TECHNICAL NOTES AND MANUALS

A Practical Guide to Public Debt Dynamics, Fiscal Sustainability, and Cyclical Adjustment of Budgetary Aggregates

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A Practical Guide to Public Debt Dynamics, Fiscal Sustainability, and Cyclical Adjustment of Budgetary Aggregates

Prepared by Julio Escolano[1]

The following is a small vade mecum of fiscal formulas, which may be of practical use in fiscal analysis. The three sections that follow derive respectively the formulas for debt dynamics, cyclical and inflation adjustment of budgetary aggregates. They also discuss other relationships for special applications, and some practical implications and usage. Box 1 introduces the notation that will be used throughout. Finally, the annex lists the main formulas for ease of reference.

I. Debt dynamics

The following formulas related to debt dynamics are based on the assumption that changes in liabilities are the result of above-the-line budgetary operations. This means that the debt path is determined by the path of overall fiscal balances (or primary balances and interest bill). In particular, the formulas abstract from the use of privatization proceeds; off-budget operations; gains and losses on (below-the-line) financial operations; or valuation changes due to exchange rate moves. For formulas usage purposes, these operations could be added to the primary balance in the equations below, as they play a similar formal role in the determination of the dynamics of debt. Also, the formulas do not consider central bank deficit financing, such as purchases of government debt (seigniorage).

The formulas can be interpreted as determining the dynamics of gross debt—the discussion that follows is by and large based on this interpretation (although if gross debt becomes negative, it will be interpreted as assets). Under the gross debt interpretation, interest payments are gross interest payments and the primary balance is defined as the overall balance plus gross interest payments (i.e., total revenue less expenditure excluding gross interest payments). Nevertheless, the debt could also be interpreted as net debt. In that case, interest payments would represent net interest payments (interest paid less interest received on assets) and the primary balance should be interpreted as the overall balance plus net interest payments.

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Box 1. Notation Glossary

Variables without a time sub-index are used sometimes in a formula, indicating that they are assumed, for simplicity, to be constant over time or that the time sub-index is unnecessary because the formula is not dynamic. The latter is the case when all variables have the same time sub-index (e.g., structural adjustment formulas). A tilde over a variable (e.g., $\tilde{b}$) denotes cyclically adjusted magnitudes, and a circumflex (e.g., $\hat{b}$) denotes inflation adjusted magnitudes. When necessary, additional notation specific to a section or formula is defined where it is used.

p_t = Primary balance in t, as a ratio to GDP at t.

b_t = Overall balance at t, as a ratio to GDP at t.

d_t = Debt at the end of period t, as a ratio to GDP at t.

π_t = Change in the GDP deflator between $t-1$ and t.

γ_t = Nominal GDP growth rate between $t-1$ and t.

g_t = Real GDP growth rate between $t-1$ and t. Notice that $1+\gamma_t = (1+g_t)(1+\pi_t)$.

i_t = Nominal interest rate in period t; paid in period t on the debt stock outstanding at the end of $t-1$.

r_t Real interest rate in period t. Defined as $r_t \equiv [(1+i_t)/(1+\pi_t)]-1$. Thus, $1+i_t = (1+r_t)(1+\pi_t)$.

Y, y = Nominal and real GDP respectively.

R, E = Nominal revenue and expenditure.

X = Net nominal budgetary aggregates (revenue less expenditure) that do not depend on GDP.

α = Output gap as a ratio to potential output. That is, actual output less potential, divided by potential.

v = Revenue-to-GDP ratio

e = Expenditure-to-GDP ratio

η = Elasticity of revenue to GDP: approximately (up to a first-order approximation) the percent increase in (nominal) revenue per percentage point of gap

κ = Elasticity of expenditure to GDP: approximately (up to a first-order approximation) the percent increase in (nominal) expenditure per percentage point of gap

For ease of notation, we define

$$\lambda_t = \frac{i_t - \gamma_t}{1+\gamma_t} \tag{1}$$

Or, when the analysis is conducted under the assumption that the constituent factors are constant over time,

$$\lambda = \frac{i-\gamma}{1+\gamma} \tag{2}$$

Notice that $1+\lambda = \frac{1+\gamma}{1+\gamma} + \frac{i-\gamma}{1+\gamma} = \frac{1+i}{1+\gamma} = \frac{(1+r)(1+\pi)}{(1+g)(1+\pi)} = \frac{1+r}{1+g}$. In particular,

$$1+\lambda = \frac{1+i}{1+\gamma} = \frac{1+r}{1+g} \tag{3}$$

and therefore

$$\lambda = \frac{i-\gamma}{1+\gamma} = \frac{r-g}{1+g} \tag{4}$$

Primary balance and debt dynamics

The main recursive equation governing the dynamics of the debt ratio is

$$d_t = (1+\lambda_t)d_{t-1} - p_t \tag{5}$$

This difference equation has solution

$$d_N = d_0 \prod_{t=1}^{N}(1+\lambda_t) - \sum_{t=1}^{N}\left[\prod_{i=t+1}^{N}(1+\lambda_i)\right]p_t \tag{6}$$

Under the assumption that λ_t is constant over time ($\lambda_t = \lambda$), the above equations can be simplified as follows.

$$d_t = (1+\lambda)d_{t-1} - p_t \tag{7}$$

$$d_N = d_0(1+\lambda)^N - \sum_{t=1}^{N}(1+\lambda)^{N-t}p_t \tag{8}$$

An important identity, derived from (8), which will be used below is the following.

$$d_0 = (1+\lambda)^{-N} d_N + \sum_{t=1}^{N}(1+\lambda)^{-t}p_t \tag{9}$$

Equation (7) can also be written

$$d_t - d_{t-1} = \lambda d_{t-1} - p_t \tag{10}$$

The latter can be generalized to several periods as follows

$$d_N - d_0 = \lambda\sum_{t=0}^{N-1}d_t - \sum_{t=1}^{N}p_t \tag{11}$$

or

$$d_N - d_0 = \lambda N\overline{d} - N\overline{p} \tag{12}$$

where the averages $\overline{d}$ and $\overline{p}$ are taken over $t = 0,\ldots, N-1$ and $t=1,\ldots,N$ respectively.

Overall balance and debt dynamics

The overall balance is related to the primary balance by the following equation.

$$b_t = p_t - \frac{i_t}{1+\gamma_t} d_{t-1} \tag{13}$$

and the main recursive equation of the debt ratio becomes

$$d_t = \frac{1}{1+\gamma_t}\, d_{t-1} - b_t \tag{14}$$

with solution

$$d_N = d_0 \prod_{t=1}^{N} (1+\gamma_t)^{-1} - \sum_{t=1}^{N} b_t \prod_{i=t+1}^{N} (1+\gamma_i)^{-1} \tag{15}$$

The corresponding solution for time-invariant nominal growth rates is the following

$$d_N = d_0 (1+\gamma)^{-N} - \sum_{t=1}^{N} (1+\gamma)^{-N+t} b_t \tag{16}$$

Assuming constant growth rates, the one-period change in the debt ratio can be expressed as

$$d_t - d_{t-1} = \frac{-\gamma}{1+\gamma}\, d_{t-1} - b_t \tag{17}$$

with multi-period generalizations

$$d_N - d_0 = \frac{-\gamma}{1+\gamma} \sum_{t=0}^{N-1} d_t - \sum_{t=1}^{N} b_t \tag{18}$$

$$d_N - d_0 = \frac{-\gamma}{1+\gamma} N\overline{d} - N\overline{b} \tag{19}$$

where the averages $\overline{d}$ and $\overline{b}$ are taken over $t = 0,\ldots, N-1$ and $t = 1,\ldots,N$ respectively.

Balances compatible with a constant debt ratio

The primary (p^*) and overall (b^*) balances that are compatible with a constant debt ratio (d^*), are given by the following equations (from equations (7) and (14) respectively)

$$p^* = \lambda d^* \qquad (20)$$

$$b^* = \frac{-\gamma}{1+\gamma} d^* \qquad (21)$$

If the overall balance is set at the constant level b^*, the actual debt ratio will asymptotically converge to d^* from any initial level (if nominal growth γ is positive). However, the same is not true for the primary balance p^*, unless the starting debt ratio is already d^* (in which case it will remain constantly at that level), or $\lambda<0$. If the starting debt ratio is not already d^* and $\lambda>0$ (consistent with the modified golden rule, see below), setting the primary balance at the constant level p^* will place the debt ratio on an explosive path: if the initial debt ratio is higher than d^*, the debt ratio will grow exponentially without limit; if the initial debt level is below d^*, the debt ratio will fall toward $-\infty$ (thus soon becoming an "asset" ratio). This is because equation (7) is not stable if $\lambda>0$ (i.e., if the modified golden rule condition holds). Thus, the usual name of debt-stabilizing primary balance for p^* is somewhat of a misnomer. To compute the constant primary balance that will bring the debt ratio to a desired level over the long term, equation (22) below should be used with a suitably large, but finite, N. If $N\rightarrow\infty$, the result will be the primary balance p^* compatible with a constant debt ratio at its current level (not at the target level).

Balances that hit a given debt ratio in finite time

Given an initial debt ratio (d_0), and a target debt ratio (d^*_N) to be achieved in N periods, the constant primary balance (p^*) that reaches the target debt ratio if maintained constant during periods $t = 1,\ldots, N$ is the following (from equation (8)).

$$p^* = \frac{\lambda}{(1+\lambda)^{-N}-1} \left((1+\lambda)^{-N} d^*_N - d_0\right) \qquad (22)$$

The corresponding formula for the constant overall balance (b^*) that reaches the target debt ratio in N periods is the following (from equation (16)).

$$b^* = \frac{-\gamma}{(1+\gamma)((1+\gamma)^N-1)} \left((1+\gamma)^N d^*_N - d_0\right) \qquad (23)$$

Notice that the primary and overall balances cannot be kept both constant (unless the debt ratio is also constant and equal to the initial debt ratio). In other words, the primary balance that solves equation (22) will generally imply a varying overall balance and the overall balance that solves equation (23) will generally imply a changing primary balance. Technically, the values p^* from equation (22) and b^* from equation (23) will not meet equation (13) unless $d_0 = d^*_N$. In the latter case, equations (22) and (23) become respectively equations (20) and (21).

Decomposition of changes in the debt ratio

Unfortunately, there is no formula that allows a clean additive decomposition of changes in the debt ratio into the most interesting underlying factors, such as interest rates, inflation, fiscal adjustment, etc. The following equations, however, come close.

From equations (1) and (5),

$$d_t - d_{t-1} = \frac{i_t}{1+\gamma_t} d_{t-1} - \frac{\gamma_t}{1+\gamma_t} d_{t-1} - p_t \tag{24}$$

which states that the change in the debt ratio equals the impact of interest (positive) and nominal growth (negative) on the debt ratio, plus the contribution of the primary balance. Moreover, notice that

$$\begin{aligned}\frac{\gamma}{1+\gamma} &= \frac{(1+\gamma)-1}{1+\gamma} = \frac{(1+g)(1+\pi)-1}{1+\gamma} = \frac{\pi+g+g\pi}{1+\gamma} = \frac{\pi+g(1+\pi)}{1+\gamma} \\ &= \frac{\pi}{1+\gamma} + \frac{g}{1+g}\end{aligned} \tag{25}$$

Therefore, from (24) and (25),

$$d_t - d_{t-1} = \frac{i_t}{1+\gamma_t} d_{t-1} - \frac{\pi_t}{1+\gamma_t} d_{t-1} - \frac{g_t}{1+g_t} d_{t-1} - p_t \tag{26}$$

The right-hand-side terms in (26) can be seen as the impact on the debt ratio from interest costs, inflation, real growth, and fiscal adjustment. However, the terms representing the interest costs and inflation are not independent, since

$$\frac{i-\pi}{1+\gamma} = \frac{[(1+r)(1+\pi)-1]-\pi}{1+\gamma} = \frac{r+\pi+r\pi-\pi}{1+\gamma} = \frac{r(1+\pi)}{(1+g)(1+\pi)} = \frac{r}{1+g} \tag{27}$$

Thus, another possible decomposition, equivalent to equation (26) but perhaps more enlightening, is the following.

$$d_t - d_{t-1} = \frac{r_t}{1+g_t} d_{t-1} - \frac{g_t}{1+g_t} d_{t-1} - p_t \tag{28}$$

Notice that this equation can be alternatively derived from (4) and (5). Equation (28) shows that the evolution of the debt ratio depends only on the real interest rate, real growth, and fiscal adjustment. Hence, it shows that inflation has an impact on the debt ratio only to the extent that it lowers the real interest rate paid by the government. Otherwise, the higher nominal interest rates associated with higher inflation will fully offset the erosion in the real value of the debt due to inflation. Inflation lowers the real interest rate paid on debt, for example, if debt issued in the past is not indexed to inflation (or denominated in foreign currency) and markets did not fully anticipate future inflation.

Stability of the debt ratio, the budget constraint, and the no-Ponzi game condition

This section discusses the relationships among the stability of the debt ratio, the government budget constraint, and the condition that the government does not run a Ponzi game with its debt financing. It also discusses two other conditions that turn out to be intimately related: that the growth-adjusted interest rate be positive (i.e., that the interest rate exceeds the growth rate of the economy) and the existence of an upper bound to the primary balance ratio to GDP. The short discussion here is conducted from a practical, heuristic standpoint. A more complete analysis can be found in Bartolini and Cottarelli (1994) and Blanchard and Weil (1992), which also explore this topic in the broader context of uncertainty.

Boundedness of the debt ratio

Markets and the public place great importance on a reasonably low and stable ratio of government debt to GDP. They tend to interpret a high and growing debt ratio as a signal of looming public insolvency. Indeed, the fiscal policy framework of many countries contains a mandate to keep the debt ratio below an upper bound—a form of fiscal rule—to reassure economic agents.

This is well founded, as an ever increasing debt ratio would eventually result in a fiscal debt crisis and default—either outright or through inflation or other means. Therefore, in scenarios of practical relevance, feasible trajectories of the debt ratio are bounded above. The condition that the debt ratio be stable (i.e., that it cannot grow forever) can be formally stated as follows.[2]

Condition 1. The debt ratio is bounded above: there is a number H such that for all periods t, $d_t \leq H$.

No-Ponzi game condition

The no-Ponzi game condition (also called transversality condition) essentially means that the government does not service its debt (principal and interest) by issuing new debt on a regular basis.

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[2] The generalization of the definition of bounded debt trajectories to uncertain environments is not straightforward. A commonly used definition is that the expected value of the debt ratio, $E(d_t)$ be bounded (Blanchard and Weil (1992)). This definition, however, has some counter-intuitive features. Among them is that the expected value of the debt ratio need not be bounded even if it is bounded in each of the possible trajectories. This could happen, for example, if the probability of trajectories that converge to increasingly higher debt ratios does not decrease fast enough. Conversely, the expected debt ratio may be bounded even if it is unbounded in all trajectories. As an example of the latter, consider two equally probable trajectories $(i = 1,2)$ with debt ratios $d_t^i = (-1)^{t+i}e^t$. The expected value of the debt ratio is zero for all periods, while each of the debt trajectories is itself unbounded. An alternative definition, which avoids these problems—at the expense of higher handling complexity—is that the debt ratio be bounded almost surely—that is, all measurable sets of unbounded trajectories must have zero probability (Bartolini and Cottarelli (1994)).

Condition 2. No-Ponzi game condition.

$$\lim_{N\to\infty}(1+\lambda)^{-N}d_N = 0 \qquad (29)$$

This equality should be interpreted as stating both that the left-hand-side limit exists and that it is equal to zero. Since it requires that, over the long term, the present value of debt must decline towards zero, it implies that, asymptotically, the debt ratio cannot grow at a rate equal or higher than the (growth-adjusted) interest rate—which is what would happen if debt and interest were systematically paid by issuing new debt. Under the no-Ponzi game condition, debt and interest payments cannot be postponed forever.

Government budget constraint

The government's budget constraint states that the net present value of all future primary balances must be sufficient to pay back the initial debt. That is, debt principal and the interest accumulated along the way will eventually have to be paid through large enough primary surpluses.

Condition 3. Government's inter-temporal budget constraint.

$$d_0 = \sum_{t=1}^{\infty}(1+\lambda)^{-t}p_t \qquad (30)$$

This condition should be interpreted as stating both that the right-hand-side limit exists and that it equals the initial debt.

Modified golden rule

Condition 4. Modified golden rule.

$$\lambda>0 \qquad (31)$$

Except when noted, the discussion here is conducted under the assumption that *Condition 4* obtains, at least asymptotically. This means that over the long term the real interest rate r (respectively the nominal interest rate i) exceeds the real growth rate g (respectively the nominal growth rate γ).

When postulated of the dynamic steady state of the economy, *Condition 4* is usually known as the "modified golden rule" and has both theoretical and empirical bases. From a theoretical standpoint, the modified golden rule derives from efficiency considerations of the growth path and the preference of economic agents for current versus future consumption (see Blanchard and Fischer (1989), Chapter 2, p. 45). Empirically, the modified golden rule generally holds for most mature economies (presumed to be around their long-term dynamic steady state) on average over sufficiently long periods. Evidence on the recent period for which reasonably comparable data exist is shown in Table 1. The exceptions in the table (Greece, Ireland, and Spain) correspond

TABLE 1. INTEREST RATE-GROWTH DIFFERENTIAL					
(Nominal growth and interest rates, geometric averages over the period, in percentage points)					
		Nominal interest rate		Interest rate-growth differential	
Country	Nominal growth	Effective[1]	Market rate on long-term benchmark bond[2]	Effective	Market rate on long-term benchmark bond
	(A)	(B)	(C)	(B)-(A)	(C)-(A)
Germany (1992 - 2008)	2.9	5.7	5.1	2.8	2.2
Ireland (1991 - 2008)	9.3	5.5	5.8	–3.8	–3.5
Greece (1992 - 2008)	9.1	8.7	9.6	–0.5	0.4
Spain (1995 - 2008)	7.1	5.8	5.4	–1.3	–1.8
France (1991 - 2008)	3.6	5.9	5.5	2.3	1.9
Italy (1991 - 2008)	4.6	7.5	7.0	2.9	2.4
Netherlands (1991 - 2008)	5.1	6.4	5.4	1.3	0.3
Austria (1991 - 2008)	4.1	5.5	5.5	1.4	1.3
Portugal (1991 - 2008)	6.5	8.0	6.9	1.5	0.4
Finland (1991 - 2008)	4.1	6.7	6.2	2.6	2.1
Sweden (1995 - 2008)	4.6	5.3	5.3	0.7	0.7
United Kingdom (1991 - 2008)	5.3	6.8	6.2	1.5	0.9
United States (1991 - 2008)	5.2	5.9	5.6	0.7	0.3
Japan (1991 - 2008)	0.8	2.9	2.5	2.2	1.7
Canada (1991 - 2007)	4.9	8.7	6.2	3.8	1.3
Average				1.2	0.7

Source: AMECO (October 22, 2009 vintage), European Commission; Datstream.
[1]Interest paid in year t as a ratio to debt outstanding at the end of year t–1.
[2]10-year benchmark central government bond, when available; closest bond available otherwise.

to economies where interest rates fell sharply after joining EMU (arguably with little relation to domestic conditions) while growth accelerated. Also, in the few economies for which earlier data are available, the evidence from the 1960s and 1970s tends to contradict *Condition 4*, as the acceleration of prices during that period was apparently unanticipated by government debt markets, driving growth–adjusted interest rates (and often real interest rates) temporarily below zero. For practical calculations, it is often assumed that $\lambda = 0.01$ (1%).

Boundedness of the primary balance

Condition 5. The primary balance as a ratio to GDP is bounded above: there is a number M such that for all periods t, $p_t \leq M$.

The assumption that the primary balance-to-GDP ratio is bounded above is reasonable: it must be bounded, for example, by 100 percent of GDP—when all GDP would be taxed away and government spending would be zero. In practice, of course, the effective primary balance ceiling—although uncertain and time- and country-specific—is more likely to be only a few percentage points of GDP, reflecting both economic and political economy constraints.

Relationships among the stated conditions

The following three propositions establish some key inter-relations among these five conditions.

Proposition 1. The no-Ponzi game condition (*Condition 2*) is equivalent to the government inter-temporal budget constraint (*Condition 3*).

Proof. The equivalence between (29) and (30) follows from taking the limit when $N \to \infty$ in identity (9).

$$d_0 = (1+\lambda)^{-N} d_N + \sum_{t=1}^{N} (1+\lambda)^{-t} p_t$$

$$d_0 = \lim_{N\to\infty} (1+\lambda)^{-N} d_N + \lim_{N\to\infty} \sum_{t=1}^{N} (1+\lambda)^{-t} p_t$$

Either both limits of the two right-hand-side terms fail to exist, or if one of them exists, so does the other. Moreover, if they exist and $\lim_{N\to\infty}(1+\lambda)^{-N} d_N = 0$ (the no-Ponzi game condition obtains), then $d_0 = \lim_{N\to\infty} \sum_{t=1}^{N} (1+\lambda)^{-t} p_t = \sum_{t=1}^{\infty} (1+\lambda)^{-t} p_t$ (the inter-temporal budget constraint obtains), and vice versa. □

Proposition 2: If $\lambda > 0$ (*Condition 4*) and the primary balance ratio is bounded above (*Condition 5*), then the no-Ponzi game condition (*Condition 2*) implies that the debt ratio is also bounded above (*Condition 1*).

Proof. Since the primary balance is bounded above, let *M* be that bound. That is, for all year t, $p_t \le M$. Assume, contrary to the proposition, that the debt ratio is not bounded above. Then, since $\lambda > 0$, for any arbitrary positive value $a > 0$, there exists a future year k for which d_k is large enough so that $\lambda d_k \ge M + \lambda a$. For ease of presentation and without loss of generality, let us redefine k as the initial period. Then,

$$\lambda d_0 \ge M + \lambda a$$

and therefore

$$M \le \lambda \mathrm{d}_0 - \lambda a$$

Moreover, since the primary balance is bounded above by *M*, for all *t*,

$$p_t \le M \le \lambda d_0 - \lambda a$$

and from equation (9), for all year N,

$$d_0 = (1+\lambda)^{-N} d_N + \sum_{t=1}^{N} (1+\lambda)^{-t} p_t$$

$$\leq (1+\lambda)^{-N} d_N + \sum_{t=1}^{N} (1+\lambda)^{-t} M$$

$$\leq (1+\lambda)^{-N} d_N + \sum_{t=1}^{N} (1+\lambda)^{-t} (\lambda d_0 - \lambda a)$$

$$= (1+\lambda)^{-N} d_N + (d_0 - a) \sum_{t=1}^{N} \lambda (1+\lambda)^{-t}$$

Taking $\lim_{N \to \infty}$ in the above expression, and using the no-Ponzi game condition (29) and the identity $\sum_{t=1}^{\infty} \lambda(1+\lambda)^{-t} = 1$ for $\lambda > 0$, results in the inequalities

$$d_0 \leq d_0 - a$$
$$a \leq 0$$

This contradicts $a > 0$, which holds by construction, and therefore, the debt ratio must be bounded above. □

The above proof of *Proposition 2* also contains a rationale for why the debt ratio must not only be stable, but also perceived as low. If the debt ratio ever went beyond the point where the largest feasible primary balance would be insufficient to pay the interest bill (i.e., stabilize the debt ratio), then, after that point, the debt ratio would grow unstoppably—leading inexorably to a fiscal crisis and default. However, the maximum feasible primary balance is not known with certainty (until after it is reached) and it varies across countries, and political and economic conjunctures.[3] Thus, when the debt ratio is high, the reaction of investors to negative news is likely to be highly non-linear. Even relatively moderate economic, political, or debt shocks could prompt a fiscal crisis if investors think that the debt ratio may be about to cross the point of "non-return."

A sort of converse of *Proposition 2* also obtains, although it requires, mainly for technical reasons, the additional assumption that the debt ratio is not only bounded above but also below. This means that the government will not accumulate assets (negative debt) that grow without limit as a ratio to GDP. This is also a reasonable policy assumption, particularly as that policy would not be optimal. This additional assumption could have been avoided by defining *Conditions 2 and 3* as inequalities, at the cost of a much more complex notation and mathematical argumentation (see Bartolini and Cottarelli (1994)). On the other hand, the converse of *Proposition 2* does not require the assumption that the primary balance has an upper bound (*Condition 5*), as did *Proposition 2*.

Proposition 3: If $\lambda > 0$ (*Condition 4*), boundedness above (*Condition 1*) and below of the debt ratio implies that the no-Ponzi game condition (*Condition 2*) holds.

Proof. Since the debt ratio is bounded above and below, there is a positive number J such that for all period N,

[3] See Blanchard (1984) for a discussion of debt sustainability based on this principle.

$$J \geq d_N \geq -J$$

$$(1+\lambda)^{-N} J \geq (1+\lambda)^{-N} d_N \geq -(1+\lambda)^{-N} J$$

Given that $\lim_{N\to\infty}(1+\lambda)^{-N} J = 0$ since $\lambda > 0$, taking the limit when $N \to \infty$ in the above inequality results in the following.

$$\lim_{N\to\infty}(1+\lambda)^{-N} J \geq \lim_{N\to\infty}(1+\lambda)^{-N} d_N \geq \lim_{N\to\infty} -(1+\lambda)^{-N} J$$

$$0 \geq \lim_{N\to\infty}(1+\lambda)^{-N} d_N \geq 0$$

$$\lim_{N\to\infty}(1+\lambda)^{-N} d_N = 0$$

This is the no-Ponzi game condition. □

Taken together, *Propositions 1, 2, and 3* imply that under sensible conditions (including that, at least in the long term, the interest rate exceeds the growth rate) the following are equivalent: (i) debt and interest are not rolled over systematically; (ii) existing debt is eventually paid in full (including accumulated interest) through future primary surpluses; and (iii) the debt ratio is kept below a ceiling.

The assumption that, at least asymptotically, the interest rate exceeds the growth rate ($\lambda > 0$), *Condition 4*) plays a key role in the previous discussion. However, as indicated above, empirically it may fail to hold for long periods in some economies. What are the implications if the growth rate exceeds the interest rate (i.e., $\lambda < 0$)?

Conventional economic theory suggest that in those cases, the inter-temporal allocation could be improved (in a welfare-enhancing sense) if private agents consumed more now through borrowing at the low interest rate and rolled over their debt—which would still decline as a ratio to their income, since growth exceeds the interest rate. Indeed, many of the historical episodes during which the growth-adjusted interest rate was negative were accompanied by large credit expansions.

A similar logic applies to the dynamics of government debt. Essentially, the government can incur a given amount of debt and postpone payment as long as $\lambda \leq 0$ without the debt snowballing. It can roll over debt and interest and still see its debt ratio decline (if $\lambda < 0$) or stay constant (if $\lambda = 0$). This is because the erosion of the debt ratio due to growth will (more than) offset the increase in the debt ratio stemming from capitalization of interest (equations (24) or (28)). Therefore, keeping the debt ratio stable no longer implies abiding by the no-Ponzi game condition. In addition, *Proposition 1* (the equivalence between the no-Ponzi game and budget constraint conditions) still applies, which means that maintaining a stable debt ratio does not require abiding by the budget constraint either. For example, the government could incur in a primary deficit for some time, accumulating a given amount of debt, and then maintain a primary balance of zero thereafter (hence, neither paying interest nor principal) and the debt ratio would not increase—rather, it would decline towards zero if $\lambda < 0$. In fact, any level of primary deficit is compatible with a stable debt ratio: this follows from equation (20) with

$\lambda < 0$. If the no-Ponzi game condition holds, then the debt ratio is not only bounded above, but it converges to zero (as a consequence of (29) with $\lambda < 0$).

Sustainability indicator

As discussed above, either equations (6) or (8) for the primary balance—or equations (15), (16), or (23) for the overall balance—can be used to determine the policy path that will bring the debt ratio to a predetermined level, asymptotically or in a given number of periods. This is a practical approach to determining an appropriate policy plan over the medium term. However, this approach is contingent on a given debt target. It does not necessarily provide an absolute benchmark to judge the degree of sustainability (or un-sustainability) of policies, particularly in a cross-country sample or for the same country over time if the appropriate target debt ratio varies. The following indicator has been suggested as a benchmark for that purpose. It is used by the European Commission in assessing long-term fiscal sustainability of EU countries in the face of aging costs (under the name of "s2" indicator).

Given a path of primary balances for all future periods ($\{p_t\}_{t=1}^{\infty}$), the sustainability indicator s is defined as the fixed annual addition at perpetuity (expressed as a ratio to the contemporaneous GDP) to the primary balances that would render the sequence of primary balances sustainable—that is, feasible as defined by the inter-temporal government budget constraint given by equation (30). Since there is no presumption that the shape of the initial given sequence of primary balances is optimal or that a fixed annual addition is the best policy approach, the indicator should be considered a benchmark and not necessarily a policy recommendation nor a measure of the adjustment needed in any particular year.

Using the budget constraint (30), the sustainability indicator (s) is defined by the following equation

$$d_0 = \sum_{t=1}^{\infty}(1+\lambda)^{-t}(p_t+s) \tag{32}$$

Or, since $\sum_{t=1}^{\infty}(1+\lambda)^{-t} = \lambda^{-1}$,

$$s = \lambda d_0 - \lambda\sum_{t=1}^{\infty}(1+\lambda)^{-t}p_t \tag{33}$$

A commonly used equivalent formulation is the following.

Let $\delta p_t \equiv p_t - p_0$, then

$$s = \lambda d_0 - p_0 - \lambda\sum_{t=1}^{\infty}(1+\lambda)^{-t}\delta p_t \tag{34}$$

The following is a practical consideration for the use of this indicator. If $t = 1$ is the current year, the indicator assesses sustainability of policies in relation to the current, inherited debt ratio. To assess the sustainability of current policies in relation to the debt ratio after the full implementation of current policy plans, $t = 0$ could be taken to be the last year of a medium-term fiscal projection. For example, consider that $t = 0$ is the last year of the WEO projection (i.e., the present year would be $t = -5$). By the end of the medium-term projection ($t = 0$), the impact of current policies will have fully run its course and fiscal aggregates should be close to their structural levels. In addition, for $t = 1,\ldots,\infty$, the primary balance can be set at $p_t = p_0 + \delta p_t$, with δp_t defined as (the negative of) the estimated long-term aging-related costs for each year (or other long-term costs, such as environmental or those from natural resource depletion).

For example, if long-term costs beyond the medium-term forecasting horizon ($t = 0$) are explicitly forecast for $t = 1,\ldots N$ and assumed constant (as a ratio to GDP) thereafter,

$$s = \lambda d_0 - p_0 - \lambda \sum_{t=1}^{N} (1+\lambda)^{-t} \delta p_t - (1+\lambda)^{-N} \delta p_N \qquad (35)$$

Where, as indicated, $t = 0$ corresponds to the last year of the WEO projection. This is meant to assess the sustainability gap that will exist by the end of the medium-term forecasting horizon (at $t = 0$), taking current policies as given until that horizon, and taking into account long-term costs forecasts. If no long-term costs are considered ($\delta p_t \equiv 0$), the above formula just gives the distance between the actual primary balance at the end of the medium-term policy-based forecast and the primary balance that would keep constant the level of debt afterwards.

II. Cyclical Adjustment

This section discusses the methodology for cyclical adjustment of budget aggregates.[4] There are at least two main purposes of performing a cyclical adjustment on budget aggregates: (i) The first purpose is to estimate the underlying fiscal position—that is, what revenue, expenditure, and balance would prevail if output were equal to potential. This may be used to assess current public finance policies abstracting from the impact of the cycle on the budget. (ii) The second purpose is to measure the discretionary fiscal policy contribution to demand—the fiscal stance. These two purposes are often served by the same cyclically adjusted measures. Ideally, however, the methodology should be modified according to the objective. For example, estimating the fiscal stance would call for excluding transfers and interest received from or paid abroad—such as foreign grants or interest payments on debt held by nonresidents—since they do not affect private domestic disposable incomes and thus domestic demand. On the other hand, these exclusions would not be appropriate when estimating the underlying fiscal position, since they are genuine fiscal revenue and expenditure items, often of a recurrent nature. The discussion here abstracts from these considerations.

[4] On this topic, see also Fedelino et al. (2009).

Throughout this section, time sub-indices are dropped since all variables in each equation refer to the same period. The convention used here is that the output gap is positive if actual output is above potential. To be specific,

$$Y = (1+\alpha)\tilde{Y} \tag{36}$$

$$y = (1+\alpha)\tilde{y} \tag{37}$$

Cyclically adjusted ratios to GDP are defined as the ratio of the corresponding adjusted magnitude to *potential* GDP. If the adjusted magnitude is desired as a ratio to actual output, the adjusted quantities calculated according to the formulas below should be divided by $1+\alpha$ For example, in the case of the cyclically adjusted balance the correction would be as follows.

$$\tilde{b}_{\text{ratio to actual GDP}} = \frac{\tilde{b}}{1+\alpha} \tag{38}$$

Relatedly, actual (unadjusted) magnitudes as a ratio to potential GDP can be obtained by multiplying by $1+\alpha$ the conventional ratio of actual magnitudes to actual GDP.

$$b_{\text{ratio to potential GDP}} = (1+\alpha)b \tag{39}$$

Cyclically adjusted fiscal magnitudes

The cyclical adjustment of revenue, expenditure, and the fiscal balance assumes that the elasticities of revenue and expenditure are constant over the interval $[\tilde{Y}, Y]$ if the gap is positive, or $[Y, \tilde{Y}]$ if the gap is negative. The adjustment equation for revenue is derived below. The derivation for expenditure is entirely similar and it is omitted.

Let the elasticity of revenue with respect to output be defined as

$$\eta = \frac{d\ln R(Y)}{d\ln Y}, \tag{40}$$

where $R(Y)$ denotes revenue as a function of output in the interval of interest. The assumption of constant elasticity implies (solving the differential equation (40))

$$\ln R(Y) = \eta \ln Y + constant\ . \tag{41}$$

Equation (41) is sometimes used to estimate elasticity values (e.g., with a regression of historical values, ideally after correcting for the effect of past tax reforms). Evaluating (41) at actual and potential output and subtracting gives

$$\ln \frac{R(Y)}{R(\tilde{Y})} = \eta \ \ln \frac{Y}{\tilde{Y}} = \eta \ln(1+\alpha), \tag{42}$$

and taking exponentials

$$R(\tilde{Y}) = R(Y)(1+\alpha)^{-\eta}\ . \tag{43}$$

This, after dividing by potential output, leads to the main adjustment equations for revenue and, similarly, expenditure and budget balance

$$\tilde{v} = v(1+\alpha)^{1-\eta} \tag{44}$$

$$\tilde{e} = e(1+\alpha)^{1-\kappa} \tag{45}$$

$$\tilde{b} = \tilde{v} - \tilde{e} \tag{46}$$

The most commonly used adjustment equations, however, are the first order approximations to the above equations for small values of α, namely

$$\tilde{v} \approx v(1+(1-\eta)\alpha) \tag{47}$$

$$e \approx e(1+(1-\kappa)\alpha) \tag{48}$$

$$\tilde{b} = \tilde{v} - \tilde{e} \approx b + v\alpha(1-\eta) - e\alpha(1-\kappa) \tag{49}$$

Poor man's cyclical adjustment

In practice, the elasticity of revenue (η) is typically found to be slightly above, but close to one. Also, the elasticity of expenditure (κ) is considered near zero for many countries. The latter is partly because, by definition, κ should reflect only the automatic stabilizers in the expenditure side of the budget (e.g., unemployment benefits), which are typically a small fraction of spending, and should not reflect discretionary actions—even if these are motivated by cyclical developments (e.g., emergency public works to provide employment). These considerations are the basis for a popular shortcut for calculating the cyclically adjusted balance.[5]

If $\eta \approx 1$ and $\kappa \approx 0$,

$$\tilde{v} \approx v \tag{50}$$

$$\tilde{e} \approx e(1+\alpha) \tag{51}$$

$$\tilde{b} \approx b - \alpha e \tag{52}$$

It is surprising that, in practice, (50), (51), and (52) approximate rather well estimates obtained by much more sophisticated methods for many countries. Notice the "rule of thumb" implied by equation (52): the cyclically adjusted balance (in percent of potential output) can be obtained from the actual balance (in percent of actual output) by subtracting one expenditure ratio for each percentage point of gap.

The above equations (50), (51), and (52) show that although the action in the real world (changes in nominal and real variables) is mainly on the revenue side (since $\eta \approx 1$ and $\kappa \approx 0$), the "worksheet action" (the cyclical adjustment) takes place on the expenditure side. This is because cyclical adjustment involves, not only the adjustment of the nominal magnitudes to their potential levels (the numerator's adjustment), but also the re-statement of the ratio in terms of potential GDP rather than actual GDP (the change in the denominator). The adjustment would take place in

[5] See Fedelino et al. (2009).

the revenue side (i.e., in the revenue ratio) if the actual revenue, spending, and balance ratios were first re-stated as ratios to potential GDP, so that actual and adjusted ratios were expressed in the same units (in this case, percentage points of potential GDP). Then, the structural balance would be approximated by subtracting to the actual balance one revenue ratio per percentage point of gap (all in percent of potential GDP): $\tilde{b} \approx b_{\text{ratio to potential GDP}} - \alpha \nu_{\text{ratio to potential GDP}}$.

OECD methodology

For OECD member countries, the OECD publishes estimates of budget elasticities that are widely used for cyclical adjustment (Girouard et al. (2005)). The methodology is based on a separate adjustment of four categories of revenue (corporate and personal income taxes, indirect taxes, and social security contributions) and one category of expenditure (unemployment benefits).

For each of these categories, an specific proxy base is identified, which is available from the national accounts or labor market statistics: gross operating surplus for corporate income, the wage bill for personal income and social security contributions, consumption for indirect taxes, and unemployment for unemployment benefits. The elasticities with respect to output for each category are estimated country-by-country by combining the elasticity of the corresponding budgetary aggregate with respect to its base with the elasticity of each base with respect to GDP. The elasticity of the budgetary aggregate with respect to its base is calculated for each country on the basis of the statutory rates and structure of its tax-benefit system. In turn, the elasticity of the proxy base with respect to GDP is estimated by econometric methods. In some cases, simplifying assumptions are made: for example, due to statistical inconsistencies and poor statistical fit, the elasticity of indirect taxes with respect to GDP is assumed to be one, as is the elasticity of unemployment benefits with respect to unemployment. A detailed discussion of the methodology and country-by-country intermediate results can be found in Girouard et al. (2005).

Let R_i, $i = 1,\ldots 4$ denote, respectively, revenue from corporate income taxes, personal income taxes, indirect taxes, and social security contributions; with corresponding elasticities with respect to GDP represented by η_i, $i = 1,\ldots 4$. Let E denote current primary expenditure and κ its elasticity with respect to GDP. Current primary expenditure is assumed to depend on GDP only through unemployment benefits. Finally, let X represent the net amount of all other budgetary aggregates (i.e., non-tax revenue less capital expenditure and net interest payments), that are assumed to be independent of GDP—that is, to have zero elasticity with respect to GDP. Then, the cyclically adjusted balance is given by the following equation.

$$\tilde{b} = \left[\sum_{i=1}^{4} R_i (1+\alpha)^{-\eta_i} - E(1+\alpha)^{-\kappa} + X\right] \Big/ \tilde{Y} \qquad (53)$$

Equivalently, this equation can be expressed in ratios to GDP as follows.

$$\tilde{b} = \sum_{i=1}^{4} v_i (1+\alpha)^{1-\eta_i} - e(1+\alpha)^{1-\kappa} + x(1+\alpha) \qquad (54)$$

Where $v_i = R_i/Y$, $i = 1,\ldots,4$, $e = E/Y$, and $x = X/Y$.

The estimated values of the elasticities (η_i, $i = 1,\ldots4$, and κ) reported in Girouard et al. (2005) are presented in Table 2. The last column of Table 2 shows the effect on the overall balance (in percentage points of GDP) of a one percentage point increase in GDP—a measure of the size of the automatic stabilizers—calculated using 2003 budgetary outturns as weights (i.e., for v_i, $i = 1,\ldots,4$, e, and x).

Commodity exporters

The adjustment of fiscal magnitudes in the case of commodity exporters depends on the particular arrangements or fiscal rules in place, and on the structure of the economy. Generally, it involves the adjustment of expenditure and non-commodity revenue as in equations (47) and (48); and the computation of a structural non-commodity balance as in (49). Then, an estimate of structural commodity revenue must be computed and added to the structural non-commodity balance to obtain the overall structural balance. The gap is sometimes estimated based on the non-commodity GDP.

The estimate of structural commodity revenue can be defined as the revenue that would accrue to the budget if the relevant commodity prices were set at their expected long-term average (a la Chile). In principle, a structural exchange rate would have to be computed by cleaning the actual exchange rate of terms-of-trade effects prompted by deviations of the actual commodity price from its structural level. If the commodity revenue is deposited in a fund and it is intended that the budget will only spend the income from the fund (but not the principal), then the structural commodity-related revenue could be computed based on a structural rate of return applied to the actual fund's balance—its expected long term rate of return (a la Norway).

III. Inflation Adjustment

Inflation distorts virtually all budgetary aggregates in the revenue and expenditure sides and consequently, the budget balance. As in the case of cyclical adjustment, it may be of interest to estimate the underlying fiscal position that would have prevailed in the absence of inflation and the fiscal policy contribution to demand. Also, as in the case of cyclical adjustment, these two objectives may require distinct adjustments. For example, lags in tax collection relative to the time when the tax obligation arises will typically result in an erosion of government revenue in real terms and in percent of GDP—this is the Tanzi effect (Tanzi (1977)). Ideally, an adjustment that sought to estimate the level of revenue that would have prevailed in the absence of inflation would correct for the Tanzi effect. On the other hand, the government losses are genuine income for taxpayers (similar to a tax cut), who benefit from the Tanzi effect—and thus, it should not be netted out in a measure of the fiscal contribution to demand. Again, the treatment here abstracts from these differences.

TABLE 2. OECD METHODOLOGY: SUMMARY OF ELASTICITIES[1]						
(Elasticities with respect to the output gap)						
	Corporate Tax	Personal Tax	Indirect Taxes	Social Security Contributions	Primary Current Expenditure	Overall Balance[2]
	η_1	η_2	η_3	η_4	κ	
United States	1.53	1.30	1.00	0.64	–0.09	0.34
Japan	1.65	1.17	1.00	0.55	–0.05	0.33
Germany	1.53	1.61	1.00	0.57	–0.18	0.51
France	1.59	1.18	1.00	0.79	–0.11	0.53
Italy	1.12	1.75	1.00	0.86	–0.04	0.53
United Kingdom	1.66	1.18	1.00	0.91	–0.05	0.45
Canada	1.55	1.10	1.00	0.56	–0.12	0.38
Australia	1.45	1.04	1.00	0.00	–0.16	0.39
Austria	1.69	1.31	1.00	0.58	–0.08	0.47
Belgium	1.57	1.09	1.00	0.80	–0.14	0.52
Czech Republic	1.39	1.19	1.00	0.80	–0.02	0.39
Denmark	1.65	0.96	1.00	0.72	–0.21	0.59
Finland	1.64	0.91	1.00	0.62	–0.18	0.48
Greece	1.08	1.80	1.00	0.85	–0.04	0.47
Hungary	1.44	1.70	1.00	0.63	–0.03	0.47
Iceland	2.08	0.86	1.00	0.60	–0.02	0.37
Ireland	1.30	1.44	1.00	0.88	–0.11	0.38
Korea	1.52	1.40	1.00	0.51	–0.04	0.22
Luxembourg	1.75	1.50	1.00	0.76	–0.02	0.47
Netherlands	1.52	1.69	1.00	0.56	–0.23	0.53
NewZealand	1.37	0.92	1.00	0.00	–0.15	0.37
Norway (mainland)	1.42	1.02	1.00	0.80	–0.05	0.53
Poland	1.39	1.00	1.00	0.69	–0.14	0.44
Portugal	1.17	1.53	1.00	0.92	–0.05	0.46
Slovak Republic	1.32	0.70	1.00	0.70	–0.06	0.37
Spain	1.15	1.92	1.00	0.68	–0.15	0.44
Sweden	1.78	0.92	1.00	0.72	–0.15	0.55
Switzerland	1.78	1.10	1.00	0.69	–0.19	0.37
OECD average	1.50	1.26	1.00	0.71	–0.10	0.44
Euro area average	1.43	1.48	1.00	0.74	–0.11	0.48
New EU members average	1.38	1.15	1.00	0.71	–0.06	0.42

Source: Girouard et al. (2005), page 22.
[1]Based on 2003 weights. Averages are unweighted.
[2]Semi-elasticity. It measures the change of the budget balance, as percentage of GDP, for a 1% change in GDP.
Share weighting based on 2003.

Inflation will affect revenue and spending through multiple channels. There is, however, no established methodology of general application to correct for the automatic effects of inflation on budgetary primary revenue and primary expenditure. This is because inflation effects are highly dependent on country-specific institutional and administrative arrangements. On the revenue side, collection lags will give rise to the Tanzi effect which could be sizable in some countries if inflation is high and the tax system does not provide for advanced payments. Pay-as-you-go withholding, monthly VAT payments, and advanced or indexed corporate payments could mitigate this effect. Also, in the absence of full inflation-indexed accounting by taxpayers, real revenue from income tax could increase due to the taxation of inflation-related profits (e.g., due to FIFO accounting of inventories)—although these "paper" profits do not increase taxpayers' real net worth. Also, under a progressive personal income tax without an inflation-indexed rate schedule, there will be bracket creeping, as taxpayers fall into higher and higher rate brackets due to the effect of inflation on their nominal incomes—even though in real terms they may not be any richer. Inflation effects on primary spending will depend on the extent of inflation indexation of public wages, pensions, and other transfers, and on the existence of payment lags or arrears.

While there is no methodology of general application, the impact of inflation on the budgetary aggregates could be significant in the presence of high inflation. Therefore, an attempt should be made to assess this impact, at least on key areas—such a tax collection lags. For this purpose, in the absence of detailed institutional information, time series regressions can be used to ascertain the average elasticities of revenue and expenditure items (possibly expressed as ratios to GDP) with respect to inflation.

Inflation adjustment of interest payments, in contrast, lends itself to a more standardized treatment. The impact of inflation on this budget line is likely to be of a first order of magnitude if debt is high. For example, if real growth is zero, debt is 50 percent of GDP, the real interest rate is 5 percent, and inflation is 10 percent, the increase in the interest bill due to inflation could be of close to 200 percent or almost 5 points of GDP—assuming that inflation is fully anticipated and debt is rolled over annually.

The measure of the fiscal balance with interest payments adjusted to eliminate the impact of inflation is called the operational balance (see Tanzi et al. (1987)). The operational balance reduces interest payments by the amount that compensates lenders for the erosion of the real value of their claims—notionally equal to the inflation rate times the outstanding debt. From the standpoint of assessing the fiscal stance, this correction is pertinent, since it can be argued that the inflation component of interest payments does not add to the income of the recipients, as it only maintains their net worth in real terms. Rational economic agents will not consider the inflation component of interest payments as net income but rather as principal amortization. In this vein, the inflation component of interest payments does not contribute to demand more than debt repayment does. Also from the standpoint of estimating the underlying balance that would prevail in the absence of inflation, eliminating the inflation compensation component is also necessary—as this expense would not arise if prices were stable.

If inflation (measured here by the change in the deflator of GDP) is fully reflected in the nominal interest rate paid on government debt, the operational balance ($\hat{b}$) is the actual overall balance increased by the inflation-induced erosion of the real value of debt. From (13) and (27) follows that it also equals the primary balance less the real component of interest payments.

$$\begin{aligned}\hat{b}_t &= b_t + \frac{\pi_t}{1+\gamma_t} d_{t-1} \\ &= p_t - \frac{i_t}{1+\gamma_t} d_{t-1} + \frac{\pi_t}{1+\gamma_t} d_{t-1} \\ &= p_t - \frac{r_t}{1+g_t} d_{t-1}\end{aligned} \tag{55}$$

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Annex. Summary of Formulas

This annex lists the main formulas introduced above for easy reference.

Growth-adjusted interest rate

$$\lambda_t = \frac{i_t - \gamma_t}{1+\gamma_t} = \frac{r_t - g_t}{1+g_t} \tag{1,4}$$

$$r_t \equiv [(1+i_t)/(1+\pi_t)]-1;\ g_t \equiv [(1+\gamma_t)/(1+\pi_t)]-1$$

Debt dynamics and primary balance: annual difference equation and its solutions
Time-varying λ_t:

$$d_t = (1+\lambda_t)\mathrm{d}_{t-1} - p_t \tag{5}$$

$$d_N = d_0\prod_{i=1}^{N}(1+\lambda_t) - \sum_{t=1}^{N}\left[\prod_{i=t+1}^{N}(1+\lambda_i)\right]p_t \tag{6}$$

Time-invariant $\lambda_t = \lambda$:

$$d_t = (1+\lambda)d_{t-1} - p_t \tag{7}$$

$$d_N = d_0(1+\lambda)^N - \sum_{t=1}^{N}(1+\lambda)^{N-t}p_t \tag{8}$$

$$d_N - d_0 = \lambda\sum_{t=0}^{N-1}d_t - \sum_{t=1}^{N}p_t \tag{11}$$

$$d_N - d_0 = \lambda N\overline{d} - N\overline{p} \tag{12}$$

$$\overline{d} \equiv \frac{1}{N}\sum_{t=0}^{N-1}d_t; \overline{p} \equiv \frac{1}{N}\sum_{t=1}^{N}p_t$$

Overall and primary balances

$$b_t = p_t - \frac{i}{1+\gamma}d_{t-1} \tag{13}$$

Debt dynamics and overall balance: annual difference equation and its solutions
Time-varying γ_t:

$$d_t = \frac{1}{1+\gamma_t}d_{t-1} - b_t \tag{14}$$

$$d_N = d_0\prod_{t=1}^{N}(1+\lambda_t) - \sum_{t=1}^{N}\left[\prod_{i=t+1}^{N}(1+\lambda_i)\right]p_t \tag{15}$$

Time-invariant $\gamma_t = \gamma$:

$$d_t = \frac{1}{1+\gamma}\, d_{t-1} - b_t$$

$$d_N = d_0(1+\gamma)^{-N} - \sum_{t=1}^{N}(1+\gamma)^{-N+t} b_t \tag{16}$$

$$d_N - d_0 = \frac{-\gamma}{1+\gamma}\sum_{t=0}^{N-1} d_t - \sum_{t=0}^{N} b_t \tag{18}$$

$$d_N - d_0 = \frac{-\gamma}{1+\gamma}\, N\bar{d} - N\bar{b} \tag{19}$$

$$\bar{d} \equiv \frac{1}{N}\sum_{t=0}^{N-1} d_t;\ \bar{b} \equiv \frac{1}{N}\sum_{t=1}^{N} b_t$$

Primary balance (p^*) and overall balance (b^*) compatible with a constant debt ratio (d^*)

$$p^* = \lambda d^* \tag{20}$$

$$b^* = \frac{-\gamma}{1+\gamma}\, d^* \tag{21}$$

Constant primary balance (p^*) and overall balance (b^*) that hit a given debt ratio (d^*_N) in a finite number of periods (N) , given initial debt (d_0)

$$p^* = \frac{\lambda}{(1+\lambda)^{-N} - 1}\left((1+\lambda)^{-N} d^*_N - d_0\right) \tag{22}$$

$$b^* = \frac{-\gamma}{(1+\gamma)((1+\gamma)^{N} - 1)}\left((1+\gamma)^{N} d^*_N - d_0\right) \tag{23}$$

Decomposition of changes in the debt ratio

$$d_t - d_{t-1} = \frac{i_t}{1+\gamma_t}\, d_{t-1} - \frac{\gamma_t}{1+\gamma}\, d_{t-1} - p_t \tag{24}$$

$$d_t - d_{t-1} = \frac{i_t}{1+\gamma_t}\, d_{t-1} - \frac{\pi_t}{1+\gamma_t}\, d_{t-1} - \frac{g_t}{1+g_t}\, d_{t-1} - p_t \tag{26}$$

$$d_t - d_{t-1} = \frac{r_t}{1+g_t}\, d_{t-1} - \frac{g_t}{1+g_t}\, d_{t-1} - p_t \tag{28}$$

No-Ponzi game condition

$$\lim_{N\to\infty}(1+\lambda)^{-N}d_N = 0 \tag{29}$$

Government's inter-temporal budget constraint

$$d_0 = \sum_{t=0}^{\infty}(1+\lambda)^{-t}p_t \tag{30}$$

Sustainability indicator

$$s = \lambda d_0 - \lambda\sum_{t=1}^{\infty}(1+\lambda)^{-t}p_t \tag{33}$$

Let $\delta p_t \equiv p_t - p_0$. Then,

$$s = \lambda d_0 - p_0 - \lambda\sum_{t=1}^{\infty}(1+\lambda)^{-t}\delta p_t \tag{34}$$

If $\delta p_t, t = 1,\ldots N$, is known, and assumed constant for $t = N, N+1,\ldots$

$$s = \lambda d_0 - p_0 - \lambda\sum_{t=1}^{N}(1+\lambda)^{-t}\delta p_t - (1+\lambda)^{-N}\delta p_N \tag{35}$$

Cyclical adjustment

Output (nominal, real) and output gap

$$Y = (1+\alpha)\tilde{Y} \tag{36}$$

$$y = (1+\alpha)\tilde{y} \tag{37}$$

Cyclically adjusted variables as a ratio to actual GDP: divide by $1+\alpha$.

$$\tilde{b}_{\text{ratio to actual GDP}} = \frac{\tilde{b}}{1+\alpha}\text{; similarly for other variables} \tag{38}$$

Actual (unadjusted) variables as a ratio to potential GDP: multiply by $1+\alpha$.

$$b_{\text{ratio to potential GDP}} = (1+\alpha)b\text{; similarly for other variables.} \tag{39}$$

Revenue function:

$$\ln R(Y) = \eta \ln Y + constant\text{; similarly for expenditure.} \tag{41}$$

$$R(\tilde{Y}) = R(Y)(1+\alpha)^{-\eta}\text{; similarly for expenditure.} \tag{43}$$

Exact cyclical adjustment

$$\tilde{v} = v(1+\alpha)^{1-\eta} \tag{44}$$

$$\tilde{e} = e(1+\alpha)^{1-\kappa} \tag{45}$$

$$\tilde{b} = \tilde{v}-\tilde{e} \tag{46}$$

Approximation for small gap (α)

$$\tilde{v} \approx v(1+(1-\eta)\alpha) \tag{47}$$

$$\tilde{e} \approx e(1+(1-\kappa)\alpha) \tag{48}$$

$$\tilde{b} = \tilde{v}-\tilde{e} \approx b+v\alpha(1-\eta)-e\alpha(1-\kappa) \tag{49}$$

If $\eta \approx 1$ and $\kappa \approx 0$

$$\tilde{v} \approx v \tag{50}$$

$$\tilde{e} \approx e(1+\alpha) \tag{51}$$

$$\tilde{b} \approx b - \alpha e \tag{52}$$

OECD methodology:

$$\tilde{b} = \left[\sum_{i=1}^{4} R_i(1+\alpha)^{-\eta_i} - E(1+\alpha)^{-\kappa} + X\right] \Big/ \tilde{Y} \tag{53}$$

$$\tilde{b} = \sum_{i=1}^{4} v_i(1+\alpha)^{1-\eta_i} - e(1+\alpha)^{1-\kappa} + x(1+\alpha) \tag{54}$$

Country-by-country estimates of η_i, $i = 1,\ldots 4$, and κ are shown in Table 2.

Inflation adjustment

Operational balance ($\hat{b}$)

$$\begin{aligned} \hat{b}_t &= b_t + \frac{\pi_t}{1+\gamma_t} d_{t-1} \\ &= p_t - \frac{i_t}{1+\gamma_t} d_{t-1} + \frac{\pi_t}{1+\gamma_t} d_{t-1} \\ &= p_t - \frac{r_t}{1+g_t} d_{t-1} \end{aligned} \tag{55}$$