A potential update method for series methods in steady seepage

W. W. Read S. R. Belward P. J. Higgins*

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Abstract

Computationally, steady seepage in arbitrary shaped aquifers reduces to a Laplacian free boundary problem. Analytic series solutions can be obtained by iteratively improving an initial estimate. Although the free water table is slowly varying in shape, accurate solutions can be extremely difficult to obtain for the long aspect ratios encountered in typical problems. In previous research, the Neumann condition along the free boundary was used as a cost function. In this paper, we present an update method that uses the Dirichlet condition as the cost function. We find that this approach is significantly more efficient and stable than the previous approach. This enables the determination of very accurate solutions for the water table at low computational cost.

*Mathematical & Physical Sciences, James Cook University, Townsville, Queensland Australia. mailto:wayne.read@jcu.edu.au

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1 Introduction

Accurate modelling of the flow processes in subsurface aquifers is of critical interest to relatively dry countries that depend heavily on groundwater resources (for example, Australia). In practical applications, most aquifers have a large aspect ratio (that is, horizontal to vertical). For these types of seepage problems, the free water table varies slowly with the horizontal length scale, and we would expect that accurate solutions could be obtained relatively easily. However, accurate resolution for the geometry of the water table are extremely difficult for numerical schemes such as the Boundary Integral Equation Method (BIEM), Finite Element and Finite Difference.

Analytic Series methods [2, 4, 5] provide comparatively fast and efficient solution methods, for most problems. These methods also have the

advantage that *exact* error bounds are immediately available. In addition, it appears that spectral convergence is achievable for the right boundary representation [6]. However, some problems are still difficult to solve, even with the series formulation. For example, solutions for low recharge rates in hillslope aquifers are very difficult to obtain as the recharge rate approaches the threshold value [5].

The steady seepage problem is formulated mathematically as a Laplacian free boundary problem, where the location of one of the boundaries is initially unknown. On the free boundary, there are two boundary conditions that must be met, and this allows an iterative solution method. An initial estimate of the free boundary is made and the resulting boundary value problem solved, using one of the free boundary conditions. The remaining free boundary condition is then used as a cost function to update the initial estimate. This process is continued until the location of the free boundary is known to sufficient accuracy.

In seepage problems, the hydraulic head (or potential) and the recharge (or normal derivative of the potential) are known along the water table. The recharge condition is linearised using the stream function, and this is used as the cost function in the series method. However, BIEM (arguably the best of the numerical methods) uses the potential condition along the water table as the cost function [1]. For seepage problems where the water table intersects the soil surface, this approach depends on solving the classic mixed boundary condition along the top boundary. However, in a significant proportion of seepage problems the water table is below the soil surface, and the problem is posed without the necessity of a mixed top boundary condition.

In this paper, we investigate the computational efficiency and accuracy of the potential cost function. In Section 2, we formulate the mathematical description of the subsurface seepage problem. The solution method is described in Section 3 and we present the results in Section 4. Finally, we discuss the results in Section 5.

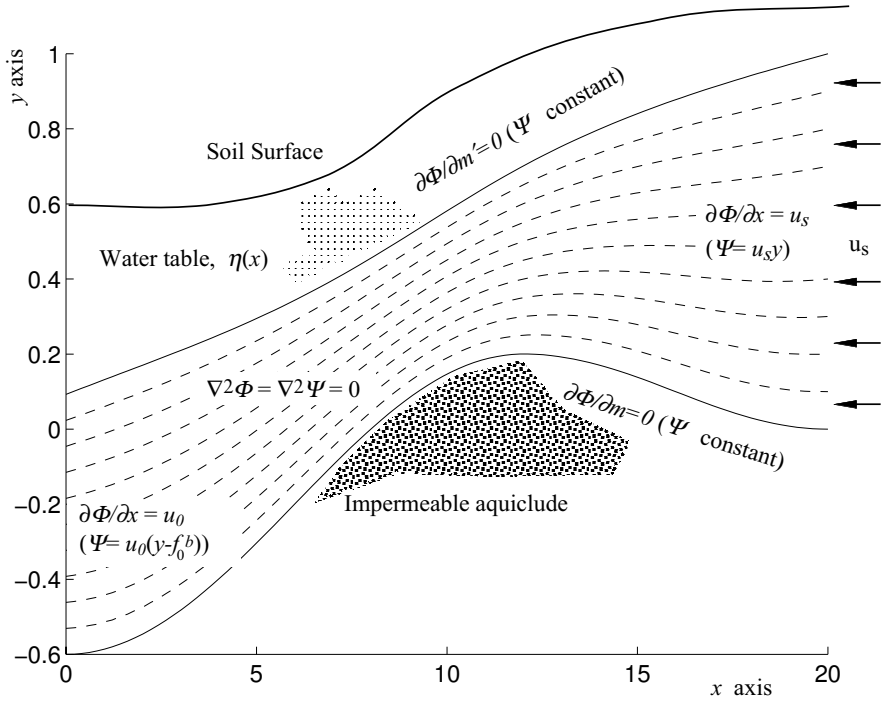

FIGURE 1: Schematic of the seepage domain.

2 Mathematical Problem Formulation

We consider subsurface seepage over an aquiclude where the seepage velocity and water table height upstream are specified. A schematic of the flow domain is given in Figure 1. In the saturated zone, the hydraulic head or potential $\Phi(x, y)$ satisfies Laplace's equation:

$$\nabla^2 \Phi = 0\,. \tag{1}$$

Along the impermeable bottom boundary $y = f^b(x)$, the normal derivative is zero:

$$\frac{\partial}{\partial m}\Phi(x, f^b(x)) = 0\,, \tag{2}$$

where m is normal to the bottom boundary.

On the vertical side boundaries at $x = 0$ and $x = s$, the seepage velocities are

$$\frac{\partial}{\partial x}\Phi(0, y) = \frac{U_0}{K} = u_0 \quad \text{and} \quad \frac{\partial}{\partial x}\Phi(s, y) = \frac{U_s}{K} = u_s\,, \tag{3}$$

where K is the hydraulic conductivity. Along the water table $y = \eta(x)$, the boundary conditions are

$$\Phi(x, \eta(x)) = \eta(x)\,, \tag{4}$$

$$\frac{\partial}{\partial m'}\Phi(x, \eta(x)) = 0\,, \tag{5}$$

where m' is normal to the water table.

2.1 Stream function formulation

We find the problem is more suitably posed using the conjugate stream function $\Psi(x, y)$, related to the potential by the Cauchy–Riemann equations:

$$\frac{\partial \Psi}{\partial x} = -\frac{\partial \Phi}{\partial y}\,, \quad \frac{\partial \Psi}{\partial y} = \frac{\partial \Phi}{\partial x}\,. \tag{6}$$

$\Psi(x, y)$ also satisfies Laplace's equation:

$$\nabla^2 \Psi(x, y) = 0\,. \tag{7}$$

Along the impermeable base $y = f^b(x)$, equation (2) becomes

$$\Psi(x, f^b(x)) = \Psi(f^b) = 0\,. \tag{8}$$

The boundary conditions in equation (3) along $x = 0$ and $x = s$ become

$$\Psi(0, y) = u_0\left(y - f_0^b\right) \quad \text{and} \quad \Psi(s, y) = u_s\left(y - f_s^b\right)\,, \tag{9}$$

where $f_0^b = f^b(0)$, $f_s^b = f^b(s)$. The mass transfer must be conserved across these two boundaries, so assuming zero recharge across the water table,

$$u_0(\eta_0 - f_0^b) = u_s(\eta_s - f_s^b)\,, \tag{10}$$

where $\eta_0 = \eta(0)$, $\eta_s = \eta(s)$. Without loss of generality, we choose

$$\eta_s = 1 \quad \text{and} \quad f_s^b = 0\,, \tag{11}$$

so that

$$u_0 = \frac{u_s}{\eta_0 - f_0^b}\,. \tag{12}$$

Along the free surface, equation (5) becomes

$$\Psi(x, \eta(x)) = u_s(\eta_s - f_s^b) = u_s\,. \tag{13}$$

2.2 Homogeneous side boundary conditions

The series solution method depends on homogeneous side boundary conditions at $x = 0$ and $x = s$. For the stream function $\Psi(x, y)$, homogeneous side boundary conditions are achieved by the transformation

$$\begin{aligned} \Psi(x,y) &= \psi(x,y) + u_0(y - f_0^b) + \frac{(u_s y - u_0(y - f_0^b))x}{s} \\ &= \psi(x,y) + \frac{u_s(y - f_0^b)(s - x)}{s(\eta_0 - f_0^b)} + \frac{u_s x y}{s}\,, \end{aligned} \tag{14}$$

using equations (11) and (12). With this transformation, $\psi(x, y)$ satisfies Laplace's equation (7) together with homogeneous boundary conditions at $x = 0$ and $x = s$:

$$\psi(0, y) = \psi(s, y) = 0\,. \tag{15}$$

The bottom boundary condition equation (8) becomes

$$\begin{aligned}\psi(x, f^b(x)) &= -\frac{u_s(f^b(x) - f_0^b)(s-x)}{s(\eta_0 - f_0^b)} - \frac{u_s x f^b(x)}{s} \\ &= h^b(x)\,. \end{aligned} \tag{16}$$

The top boundary condition equation (13) becomes

$$\begin{aligned}\psi(x, \eta(x)) &= u_s - \frac{u_s(\eta(x) - f_0^b)(s-x)}{s(\eta_0 - f_0^b)} - \frac{u_s x \eta(x)}{s} \\ &= h^t(x)\,. \end{aligned} \tag{17}$$

Using the Cauchy–Riemann equations (6), the corresponding transformation for the potential function $\Phi(x, y)$ is

$$\Phi(x, y) = \phi(x, y) + \frac{u_s\left(2(sx - f_0 y) + (1 + f_0^b - \eta_0)(y^2 - x^2)\right)}{2s(\eta_0 - f_0^b)}\,, \tag{18}$$

where $\phi(x, y)$ has homogeneous side boundary conditions ($\partial\phi/\partial x = 0$) at $x = 0$ and $x = s$.

3 Series Solution

We obtain the series solution by applying separation of variables to the stream function as fully detailed in [2, 4]. The series solution is

$$\psi(x, y) = \sum_{n=1}^{\infty} \left[a_n u_n(x, y) + b_n v_n(x, y)\right]\,, \tag{19}$$

where

$$u_n(x, y) = \sinh\left(\frac{n\pi y}{s}\right) \sin\left(\frac{n\pi x}{s}\right)\,, \tag{20}$$

$$v_n(x, y) = \cosh\left(\frac{n\pi y}{s}\right) \sin\left(\frac{n\pi x}{s}\right)\,. \tag{21}$$

The bottom and top boundary conditions (equations (16) and (17)) become

$$h^b(x) = \sum_{n=1}^{\infty} \left[a_n u_n^b(x) + b_n v_n^b(x)\right] , \tag{22}$$

$$h^t(x) = \sum_{n=1}^{\infty} \left[a_n u_n^t(x) + b_n v_n^t(x)\right] , \tag{23}$$

where

$$u_n^t(x) = u_n(x, \eta(x)) , \quad u_n^b(x) = u_n(x, f^b(x)) , \tag{24}$$

$$v_n^t(x) = v_n(x, \eta(x)) , \quad v_n^b(x) = v_n(x, f^b(x)) . \tag{25}$$

We assume $h^b(x)$ and $u_n^b(x)$ lie in $\text{span}\{v_1^b(x), v_2^b(x), \ldots\}$ and then expand them in terms of this basis:

$$h^b(x) = \sum_{i=0}^{\infty} k_i^{hb} v_i^b(x) , \quad u_n^b(x) = \sum_{j=1}^{\infty} k_{jn}^{ub} v_j^b(x) , \tag{26}$$

where k_i^{hb}, k_{jn}^{ub} are the expansion coefficients. After some manipulation we obtain

$$k_i^{hb} = \sum_{j=1}^{\infty} k_{ij}^{ub} a_j + b_i . \tag{27}$$

The same procedure produces a similar set of equations for the top boundary condition. The series is truncated after a suitable number of terms N, and given the expansion coefficients k_i^{hb}, k_i^{ht}, k_{ij}^{ub}, k_{ij}^{ut}, the series coefficients can be evaluated. (Complete details of this procedure are given in [3, 4].)

The expansion coefficients can be evaluated in a variety of ways. The Gramm–Schmidt process can be used to generate an orthonormal set of basis functions, with the expansion coefficients evaluated using the orthogonality relationship. Equivalently, least squares can be used to generate the Normal equations, which can then be solved directly for the expansion coefficients using QR factorization. Although this procedure is theoretically the same

as the Gramm–Schmidt process, computationally it is more stable and more efficient. The approach we use in this paper is based on the pseudo-spectral approach. The expansion coefficients are forced to collocate at N points. As the underlying eigenfunctions are sine functions, we choose an equally spaced grid that does not include the endpoints [6].

3.1 Free boundary approximation

The free boundary location $\eta(x)$ is not known and must be determined as part of the solution process. There is an almost arbitrary choice of representations available. In this paper, we use local cubic spline interpolants that are forced to collocate at the same points as the series solution. We use these interpolants, rather than a global sine series (which would give exponential convergence [6]), to compare the potential update results with previously obtained results. The free boundary is then updated at $N-1$ points using a cost function—the water table at $x = s$ is fixed at $\eta_s = 1$. For the initial estimate of $\eta(x)$, we use a horizontal line:

$$\eta^{(0)}(x) = 1\,. \tag{28}$$

3.2 Update method

At any point in the iterative process, say the ith iteration, we obtain an improved estimate of the water table $\eta^{(i)}(x)$ using the quasi–Newton method

$$\eta^{(i+1)}(x) = \eta^{(i)}(x) - cf^{(i)}\left(y,\ \Psi^{(i)}(x,y),\ \Phi^{(i)}(x,y)\right)_{y=\eta^{(i)}}\,, \tag{29}$$

where $f^{(i)}(y, \Psi^{(i)}, \Phi^{(i)})$ is a cost function based on equations (5), (13) and c is a constant chosen to enhance the convergence to the true water table $\eta(x)$.

c corresponds to the inverse Jacobian in Newton's method. In previous research [5, 6, 7] we have used

$$f\left(y,\ \Psi^{(i)}(x,y),\ \Phi^{(i)}(x,y)\right)_{y=\eta^{(i)}} = 1 - \Psi^{(i)}(x,\eta^{(i)})\,. \tag{30}$$

In this paper we consider

$$\begin{aligned} f\left(y,\ \Psi^{(i)}(x,y),\ \Phi^{(i)}(x,y)\right)_{y=\eta^{(i)}} &= \eta^{(i)}(x) - \Phi(x,\eta^{(i)})\,, \\ \text{say}\quad f^{(i)}(\eta,\Phi) &= \eta^{(i)} - \Phi^{(i)}(\eta^{(i)})\,. \end{aligned} \tag{31}$$

The cost function equation (29) becomes

$$\begin{aligned} \eta^{(i+1)} &= \eta^{(i)} - c\left(\eta^{(i)} - \Phi^{(i)}(\eta^{(i)})\right) \\ &= (1-c)\eta^{(i)} + c\Phi^{(i)}(\eta^{(i)})\,. \end{aligned} \tag{32}$$

Note that $c = 0$ corresponds to no update, whereas $c = 1$ corresponds to a pure hydraulic potential update of the form used by BIEM.

4 Results

We use a cubic spline to represent the bottom boundary—the equally spaced spline knots are obtained from

$$f(x) = \begin{cases} \frac{4}{10}\left(\cos(x-8)\pi/12 + 1\right)\,, & 0 \le x \le 12\,, \\ \frac{1}{10}\left(\cos(x-12)\pi/8 + 1\right)\,, & 12 < x \le 20\,. \end{cases} \tag{33}$$

This equation corresponds to the bottom boundary used to produce Figure 1. The upstream velocity and base length were $u_s = 0.0275$ and $s = 20$, respectively.

The accuracy of the free boundary condition is determined by examining the boundary errors. The root mean square error ε_N^α in the approximation of $h^\alpha(x)$ by $\psi_N^\alpha(x)$ is

$$\varepsilon_N^\alpha = \left(\frac{1}{s}\int_0^s \left(h^\alpha(x) - \psi_N^\alpha(x)\right)^2\ dx\right)^{1/2} \tag{34}$$

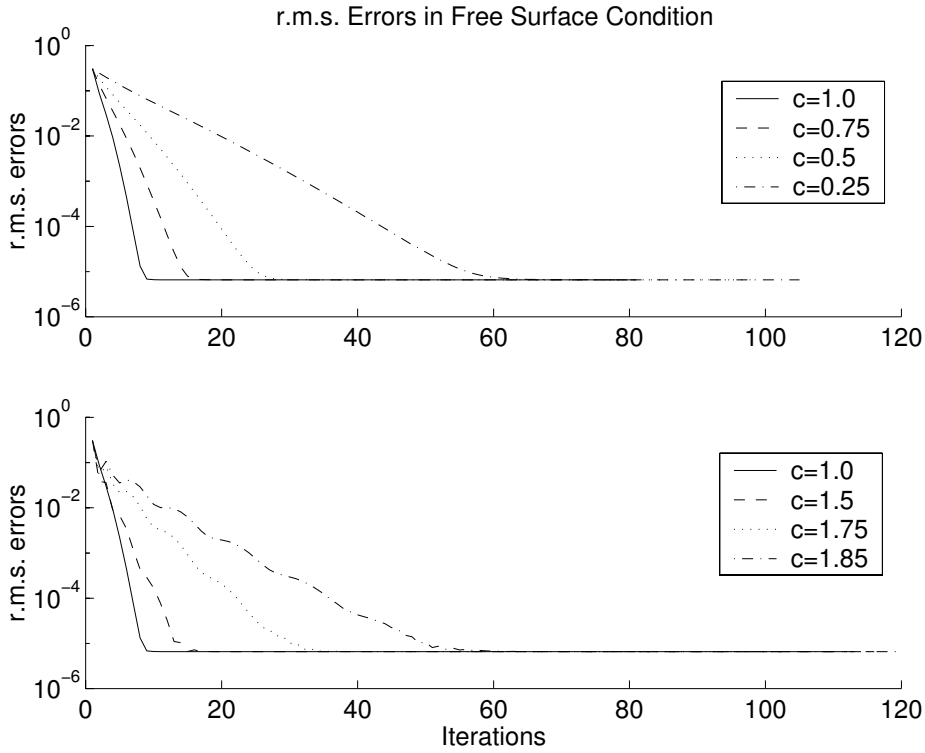

FIGURE 2: RMS error vs iteration count, $0 < c < 2$.

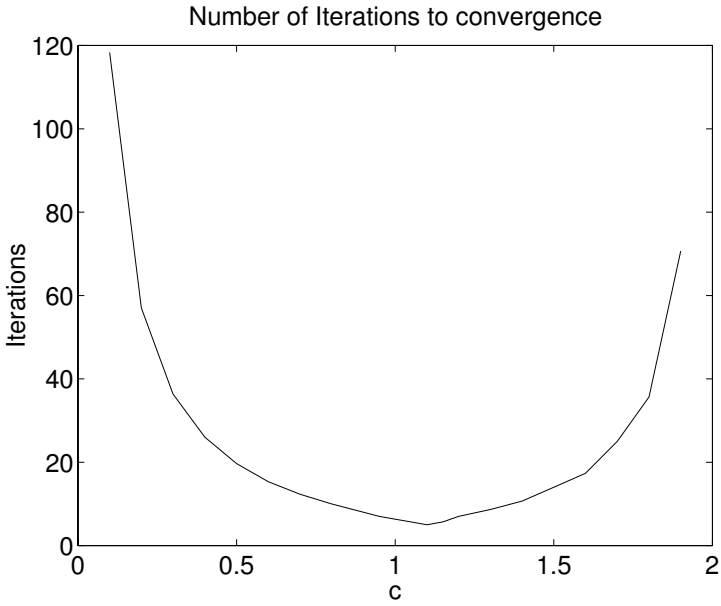

FIGURE 3: Iteration count versus c, $0.1 \leq c \leq 1.9$.

The RMS error in the free boundary approximation is determined in the same way. We use $N = 100$ terms in the series solution (and the number of spline segments in the water table approximation)—this ensures the RMS errors in the top and bottom boundary approximations are less than 10^{-5}. Figure 2 shows plots of the RMS error in the free boundary condition versus the number of iterations for $c = 0.25, 0.5, \ldots, 1.75, 1.85$ respectively. The method failed to converge, when c exceeded ≈ 1.9.

We note that the minimum number of iterations for convergence occurs when $c \approx 1$. Next, we plot the number of iterations to convergence versus c in Figure 3, for 50 equally spaced values of c between 0.1 and 1.9. Clearly, there is a minimum value at approximately $c = 1.1$

5 Discussion

In previous research, accurate solutions for the water table location typically took hundreds of iterations, particularly for the more difficult problems [5]. In addition, convergence could not be achieved when more than ten spline segments were used—previous methods become unstable even when the updates are averaged. Typical values of c for *all* problems, difficult or straightforward, are in the range 10^{-1} to 10^{-3}.

In this paper, we observe similar behaviour of the potential update scheme for smaller values of c—that is, the number of iterations to convergence rapidly increases. For values of c greater than one, the number of iterations increases and the method becomes unstable. We note that roughly ten iterations are required to achieve convergence for the potential update method when c is chosen close to one. This alone is a substantial improvement on previous results, without taking into account the extra stability inherent in the new scheme.

Values of c between 0 and 1 effectively "slow down" the convergence of the scheme by using explicit information from the water table location at the previous iteration. Values of c greater than 1 magnify the emphasis of current value of the potential, and the stability of the scheme deteriorates rapidly. The constant c replaces the inverse Jacobian in the full Newton's method. We conjecture that the optimal value of $c \approx 1$ approaches (in some sense) the inverse Jacobian, for this update method. This is probably why the scheme works so well for BIEM. Clearly, this topic is an area for future research.

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