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A POTENTIAL REPRESENTATION FOR TWO-DIMENSIONAL WAVES IN ELASTIC MATERIALS OF HARMONIC TYPE*

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1. Introduction. In the present note we consider two-dimensional finite dynamical deformations for the class of homogeneous, isotropic elastic materials introduced by F. John in [1] and referred to by him as materials of *harmonic type*. The theory of such materials, developed in [1] and [2], appears to be simpler in many respects than that of more general elastic materials, and it may offer the possibility of investigating some features of nonlinear elastic behavior more explicitly than is possible in general.

For plane motions of such materials, we derive here a representation for the displacements in terms of two potentials which is analogous to the theorem of Lamé in classical linear elasticity (see [3]) for the case of plane strain. The two nonlinear differential equations satisfied by the potentials reduce upon linearization to the wave equations associated with irrotational and equivoluminal waves in the linear theory.

In the following section we state without derivation the equations governing two-dimensional waves in an elastic material of harmonic type. The reader is referred to [1] for details. In Sec. 3 we derive the representation in terms of potentials described briefly above.

2. Harmonic materials. Let x, y, z be coordinates in a fixed rectangular Cartesian frame, and let R be a region in the x, y-plane. In its undeformed state an elastic body is assumed to occupy a cylindrical region with generators parallel to the z-axis and whose cross-section in the plane $z = 0$ is R. We consider deformations of this body in which a particle at (x, y, z) in the undeformed state moves to the point $(x + u, y + v, z)$ at time t, where $u = u(x, y, z, t)$ and $v = v(x, y, z, t)$ are displacements at time t in the x- and y-directions, respectively.

For the present case of plane strain, the strain energy W per unit undeformed volume for elastic materials of harmonic type is of the form

$$W = 2\mu[F(r) - s], \tag{2.1}$$

where

$$r = [(2 + u_x + v_y)^2 + (u_y - v_x)^2]^{1/2}, \tag{2.2}$$

$$s = (1 + u_x)(1 + v_y) - u_y v_x , \tag{2.3}$$

and the constant $\mu > 0$ is the shear modulus of linear elasticity. In (2.2), (2.3) and subsequently, subscripts x and y indicate partial differentiation. In the undeformed state we have $r = 2$ and $s = 1$. The three times continuously differentiable function F in (2.1) is characteristic of the given material and is subject to the following restrictions:

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$$F(2) = 1, \qquad F'(2) = 1, \qquad F''(2) = \tfrac{1}{2}c_1^2/c_2^2, \tag{2.4}$$

where c_1, c_2 are the respective speeds of irrotational (dilatation) and equivoluminal (shear) waves in the linear theory. The first of (2.4) corresponds to the requirement that $W = 0$ for rigid body motions, while the remaining conditions assure consistency with the linear theory of isotropic materials. It is convenient to define

$$G(r) = 2F'(r)/r, \qquad r > 0; \tag{2.5}$$

we assume[1] that $G(r)$ is positive for all $r > 0$.

If body forces are absent, the differential equations of motion are

$$\rho u_{tt} = A_x + B_y, \qquad \rho v_{tt} = A_y - B_x, \tag{2.6}$$

where ρ is the constant mass per unit undeformed volume and

$$A = 2\mu F'(r)\cos\theta, \qquad B = -2\mu F'(r)\sin\theta, \tag{2.7}$$

$$\cos\theta = \frac{2 + u_x + v_y}{r}, \qquad \sin\theta = \frac{v_x - u_y}{r}. \tag{2.8}$$

In the equilibrium case A and B are conjugate harmonic functions and consequently the local rotation angle θ satisfies Laplace's equation. This property is responsible for the term *harmonic* for materials of this type.

The "Lagrange stresses" associated with the deformation are computed as follows:

$$\begin{aligned} q_{11} &= 2\mu[F'(r)\cos\theta - 1 - v_y], & q_{21} &= 2\mu[F'(r)\sin\theta + u_y], \\ q_{12} &= 2\mu[-F'(r)\sin\theta + v_x], & q_{22} &= 2\mu[F'(r)\cos\theta - 1 - u_x]. \end{aligned} \tag{2.9}$$

Here q_{11} is the component of traction in the x-direction, measured per unit undeformed area, on a surface element whose orientation in the undeformed state was normal to the x-axis; q_{21} is the component of traction in the y-direction, also measured per unit undeformed area, on such a surface element.[2] Similar interpretations apply to q_{22} and q_{12}.

3. Potential representation. We now suppose that u, v are twice continuously differentiable with respect to x, y, t for $t \geq 0$, $(x, y) \in R$ and satisfy the differential equations of motion (2.6). For simplicity we also assume that $u = u_t = v = v_t = 0$ in R for $t = 0$. Integrating (2.6) twice with respect to t, we obtain

$$\begin{aligned} u(x, y, t) &= \frac{\partial}{\partial x}\int_0^t (t - \tau)\frac{A(x, y, \tau)}{\rho}\,d\tau + \frac{\partial}{\partial y}\int_0^t (t - \tau)\frac{B(x, y, \tau)}{\rho}\,d\tau, \\ v(x, y, t) &= \frac{\partial}{\partial y}\int_0^t (t - \tau)\frac{A(x, y, \tau)}{\rho}\,d\tau - \frac{\partial}{\partial x}\int_0^t (t - \tau)\frac{B(x, y, \tau)}{\rho}\,d\tau. \end{aligned} \tag{3.1}$$

If we define

$$\Phi = \int_0^t (t - \tau)\frac{A}{\rho}\,d\tau, \qquad \Psi = -\int_0^t (t - \tau)\frac{B}{\rho}\,d\tau, \tag{3.2}$$

(3.1) may be written

$$u = \Phi_x - \Psi_y, \qquad v = \Phi_y - \Psi_x. \tag{3.3}$$

[1] See the discussion in § 2.2 of [1].

[2] The notation for the q's is that of [1]; note that $q_{12} \neq q_{21}$.

From (3.2) and (2.7) we have

$$\Phi_{tt} = \frac{A}{\rho} = \frac{2\mu}{\rho} F'(r) \cos\theta, \qquad \Psi_{tt} = -\frac{B}{\rho} = \frac{2\mu}{\rho} F'(r) \sin\theta, \tag{3.4}$$

or, from (2.2), (2.8), (2.5) and (3.3),

$$\rho\Phi_{tt} = \mu G(r)(2 + \Delta\Phi), \qquad \rho\Psi_{tt} = \mu G(r)\,\Delta\Psi, \tag{3.5}$$

where Δ stands for $\partial^2/\partial x^2 + \partial^2/\partial y^2$. Since from (3.2), (2.7), (2.8), (2.4), $\Phi = \mu t^2/\rho$ in the undeformed state, it is natural to write

$$\Phi = \frac{\mu t^2}{\rho} + \varphi, \qquad \Psi = \psi. \tag{3.6}$$

We have thus shown that

$$u = \varphi_x - \psi_y\,, \qquad v = \varphi_y + \psi_x \tag{3.7}$$

where φ and ψ satisfy the differential equations

$$\frac{1}{c_2^2}\varphi_{tt} = G(r)\,\Delta\varphi + 2[G(r) - 1], \qquad \frac{1}{c_2^2}\psi_{tt} = G(r)\,\Delta\psi, \tag{3.8}$$

$$r = [(2 + \Delta\varphi)^2 + (\Delta\psi)^2]^{1/2}, \tag{3.9}$$

and use has been made of the fact that $c_2^2 = \mu/\rho$.

If the initial displacement and velocity vectors do not vanish, they may be represented in terms of potentials by means of the Helmholtz theorem [3] specialized to two dimensions. The remainder of the above argument is then carried out with little modification.[3]

On the other hand, suppose that φ, ψ are three times continuously differentiable with respect to x, y, t and define u, v by (3.7). A simple direct calculation shows that u, v satisfy (2.6) if φ, ψ satisfy (3.8).

We may express the Lagrange stresses q in terms of φ and ψ as follows. Using (3.7), (3.9), (2.7), (2.8) in the first of (2.9), we obtain

$$q_{11} = 2\mu[\tfrac{1}{2}G(r)(2 + \Delta\varphi) - 1 - \varphi_{yy} - \psi_{xy}].$$

By the first of (3.8), this can be written

$$q_{11} = 2\mu\left(\frac{1}{2c_2^2}\varphi_{tt} - \varphi_{yy} - \psi_{xy}\right). \tag{3.10}$$

For the other q's, we obtain similarly

$$q_{12} = 2\mu\left(-\frac{1}{2c_2^2}\psi_{tt} + \psi_{xx} + \varphi_{xy}\right), \tag{3.11}$$

$$q_{21} = 2\mu\left(\frac{1}{2c_2^2}\psi_{tt} - \psi_{yy} + \varphi_{xy}\right), \tag{3.12}$$

$$q_{22} = 2\mu\left(\frac{1}{2c_2^2}\varphi_{tt} - \varphi_{xx} + \psi_{xy}\right). \tag{3.13}$$

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[3] The derivation of (3.7), (3.8) given above is similar to that employed in the linear theory in three dimensions by Somigliana [4].

Eqs. (3.7), (3.8) and (3.10)–(3.13) comprise the representation formulas which we set out to obtain.

We consider the linearization of (3.8). According to (2.4), (2.5),

$$G(r) = 1 + \tfrac{1}{2}(c_1^2/c_2^2 - 1)(r - 2) + O[(r - 2)^2] \quad \text{as} \quad r \to 2. \tag{3.14}$$

Linearizing (3.8) with respect to φ, ψ thus yields

$$\frac{1}{c_1^2}\varphi_{tt} = \Delta\varphi, \qquad \frac{1}{c_2^2}\psi_{tt} = \Delta\psi; \tag{3.15}$$

these are the respective equations for dilatation and shear waves in the linear theory.

Since the right-hand sides of (3.10)–(3.13) are linear in φ and ψ, they must be identical with the corresponding expressions of the linear theory. It may be noted that if (3.15) is used to simplify (3.11), (3.12), we recover the fact that $q_{12} = q_{21}$ in the linear theory.

As a final remark, we observe that the two nonlinear differential equations (3.8) for the potentials φ, ψ may be combined into a single *complex* equation. Thus if

$$\chi = \varphi + i\psi, \tag{3.16}$$

then Eqs. (3.8) are equivalent to

$$\frac{1}{c_2^2}\chi_{tt} = G(|2 + \Delta\chi|)(2 + \Delta\chi) - 2. \tag{3.17}$$

References

[1] F. John, *Plane strain problems for a perfectly elastic material of harmonic type*, Comm. Pure Appl. Math. **13**, 239–296 (1960)

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