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A PRECISE UPPER BOUND FOR THE ERROR OF INTERPOLATION OF STOCHASTIC PROCESSES

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Abstract. We obtain a precise upper bound for the truncation error of interpolation of functions of the Paley–Wiener class with the help of finite Whittaker–Kotelnikov–Shannon sums. We construct an example of an extremal function for which the upper bound is achieved. We study the error of interpolation and the rate of the mean square convergence for stochastic processes of the weak Cramér class. The paper contains an extensive list of references concerning the upper bounds for errors of interpolation for both deterministic and stochastic cases. The final part of the paper contains a discussion of new directions in this field.

1. Introduction

Recovering a continuous signal from its discrete readings and estimation of the amount of information lost due to the discretization procedure is one of the fundamental problems in the theory of interpolation and approximation.

Let $\mathbf{X}$ be a normed space equipped with a norm $\|\cdot\|_{\mathbf{X}}$. Assume that the structure of $\mathbf{X}$ admits the approximation

$$f(x) = \sum_{n\in\mathbb{Z}} f(t_n)S(x,t_n), \qquad f \in \mathbf{X}, \tag{1}$$

where $x \in \mathbb{R}$, $\{t_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ is the sampling set, and S is the sampling function. The latter relation is one of the basic tools in signal processing.

It is a common approach in various numerical applications to consider the truncated version of (1), namely

$$Y_{\mathfrak{J}}(f;x) = \sum_{n\in\mathfrak{J}} f(t_n)S(x,t_n), \qquad \mathfrak{J} \subset \mathbb{Z}, \tag{2}$$

where $\mathfrak{J}$ is a finite set.

The classical problem is to obtain an upper bound $\varphi_{\mathfrak{J}}(f;x)$ for the truncation error $T_{\mathfrak{J}}(f;x)$, namely

$$\|T_{\mathfrak{J}}(f;x)\| = \|f(x) - Y_{\mathfrak{J}}(f;x)\| \le \varphi_{\mathfrak{J}}(f;x), \qquad f \in X.$$

It is very important for numerical applications to find simple upper bounds that do not involve infinite products, iterative procedures, or unknown values of the function.

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The problem of evaluating an optimal/minimal size $|\mathfrak{J}|$ of the interpolating sum in (2) for a given truncation error $\varepsilon > 0$ of approximation (interpolation) is solved in some earlier papers by finding a minimal $|\mathfrak{J}|$ such that

$$\sup_x \varphi_{\mathfrak{J}}(f;x) \le \varepsilon.$$

More details concerning this approach can be found in §3.5 and §4.5 in the survey paper [5] or in §IV.C and §VI of the classical paper [15], or in §III.A of [17]. Chapter 11 of the book [12] contains a useful discussion of the analysis of errors. More details on the upper bounds for the truncation errors for various procedures of interpolation of a signal and on different approaches to this topic are given in Section 5.

We mention papers [9, 13, 25] containing results for the most general case of nonperiodic readings and their stochastic counterparts [18]. However the deviation between the estimates and optimal solutions is not studied in these papers (optimal solutions correspond to $T_{\mathfrak{J}}(f;x)$ rather than to $\varphi_{\mathfrak{J}}(f;x)$).

The main aim of this paper is to obtain optimal solutions. We propose to consider the following function:

$$\varphi_{\mathfrak{J}}(f;x) = \varepsilon_{|\mathfrak{J}|}\|f\|. \tag{3}$$

This means that our aim is to obtain a pointwise upper bound that holds for all arguments x and such that the constant $\varepsilon_{|\mathfrak{J}|}$ cannot be improved; that is, there exist at least one function f and at least one real number x such that

$$\|T_{\mathfrak{J}}(f;x)\| = \varepsilon_{|\mathfrak{J}|}\|f\|. \tag{4}$$

We denote this interpolation procedure by $f(x) \overset{\varepsilon}{\approx} Y_{\mathfrak{J}}(f;x)$.

At first glance, the described problem is similar to the so-called aliasing problem (see [5, 12]). Nevertheless, these problems are totally different. The difference between f and the *nontruncated* series (1) is studied in the aliasing problem. Thus the errors in this problem appear due to the difference between the real spectrum and the one we use for the model.

For the sake of simplicity we consider the one-dimensional case in the paper. We find an optimal value of ε^* and construct a function for which the upper bound (4) is attained. As far as we know, this is the first result for the above setting where the optimal value of $\varepsilon_{|\mathfrak{J}|}$ is evaluated in an explicit form. We show that

$$\lim_{|\mathfrak{J}|\to\infty} Y_{\mathfrak{J}}(f;x) = f(x)$$

for the optimal $\varepsilon^* = \varepsilon^*_{|\mathfrak{J}|}$ and for several types of convergence (pointwise, uniform, etc.). We also consider the stochastic approximation of random processes $\xi(x)$ belonging to a weak Cramér class. We use results obtained in Section 2 for the deterministic case to solve the interpolation problem $\xi(x) \overset{\varepsilon}{\approx} Y_{\mathfrak{J}}(\xi;x)$ in the $L^2(\Omega)$ sense. Finally, we discuss problems of the mean square convergence of $Y_{\mathfrak{J}}(\xi;x)$ to $\xi(x)$.

2. The exact upper bound in the sampling theorem for deterministic signals

Consider the Paley–Wiener class of all complex-valued functions of $L^2(\mathbb{R})$ for which the support of the Fourier spectrum is $[-\pi,\pi]$. Following [12, §6] we denote this class by PW^2_π. According to the classical Whittaker–Kotelnikov–Shannon theorem, every function f of the class PW^2_π can be uniquely reconstructed from its values at integer points; namely,

$$f(x)=\sum_{n=-\infty}^{\infty}\operatorname{sinc}(x-n)f(n) \tag{5}$$

where

$$\operatorname{sinc}(t):=\begin{cases}\frac{\sin(\pi t)}{\pi t}, & t\neq 0,\\ 1, & t=0.\end{cases}$$

The truncated version of (5), that is,

$$Y_N(f;x):=\sum_{|x-n|\le N}\operatorname{sinc}(x-n)f(n), \tag{6}$$

is commonly used in applications. Our goal is to obtain bounds of the following form:

$$\|f(x)-Y_N(f;x)\|\le\varepsilon\cdot\|f\|, \tag{7}$$

where $\|\cdot\|$ denotes some L_p-norm (the norms on the left-hand and right-hand sides of (7) are possibly different). Consider the norm

$$\|f\|_\infty=\inf\{a>0\colon \text{such that } |f(x)|\le a \text{ for all } x\in\mathbb{R}\}$$

for the left-hand side and the norm $\|f\|_2=\left(\int_{\mathbb{R}}|f(x)|^2\,dx\right)^{1/2}$ for the right-hand side of (7).

Our aim is to find the *minimal* constant $\varepsilon^*=\varepsilon^*_N$ such that, given a number N, inequality (7) holds for all functions of PW^2_π and for the above norms. Another interesting problem is to find the so-called *extremal function* f^* for which inequality (7) becomes an equality for the number ε^*.

Thus we deal with the precise bounds for the truncation error in an approximation. Therefore we want to find the minimal ε such that the difference

$$T_N(f,x):=f(x)-Y_N(f;x)$$

satisfies the inequality

$$\|T_N(f,x)\|_\infty\le\varepsilon\|f\|_2$$

for a given number of terms in (6), for all functions f of PW^2_π, and for all $x\in\mathbb{R}$. Thus we obtain a uniform estimate in $\mathbb{R}$, while other papers deal with estimates that are uniform only in bounded subsets of $\mathbb{R}$.

Theorem 1. *Let $f\in PW^2_\pi$. Then*

$$\|T_N(f,\cdot)\|_\infty\le\left(1-\frac{8}{\pi^2}\sum_{n=1}^{N}\frac{1}{(2n-1)^2}\right)^{1/2}\|f\|_2. \tag{8}$$

Inequality (7) *cannot be improved*; *the extremal function is given by*

$$f^*_N(x):=\sum_{|n-2^{-1}|>N}\frac{\sin(\pi x)}{\pi^2\left(n-\frac{1}{2}\right)(n-x)}. \tag{9}$$

Proof. Applying the Cauchy–Schwarz inequality and Parceval equality ([12, §6.7])

$$\|f\|_2=\left(\sum_{n=-\infty}^{\infty}|f(n)|^2\right)^{1/2},\qquad f\in PW^2_\pi,$$

we obtain an upper bound for the pointwise error:

$$
\begin{aligned}
|f(x)-Y_N(f;x)| &= \Big|\sum_{|x-n|>N} \operatorname{sinc}(x-n)f(n)\Big| \le \sum_{|x-n|>N} |\operatorname{sinc}(x-n)f(n)| \\
&\le \Big(\sum_{|x-n|>N} |f(n)|^2\Big)^{1/2}\Big(\sum_{|x-n|>N} \operatorname{sinc}^2(x-n)\Big)^{1/2} \qquad (10)\\
&\le \|f\|_2\Big(\sum_{|x-n|>N} \operatorname{sinc}^2(x-n)\Big)^{1/2}.
\end{aligned}
$$

To evaluate the minimal $\varepsilon=\varepsilon_N$ we consider the behavior of the function

$$\Psi_N(x) := \sum_{|x-n|>N} \operatorname{sinc}^2(x-n) = \sin^2(\pi x)\sum_{|x-n|>N}\frac{1}{\pi^2(x-n)^2} \tag{11}$$

with respect to the argument x. It follows from

$$\frac{1}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(x-n)^2} = \frac{1}{\sin^2(\pi x)} \tag{12}$$

that

$$\Psi_N(x) := 1-\sin^2(\pi x)\sum_{|x-n|\le N}\frac{1}{\pi^2(x-n)^2}. \tag{13}$$

Note that $\sin^2(\pi x)=\sin^2[\pi(1-x)]$ and $\sum_{|x-n|\le N}(x-n)^{-2}$ does not change if $(1-x)$ is substituted for x. Thus $\Psi_N(x)$ is a symmetric function about $x=\frac{1}{2}$. Thus we can restrict our consideration of $\Psi_N(x)$ to the interval $[0,\frac{1}{2}]$, since $\Psi_N(x)$ is a periodic function with period 1. In order to obtain the minimal number ε in (7), we find the maximum of the function $\Psi_N(x)$.

The function $\Psi_N(x)$ can be rewritten in the following form:

$$1+\frac{\sin^2(\pi x)\Psi(1,N+1-x)}{\pi^2}-\frac{\sin^2(\pi x)\Psi(1,-N+1-x)}{\pi^2},$$

where $\Psi(n,x)$ is the polygamma function of order n; that is, $\Psi(n,x)$ is the nth derivative of the digamma function

$$\Psi(x) := (\ln(\Gamma(x)))' = \frac{\Gamma'(x)}{\Gamma(x)}.$$

The graphs of $\Psi_N(x)$ can be depicted for various N with the help of **Maple 8**. All the graphs depicted have the same unimodal form shown in Figure 1 for $N=20$. Therefore it is natural to conjecture that $x=\frac{1}{2}$ is the point of supremum of the function $\Psi_N(x)$.

Let us prove that this is the case, indeed. First we rewrite (11) as follows:

$$\Psi_N(x) = \sum_{k>N}\frac{\sin^2(\pi x)}{\pi^2}\left(\frac{1}{(k-x)^2}+\frac{1}{(x+k-1)^2}\right) :\ = \sum_{k>N}\varphi_k(x). \tag{14}$$

Then we show that each function $\varphi_k(x)$, $k>1$, is increasing in x on the interval $[0,\frac{1}{2}]$. The function $\Psi_N(x)$, as the sum of increasing terms, also is increasing on $[0,\frac{1}{2}]$ and, moreover,

$$\max_{x\in[0,\frac{1}{2}]}\Psi_N(x) = \Psi_N\left(\frac{1}{2}\right). \tag{15}$$

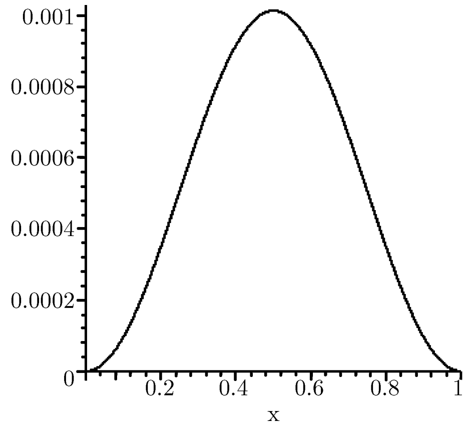

FIGURE 1. $\Psi_N(x)$, $N = 20$

Indeed,

$$\begin{aligned}\frac{\partial \varphi_k(x)}{\partial x} &= \frac{2\sin(\pi x)}{\pi^2 (k-x)^3 (k-1+x)^3} \\ &\quad \times \left\{ \pi (k-x)(k-1+x) \left[(k-x)^2 + (k-1+x)^2 \right] \cos(\pi x) \right. \\ &\qquad \left. - (1-2x) \left[(k-x)^2 + (k-x)(k-1+x) + (k-1+x)^2 \right] \sin(\pi x) \right\}.\end{aligned}$$

Since $\cos(x)$ is concave in the first quadrant, it follows that $\cos(\pi x) \geq 1 - 2x$, $x \in [0, \frac{1}{2}]$. Using the estimate $\sin(\pi x) \leq \pi$ we get

$$\begin{aligned}\frac{\partial \varphi_k(x)}{\partial x} &\geq \frac{2\sin(\pi x)(1-2x)}{\pi (k-x)^3 (k-1+x)^3} \\ &\quad \times \left\{ \left[(k-x)^2 + (k-1+x)^2 \right] \left[(k-x)(k-1+x) - 1 \right] - (k-x)(k-1+x) \right\} \\ &\geq \frac{2\sin(\pi x)(1-2x)}{\pi (k-x)^3 (k-1+x)^3} \\ &\quad \times \{ 2(k-x)(k-1+x)[(k-x)(k-1+x) - 1] - (k-x)(k-1+x) \} \\ &= \frac{4\sin(\pi x)(1-2x)}{\pi (k-x)^2 (k-1+x)^2} \{ (k-x)(k-1+x) - 3/2 \} \geq 0.\end{aligned}$$

The latter inequality holds in view of

$$(k-x)(k-1+x) - 3/2 = (k-1/2)^2 - (1/2-x)^2 - 3/2 \geq (k-1/2)^2 - 7/4 > 0,$$

which is true for all $k > 1$.

This proves that the function $\varphi_k(x)$ is increasing on the interval $[0, \frac{1}{2}]$. Therefore $\Psi_N(x)$ is increasing, too. Now (13) implies that

$$\Psi_N\left(\frac{1}{2}\right) = 1 - \frac{2}{\pi^2} \sum_{n=1}^{N} \frac{1}{\left(n - \frac{1}{2}\right)^2} = 1 - \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2},$$

whence

$$\varepsilon_N^* := \sqrt{\sup_{x \in \mathbb{R}} \Psi_N(x)} = \sqrt{1 - \frac{8}{\pi^2} \sum_{n=1}^{N} \frac{1}{(2n-1)^2}}.$$

Therefore the left-hand side of (10) does not exceed $\varepsilon_N^* \|f\|_2$.

To show that the above number ε_N^* cannot be improved in the problem under consideration, we introduce the function $f_N(n)$ assuming the following values at integer arguments:

$$f_N(n) := \begin{cases} 0, & \text{for } n = -N+1, \dots, N, \\ \operatorname{sinc}\left(n - \frac{1}{2}\right) \equiv \frac{(-1)^{n+1}}{\pi(n-2^{-1})}, & \text{otherwise.} \end{cases}$$

Then we define the extremal function $f_N^*(x)$ according to the Whittaker–Kotelnikov–Shannon formula (5):

$$\begin{aligned} f_N^*(x) &:= \sum_{n=-\infty}^{\infty} \operatorname{sinc}(x-n) f_N(n) = \sum_{|n-2^{-1}|>N} \frac{(-1)^{n+1}\operatorname{sinc}(x-n)}{\pi\left(n-\frac{1}{2}\right)} \\ &= \sum_{|n-2^{-1}|>N} \frac{\sin(\pi x)}{\pi^2\left(n-\frac{1}{2}\right)(n-x)}. \end{aligned}$$

The function $f_N^*(x)$ belongs to the class PW_π^2 since $\{\operatorname{sinc}(x-n)\}_{n\in\mathbb{Z}}$ is an orthogonal basis in PW_π^2. Note also that $f_N(n)$ is the nth Fourier coefficient of $f_N^*(x)$ ([12, §6.7]) and, moreover, the series of squared Fourier coefficients converges:

$$\left(\sum_{n=-\infty}^{\infty} |f_N(n)|^2\right)^{1/2} = \left(\sum_{n=N+1}^{\infty} \frac{8}{\pi^2(2n-1)^2}\right)^{1/2} \le \frac{\sqrt{2}}{\pi\sqrt{N}} < \infty.$$

Inequalities (10) become equalities for the function $f_N^*(x)$ at $x = \frac{1}{2}$. This shows, in particular, that, given N, the constant $\varepsilon^* = \varepsilon_N^*$ in (8) cannot be improved for the function $f_N(x)$. $\square$

Remark 1. The above discussion implies that the set of functions for which inequalities (8) become equalities for ε^* coincides with

$$\left\{\kappa f_N^*(x+m) \colon \kappa \in \mathbb{C},\, m \in \mathbb{Z}\right\}.$$

Now we consider some corollaries of Theorem 1. Note that Theorem 1 can be used to prove that

$$\lim_{N\to\infty} Y_N(f;x) = f(x) \tag{16}$$

in a certain sense. First we define the space of functions

$$PW_{\pi,C}^2 := \left\{f \in PW_\pi^2 \colon \|f\|_2 < C\right\},$$

where $C > 0$.

Corollary 1.1.

$$\lim_{N\to\infty} \sup_{f\in PW_{\pi,C}^2} \|f(x) - Y_N(f;x)\|_\infty = 0.$$

Proof. The equality

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

easily implies that $\varepsilon_N^* \to 0$ as $N \to \infty$. Combining this result and (8) we prove that (16) holds uniformly on $PW_{\pi,C}^2 \times \mathbb{R}$. $\square$

The following assertion contains a precise solution of the problem of finding the minimal number of terms in the approximating sum $Y_N(f;x)$ in (6) if the approximation error ε is fixed.

Corollary 1.2. *For a given level ε of the approximation error, the minimal number $N^*(\varepsilon)$ of terms in the truncated sum is equal to*

$$N^*(\varepsilon) = \min\left\{N\colon \frac{(1-\varepsilon^2)\pi^2}{8} \le \sum_{n=1}^{N} \frac{1}{(2n-1)^2}\right\}.$$

3. Stochastic processes of the weak Cramér class

In Section 4, we obtain the interpolation formula for the class of nonstationary stochastic processes by using the Piranashvili approach [22]. In this section, we briefly discuss some results for stochastic processes of the *weak Cramér class* that are generalizations of the corresponding results of Rozanov [29] and Rao [27] (more detail and references can be found in [16]).

Let $\mathcal{S}_\Lambda$ be a σ-ring of subsets of $\Lambda \subseteq \mathbb{R}$ and $F\colon \mathcal{S}_\Lambda \times \mathcal{S}_\Lambda \mapsto \mathbb{C}$ be a positive definite bimeasure. The Fréchet variation or semivariation of F on (A, B) is defined by

$$\|F\|(A,B) := \sup\left\{\left|\sum_{i=1}^{n}\sum_{j=1}^{m} a_i \bar{b}_j F(A_i, B_j)\right| \colon |a_i| \le 1, |b_j| \le 1,\right.$$
$$\left. \{A_i\}_1^n \in \mathcal{S}(A), \{B_j\}_1^m \in \mathcal{S}(B); A_i A_l = B_j B_k = \varnothing, n \ge 1\right\}$$

where $\mathcal{S}(\Delta) = \{\Delta \cap B\colon B \in \mathcal{S}_\Lambda\}$.

Let $\{\xi(x), x \in \mathbb{R}\}$ be a second-order stochastic process on a certain probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. For the sake of simplicity we assume that $\mathsf{E}\xi(x) = 0$ and that the correlation function $B(x,y) = \mathsf{E}\xi(x)\overline{\xi(y)}$ of the process ξ is determined by a family

$$\{f(x,\lambda)\colon x \in \mathbb{R}, \lambda \in \Lambda\}$$

of $\mathcal{S}_\Lambda$-measurable functions as follows:

$$B(x,y) = \int_\Lambda \int_\Lambda^* f(x,\lambda)\overline{f(y,\mu)}\, F_\xi(d\lambda, d\mu) \tag{17}$$

where "$*$" means that the integral on the right-hand side of (17) exists in the strong Morse–Transue sense with respect to the bimeasure $F_\xi(d\lambda, d\mu)$ that has a bounded variation. We say in this case that the correlation function (17) belongs to the *weak Cramér class*. A stochastic process $\xi(x)$ belongs to the weak Cramér class if its correlation function belongs to the weak Cramér class. The following spectral representation

$$\xi(x) = \int_\Lambda f(x,\lambda)\, Z_\xi(d\lambda) \tag{18}$$

is well known for $\xi(x)$ whose correlation function is of the form (17). Representation (18) means that there exists a stochastic measure $Z_\xi\colon S_\Lambda \mapsto L^2(\Omega)$ such that (18) holds, where the integral is defined in the Dunford–Schwartz sense. The converse is also true, namely if (18) holds with a stochastic measure Z_ξ, then the correlation function of ξ is of the form (17). The semivariation of Z_ξ on A is defined by

$$\|Z_\xi\|(A) := \sup\left\{\left\|\sum_{i=1}^{n} a_i Z_\xi(A_i)\right\|_{L_2} \colon |a_i| \le 1, \{A_i\}_1^n \in \mathcal{S}(A), A_i A_l = \varnothing, n \ge 1\right\},$$

where $\|\cdot\|_{L^2}$ means the norm in $L^2(\Omega)$. The semivariation of the bimeasure F_ξ and that of the stochastic measure Z_ξ are such that $\|Z_\xi\|(A)^2 = \|F_\xi\|(A,A)$ [16].

We denote by $L^2_{MT}(\Lambda; F_\xi)$ the Hilbert space of complex-valued functions on Λ that are square integrable (in the Morse–Transue sense) with respect to the measure F_ξ. Note that the space $L^2_{MT}(\Lambda; F_\xi)$ is isometric to $L^2(\Lambda;\Omega)$, where $L^2(\Lambda;\Omega)$ consists of all stochastic measures

$$\zeta\colon S_\Lambda \times \mathfrak{F} \mapsto \mathbb{C}$$

that are $L^2(\Omega)$-bounded, that is, $\mathsf{E}|\zeta(A)|^2 < \infty$, $A \in \mathcal{S}_\Lambda$.

4. Interpolation of stochastic processes

Our next aim is to investigate the approximation procedure

$$\xi(x) \overset{\varepsilon}{\approx} Y_N(\xi;x) = \sum_{|x-n|\le N} \operatorname{sinc}(x-n)\xi(n) \quad \text{in the } L^2(\Omega) \text{ sense}$$

for a stochastic process $\xi(x)$ of the weak Cramér class. More precisely, we fix N and find $\varepsilon = \varepsilon_N$ such that

$$\|\xi(x) - Y_N(\xi;x)\|^2_{L_2} < \varepsilon \|F_\xi\|(\Lambda,\Lambda).$$

Furthermore, we fix ε and find the mean square optimal interpolator $Y_N(\xi;x)$.

Put

$$\|f(x,\cdot)\|_{\infty,F_\xi} := \inf\bigl\{\alpha\colon \|F_\xi\|(A_\alpha,A_\alpha) = 0,\ A_\alpha = \{\lambda\in\Lambda\colon |f(x,\lambda)| \ge \alpha\}\bigr\}.$$

Theorem 2. *Let $\{\xi(x), x\in\mathbb{R}\}$ be a stochastic process of the weak Cramér class. Assume that $f(\cdot,\lambda)$, as a function of the first argument, belongs to PW^2_π for almost all λ. Then the upper bound of the mean square truncation error $\mathfrak{T}_N(\xi;x) := \|\xi(x) - Y_N(\xi;x)\|^2_{L_2}$ for the discrete approximation procedure $\xi(x) \overset{\varepsilon}{\approx} Y_N(\xi;x)$ is determined by the following inequality:*

$$\|\mathfrak{T}_N(\xi;\cdot)\|_\infty \le C_f^2 \left(1 - \frac{8}{\pi^2}\sum_{n=1}^{N} \frac{1}{(2n-1)^2}\right) \|F_\xi\|(\Lambda,\Lambda), \tag{19}$$

where

$$C_f = \bigl\| \, \|f(\cdot,\cdot)\|_2 \, \bigr\|_{\infty,F_\xi}$$

and the norm $\|\cdot\|_2$ is considered with respect to the first argument of the function f, while the norm $\|\cdot\|_{\infty,F_\xi}$ is considered with respect to its second argument.

Proof. Relation (18) implies that

$$\xi(x) = \int_\Lambda f(x,\lambda)\, Z_\xi(d\lambda).$$

Thus $\xi(x)$ admits the representation

$$\xi(x) = Y_N(\xi;x) + \int_\Lambda \sum_{|x-n|>N} \operatorname{sinc}(x-n) f(n,\lambda)\, Z_\xi(d\lambda)$$

in the $L^2(\Omega)$ sense. Using the isometry between the space $L^2(\Lambda;\Omega)$ and the Hilbert space $L^2_{MT}(\Lambda;F_\xi)$ we get

$$\begin{aligned}
\mathfrak{T}_N(\xi;x) &= \int_\Lambda \int_\Lambda^* \sum_{|x-n|>N} \operatorname{sinc}(x-n) f(n,\lambda) \\
&\qquad\qquad \times \sum_{|x-m|>N} \operatorname{sinc}(x-m)\overline{f(m,\mu)}\, F_\xi(d\lambda,d\mu) \\
&= \int_\Lambda \int_\Lambda^* T_N(f;x,\lambda)\overline{T_N(f;x,\mu)}\, F_\xi(d\lambda,d\mu) \\
&\le \|T_N(f;x,\cdot)\|^2_{\infty,Z_\xi} \|F_\xi(\Lambda,\Lambda)\|,
\end{aligned} \tag{20}$$

where $T_N(f;x,\lambda) = f(x,\lambda) - Y_N(f(\cdot,\lambda);x)$.

The last inequality in (20) follows from the isometry between the spaces $L^2(\Lambda;\Omega)$ and $L^2_{MT}(\Lambda;F_\xi)$ since

$$\|Z_\xi\|(A) = \sup\left\{\left\|\int_A f(\lambda)Z_\xi(d\lambda)\right\|_{L_2} : \|f\|_{\infty,Z_\xi} \le 1,\ f \text{ is } Z_\xi\text{-integrable}\right\},$$

where

$$\|f(x,\cdot)\|_{\infty,Z_\xi} := \inf\bigl\{\alpha\colon \|Z_\xi\|(A_\alpha)=0,\ A_\alpha=\{\lambda\in\Lambda\colon |f(x,\lambda)|\ge\alpha\}\bigr\}$$

(see [16]).

A set A is called a Z_ξ-null set if $\|Z_\xi\|(A)=0$; similarly, A is called an F_ξ-null set if $\|F_\xi\|(A,A)=0$. Since Z_ξ-negligible sets and F_ξ-negligible sets coincide,

$$\|f(x,\cdot)\|_{\infty,Z_\xi} \equiv \|f(x,\cdot)\|_{\infty,F_\xi}.$$

Now we use the upper bound (8) for the truncation error:

$$\|T_N(f;x,\cdot)\|_{\infty,F_\xi} \le \bigl\|\|T_N(f;\cdot,\cdot)\|_\infty\bigr\|_{\infty,F_\xi} \le \varepsilon_N^*\bigl\|\|f(\cdot,\cdot)\|_2\bigr\|_{\infty,F_\xi}.$$

Finally, relation (20) and the preceding discussion imply that

$$\|\mathfrak{T}_N(\xi;x)\|_\infty \le C_f^2(\varepsilon_N^*)^2\|F_\xi\|(\Lambda,\Lambda) = C_f^2\left(1-\frac{8}{\pi^2}\sum_{n=1}^N\frac{1}{(2n-1)^2}\right)\|F_\xi\|(\Lambda,\Lambda),$$

whence the upper bound (19) follows. □

Theorem 3. *Inequality* (19) *cannot be improved in the class of stochastic processes that have the spectral representation of the form* (18) *with $f\in PW_\pi^2$. The extremal process for* (19) *is given by*

$$\eta_\varrho(x) = \sum_{|n-2^{-1}|>N}\frac{\varrho\sin(\pi x)}{\pi^2\left(n-\frac12\right)(n+m-x)},$$

where $m\in\mathbb{Z}$ and the random variable ϱ has finite second moment.

Proof. It follows from the deterministic case that, given a fixed λ, the function

$$|T_N(f;x,\lambda)|$$

attains its maximal value at the point $x=\frac12$ if $f(n,\lambda)$ is multiplicative, that is, if

$$f(n,\lambda)=f_N(n)g(\lambda),$$

where $f_N(x)$ is defined by (9). Putting $g(\lambda)\equiv 1$ we get $f(x,\lambda)=f_N(x)$ and the corresponding value

$$C_f^2 = 1-\frac{8}{\pi^2}\sum_{n=1}^N\frac{1}{(2n-1)^2}.$$

The stochastic process that corresponds to this function $f(x,\lambda)$ is given by

$$\int_\Lambda\left(\sum_{n=-\infty}^{\infty}\operatorname{sinc}(x-n)f_N(n)\right)Z_\xi(d\lambda) = \sum_{|n-2^{-1}|>N}\frac{\int_\Lambda Z_\xi(d\lambda)\sin(\pi x)}{\pi^2(n-\frac12)(n-x)}.$$

Thus the extremal stochastic process can be represented as follows:

$$\eta_\varrho(x) := \sum_{|n-2^{-1}|>N}\frac{\varrho\sin(\pi x)}{\pi^2\left(n-\frac12\right)(n-x)},$$

where $\varrho = \int_\Lambda Z_\xi(d\lambda)$ is a random variable such that

$$\mathsf{E}|\varrho|^2 = \mathsf{E}Z_\xi(\Lambda)\overline{Z_\xi(\Lambda)} = \|F(\Lambda,\Lambda)\| < +\infty.$$

It remains to use Remark 1. □

As one can see from the above representation, the extremal process $\eta_\varrho(x)$ is determined by a single random variable ϱ.

We introduce the subclass $\mathfrak{G}_C$, $C>0$, of weak Cramér stochastic processes by

$$\mathfrak{G}_C := \left\{\xi\colon C_f^2\|F_\xi\|(\Lambda,\Lambda) < C\right\},$$

where C_f is defined in the statement of Theorem 2, and F_ξ is the bimeasure for the weak Cramér class of correlation functions (16) that corresponds to ξ.

Corollary 3.1. *The convergence*

$$\lim_{N\to\infty}\sup_{\xi\in\mathfrak{G}_C}\|\xi(x) - Y_N(\xi,x)\|_{L_2} = 0$$

is uniform for $x\in\mathbb{R}$.

5. Concluding remarks

A. We have already mentioned in Section 1 that the upper bounds for the truncation error are the main tool to solve interpolation problems such as $f(x) \overset{\varepsilon}{\approx} Y_N(f;x)$. Earlier papers dealt with truncations of the usual Whittaker–Kotelnikov–Shannon series (5), that is, with

$$Y_N^0(f;x) := \sum_{|n|\le N}\operatorname{sinc}(x-n)f(n)$$

for both deterministic and stochastic signals f belonging to a certain space of functions. A nice upper bound of the truncation error $T_N(f;x) = f(x) - Y_N^0(f;x)$ allows one to prove various types of convergence (namely pointwise, almost sure, uniform convergence, etc.) of the approximating Whittaker–Kotelnikov–Shannon sequence $Y_N^0(f;x)$ to the original signal f. There are two main approaches for obtaining results of this type. The first approach originated with Belyaev [1] in 1959 who proved that the bound

$$\mathfrak{T}_N(\xi;x) \le \frac{16\pi^2(2+|x|)^2\mathsf{E}|\xi(x)|^2}{(\pi-w)^2N^2}, \qquad w<\pi,$$

is exact in the class of weakly stationary stochastic processes with spectra whose supports belong to $[-w,w]$ (with a band-limited spectrum, in other words). Another result of the paper [1] is that the paths of second-order stationary processes with a band-limited spectrum belong almost surely to the class of functions of exponential type with bounded exponent[1] (see [25]). Using the one-to-one correspondence between functions of exponential type with a finite index and the band-limited spectrum, Piranashvili [22] generalized Belyaev's results and obtained an upper bound of the truncation error of order $\mathcal{O}(N^{-2})$ for all bounded x-subsets of $\mathbb{R}$. Further upper bounds for the truncation error are obtained in [7, 10, 19, 23, 24, 25, 27, 28, 30, 31, 32, 33, 36] under various conditions on the signal function (typically, the conditions are close to the assumption that the signal function is of exponential type). The proofs use similar approaches but specific conditions on the signal function (Lipschitz, Paley–Wiener, Bernstein, a restriction on the spectrum, superdiscretization, the polynomially bounded correlation etc.) lead to different forms of upper bounds. Note that the time shifted function $Y_N(f;x)$ is considered in [9, 18, 26] instead of $Y_N^0(f;x)$.

[1] A relation between functions with a band-limited spectrum and functions of exponential type with bounded exponent is described in the prominent Paley–Wiener theorem.

Other upper bounds of the truncation error deal with a narrow time interval containing the origin. The upper bounds for the truncation error in [2, 3, 4, 11, 14, 34, 35] use the estimation from above of the sum $\sum_{n=a}^{b} |\mathrm{sinc}(x-n)|^q$, $q > 0$, and convexity of the function $\sin(x)$. The analysis of the estimate of the truncation error is done in [20, 21] for $x \in [-N, N]$.

One of the best results known for weakly stationary processes with band-limited spectrum on $[-\pi+\gamma, \pi-\gamma]$, $\gamma \in (0,\pi)$, is due to Cambanis and Masry [6]:

$$\mathfrak{T}_N(\xi; x) \le \frac{2x^2 \mathsf{E}|\xi(x)|^2 \mathrm{sinc}^2(x)}{\sin^2(\gamma/2) N^2}, \qquad x \in [-2, 2].$$

Our method described above allows one to obtain simple upper bounds (8) and (19) being *uniform, minimal, and time shifted*. These bounds of the interpolation error hold for functions of the class PW_π^2 and for stochastic processes of the weak Cramér class whose kernel functions in the spectral representations (18) belong to the class PW_π^2. Note that the earlier papers cited above do not contain examples of *extremal functions* for the approximation $f(x) \overset{\varepsilon}{\approx} Y_N(f; x)$, $f \in PW_\pi^2$.

B. The method used in the proof of Theorems 1–3 and their corollaries raises the following questions.

(i) Does our method work in a more realistic model than that of regular/homogeneous readings for signals with band-limited spectra?

(ii) If the answer to the question (i) is positive, then what are the conditions to be posed on the signal function? Following the classical Yen paper [37] (written in a heuristic style) Flornes et al. [9] proposed to make a correction to the discrete approximation of the Whittaker–Kotelnikov–Shannon sum such that the deviation of the new nodes from the uniform nodes does not exceed a certain constant (this allows one to deal with the interpolation which is stable in the *Landau* sense according to the $\frac{1}{4}$-Kadetz theorem). Yen [37] considered other approaches to the nonregular time discretization ("jitter"). A multivariate analog of these results is obtained by Pogány [26] for the deterministic case and by Olenko and Pogány [18] for random fields of the weak Cramér class.

The contour integration, integral Cauchy theorem, etc. are used in some papers to obtain upper bounds of the truncation error (see, for example, Butzer et al. [5], Higgins [12], Seip [30, 31], Pogány [26], Yao and Thomas [36], and Helms and Thomas [11]. One can conjecture that these methods are also useful to obtain the minimal $\varepsilon^* = \varepsilon_N^*$.

It would be interesting to check our method in all the cases mentioned above. Even an incomplete test of the method related to the abstract harmonic analysis studied in [8] would be of essential interest for the approximation $f(x) \overset{\varepsilon}{\approx} Y_N(f; x)$. Among other topics, Dodson and Beaty [8] study the problem of extremal functions.

Finally, an interesting problem arises if one switches from the mean square convergence to the almost sure convergence. The paper [25] contains some results concerning the rate of convergence for the almost sure convergence.

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