+1) \times 3$ Boolean matrix defined for T_k, where

$$E_k(i,j) = \begin{cases} 1 & \text{if the node } i \text{ in } \omega_z \text{ is the vertex } j \text{ in } T_k \\ 0 & \text{otherwise} \end{cases},$$

then $\boldsymbol{v}_k = E_k^T \boldsymbol{v}_z$, and

$$\partial_x v(x,y) = \frac{1}{h_z}\boldsymbol{v}_z^T E_k \boldsymbol{a}_k, \text{ and } \partial_y v(x,y) = \frac{1}{h_z}\boldsymbol{v}_z^T E_k \boldsymbol{b}_k. \tag{3.3}$$

Let $G^h_1 v$ and $G^h_2 v$ stand for the recovered $x-$ and $y-$derivatives, respectively. Establishing the boundedness of G^h in the sense of (1.7) would be easy if $G^h_l v(z)$ can be expressed as a liner combination of the first derivatives of v on the triangles of ω_z, for $l = 1, 2$. So, we will try to find a set of *bounded values* $\alpha_{z,l,1}, \ldots, \alpha_{z,l,m}$, and $\beta_{z,l,1}, \ldots, \beta_{z,l,m}$ such that

$$G^h_l v(z) = \sum_{k=1}^{m}[\beta_{z,l,k}(\partial_x v)_k + \alpha_{z,l,k}(\partial_y v)_k], \quad l = 1, 2, \tag{3.4}$$

where $(\partial_x v)_k$ and $(\partial_y v)_k$ are the derivatives of v on triangle T_k. Using equations (2.2) and (3.3) we have

$$\frac{1}{h_z}\boldsymbol{v}_z^T \sum_{k=1}^{m}[\beta_{z,l,k}E_k\boldsymbol{a}_k + \alpha_{z,l,k}E_k\boldsymbol{b}_k] = \frac{1}{h_z}\boldsymbol{v}_z^T A_z B_z^{-1} e_{l+1}, \quad l = 1, 2.$$

Setting

$$M_z = \begin{bmatrix} E_1\boldsymbol{a}_1 & \cdots & E_m\boldsymbol{a}_m & E_1\boldsymbol{b}_1 & \cdots & E_m\boldsymbol{b}_m \end{bmatrix} \tag{3.5}$$

and

$$\gamma_{z,l} = \begin{bmatrix} \alpha_{z,l,1} & \cdots & \alpha_{z,l,m} & \beta_{z,l,1} & \cdots & \beta_{z,l,m} \end{bmatrix}^T,$$

we get

$$\frac{1}{h_z} v_z^T M_z \gamma_{z,l} = \frac{1}{h_z} v_z^T A_z B_z^{-1} e_{l+1}, \qquad l = 1, 2.$$

This is true for all $v \in S_h$, and so

$$M_z \gamma_{z,l} = A_z B_z^{-1} e_{l+1}, \qquad l = 1, 2. \tag{3.6}$$

Note that the order of M_z is $(n+1) \times (2m)$.

Lemma 3.1 *Consider a mesh node $z \in \overline{\Omega}$. If the patch ω_z corresponding to z has no degenerate triangles and B_z is invertible, then* $\operatorname{Rank} M_z = n$, *and the system in* (3.6) *has infinitely many solutions.*

Proof. Since ω_z is simply connected, and using Euler's theorem, $(n+1) - e + m = 1$, where e is the number of edges in ω_z. Hence, $(n+1) - 2m = e - 3m + 1$. By a simple induction argument on m, we can show that $e - 3m + 1 < 0$ for $m \geq 3$. Hence, the system in (3.6) is underdetermined.

To prove that $\operatorname{Rank} M_z = n$, we consider the homogeneous linear system

$$M_z^T \boldsymbol{w} = 0, \tag{3.7}$$

with $\boldsymbol{w} = \begin{bmatrix} w_0 & w_1 & \cdots & w_n \end{bmatrix}^T$. We can view $w_0, w_1, \ldots, w_n$ as the nodal values of some function $w \in S_h$ at the nodes of ω_z. With this in mind, equations j and $j+m$ of the homogeneous system in (3.7) leads to $\nabla w = 0$ on T_k, for $k = 1, 2, \ldots, m$, and so w must be constant on ω_z. Since w is piecewise linear, the only solution to this homogeneous system is $w_0 = w_1 = \ldots = w_n$. Therefore, the dimension of the null space of M_z^T is 1, and $\operatorname{Rank} M_z^T = \operatorname{Rank} M_z = n$. Also, this implies that the only row operation on M_z that leads to a row of zeros is adding all the rows together. Since $\mathrm{G^h}$ recovers the exact gradient for any polynomial $p \in P_2(\omega_z)$, it is easy to verify that the sum of the rows of the column $A_z B_z^{-1} e_{l+1} = 0$ for $l = 1, 2$, and so the homogeneous system in (3.6) is consistent for $l = 1, 2$. □

Among all the solutions of (3.6), we consider the one with the minimum length given by

$$\gamma_{z,l}^* = M_z^\dagger A_z B_{z,l+1}^{-1}, \qquad l = 1, 2, \tag{3.8}$$

where $M_z^\dagger$ is the pseudoinverse of M_z. As before, let $\omega_T = \bigcup \{\omega_z : z \text{ is a vertex of } T\}$ be a patch corresponding to triangle $T \in \overline{\Omega}$.

Theorem 3.2 *Let $0 < C_1 \le \sigma_6(A_z) \le \sigma_1(A_z) \le C_2$ and $0 < C_3 \le \sigma_n(M_z)$ for every mesh node $z \in \overline{\Omega}$ and for some constants C_1, C_2 and C_3 that are independent of h. Then there exist a constant C, independent of h, such that*

$$\| \mathrm{G}^{\mathrm{h}} v\|_{L^2(T)} \le C|v|_{1,\omega_T}. \tag{3.9}$$

for all $T \subset \overline{\Omega}$, and for all $v \in S_h$

Proof. Consider a mesh triangle $T \subset \overline{\Omega}$, and let z be one of its vertices. Let ω_T be the patch corresponding to T, and let v be any function in S_h. Using equations (3.4) and (3.8) we get

$$\begin{aligned} |\mathrm{G}^{\mathrm{h}}_l v(z)| &\le \|\gamma_l^*\|_1 |v|_{1,\infty,\omega_z} \le c_1 \|\gamma_l^*\|_2 |v|_{1,\infty,\omega_T} \\ &\le c_2 \|M^\dagger\|_2 \|A\|_2 \|B^{-1}\|_2 |v|_{1,\infty,\omega_T} \\ &\le \frac{c_2 C_2}{C_3 C_1^2} |v|_{1,\infty,\omega_T} \end{aligned}$$

for $l = 1, 2$. By linearity of $\mathrm{G}^{\mathrm{h}} v$ on T, we have

$$\| \mathrm{G}^{\mathrm{h}} v\|_{L^\infty(T)} \le C|v|_{1,\infty,\omega_T}.$$

Hence,

$$\begin{aligned} \| \mathrm{G}^{\mathrm{h}} v\|_{L^2(T)} &\le \sqrt{|T|} \| \mathrm{G}^{\mathrm{h}} v\|_{L^\infty(T)} \le C \operatorname{diam}(T) |v|_{1,\infty,\omega_T} \\ &\le C \frac{\operatorname{diam}(T)}{\operatorname{diam}(\omega_T)} |v|_{1,\omega_T} \le C|v|_{1,\omega_T}. \end{aligned} \tag{3.10}$$

The first inequality in (3.10) is obtained using an inverse estimate. □

It is obvious that the bounds assumed about the singular values of A_z and M_z in Theorem 3.2 are mesh dependent. To simplify the situation, consider the unit disc

$$\overline{B(0,1)} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}.$$

Let $\mathcal{T}$ be a triangulation of $\overline{B(0,1)}$ that is similar to one of the patterns shown in Fig. 4. Let $z_0 = (0,0), z_1, \ldots, z_n$ be the nodes of $\mathcal{T}$ with at least one of them on the unit circle. Let $T_1, T_2, \ldots, T_n$ be the triangles of $\mathcal{T}$ with θ_m and θ_M being the smallest and the largest angles in any one of these triangles. As before, write $n = n_1 + n_2$ with n_1 denoting the number of nodes directly linked to z_0. If $n_1 = 4$, let D be the set of the diagonals of the quadrilateral $z_1 z_2 z_3 z_4$, and set $d = \min_{5 \le i \le n} \operatorname{dist}(z_i, D)$. Let A and M be the matrices defined for ω as in equations (2.1) and (3.5), respectively, and set $B = A^T A$.

Lemma 3.3 *Let $\mathcal{T}$ be any triangulation of $\overline{B(0,1)}$ that is similar to one of the patterns shown in Fig. 4. Assume that $\mathcal{T}$ satisfies the following conditions for some positive constants N, δ, ϕ_m, and ϕ_M:*

1. *ω satisfies the angle condition,*

2. *$n_1 < N < \infty$,*

3. *$0 < \delta \leq d$, and*

4. *$0 < \phi_m \leq \theta_m$, and $\theta_M \leq \phi_M < \pi$,*

Then, there exist constants C_1, C_2, and C_3, that depend only on N, δ, ϕ_m, and ϕ_M, such that

$$0 < C_1 \leq \sigma_6(A) \leq \sigma_1(A) \leq C_2, \quad \text{and} \quad 0 < C_3 \leq \sigma_n(M).$$

Proof. Note that the condition about d is to ensure that ω satisfies the line condition when $n_1 = 4$. We now show that $\sigma_1(A) \leq C_2$. Using the definition of A, it is easy to verify that $|B(i,j)| \leq (n+1)$ for $1 \leq i,j \leq 6$. Hence, $\sigma_1(A) = \sqrt{\sigma_1(B)} \leq \sqrt{\sigma_1(|B|)} \leq \sqrt{6(n+1)} \leq \sqrt{6(N+5)} = C_2$. To show that $0 < C_1 \leq \sigma_6(A)$ proceed by contradiction. Note that $\overline{B(0,1)}$ and $[\phi_m, \phi_M]$ are compact, and ω has at least one of its nodes on $\partial B(0,1)$. Based on that, a standard argument can be used to establish the existence of a patch ω that satisfies the given conditions and for which B is singular, i.e. $\sigma_6(A) = 0$. Since ω satisfies the angle and the line conditions, B is nonsingular by Theorem 2.6, and this is a contradiction. Similarly, we can show that $0 < C_3 \leq \sigma_n(M)$. Note that, under the given conditions, none of the given triangles in ω is degenerate and so Rank $M = n$ by Lemma 3.1, i.e. $\sigma_n(M)$ can not be zero. □

Theorem 3.4 *Let $\mathcal{T}_h$ be a triangulation of Ω that satisfy the following conditions for any h.*

1. *If n_1 is the number of nodes directly attached to a mesh node z, then $4 \leq n_1 \leq N$ for $z \in \Omega$, and $1 \leq n_1 \leq N$ for $z \in \partial\Omega$, where N is some finite positive integer.*

2. *If $n_1 = 4$ for a mesh node $z \in \Omega$, then the sum of any two adjacent angles at z is π.*

3. *If $n_1 > 4$ for a mesh node $z \in \Omega$, then the sum of any two adjacent angles at z is at most $\pi - \phi$ for some $\phi > 0$.*

4. *If $z \in \partial\Omega$, then the sum of any two adjacent angles at z is at most π.*

5. *If $\theta_{m,h}$ and $\theta_{M,h}$ are the smallest and largest angles in any mesh $T \in \mathcal{T}_h$, then*

$$0 < \phi_m \leq \theta_{m,h} \leq \theta_{M,h} \leq \phi_m < \pi$$

for some constants $0 < \phi_m \leq \phi_M < \pi$.

Then, there exist constants C_1, C_2, and C_3, that depend only on N, ϕ, ϕ_m, and ϕ_M, such that

$$0 < C_1 \leq \sigma_6(A_z) \leq \sigma_1(A_z) \leq C_2, \quad \textit{and} \quad 0 < C_3 \leq \sigma_n(M_z),$$

for all $z \in \Omega$

Proof. Let $z \in \Omega$, and let ω_z be its corresponding patch. Let $\omega = \dfrac{\omega_z - z}{h_z}$. The given conditions implies that ω satisfies the conditions in Lemma 3.3, and the theorem conclusion is true for all $z \in \Omega$.

Remark 3.5 Examining the assumptions in Theorem 3.4, we can see that the important ones are the those about the angles. The second assumption is easy to achieve during mesh generation. As was shown in [2], it is desirable to avoid having large angles, and so the assumptions from 3 to 5 seems to be practical.

Lemma 3.6 *Let $\mathcal{T}_h$ be a triangulation of Ω that satisfies the assumptions in Theorem 3.4 for any h. Also assume that for any mesh node $z \in \partial\Omega$ the following conditions are satisfied:*

1. *z is connected to at least one mesh node $\hat{z} \in \Omega$ directly or through a node $\tilde{z} \in \partial\Omega$, and*

2. *ω_z is constructed such that it contains a patch $\omega_{\hat{z}}$ corresponding to a mesh node $\hat{z} \in \Omega \cap \omega_{\hat{z}}$.*

Then, there exist constants C_1, C_2, and C_3, independent of h, such that

$$0 < C_1 \leq \sigma_6(A_z) \leq \sigma_1(A_z) \leq C_2, \quad \textit{and} \quad 0 < C_3 \leq \sigma_n(M_z),$$

for all $z \in \partial\Omega$.

Proof. Let $z \in \partial\Omega$, and let ω_z be its corresponding patch that satisfies the second assumption. Let $\omega = \dfrac{\omega_z - z}{h_z}$, and let $\omega_1 = \dfrac{\omega_{\hat{z}} - z}{h_z}$. By the first assumption, the number of nodes in ω is bounded by $1+N^2$ when z is directly attached to $\hat{z}$. So, if z is connected to $\hat{z} \in \Omega$ through $\tilde{z} \in \partial\Omega$ then, the number of nodes in ω is at most $1 + N(1 + N^2)$. Using an argument similar to that used in Lemma 3.3, it is easy to find the constant C_2. To prove the existence of C_1, proceed by contradiction. Again, based on the compactness of $\overline{B(0,1)}$ and $[\phi_m, \phi_M]$, a standard argument can be used to establish the existence of a patch ω for which B is singular, and $\omega \supset \omega_1$ with ω_1 satisfying the angle and the line conditions, by the assumptions in Theorem 3.4. we can write $A = [A_1^T A_2^T]^T$, where A_1 corresponds to ω_1. Then, $B = A^T A = A_1^T A_1 + A_2^T A_2 = B_1 + B_2$, where $B_1 = A_1^T A_1$ and $B_2 = A_2^T A_2$. Note that B is positive semidefinite, as B is singular, and B_2 is at least positive semidefinite. So, we can find a vector $c \in \mathbb{R}^6$ such that $c^T Bc = c^T B_1 c + c^T B_2 c = 0$, and, as a result, $c^T B_1 c = 0$. This means that B_1 is singular, and this is a contradiction as ω_1 satisfies the angle and the line conditions. Similarly, we can establish the existence of C_3. □

Remark 3.7 The first assumption in Lemma 3.6 can be relaxed, and in this case C_2 might get lager as the number of nodes in patches corresponding to nodes on $\partial\Omega$ will increase, and C_1 might get smaller as the $h_{\hat{z}}/h_z$ will decrease.

From Theorem 3.4 and Lemma 3.6, we have the following corollary.

Corollary 3.8 *Let $\mathcal{T}_h$ be a triangulation of Ω that satisfies the conditions in Theorem 3.4 for any h, then there exist constants C_1, C_2, and C_3, independent of h, such that*

$$0 < C_1 \leq \sigma_6(A_z) \leq \sigma_1(A_z) \leq C_2, \quad \textit{and} \quad 0 < C_3 \leq \sigma_n(M_z),$$

for all $z \in \overline{\Omega}$.

4 Superconvergence Property of PPR-Recovered Gradient

We begin with the following main theorem.

Theorem 4.1 *Let $\mathcal{T}_h$ be a triangulation of Ω that satisfies the condition (α, σ), and the assumptions in both of Theorem 3.4 and Lemma 3.6. If $u \in W^3_\infty(\Omega)$, then*

$$\|\nabla u - \mathrm{G^h} u_h\|_{L^2(\Omega)} \leq Ch^{1+\rho}\|u\|_{3,\infty,\Omega},$$

where $\rho = \min(\alpha, \frac{1}{2}, \frac{\sigma}{2})$.

Proof. Under the given assumptions, we have

$$\|\,\mathrm{G^h} v\|_{L^2(T)} < C_1\|\nabla v\|_{L^2(\omega_T)} \text{ for all } v \in S_h \text{ and } T \in \mathcal{T}_h \tag{4.1}$$

for some C_1 that is independent of h, and

$$\|\nabla(I_h u - u_h)\|_{L^2(\Omega)} \leq C_2 h^{1+\rho}\|u\|_{3,\infty,\Omega}, \tag{4.2}$$

where $\rho = \min(\alpha, \frac{1}{2}, \frac{\sigma}{2})$. We write

$$\nabla u - \mathrm{G^h} u_h = (\nabla u - \mathrm{G^h}(I_h u)) + (\mathrm{G^h}(I_h u - u_h)) \tag{4.3}$$

As it was shown in [11],

$$\|\nabla u - \mathrm{G^h}(I_h u)\|_{L^\infty(\Omega)} \leq Ch^2 |u|_{3,\infty,\Omega}. \tag{4.4}$$

Hence,

$$\|\nabla u - \mathrm{G^h}(I_h u)\|_{L^2(\Omega)} \leq Ch^2 \sqrt{|\Omega|} \|u\|_{3,\infty,\Omega}. \tag{4.5}$$

For the second part in equation (4.3), and by virtue of (4.1), we have

$$\begin{aligned}
\| \mathrm{G^h}(I_h u - u_h)\|^2_{L^2(\Omega)} &= \sum_{T \in \mathcal{T}_h} \| \mathrm{G^h}(I_h u - u_h)\|^2_{L^2(T)} \\
&\leq \sum_{T \in \mathcal{T}_h} C_1^2 \|\nabla (I_h u - u_h)\|^2_{L^2(\omega_T)} \\
&\leq C \|\nabla (I_h u - u_h)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Consequently, and by using (4.2),

$$\| \mathrm{G^h}(I_h u - u_h)\|_{L^2(\Omega)} \leq Ch^{1+\rho} \|u\|_{3,\infty,\Omega}. \tag{4.6}$$

Using (4.5) and (4.6) in (4.3), we have the required result. □

Remark 4.2 We should mention that the conclusion of Theorem 4.1 is true under any conditions that guarantee the results in (4.1) and (4.2). In other words, $\mathrm{G^h}$, provided that it is bounded, can sense any superconvergence in $\nabla(I_h u - u_h)$, and produces a superconvergent recovered gradient.

Consider the global a posteriori error estimator η_h defined by

$$\eta_h = \| \mathrm{G^h} u_h - \nabla u_h \|_{L^2(\Omega)}.$$

Under the assumptions in Theorem 4.1, it is easy to prove that η_h is asymptotically exact, as shown in the following corollary.

Corollary 4.3 *If, in addition to the assumptions in Theorem 4.1,*

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \geq c(u) h, \tag{4.7}$$

then

$$\left| \frac{\eta_h}{\|\nabla(u - u_h)\|_{L^2(\Omega)}} - 1 \right| \leq Ch^\rho.$$

Proof. By Theorem 4.1, and the assumption in (4.7), we have

$$\left| \frac{\eta_h}{\|\nabla(u - u_h)\|_{L^2(\Omega)}} - 1 \right| \leq \frac{\| \mathrm{G^h} u_h - \nabla u_h\|_{L^2(\Omega)}}{\|\nabla(u - u_h)\|_{L^2(\Omega)}} \leq \frac{Ch^{1+\rho}\|u\|_{3,\infty,\Omega}}{c(u) h} = Ch^\rho.$$ □

5 Numerical Results

In this section we will go over some numerical results that demonstrate the superconvergence property of G^h and the asymptotic exactness of the a posteriori error estimator based on it. Also, a comparison is held between SPR and PPR with special attention paid to regions near the boundary. The results are presented through two examples in which we consider the model problem

$$-\triangle u = f \text{ in } \Omega, \text{ and } u = g \text{ on } \partial\Omega.$$

Let $\mathcal{T}_h$ be a triangular partition of Ω, and let $\mathcal{N}_h$ be the set of the mesh nodes in $\overline{\Omega}$. It is known that the recovery operators loose their good properties in regions near $\partial\Omega$. For that, we write $\mathcal{N}_h = \mathcal{N}_{h,1} \cup \mathcal{N}_{h,2}$, where

$$\mathcal{N}_{h,1} = \{z \in \mathcal{N}_h : \text{dist}(z, \partial\Omega) \geq H\},$$

and H is some fixed positive constant. Based on that, we write $\overline{\Omega} = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \bigcup\{T \in \mathcal{T}_h : T \text{ has all of its vertices in } \mathcal{N}_{h,1}\}.$$

Let $\mathcal{A} \in \overline{\Omega}$ be the union of a set of mesh triangles in $\mathcal{T}_h$. The a posterior error estimator in $\mathcal{A}$ is $\eta_{h,\mathcal{A}} = \|R_h u_h - \nabla u_h\|_{L^2(\mathcal{A})}$, where R_h denotes the recovery operator associated with SPR or PPR. To measure the accuracy of $\eta_{h,\mathcal{A}}$, we use the effectivity index $\theta_{h,\mathcal{A}}$ defined by

$$\theta_{h,\mathcal{A}} = \frac{\eta_{h,\mathcal{A}}}{\|\nabla(u - u_h)\|_{L^2(\mathcal{A})}}.$$

One way to locally study the accuracy of the a posteriori error estimator in $\mathcal{A}$, is to use the mean, $\mu_{h,\mathcal{A}}$, and the standard deviation, $\sigma_{h,\mathcal{A}}$, of the effectivity indices defined for the mesh triangles in $\mathcal{A}$ and see how they change with h. If the estimator is locally exact in asymptotic sense, then $\mu_{h,\mathcal{A}} \to 1$ and $\sigma_{h,\mathcal{A}} \to 0$ as $h \to 0$. Of course

$$\mu_{h,\mathcal{A}} = \frac{1}{N_{h,\mathcal{A}}} \sum_{T \subset \mathcal{A}} \theta_{h,T},$$

and

$$\sigma^2_{h,\mathcal{A}} = \frac{1}{N_{h,\mathcal{A}}} \sum_{T \subset \mathcal{A}} (\theta_{h,T} - \mu_{h,\mathcal{A}})^2,$$

where $N_{h,\mathcal{A}}$ is the number of mesh triangles in $\mathcal{A}$.

Example 1. In this example $\Omega = (0,1)^2$, the solution is $u = \sin(\pi x)\sin(\pi y)$, and H is taken to be $1/8$. For mesh generation we consider two cases.

In the first case, the successive meshes are obtained by decomposing the unit square into $N \times N$ equal squares and then divide every square into two triangles such that the triangles are arranged in Chevron pattern. This is done for $N = 16, 32$, and 64. Before we go over the results for this case, we should mention that Theorem 1.4 is not applicable and that G^h is bounded. As shown in Fig. 5, we can see that $\nabla(I_h u - u_h)$ has superconvergence that enables G^h to produce superconvergent recovered gradient, as mentioned in Remark 4.2. This is not the case with SPR, as it does not preserve polynomials of order 2; a property that is crucial in proving a result similar to the one in (4.4) for the operator associated with SPR. Consequently, the behavior of the a posteriori error estimator based on SPR is inferior to that based on PPR, as shown in Fig. 6. We can see that the error estimator based on SPR is over estimating the actual error. Also, the statistics depicted in this figure shows how fast $\mu_{h,\mathcal{A}} \to 1$ and $\sigma_{h,\mathcal{A}} \to 0$ when PPR is used.

In the second case, we start with an initial mesh generated by Delauny triangulation at $h = 0.1$, and in successive iterations, the new mesh is obtained from the old one by regular refinement. The results are shown in Fig. 7 and Fig. 8, where we can note two things. First, although PPR and SPR have almost the same global behavior in Ω_1, the statistics shows that PPR is slightly better when we consider the local behavior. Secondly, the global and local properties of PPR is much better when it comes to Ω_2.

Example 2. In this example $\Omega = (-1,1)^2 \setminus [1/2,1)^2$. Using a polar coordinate system at (1/2,1/2), the solution is taken to be $u = r^{\frac{1}{3}} \sin\left(\frac{2\theta - \pi}{3}\right)$. As before, H is 1/8, and we start with an initial mesh generated with Delauny triangulation at $h = 0.2$, and in successive iterations, the mesh is regularly refined. The numerical results for this example are shown in Fig. 9 and Fig. 10. As we know ∇u is singular at the reentrant corner (1/2,1/2). To reduce the pollution effect due to this singularity, the region within 0.1 from (1/2,1/2) is refined more than the rest of the domain in the initial mesh. Also, we expect both of PPR and SPR to behave badly near this point. Of course, this will affect the convergence rates for the recovered gradients, especially in Ω_1, but still PPR yields some what better results. Considering the local properties, we can see that PPR is still doing better than SPR.

In conclusion, under mild conditions, we have shown that G^h is bounded in the sense of (1.7). As a result, G^h can detect any superconvergence in $\nabla(I_h u - u_h)$, and reflects it in the recovered gradient. Consequently, the a posteriori error estimator based on it is asymptotically exact, at least globally. The examples indicate that PPR is, at least, as good as SPR inside the domain, while, near the boundary, PPR seems to be superior.

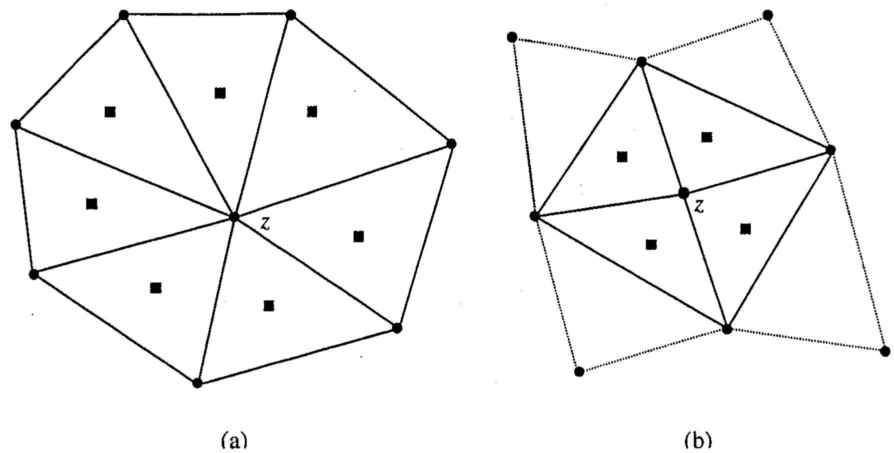

Figure 1: Patch required for gradient recovery. Sampling points for SPR are marked with ■, while those needed for PPR are marked with •

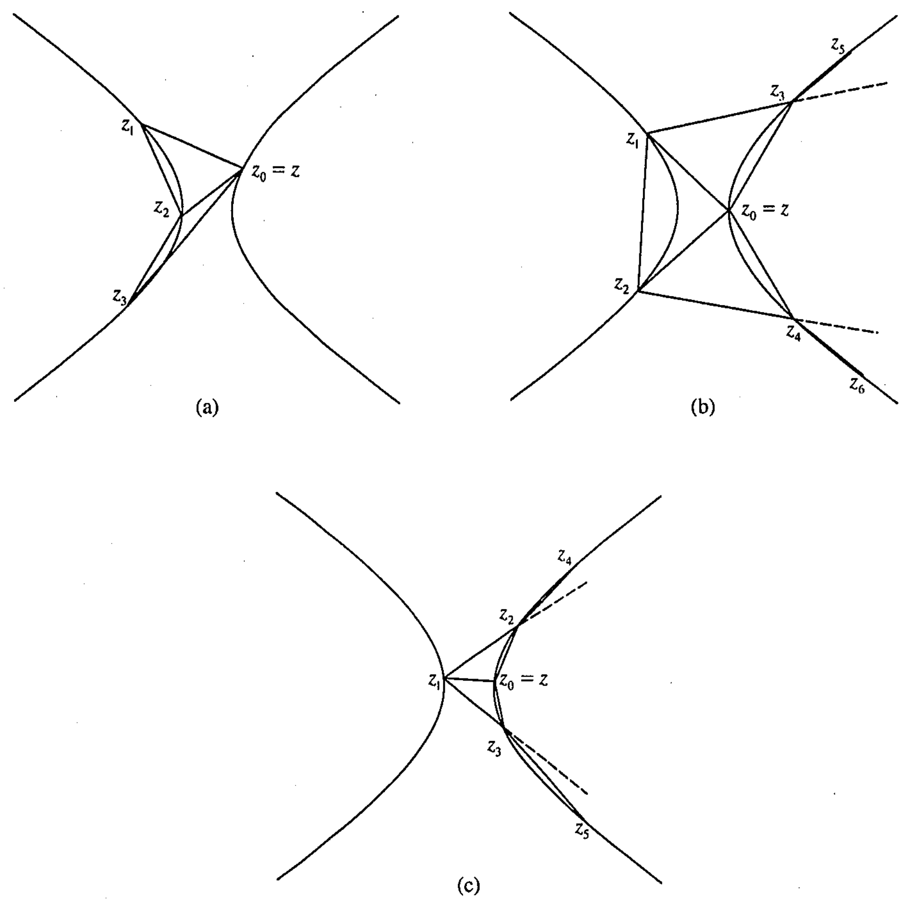

Figure 2: Nodes in ω_z can not be distributed on two branches of a hyperbola when ω_z satisfies the angle condition.

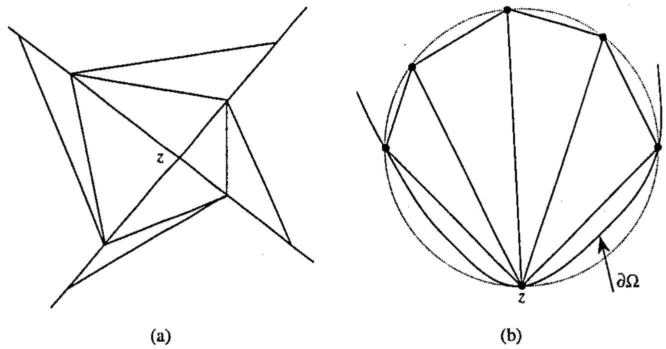

Figure 3: Patches that lead to singular B_z

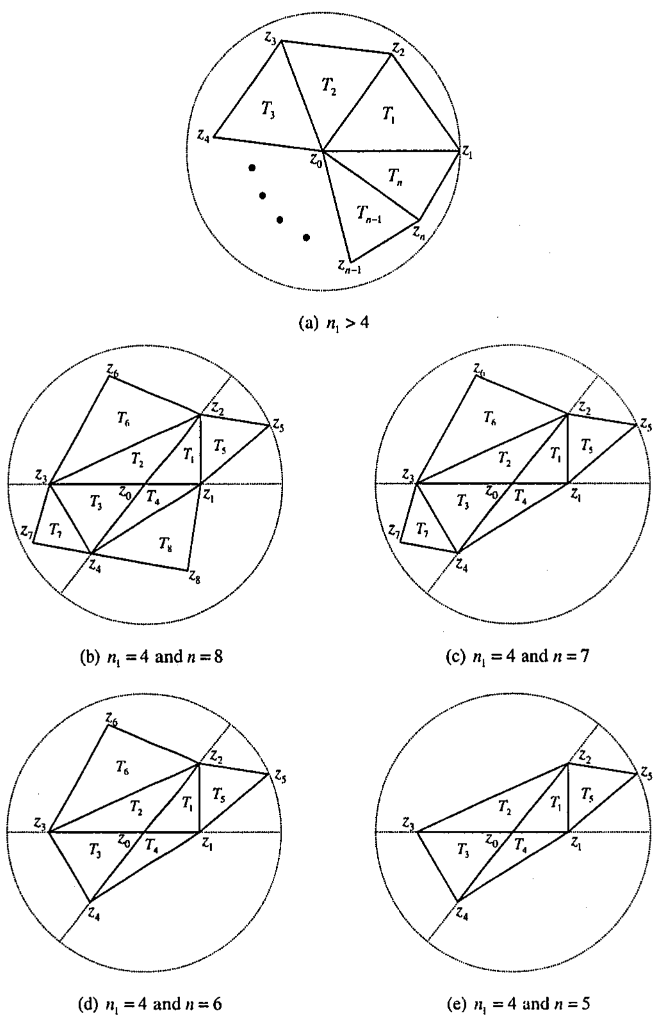

Figure 4: Triangulations for of $\overline{B(0,1)}$ for $n_1 \geq 4$.

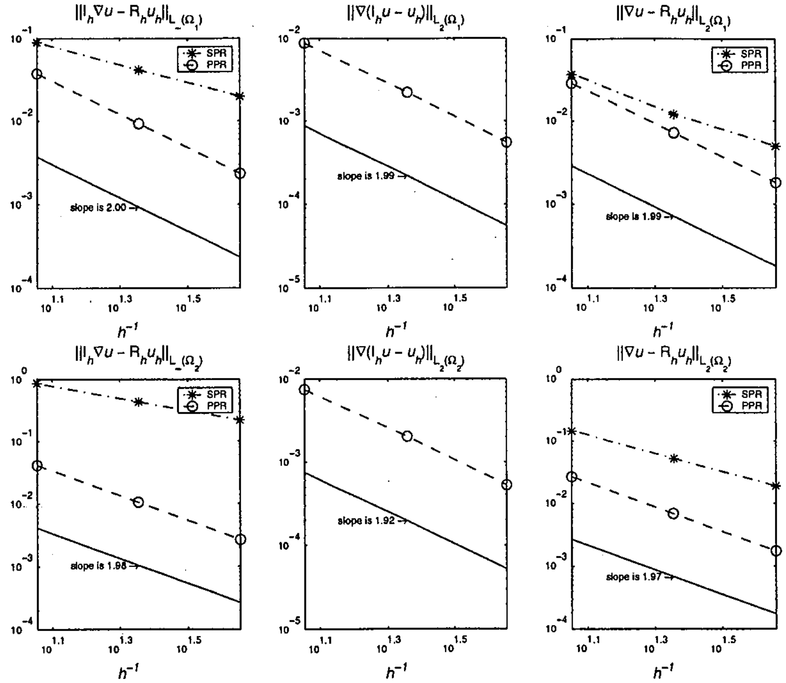

Figure 5: Convergence rates for $R_h u_h$ - Example 1 (Chevron mesh).

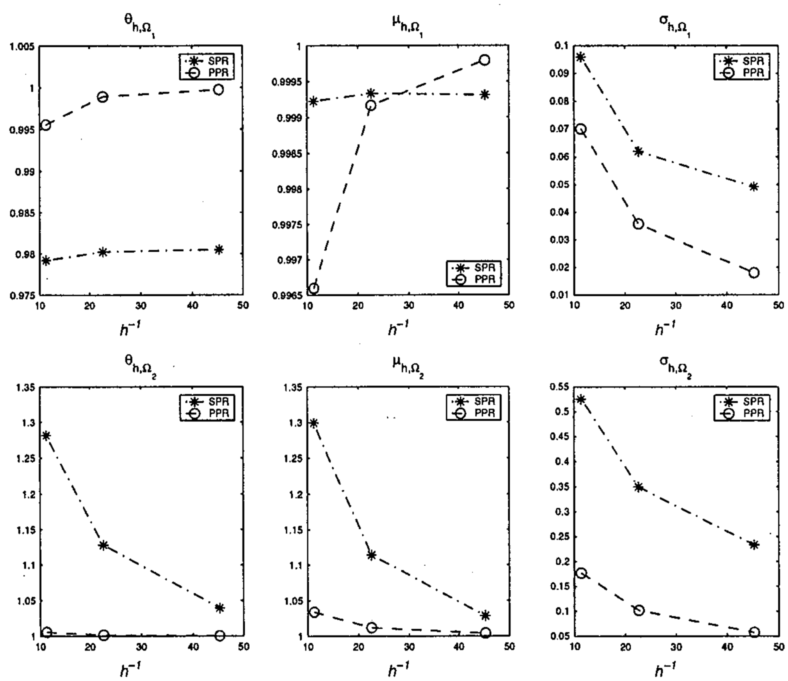

Figure 6: Properties of θ_h - Example 1 (Chevron mesh).

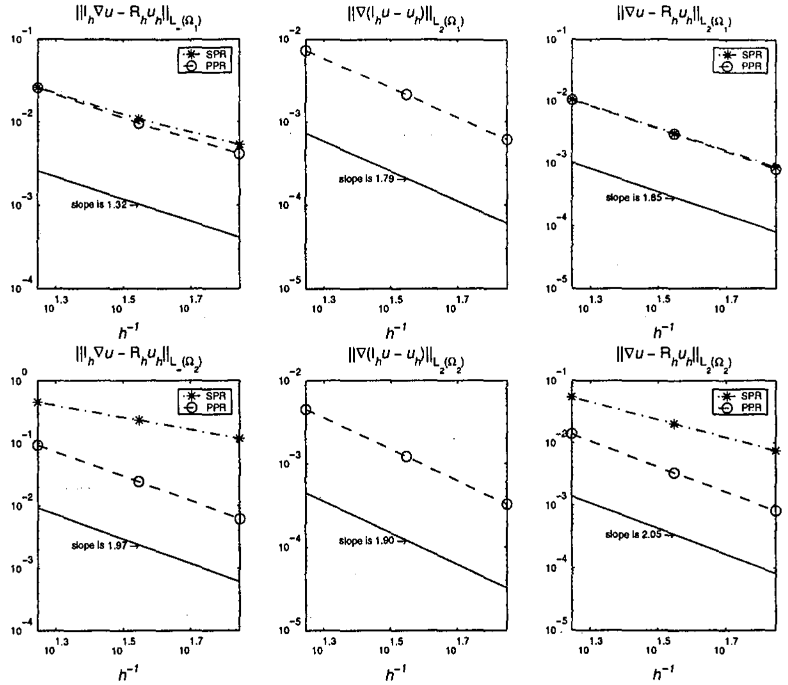

Figure 7: Convergence rates for $R_h u_h$ - Example 1 (Delauny triangulation).

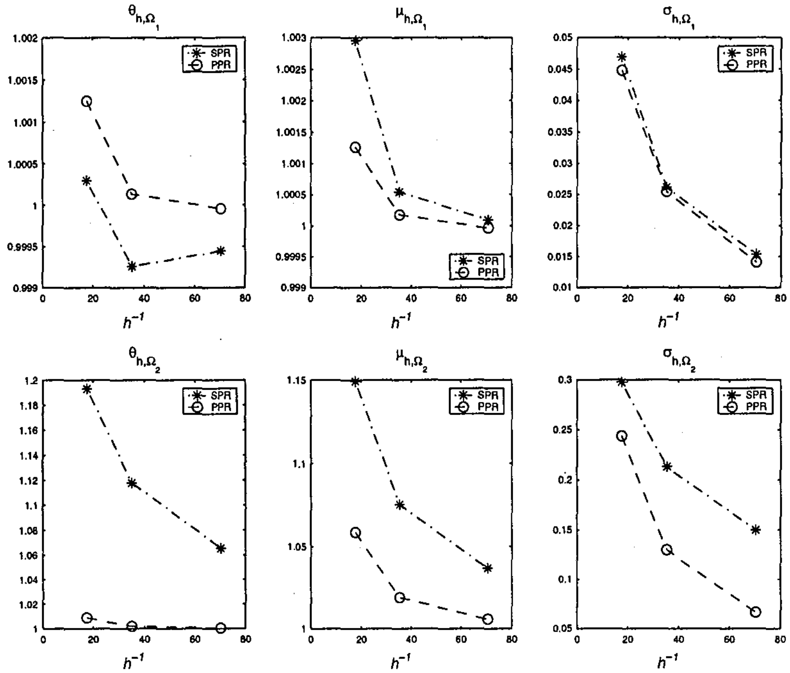

Figure 8: Properties of θ_h - Example 1 (Delauny triangulation).

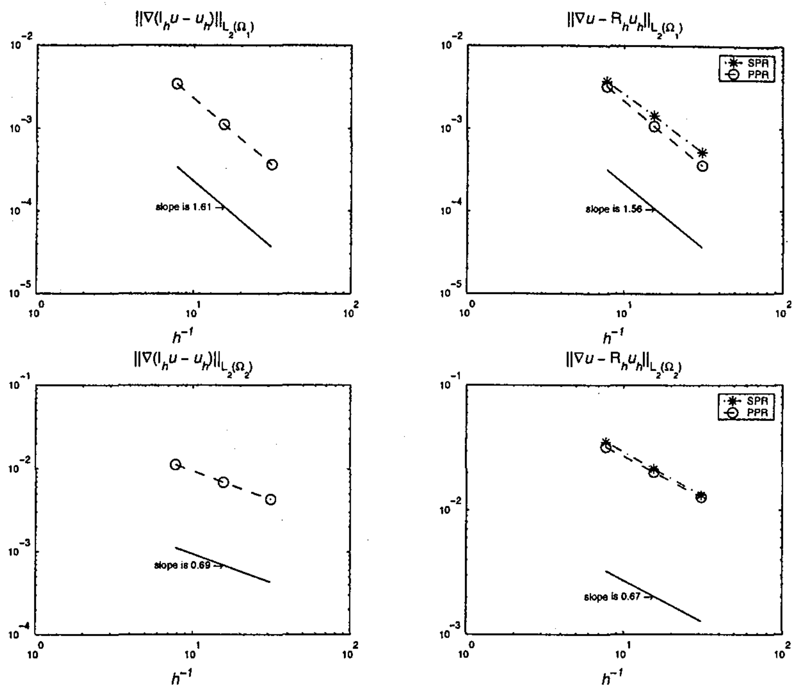

Figure 9: Convergence rates for $R_h u_h$ - Example 2.

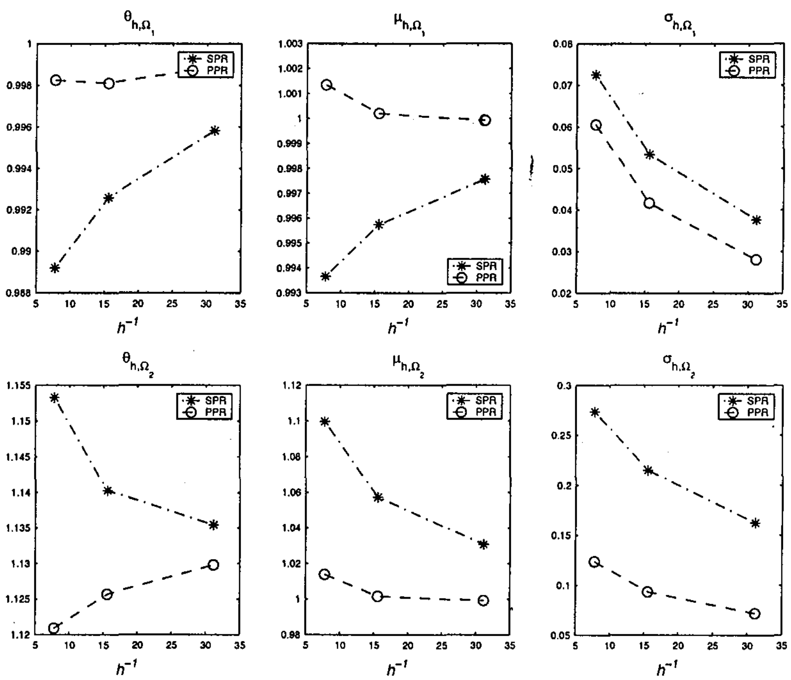

Figure 10: Properties of θ_h - Example 2.

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