A POSTERIORI ERROR ESTIMATES FOR MAXWELL EQUATIONS

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Abstract. Maxwell equations are posed as variational boundary value problems in the function space $H(\text{curl})$ and are discretized by Nédélec finite elements. In [4], a residual type a posteriori error estimator was proposed and analyzed under certain conditions onto the domain. In the present paper, we prove the reliability of that error estimator on Lipschitz domains. The key is to establish new error estimates for the commuting quasi-interpolation operators introduced recently in [25]. Similar estimates are required for additive Schwarz preconditioning. To incorporate boundary conditions, we establish a new extension result.

1. Introduction

Maxwell equations are partial differential equations describing electromagnetic phenomena. In comparison to other fields, their numerical treatment by finite element methods is relatively new. A reason is that they require the vector valued function space $H(\text{curl})$, what has many consequences for the whole numerical analysis. A recent monograph is [18].

The key for the numerical analysis for Maxwell equations is most often the de Rham complex [8]. It is the basis for the construction of finite elements [19, 20, 31, 14, 1, 26, 32] and the a priori error estimates, preconditioners [16, 3, 28, 22], and eigenvalue problems [6, 7].

The principle of energy-based a posteriori error estimators [30, 2] is the localization of error contributions. For the residual error estimator, the Clément operator is applied to subtract a global function. By a partition of unity method, the rest can be split into local functions. The same concept is needed for two-level domain decomposition methods.

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After subtracting a coarse grid function, the remainder can be split into local functions on overlapping sub-domains [29].

Residual based a posteriori error estimators for Maxwell equations were introduced in [4]. In [17], scattering problems were treated. In these papers, proper element and inter-element jump terms have been derived. In [21, 12] the heterogeneous Maxwell equation is addressed. An alternative are hierarchical error estimators [5], or equilibrated residual error estimators [9]. In the present paper, we prove the reliability of residual error estimators on Lipschitz domains. The key is to establish new error estimates for the commuting quasi-interpolation operators introduced recently in [25]. These operators are no projectors. In [27], the operators have been modified to obtain the projection property as well.

Notation: We write $a \preceq b$, when $a \leq cb$, where c is a constant independent of a, b, the coefficients ν and κ of the equation, and the mesh-size h. The constant may depend on the shape of the finite elements. We write $a \succeq b$ for $b \preceq a$, and we write $a \simeq$ for $a \preceq b$ and $b \preceq a$.

The rest of the paper is organized as follows. In Section 2, the variational problem, the error estimator and the main theorem is presented. The commuting quasi-interpolation operators are defined in Section 3, and the new approximation properties are proven in Section 4. Necessary extension results for $H(\text{curl})$ and $H(\text{div})$ are proven in Appendix A.

2. The residual error estimator

Let Ω be a bounded, polyhedral Lipschitz domain in $\mathbb{R}^3$. Its boundary $\Gamma = \partial\Omega$ is decomposed into the Dirichlet part Γ_D and the Neumann part Γ_N. As usual, define the space $H(\text{curl}, \omega) = \{v \in [L_2(\omega)]^3 : \text{curl}\, v \in [L_2(\omega)]^3\}$ for some domain ω, and write $H(\text{curl})$ for $\omega = \Omega$. Let $V := H_D(\text{curl}) := \{v \in H(\text{curl}) : v_t = 0 \text{ on } \Gamma_D\}$. Similarly, we define $H^1_D = \{v \in H^1 : v = 0 \text{ on } \Gamma_D\}$. We write v_t and v_n for the tangential and normal traces, respectively.

Several formulations of Maxwell equations lead to the variational problem: find $u \in V$ such that

$$A(u, v) = f(v) \qquad \forall\, v \in V \tag{1}$$

with the bilinear-form

$$A(u, v) := \int_\Omega \nu(x)\; \text{curl}\, u\; \text{curl}\, v\, dx + \int_\Omega \kappa(x)\, uv\, dx$$

and the linear form $f(.)$ defined as

$$f(v) := \int_\Omega jv\,dx.$$

The coefficients $\nu(x)$ and $\kappa(x)$ are modified material parameters. In time-stepping methods, $\kappa(x)$ includes the time step Δt, while in time harmonic formulations, the equation becomes complex-valued with $\kappa(x) = i\omega\sigma - \omega^2\varepsilon$, where σ and ε are positive coefficient functions. We assume that the bilinear-form $A(.,.)$ is continuous and $\inf-\sup$ stable with respect to the norm

$$\|v\|_V^2 := \nu \,\|\operatorname{curl} v\|_{L_2}^2 + \kappa\,\|v\|_{L_2}^2,$$

where ν and κ are positive constants. The given current density $j \in [L_2]^3$ satisfies $\operatorname{div} j = 0$ and $j_n = 0$.

Let the domain Ω be covered with a shape regular triangulation. We define

$$\begin{aligned}
\text{the set of vertices} \qquad & \mathcal{V} = \{V_i\}, \\
\text{the set of edges} \qquad & \mathcal{E} = \{E = [V_{E_1}, V_{E_2}]\}, \\
\text{the set of faces} \qquad & \mathcal{F} = \{F = [V_{F_1}, V_{F_2}, V_{F_3}]\}, \\
\text{the set of tetrahedra} \qquad & \mathcal{T} = \{T = [V_{T_1}, V_{T_2}, V_{T_3}, V_{T_4}]\}.
\end{aligned}$$

For each edge E we define a unique tangential vector t_E, and for each face F we define a unique normal vector n_F. For each edge E, face F, and element T the local mesh-size h_E, h_F, and h_T is defined by the diameter, and for a vertex V the mesh-size h_V is defined as $\max_{T:V\in T} h_T$. Note that all geometric entities are closed sets. We need several domains associated with the entities of the mesh. First, define the small patches associated with vertices, edges and faces as

$$\Omega_V = \bigcup_{T:V\in T} T, \qquad \Omega_E = \bigcup_{T:E\subset T} T, \qquad \Omega_F = \bigcup_{T:F\subset T} T,$$

see Figure 1. We will need the influence domains of the interpolation operators. For this, let $\omega_V \subset \Omega_V$ be a domain with three dimensional measure $|\omega_V| \simeq h_V^3$. It can be a ball with center V, and a radius proportional to the local mesh-size. We assume that $\operatorname{dist}\{\omega_{V_i}, \omega_{V_j}\} \succeq |V_i - V_j|$. Furthermore, let

$$\omega_E = [\omega_{E_1}, \omega_{E_2}], \qquad \omega_F = [\omega_{F_1}, \omega_{F_2}, \omega_{F_3}], \qquad \omega_T = [\omega_{T_1}, \omega_{T_2}, \omega_{T_3}, \omega_{T_4}]$$

be the convex hulls of the domains associated with the vertices of the edge E, the face F, and the element T, see Figure 2. We assume that $\omega_E \subset \cup_{V\in E}\Omega_V$, $\omega_F \subset \cup_{V\in F}\Omega_V$, and $\omega_T \subset \cup_{V\in T}\Omega_V$. Note that we write

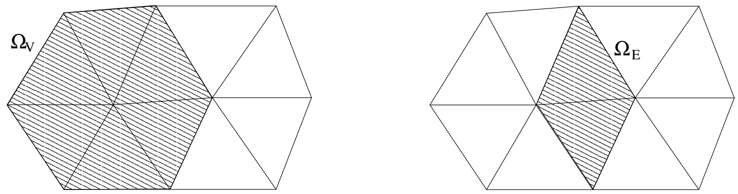

FIGURE 1. Element patches Ω_V and Ω_E

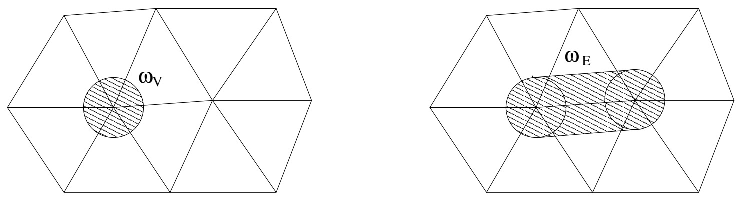

FIGURE 2. Domains ω_V and ω_E

ω_i as abbreviation for ω_{V_i} to avoid more levels of subscripts. Finally, we define the domains

$$\widetilde{\Omega}_V = \bigcup_{V' \in \Omega_V} \Omega_{V'} \qquad \text{and} \qquad \widetilde{\Omega}_T = \bigcup_{V' \in T} \widetilde{\Omega}_{V'}$$

containing the neighbor elements of neighbor elements of a vertex V and an element T, respectively.

Nédélec [19, 20] finite elements are the natural choice for the approximation of equation (1). For example, the k^{th} order element of the first family of Nédélec elements generates the space

$$\mathcal{N}_h^k = \{ v \in V : v|_T = a_T + b_T \times x \text{ with } a_T, b_T \in [P^k(T)]^3 \}.$$

The lowest order element ($k = 0$) of this family is the popular edge element. We assume that the finite element space $V_h \subset V$ contains the lowest order Nédélec space $\mathcal{N}_h^0$. The finite element approximation to (1) is to find $u_h \in V_h$ such that

$$A(u_h, v_h) = f(v_h) \qquad \forall\, v_h \in V_h.$$

The goal is to derive computable a posteriori error estimators $\eta(u_h, j)$ for the error $\|u - u_h\|_V$. In [4], a residual error estimator was derived. As usual, it contains element residuals and jump terms on faces:

$$\begin{aligned} \eta_T^2(u_h, j) \;:=\; & \frac{h_T^2}{\nu} \| \operatorname{curl} \nu \operatorname{curl} u_h + \kappa u_h - j \|_{L_2(T)}^2 + \frac{h_T^2}{\kappa} \| \operatorname{div} \kappa u_h \|_{L_2(T)}^2 + \\ & \sum_{F \subset T} \Big\{ \frac{h_F}{\nu} \| [\nu \operatorname{curl} u_h]_t \|_{L_2(F)}^2 + \frac{h_F}{\kappa} \| [\kappa u_h]_n \|_{L_2(F)}^2 \Big\}. \end{aligned}$$

In [4], the efficiency estimate of the error estimator was proven:

$$\|u - u_h\|_V + h.o.t.(j) \succeq \eta(u_h, j)$$

The reliability estimate

$$\|u - u_h\|_V \preceq \eta(u_h, j)$$

was proven under the assumption of an H^1-regular Helmholtz decomposition. This assumption is satisfied for convex or smooth domains, but does not hold true for general Lipschitz domains. The main result of this paper is to prove the reliability estimate for problems on Lipschitz domains. In [25], a Clément-type quasi-interpolation operator was introduced, and a priori estimates were proven. Now, we prove a new approximation error estimate needed for the a posteriori error analysis:

Theorem 1. *There exists an operator $\Pi_h : H_D(\mathrm{curl}) \to \mathcal{N}_h^0$ with the following properties: For every $u \in H_D(\mathrm{curl})$ there exists $\varphi \in H_D^1$ and $z \in [H_D^1]^3$ such that*

$$u - \Pi_h u = \nabla\varphi + z. \tag{2}$$

The decomposition satisfies

$$\begin{aligned} h_T^{-1}\|\varphi\|_{L_2(T)} + \|\nabla\varphi\|_{L_2(T)} &\leq c\,\|u\|_{L_2(\widetilde{\Omega}_T)}, \\ h_T^{-1}\|z\|_{L_2(T)} + \|\nabla z\|_{L_2(T)} &\leq c\,\|\operatorname{curl} u\|_{L_2(\widetilde{\Omega}_T)}. \end{aligned}$$

The constant c depends only on the shape of the elements in the enlarged element patch $\widetilde{\Omega}_T$, but does not depend on the global shape of the domain Ω, or the size of the patch $\widetilde{\Omega}_T$.

The proof of the theorem is postponed to Section 4. We note that ∇z is the matrix $\left(\frac{\partial z_i}{\partial x_j}\right)_{i,j=1,\ldots,n}$.

Corollary 2. *The residual error estimator is reliable.*

Proof. The proof is standard for residual error estimators. The inf − sup stability of $A(.,.)$ and Galerkin orthogonality implies

$$\|u - u_h\|_V \preceq \sup_{v\in V} \frac{A(u - u_h, v)}{\|v\|_V} = \sup_{v\in V} \frac{f(v - \Pi_h v) - A(u_h, v - \Pi_h v)}{\|v\|_V}.$$

We apply Theorem 1 to decompose $v - \Pi_h v = \nabla\varphi + z$ satisfying the corresponding norm estimates, and bound

$$
\begin{aligned}
& f(v - \Pi_h v) - A(u_h, v - \Pi_h v) \\
= \quad & \int_\Omega j(\nabla\varphi + z) - \int_\Omega \nu \text{ curl } u_h \text{ curl}\, z - \int_\Omega \kappa u_h(\nabla\varphi + z)\, dx \\
= \quad & \sum_{T\in\mathcal{T}} \int_T (j - \operatorname{curl} \nu \operatorname{curl} u_h - \kappa u_h) z\, dx + \sum_{T\in\mathcal{T}} \int_T \operatorname{div} \kappa u_h \varphi\, dx \\
& + \sum_{F\in\mathcal{F}} \int_F [\nu \operatorname{curl} u_h]_t z_t\, ds + \sum_{F\in\mathcal{F}} \int_F [\kappa u_h]_n \varphi\, ds \\
\leq \quad & \sum_{T\in\mathcal{T}} \frac{h_T}{\sqrt{\nu}} \| j - \operatorname{curl} \nu \operatorname{curl} u_h - \kappa u_h \|_{L_2(T)} \frac{\sqrt{\nu}}{h_T} \|z\|_{L_2(T)} \\
& + \sum_{T\in\mathcal{T}} \frac{h_T}{\sqrt{\kappa}} \| \operatorname{div} \kappa u_h \|_{L_2(T)} \frac{\sqrt{\kappa}}{h_T}\, \|\varphi\|_{L_2(T)} \\
& + \sum_{F\in\mathcal{F}} \sqrt{\frac{h_F}{\nu}}\, \| [\nu \operatorname{curl} u_h]_t \|_{L_2(F)} \sqrt{\frac{\nu}{h_F}} \|z\|_{L_2(F)} \\
& + \sum_{F\in\mathcal{F}} \sqrt{\frac{h_F}{\kappa}} \| [\kappa u_h]_n \|_{L_2(F)} \sqrt{\frac{\kappa}{h_F}} \|\varphi\|_{L_2(F)} \\
\leq \quad & \eta(u_h, j)\, \left(\nu \| \operatorname{curl} v\|_{L_2}^2 + \kappa \|v\|_{L_2}^2 \right)^{1/2}.
\end{aligned}
$$

In the last step, we have used the trace theorem $\frac{1}{h_F}\|z\|^2_{L_2(F)} \preceq \frac{1}{h_F^2}\|z\|^2_{L_2(T)} + \|\nabla z\|^2_{L_2(T)}$, where T is an element containing the face F. □

3. Commuting quasi-interpolation operators

To study interpolation operators in $H(\text{curl})$ it is of advantage to consider the whole sequence of spaces H^1, $H(\text{curl})$, $H(\text{div})$ and L_2. The corresponding lowest order finite elements are continuous and piecewise linear elements $\mathcal{L}_h^1$ with the vertex basis $\{\varphi_V\}$ for H^1, the Nédélec elements $\mathcal{N}_h^0$ with the edge basis $\{\varphi_E\}$ for $H(\text{curl})$, the Raviart Thomas elements $\mathcal{RT}_h^0$ with the face basis $\{\varphi_F\}$ in $H(\text{div})$, and piece-wise constant elements $\mathcal{S}_h^0$ with the element basis $\{\varphi_T\}$ for L_2. The basis functions are chosen biorthogonal to the canonical degrees of freedom, i.e., $\varphi_{V_j}(V_i) = \delta_{i,j}$, $\int_{E_i} \varphi_{E_j} \cdot t_i\, ds = \delta_{i,j}$, $\int_{F_i} \varphi_{F_j} \cdot n_i\, ds = \delta_{i,j}$, and $\int_{T_i} \varphi_{T_j}\, dx = \delta_{i,j}$.

In [25], quasi-interpolation operators for all these spaces were constructed which satisfy the commuting diagram properties

$$\nabla \Pi_h^V = \Pi_h^E \nabla, \qquad \operatorname{curl} \Pi_h^E = \Pi_h^F \operatorname{curl}, \qquad \operatorname{div} \Pi_h^F = \Pi_h^T \operatorname{div},$$

which are visualized in the de Rham complex as

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & H(\operatorname{curl}) & \xrightarrow{\operatorname{curl}} & H(\operatorname{div}) & \xrightarrow{\operatorname{div}} & L^2 \\ \big\downarrow \Pi_h^V & & \big\downarrow \Pi_h^E & & \big\downarrow \Pi_h^F & & \big\downarrow \Pi_h^T \\ \mathcal{L}_h^1 & \xrightarrow{\nabla} & \mathcal{N}_h^0 & \xrightarrow{\operatorname{curl}} & \mathcal{RT}_h^0 & \xrightarrow{\operatorname{div}} & \mathcal{S}_h^0 . \end{array} \tag{3}$$

For smooth functions, classical nodal interpolation operators can be applied. These are defined as

$$\begin{aligned} (I_h^V w)(x) &:= \sum_{V \in \mathcal{V}} w(V)\ \varphi_V(x), \\ (I_h^E v)(x) &:= \sum_{E \in \mathcal{E}} \int_E v \cdot t_E \, ds \ \ \varphi_E(x), \\ (I_h^F q)(x) &:= \sum_{F \in \mathcal{F}} \int_F q \cdot n_F \, ds \ \ \varphi_F(x), \\ (I_h^T s)(x) &:= \sum_{T \in \mathcal{T}} \int_T s \, dx \ \ \varphi_T(x). \end{aligned}$$

A quasi-interpolation operator for H^1 functions is defined by local averaging. For each vertex V, fix a function $f_V \in L_2(\omega_V)$ such that $\int_{\omega_V} f_V(y)\, dy = 1$ and $\|f_V\|_{L_2} \simeq h^{-3/2}$. One possible choice is $f = \frac{1}{|\omega_V|}$. Then, the quasi-interpolation operator is defined as

$$\Pi_h^V w = \sum_V \Big(\int_{\omega_V} f_V(y) w(y) dy \Big)\ \varphi_V .$$

The quasi-interpolation operator is well defined for $w \in L_2(\Omega)$. Due to the integral constraint on f_V, the quasi-interpolation operator preserves constant functions.

To deal with boundary conditions, we propose a modification for the vertices on the boundary. Let $\widetilde{\Omega}$ be an enlarged domain, and let Ω_D be an outer neighborhood of the essential boundary Γ_D, see Figure 4 in Appendix A. The function $w \in H_D^1(\Omega)$ is continuously extended to $\widetilde{w} \in \widetilde{\Omega}$. The extension is such that $\widetilde{w} = 0$ in Ω_D. In Appendix A we introduce such extension procedures for all involved function spaces.

If V is a vertex on the essential boundary Γ_D, we choose $\omega_V \subset \Omega_D$, again with $|\omega_V| \simeq h_V^3$. Thus, the interpolation function preserves zero

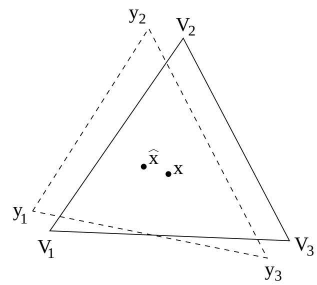

FIGURE 3. Moved point $\hat{x}$

boundary values. If V is on the natural boundary, we may choose $\omega_V \subset \widetilde{\Omega}$ such that $\widetilde{w}|_{\omega_V}$ depends on $w|_{\Omega_V}$, only.

This class of averaging operators was extended to the other function spaces in [25]. Now, we give a different definition for the same operators. We define the quasi-interpolation operator as the composition of the classical interpolation operator, and a smoothing operator S

$$\Pi_h = I_h S.$$

Let the point x be contained in the tetrahedral element $T = [V_{T_1}, V_{T_2}, V_{T_3}, V_{T_4}]$. By means of its barycentric coordinates $\lambda_1(x), \ldots, \lambda_4(x)$, it is represented as

$$x = \sum_{j=1}^{4} \lambda_j(x) V_{T_j}.$$

Now, let $y_j \in \omega_{T_j}$. Define $\hat{x}$ by the same barycentric coordinates with respect to the tetrahedron $[y_1, \ldots, y_4]$:

$$\hat{x}(x, y_1, y_2, y_3, y_4) = \sum_{j=1}^{4} \lambda_j(x) y_j,$$

see Figure 3.

We define the smoothing operator S^V for H^1 functions as

(4)

$$(S^V w)(x) := \int\limits_{\omega_{T_1}} \int\limits_{\omega_{T_2}} \int\limits_{\omega_{T_3}} \int\limits_{\omega_{T_4}} f_{T_1}(y_1) f_{T_2}(y_2) f_{T_3}(y_3) f_{T_4}(y_4) w(\hat{x})\, dy_4\, dy_3\, dy_2\, dy_1.$$

If x coincides with a vertex of the element, say, $x = V_{T_1}$, then $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$, and thus, $\hat{x} = y_1$. In this case, the smoothing operator simplifies to

$$\begin{aligned}(S^V w)(V_{T_1}) &= \int\limits_{\omega_{T_1}} f_{T_1}(y_1)w(y_1)dy_1 \int\limits_{\omega_{T_2}} f_{T_2}(y_2)dy_2 \int\limits_{\omega_{T_3}} f_{T_3}(y_3)dy_3 \int\limits_{\omega_{T_4}} f_{T_4}(y_4)dy_4 \\ &= \int\limits_{\omega_{T_1}} f_{T_1}(y_1)w(y_1)dy_1.\end{aligned}$$

The nodal interpolation operator I_h^V requires these vertex values, only. Thus, the quasi-interpolation operator $\Pi_h^V = I_h^V S^V$ is

$$\Pi_h^V w = \sum_{V\in\mathcal{V}} (S^V w)(V_i)\varphi_V = \sum_{V\in\mathcal{V}} \int\limits_{\omega_V} f_V(y)w(y)dy \;\; \varphi_V.$$

Similarly, if x is on an edge, only the two barycentric coordinates of the vertices on the edge are non-zero, and the quadruple integral simplifies to a double integral. For faces, the integral simplifies to a triple integral involving the vertices of the face. This property ensures continuity of $S^V w$ between neighboring elements.

The smoothing operators for the $H(\mathrm{curl})$ is defined by the co-variant transformation

$$(S^E u)(x) := \int\limits_{\omega_{T_1}} \int\limits_{\omega_{T_2}} \int\limits_{\omega_{T_3}} \int\limits_{\omega_{T_4}} f_{T_1} f_{T_2} f_{T_3} f_{T_4} \left(\frac{d\hat{x}}{dx}\right)^T u(\hat{x})\, dy_4\, dy_3\, dy_2\, dy_1, \tag{5}$$

the smoothing for $H(\mathrm{div})$ involves the Piola-transformation

$$(S^F q)(x) := \int\limits_{\omega_{T_1}} \int\limits_{\omega_{T_2}} \int\limits_{\omega_{T_3}} \int\limits_{\omega_{T_4}} f_{T_1} f_{T_2} f_{T_3} f_{T_4} \det\left(\frac{d\hat{x}}{dx}\right) \left(\frac{d\hat{x}}{dx}\right)^{-T} q(\hat{x})\, dy_4\, dy_3\, dy_2\, dy_1, \tag{6}$$

and for the L_2-case it becomes

$$(S^T s)(x) := \int\limits_{\omega_{T_1}} \int\limits_{\omega_{T_2}} \int\limits_{\omega_{T_3}} \int\limits_{\omega_{T_4}} f_{T_1} f_{T_2} f_{T_3} f_{T_4} \det\left(\frac{d\hat{x}}{dx}\right) s(\hat{x})\, dy_4\, dy_3\, dy_2\, dy_1. \tag{7}$$

The $H(\text{curl})$ quasi-interpolation operator is

$$\begin{aligned}\Pi_h^E u &= I_h^E S^E u = \sum_{E\in\mathcal{E}} \int_E (S^E u)_t \, ds \;\; \varphi_E \\ &= \sum_{E\in\mathcal{E}} \int_{\omega_{E_1}} \int_{\omega_{E_2}} f_{E_1} f_{E_2} \int_{V_1}^{V_2} \Big[\Big(\frac{d\hat{x}}{dx} \Big)^T u(\hat{x}) \Big]_t \, ds \; dy_1 dy_2 \;\; \varphi_E \\ &= \sum_{E\in\mathcal{E}} \int_{\omega_{E_1}} \int_{\omega_{E_2}} f_{E_1} f_{E_2} \int_{y_1}^{y_2} u_t \, ds \; dy_1 dy_2 \;\; \varphi_E.\end{aligned}$$

Instead of taking the line integral of the tangential component from V_{E_1} to V_{E_2}, one integrates over all lines from ω_{E_1} to ω_{E_2}, and averages. This was the definition in [25]. Similarly, the $H(\text{div})$ quasi-interpolation operator is a triple-integral over the normal flux over moved faces:

$$\Pi_h^F q = \sum_{F\in\mathcal{F}} \int_{\omega_{F_1}} \int_{\omega_{F_2}} \int_{\omega_{F_3}} f_{F_1} f_{F_2} f_{F_3} \int_{[y_1, y_2, y_3]} q_n \, ds \; dy_1 dy_2 dy_3 \;\; \varphi_F.$$

Lemma 3. *The smoothing operators commute in the sense of*

$$\begin{aligned}\nabla S^V &= S^E \nabla, \\ \operatorname{curl} S^E &= S^F \operatorname{curl}, \\ \operatorname{div} S^F &= S^T \operatorname{div}.\end{aligned}$$

Proof. We prove the first relation. The other ones use the proper transformation rules for the co-variant and the Piola-transformation.

$$\begin{aligned}(\nabla S^V w)(x) &= \int\!\!\int\!\!\int\!\!\int f_{T_1} \dots f_{T_4} \nabla (w(\hat{x})) dy_4 dy_3 dy_2 dy_1 \\ &= \int\!\!\int\!\!\int\!\!\int f_{T_1} \dots f_{T_4} \Big(\frac{d\hat{x}}{dx} \Big)^T (\nabla w)(\hat{x})) dy_4 dy_3 dy_2 dy_1 \\ &= (S^E \nabla w)(x)\end{aligned}$$

□

Corollary 4. *The quasi-interpolation operators commute in the sense of*

$$\begin{aligned}\nabla \Pi_h^V &= \Pi_h^E \nabla, \\ \operatorname{curl} \Pi_h^E &= \Pi_h^F \operatorname{curl}, \\ \operatorname{div} \Pi_h^F &= \Pi_h^T \operatorname{div}.\end{aligned}$$

Proof. The nodal interpolation operators commute, so also the composition $\Pi_h = I_h S$. □

Remark 5. *There are several possibilities to choose the weighting functions f_V such that the H^1 operator preserves finite element functions. But, the operators for the other spaces will in general not inherit this projection property. For the purpose of a posteriori error analysis, the projection property is not required. In* [27]*, the operators are modified to obtain projections.*

4. Interpolation error estimates for the Π^E

Before proving Theorem 1, we first analyze the decomposition of the interpolation error into local $H(\text{curl})$ functions.

Theorem 6. *There exists a decomposition of the interpolation error*

$$u - \Pi_h^E u = \sum_{V \in \mathcal{V}} u_V \qquad \textit{with} \qquad u_V \in H_D(\text{curl}, \Omega_V),$$

where $H_D(\text{curl}, \Omega_V) = \{v \in H_D(\text{curl}) : v = 0 \textit{ in } \Omega \setminus \Omega_V\}$. *This decomposition satisfies the local estimates*

$$\begin{aligned} \|u_V\|_{L_2(\Omega_V)} &\preceq \|u\|_{L_2(\widetilde{\Omega}_V)}, \\ \|\operatorname{curl} u_V\|_{L_2(\Omega_V)} &\preceq \|\operatorname{curl} u\|_{L_2(\widetilde{\Omega}_V)}. \end{aligned}$$

Proof. We decompose the interpolation error as

$$u - \Pi_h^E u = (u - S^E u) + (S^E u - I_h^E S^E u), \tag{8}$$

and bound the two terms on the right hand side in Lemma 7 and Lemma 10 below. □

Lemma 7. *There exists a decomposition*

$$u - S^E u = \sum_{V \in \mathcal{V}} u_V \qquad \textit{with} \qquad u_V \in H_D(\text{curl}, \Omega_V) \tag{9}$$

which satisfies the continuity estimates

$$\begin{aligned} \|u_V\|_{L_2(\Omega_V)} &\preceq \|u\|_{L_2(\widetilde{\Omega}_V)}, \\ \|\operatorname{curl} u_V\|_{L_2(\Omega_V)} &\preceq \|\operatorname{curl} u\|_{L_2(\widetilde{\Omega}_V)}. \end{aligned}$$

Proof. We formally extend the quadruple integral of the smoothing operators to an N-dimensional integral, where N is the global number of vertices:

$$S^V w(x) = \int\limits_{\omega_1} \cdots \int\limits_{\omega_N} f_1(y_1) \cdots f_N(y_n) w(\hat{x}) \, dy_N \cdots dy_1$$

Formally, we write $\hat{x} = \hat{x}(x, y_1, \ldots y_N)$. Indeed, $\hat{x}$ depends only on the four (three, two, one) y_i corresponding to the vertices of the element (face, edge, vertex, respectively) containing the point x. The other integrals $\int_{\omega_k} f_k(y_k)dy_k$ are just constant factors 1. This extended notation allows the definition of partial smoothing operators

$$S_i^V w = \int\limits_{\omega_1} \cdots \int\limits_{\omega_i} f_1(y_1) \cdots f_i(y_i) w(\hat{x}(y_1, \ldots y_i, V_{i+1}, V_N))\, dy_i \cdots dy_1.$$

We can apply telescoping

$$w - S^V w = \sum_{i=1}^{N} \big(S_{i-1}^V w - S_i^V w \big).$$

These terms are indeed a local decomposition of w. Let $w_i := S_{i-1}^V w - S_i^V w$. If x does not belong to the interior of Ω_{V_i}, then $\hat{x}$ does not depend on y_i, which implies that $w_i(x) = 0$. In the same way, we define partial smoothing operators for the other spaces. Again, the partial smoothing operators commute. It remains to show the L_2-bounds for the decomposition, namely

$$\|S_{i-1}u - S_i u\|_{L_2(\Omega_V)} \preceq \|u\|_{L_2(\widetilde{\Omega}_V)}.$$

The commutativity immediately implies such bounds for the semi-norms, e.g.,

$$\|\operatorname{curl}(S_{i-1}^E u - S_i^E u)\|_{L_2(\Omega_V)} = \|(S_{i-1}^F - S_i^F) \operatorname{curl} u\|_{L_2(\Omega_V)} \preceq \|\operatorname{curl} u\|_{L_2(\widetilde{\Omega}_V)}.$$

The L_2 continuity is proven element-wise for S_i. We show that

$$\|S_i^V w\|_{L_2(T)} \preceq \|w\|_{L_2(\omega_T)}.$$

The operator S_i^V performs smoothing for the vertices T_j of the element with $T_j \leq i$, but keeps vertices T_j with $j > i$ constant. To keep the complexity of the notation reasonable, we assume (w.l.o.g) that smoothing is performed for the first two vertices, i.e., $T_1 \leq i$, $T_2 \leq i$, $T_3 > i$, and $T_4 > i$. Then, smoothing gives on the element T

$$(S_i^V w)(x) = \int\limits_{\omega_{T_1}} \int\limits_{\omega_{T_2}} f_{T_1}(y_1) f_{T_2}(y_2) w(\hat{x}(x, y_1, y_2, V_{T_3}, V_{T_4}))\, dy_2 dy_1.$$

We apply the Hölder inequality for $L_1 - L_\infty$ to bound

$$
\begin{aligned}
&\|S_i^V w\|^2_{L_2(T)} \\
&= \int_T \Big(\int_{\omega_{T_1}} \int_{\omega_{T_2}} f_{T_1}(y_1) f_{T_2}(y_2) w(\hat{x}(x, y_1, y_2, V_{T_3}, V_{T_4}))\, dy_2 dy_1 \Big)^2 dx \\
&\leq \int_T \Big(\int_{\omega_{T_1}} \int_{\omega_{T_2}} |f_{T_1}(y_1)|\, |f_{T_2}(y_2)|\, dy_2 dy_1 \Big)^2 \\
&\quad \sup_{\substack{y_1 \in \omega_{T_1} \\ y_2 \in \omega_{T_2}}} |w(\hat{x}(x, y_1, y_2, V_{T_3}, V_{T_4}))|^2 \, dx \\
&= \|f_{T_1}\|^2_{L_1(\omega_{T_1})} \|f_{T_2}\|^2_{L_1(\omega_{T_2})} \sup_{\substack{y_1 \in \omega_{T_1} \\ y_2 \in \omega_{T_2}}} \int_T w(\hat{x}(x, y_1, y_2, V_{T_3}, V_{T_4}))^2 \, dx.
\end{aligned}
$$

There holds $\|f_{T_1}\|_{L_1(\omega_{T_1})} \leq \|f_{T_1}\|_{L_2(\omega_{T_1})} |\omega_{T_1}|^{1/2} \preceq 1$. The integral in the last term is transformed to the moved tetrahedron $\hat{x}(T, y_1, y_2, V_{T_3}, V_{T_4})$

$$
\begin{aligned}
&\int_T w(\hat{x}(x, y_1, y_2, V_{T_3}, V_{T_4}))^2 \, dx \\
&= \int_{\hat{x}(T, y_1, y_2, V_{T_3}, V_{T_4})} w(\xi)^2 \det \Big(\frac{d\hat{x}}{dx} \Big)^{-1} d\xi \\
&\preceq \|w\|^2_{L_2(\hat{x}(T, y_1, y_2, V_{T_3}, V_{T_4}))} \leq \|w\|^2_{L_2(\omega T)}.
\end{aligned}
$$

We have used that $\frac{d\hat{x}}{dx}$ as well as its inverse is bounded by a constant due to the sufficiently separated domains ω_V. The L_2-estimates for the other smoothing operators are proven in the same way. □

We have already observed that the smoothing operator S^V provides well defined vertex values. Similarly, also the other smoothing operators provide well defined values at some of the lower dimensional objects.

Lemma 8. *The smoothed functions have well defined boundary values in the following sense:*

$$
\begin{aligned}
\|S^V w\|^2_{L_2(V)} &\preceq h^{-3} \|w\|^2_{L_2(\omega_V)} \\
\|S^V w\|^2_{L_2(E)} &\preceq h^{-2} \|w\|^2_{L_2(\omega_E)} \\
\|S^V w\|^2_{L_2(F)} &\preceq h^{-1} \|w\|^2_{L_2(\omega_F)} \\
\|(S^E u)_t\|^2_{L_2(E)} &\preceq h^{-2} \|u\|^2_{L_2(\omega_E)} \\
\|(S^E u)_t\|^2_{L_2(F)} &\preceq h^{-1} \|u\|^2_{L_2(\omega_F)} \\
\|(S^F q)_n\|^2_{L_2(F)} &\preceq h^{-1} \|q\|^2_{L_2(\omega_F)}
\end{aligned}
$$

Proof. We prove $\|S^V w\|^2_{L_2(F)} \preceq h^{-1}\|w\|^2_{L_2(\omega_F)}$. The other estimates follow with the same arguments. The face F is split into three parts $F_{\lambda_1}, F_{\lambda_2}, F_{\lambda_3}$ according to

$$F_{\lambda_i} = \{x : \lambda_i(x) = \max\{\lambda_1(x), \lambda_2(x), \lambda_3(x)\}\}.$$

We apply Cauchy-Schwarz on ω_{F_1}, and the $L_1 - L_\infty$ Hölder inequality on ω_{F_2} and ω_{F_3} to bound

$$\begin{aligned}
&\|S^V w\|^2_{L_2(F_{\lambda_1})} \\
&= \int_{F_{\lambda_1}} \Big(\int_{\omega_{F_1}} \int_{\omega_{F_2}} \int_{\omega_{F_3}} f_1(y_1) f_2(y_2) f_3(y_3) w(\hat{x}(x, y_1, y_2, y_3)) dy_3 dy_2 dy_1 \Big)^2 dx \\
&\leq \|f_1\|^2_{L_2} \|f_2\|^2_{L_1} \|f_3\|^2_{L_1} \sup_{y_2, y_3} \int_{F_{\lambda_1}} \int_{\omega_{F_1}} |w(\hat{x}(x, y_1, y_2, y_3))|^2 \, dy_1 dx \\
&\preceq h_T^{-3} \sup_{y_2, y_3} \int_{F_{\lambda_1}} \int_{\hat{x}(x, \omega_{F_1}, y_2, y_3)} w(\eta)^2 \, \det \left(\frac{d\hat{x}}{dy_1} \right)^{-1} d\eta \, dx.
\end{aligned}$$

The transformation is $\hat{x}(x, y_1, y_2, y_3) = \sum_{i=1}^3 \lambda_i(x)\, y_i$. Thus, $\frac{d\hat{x}}{dy_1} = \lambda_1(x) I$. On F_{λ_1} there is $\lambda_1 \in [\frac{1}{3}, 1]$, and thus $\det \frac{d\hat{x}}{dy_1} \simeq 1$. Insert this to obtain

$$\|S^V w\|^2_{L_2(F_{\lambda_1})} \preceq h_T^{-3} \int_{F_{\lambda_1}} \int_{\omega_F} w(\eta)^2 \, d\eta \, dx \preceq h_T^{-1} \, \|w\|^2_{L_2(\omega_F)}$$

The L_2-norm on the other two parts F_{λ_2} and F_{λ_3} follow from permutation. $\square$

Lemma 9. *There exists an extension operator*

$$E^E : H^1_0(E) \to H^1_0(\Omega_E)$$

which is continuous in the sense

$$\begin{aligned}
\|E^E w\|_{H^1(\Omega_E)} + h^{1/2} \|E^E w\|_{H^1(F)} &\preceq h \|w\|_{H^1(E)} \\
\|E^E w\|_{L_2(\Omega_E)} + h^{1/2} \|E^E w\|_{L_2(F)} &\preceq h \|w\|_{L_2(E)}.
\end{aligned}$$

Here, F is an arbitrary face inside Ω_E. There exists an extension operator

$$E^F : H^1_0(F) \to H^1_0(\Omega_F)$$

which is continuous in the sense

$$\begin{aligned}
\|E^F w\|_{H^1(\Omega_F)} &\leq h^{1/2} \|w\|_{H^1(F)}, \\
\|E^F w\|_{L_2(\Omega_F)} &\leq h^{1/2} \|w\|_{L_2(F)}.
\end{aligned}$$

Proof. Let $w \in H_0^1(E)$. We construct the extension onto an element T sharing the edge E. Let λ_{E_1} and λ_{E_2} the two barycentric coordinates of the vertices connected by the edge, and set $\lambda_E = \lambda_{E_1} + \lambda_{E_2}$.

The extension $E^E w$ is defined by

$$E^E w(x) = \lambda_E w(\hat{x}) \qquad \text{with} \qquad \hat{x} = \sum_{i=1}^{2} \frac{\lambda_{E_i}}{\lambda_E} V_{E_i}.$$

Product and chain rule lead to

$$\nabla E^E w(x) \quad = \quad \nabla \lambda_E \, w(\hat{x}) + \lambda_E \nabla_t w(\hat{x}) \frac{d\hat{x}}{dx}.$$

Observe that $|\nabla \lambda_i| \preceq h^{-1}$, and $\lambda_E \frac{d\hat{x}}{dx} = \lambda_E \frac{d}{dx} \frac{\lambda_{E_1}(V_{E_1} - V_{E_2})}{\lambda_E} = (\nabla \lambda_{E_1} - \frac{\lambda_{E_1}}{\lambda_E} \nabla \lambda_E)(V_{E_1} - V_{E_2})$. From $|V_{E_1} - V_{E_2}| \preceq h$ there follows $|\lambda_E \frac{d\hat{x}}{dx}| \preceq 1$. This leads to

$$|\nabla E^E w(x)| \preceq h^{-1} |w(\hat{x})| + |\nabla_t w(\hat{x})|.$$

With the transformation of integrals and a Friedrichs´ inequality on the edge we observe

$$\|\nabla E^E w\|_{L_2(T)}^2 \preceq h^{-1} \|w(\hat{x}(x))\|_{L_2(T)} + \|\nabla_t w(\hat{x}(x))\|_{L_2(T)}^2 \preceq h \|\nabla_t w\|_{L_2(E)}.$$

The L_2 estimate and the estimates on faces is left to the reader. Similarly, we define the extension operator from faces by

$$E^F w(x) = \lambda_F w(\hat{x}) \qquad \text{with} \qquad \hat{x} = \sum_{i=1}^{3} \frac{\lambda_{F_i}}{\lambda_F} V_{F_i},$$

where F_1, F_2, and F_3 are the vertices of the face, and $\lambda_F = \sum_{i=1}^3 \lambda_{F_i}$. The continuity estimates follow with the same arguments. □

Lemma 10. *There exists a decomposition*

$$S^E u - I_h^E S^E u = \sum_{V \in \mathcal{V}} u_V \qquad \text{with} \qquad u_V \in H_D(\operatorname{curl}, \Omega_V) \tag{10}$$

which satisfies the continuity estimates

$$\begin{aligned} \|u_V\|_{L_2(\Omega_V)} &\preceq \|u\|_{L_2(\widetilde{\Omega}_V)}, \\ \|\operatorname{curl} u_V\|_{L_2(\Omega_V)} &\preceq \|\operatorname{curl} u\|_{L_2(\widetilde{\Omega}_V)}. \end{aligned}$$

Proof. Since $S^E u \in L_2(E)$, the nodal edge interpolator is well defined. Set $u_2 := S^E u - I_h^E S^E u$. It satisfies the continuity estimates

$$\begin{aligned} h \|u_{2,t}\|_{L_2(E)} &\preceq \|u\|_{\omega_E}, \\ h^{1/2} \|u_{2,t}\|_{L_2(F)} &\preceq \|u\|_{\omega_F}, \\ \|u_2\|_{L_2(T)} &\preceq \|u\|_{\omega_T}. \end{aligned}$$

Integrating the tangential component of u_2 along the edge $E = [E_1, E_2]$ results in

$$\Phi_E(x) := \int_{E_1}^{x} u_{2,t} \, ds.$$

Due to zero mean, $\Phi_E \in H_0^1(E)$. Using the extension from edges of Lemma 9, we construct

$$u_3 = u_2 - \sum_{E \in \mathcal{E}} \nabla E^E \Phi_E.$$

Each of the terms $\nabla E^E \Phi_E$ can be included in one of the terms of the decomposition (10). The rest u_3 satisfies

$$\begin{aligned} h^{1/2} \|u_{3,t}\|_{L_2(F)} &\preceq \|u\|_{\omega_F}, \\ \|u_3\|_{L_2(T)} &\preceq \|u\|_{\omega_T}. \end{aligned}$$

By commutativity, the according estimates are also obtained for $\operatorname{curl} u$:

$$\begin{aligned} h^{1/2} \|(\operatorname{curl} u_3)_n\|_{L_2(F)} &\preceq \|\operatorname{curl} u\|_{\omega_F}, \\ \|\operatorname{curl} u_3\|_{L_2(T)} &\preceq \|\operatorname{curl} u\|_{\omega_T}. \end{aligned}$$

Next, we extend from faces. For this, decompose $u_{3,t} \in H_0(\operatorname{curl}, F)$ into

$$u_{3,t}|_F = (\nabla \Phi_F + z_F)_t$$

such that $\Phi_F \in H_0^1(F)$ and $z_F \in [H_0^1(F)]^3$ satisfy

$$\begin{aligned} \|\nabla_t \phi_F\|_{L_2} + \|z_F\|_{L_2} &\preceq \|u_{3,t}\|_{L_2}, \\ \|\nabla_t z_F\|_{L_2(F)} &\preceq \|\operatorname{curl} u_{3,t}\|. \end{aligned}$$

This is possible due to the two-dimensional version of [22], Lemma 2.2. Both functions, Φ_F and z_F are extended by E^F onto the adjacent elements. These terms match the decomposition (10) and satisfy the continuity estimates

$$\|\nabla E^F \Phi_F + E^F z_F\|_{L_2(\Omega_F)} \preceq h^{1/2} \|(S^E u)_t\|_{L_2(F)} \preceq \|u\|_{\omega_F}$$

and

$$\|\operatorname{curl} E^F z_F\|_{L_2(\Omega_F)} \preceq h^{1/2} \|\operatorname{curl}(S^E u)_t\|_{L_2(F)} \preceq \|\operatorname{curl} u\|_{\omega_F}.$$

Finally, define

$$u_4 = u_3 - \sum_{F \in \mathcal{F}} \left\{ \nabla E^F \Phi_F + E^F z_F \right\}$$

which has vanishing tangential trace on all faces, and thus splits into local terms. $\square$

By the same techniques, one proves also a decomposition result for the space $H(\mathrm{div})$. It might be useful for the analysis of a posteriori error estimators for mixed methods involving the space $H(\mathrm{div})$ such as in [10].

Theorem 11. *There exists a decomposition of the interpolation error*

$$q - \Pi_h^F q = \sum_{V \in \mathcal{V}} q_V \qquad \text{with} \qquad q_V \in H_D(\mathrm{div}, \Omega_V),$$

where $H_D(\mathrm{div}, \Omega_V) = \{v \in H_D(\mathrm{div}) : v = 0 \text{ in } \Omega \setminus \Omega_V\}$. *This decomposition satisfies the local estimates*

$$\begin{aligned} \|q_V\|_{L_2(\Omega_V)} &\preceq \|q\|_{L_2(\widetilde{\Omega}_V)}, \\ \|\operatorname{div} q_V\|_{L_2(\Omega_V)} &\preceq \|\operatorname{div} q\|_{L_2(\widetilde{\Omega}_V)} \end{aligned}$$

.

Now, we are ready to prove our main result:

Proof of Theorem 1. Let $u = \sum u_V$ be the decomposition of Theorem 6. First, assume that V is an inner vertex or a vertex on the Dirichlet boundary. Then $u_V \in H_0(\mathrm{curl}, \Omega_V)$. According to [22], Lemma 2.2, there exists a decomposition

$$u_V = \nabla \varphi_V + z_V$$

with $\varphi_V \in H_0^1(\Omega_V)$ and $z_V \in [H_0^1(\Omega_V)]^3$. The decomposition is bounded by

$$\begin{aligned} h_V^{-1} \|\varphi_V\|_{L_2(\Omega_V)} + \|\nabla \varphi_V\|_{L_2(\Omega_V)} &\preceq \|u_V\|_{L_2(\Omega_V)}, \\ h_V^{-1} \|z_V\|_{L_2(\Omega_V)} + \|\nabla z_V\|_{L_2(\Omega_V)} &\preceq \|\operatorname{curl} u_V\|_{L_2(\Omega_V)}, \end{aligned}$$

where the involved constants depend only on the shape of the local domain Ω_V. If the vertex is on the Neumann boundary, than $u_{V,t}$ does not necessarily vanish on the boundary of Ω_V which is also the domain boundary. Since the domain is Lipschitz, the whole patch Ω_V can be mirrored over the domain boundary to obtain $\widetilde{\Omega}_V$. The function is extended by the co-variant transformation to $H_0(\mathrm{curl}, \widetilde{\Omega}_V)$. Now, the above decomposition can be applied.

We define

$$\varphi = \sum_{V \in \mathcal{V}} \varphi_V \qquad \text{and} \qquad z = \sum_{V \in \mathcal{V}} z_V$$

to obtain the claimed decomposition (2)

$$u - \Pi_h^E u = \nabla \varphi + z.$$

The norm bounds follow from the finite number of overlapping patches. □

Appendix A. Commuting extension operators

We establish extension operators for the spaces $H(\mathrm{curl})$ and $H(\mathrm{div})$ which are bounded in the L_2 norm and in the corresponding semi-norms. The extended function vanishes on an outer neighborhood of the Dirichlet boundary. We introduce a continuous bijection $x \mapsto \tilde{x}(x)$ between the inner (Ω_i) and outer (Ω_o) neighborhoods of the boundary $\partial\Omega$, see Figure 4. The transformation shall fulfill

$$\tilde{x}(x) = x \qquad \forall\, x \in \Gamma_N$$

and is bounded in the sense

$$\Big\| \frac{d\tilde{x}}{dx} \Big\|_{L_\infty} \leq c \qquad \text{and} \qquad \Big\| \Big(\frac{d\tilde{x}}{dx} \Big)^{-1} \Big\|_{L_\infty} \leq c.$$

On Dirichlet boundaries, we shift the exterior domain Ω_o away from the boundary to obtain the domain Ω_D between Γ_D and $\tilde{x}(\Gamma_D)$. Let $\widetilde{\Omega} = \overline{\Omega} \cup \overline{\Omega}_D \cup \Omega_o$.

We sketch this construction for general Lipschitz domains. Let $U_1, \ldots, U_M$ be an open covering of the boundary $\partial\Omega$. Assume that a strip S of with s along $\partial\Omega$ is contained in $\cup U_i$. Let $(e_{\xi_i}, e_{\eta_i}, e_{\zeta_i})$ be local coordinate systems, and let $\varphi_i(\xi_i, \eta_i)$ be Lipschitz functions such that $U_i \cap \Omega = \{(\xi_i, \eta_i, \zeta_i) \in U_i : \zeta_i > \varphi_i(\xi_i, \eta_i)\}$. Define the limited distance function to the non-Dirichlet boundary as $d(x) := \min\{s/2, \mathrm{dist}\{x, \Gamma_N\}\}$. Now, we can define the mirroring operator with shift for the Dirichlet boundary: Assume $x \in U_i \cap \Omega$ has the local coordinates (ξ_i, η_i, ζ_i). The vertical projection to the boundary $x_i^b(x)$ is defined by the local coordinates $(\xi_i, \eta_i, \varphi_i(\xi_i, \eta_i))$, and $\tilde{x}_i(x) = x_b - (|x_i^b - x| + d(x_b)) e_{\zeta_i}$. Finally, introduce a partition of unity $\{\psi_i\}$ such that $\sum \psi_i = 1$ on $\partial\Omega$, and set $\tilde{x}(x) := \sum \psi_i(x_i^b) \tilde{x}_i(x)$.

The extension for H^1 functions is defined by mirroring:

$$\tilde{w}(x) = \left\{ \begin{array}{cl} w(x) & x \in \Omega, \\ 0 & x \in \Omega_D, \\ w(\tilde{x}^{-1}(x)) & x \in \Omega_o. \end{array} \right.$$

Using the chain rule, its piece-wise gradient evaluates to

$$\nabla \tilde{w}(x) = \left\{ \begin{array}{cl} \nabla w(x) & x \in \Omega, \\ 0 & x \in \Omega_D, \\ (\tilde{x}')^{-T} (\nabla w)(\tilde{x}^{-1}(x)) & x \in \Omega_o. \end{array} \right.$$

Since the extension has continuous traces on the interfaces between Ω_i, Ω_o, and Ω_D, the piece-wise gradient is also the global gradient of $\tilde{w}$.

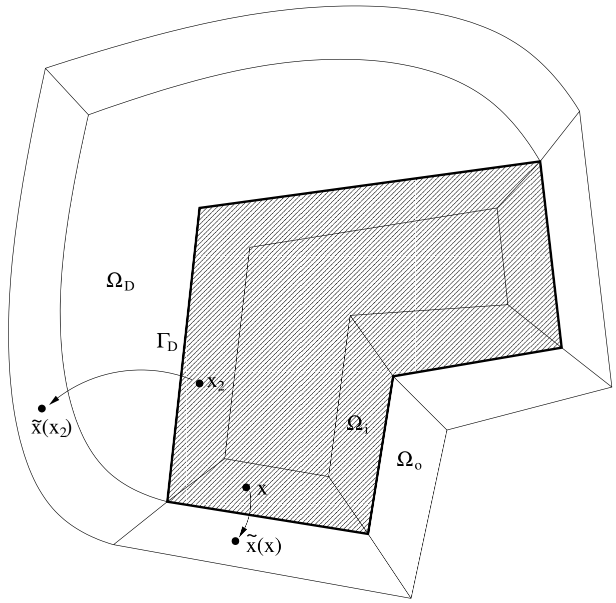

FIGURE 4. Transformation for extension

We have assumed that $\tilde{x}'$ as well as its inverse is in L_∞. This ensures that the extension is bounded with respect to the L_2-norm. It also ensures that the gradient of the extension is bounded by the L_2-norm of the gradient, i.e., the extension is bounded in the H^1-semi-norm.

Motivated by the commuting diagram, the extension $\tilde{u}$ of an $H(\text{curl})$ function u is defined like the extension of gradients:

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega, \\ 0 & x \in \Omega_D, \\ (\tilde{x}')^{-T} u(\tilde{x}^{-1}(x)) & x \in \Omega_o. \end{cases}$$

With this so called co-variant transformation for the function u, the transformation of its curl evaluates to the Piola-transformation:

$$\operatorname{curl} \tilde{u}(x) = \begin{cases} \operatorname{curl} u(x) & x \in \Omega, \\ 0 & x \in \Omega_D, \\ \det(\tilde{x}')^{-1} (\tilde{x}') \operatorname{curl} u(\tilde{x}^{-1}(x)) & x \in \Omega_o. \end{cases}$$

The extension $\tilde{u}$ has continuous tangential traces ensuring that $\tilde{u} \in H(\text{curl}, \widetilde{\Omega})$. Since the curl of the extended function depends continuously only on the curl of the original function, the extension is bounded in the curl semi-norm. In the same fashion, we define the extension of

$H(\mathrm{div})$ functions q by the Piola-transformation:

$$\tilde{q}(x) = \begin{cases} q(x) & x \in \Omega, \\ 0 & x \in \Omega_D, \\ \det(\tilde{x}')^{-1}(\tilde{x}')q(\tilde{x}^{-1}(x)) & x \in \Omega_o. \end{cases}$$

This one provides continuous normal traces. Now, forming the divergence leads to

$$\operatorname{div} \tilde{q}(x) = \begin{cases} \operatorname{div} q(x) & x \in \Omega, \\ 0 & x \in \Omega_D, \\ \det(\tilde{x}')^{-1} \operatorname{div} q(\tilde{x}^{-1}(x)) & x \in \Omega_o. \end{cases}$$

This one we take also for the extension of L_2-functions.

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