A presentation of the dual symmetric inverse monoid

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Abstract

The dual symmetric inverse monoid $\mathscr{I}_n^*$ is the inverse monoid of all isomorphisms between quotients of an n-set. We give a monoid presentation of $\mathscr{I}_n^*$ and, along the way, establish criteria for a monoid to be inverse when it is generated by completely regular elements.

1 Introduction

Inverse monoids model the partial or local symmetries of structures, generalizing the total symmetries modelled by groups. Key examples are the *symmetric inverse monoid* $\mathscr{I}_X$ on a set X (consisting of all bijections between subsets of X), and the *dual symmetric inverse monoid* $\mathscr{I}_X^*$ on X (consisting of all isomorphisms between subobjects of X in the category $\mathbf{Set}^{\mathrm{opp}}$), each with an appropriate multiplication. They share the property that every inverse monoid may be faithfully represented in some $\mathscr{I}_X$ and some $\mathscr{I}_X^*$. The monoid $\mathscr{I}_X^*$ may be realized in many different ways; in [2], it was described as consisting of bijections between quotient sets of X, or *block bijections* on X, which map the blocks of a "domain" equivalence (or partition) on X bijectively to blocks of a "range" equivalence. These objects may also be regarded as special binary relations on X called *biequivalences*. The appropriate multiplication involves the join of equivalences—details are found in [2], and an alternative description in [4, pp. 122–124].

1.1 Finite dual symmetric inverse monoids

In this paper we focus on finite X, and write $\mathbf{n} = \{1, \dots n\}$ and $\mathscr{I}_n^* = \mathscr{I}_{\mathbf{n}}^*$. In a graphical representation described in [5], the elements of $\mathscr{I}_n^*$ are thought of as graphs on a vertex set $\{1, \dots, n\} \cup \{1', \dots, n'\}$ (consisting of two copies of $\mathbf{n}$) such that each connected component has at least one dashed and one undashed vertex. This representation is not unique—two graphs are regarded as equivalent if they have the same connected components—but it facilitates visualization and is intimately connected to the combinatorial structure. Conventionally, we draw the graph of an element of $\mathscr{I}_n^*$ such that the vertices $1, \dots, n$ are in a horizontal row (increasing from left to right), with vertices $1', \dots, n'$ vertically below. See Fig. 1 for the graph of a block bijection $\theta \in \mathscr{I}_8^*$ with domain $(1, 2 \,|\, 3 \,|\, 4, 6, 7 \,|\, 5, 8)$ and range $(1 \,|\, 2, 4 \,|\, 3 \,|\, 5, 6, 7, 8)$. In an obvious notation, we also write $\theta = \left(\begin{smallmatrix} 1,2 \\ 2,4 \end{smallmatrix} \middle| \begin{smallmatrix} 3 \\ 5,6,7,8 \end{smallmatrix} \middle| \begin{smallmatrix} 4,6,7 \\ 1 \end{smallmatrix} \middle| \begin{smallmatrix} 5,8 \\ 3 \end{smallmatrix}\right)$.

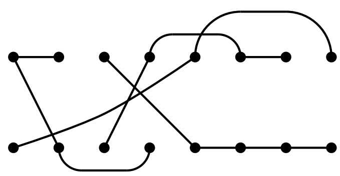


Figure 1: A graphical representation of a block bijection $\theta \in \mathscr{I}_8^*$.

To multiply two such diagrams, they are stacked vertically, with the "interior" rows of vertices coinciding; then the connected components of the resulting graph are constructed and the interior vertices are ignored. See Fig. 2 for an example.

It is clear from its graphical representation that $\mathscr{I}_n^*$ is a submonoid of the *partition monoid*, though not one of the submonoids discussed in [3]. Maltcev [5] shows that $\mathscr{I}_n^*$ with the zero of the partition monoid adjoined is a maximal inverse subsemigroup of the partition monoid, and gives a set of generators for $\mathscr{I}_n^*$. These generators are completely regular; later in this paper, we present auxiliary results on the generation of inverse semigroups by completely regular elements. Although these results are of interest in their own right, our main goal is to obtain a presentation, in terms of generators and relations, of $\mathscr{I}_n^*$. Our method makes use of known presentations of some special subsemigroups of $\mathscr{I}_n^*$. We now describe these subsemigroups, postponing their presentations until a later section.

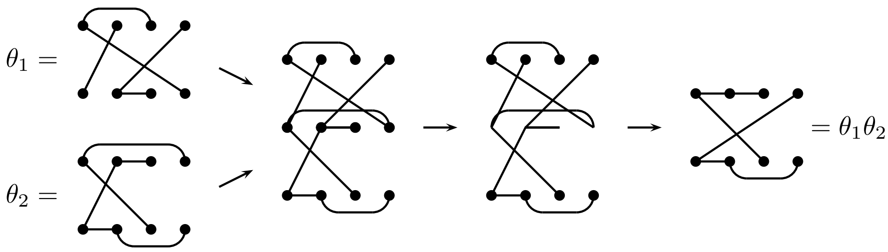


Figure 2: The product of two block bijections $\theta_1, \theta_2 \in \mathscr{I}_4^*$.

The group of units of $\mathscr{I}_n^*$ is the symmetric group $\mathcal{S}_n$, while the semilattice of idempotents is (isomorphic to) $\mathscr{E}_n$, the set of all equivalences on $\mathbf{n}$, with multiplication being join of equivalences. Another subsemigroup consists of those block bijections which are induced by permutations of $\mathbf{n}$ acting on the equivalence relations; this is the *factorizable part* of $\mathscr{I}_n^*$, which we denote by $\mathscr{F}_n$, and which is equal to the set product $\mathscr{E}_n\mathcal{S}_n = \mathcal{S}_n\mathscr{E}_n$. In [2] these elements were called *uniform*, and in [5] *type-preserving*, since they have the characteristic property that corresponding blocks are of equal cardinality. We will also refer to the *local submonoid* $\varepsilon\mathscr{I}_X^*\varepsilon$ of $\mathscr{I}_X^*$ determined by a non-identity idempotent ε. This subsemigroup consists of all $\beta \in \mathscr{I}_X^*$ for which ε is a (left and right) identity. Recalling that the idempotent ε is an equivalence on X, it is easy to see that there is a natural isomorphism $\varepsilon\mathscr{I}_X^*\varepsilon \to \mathscr{I}_{X/\varepsilon}^*$. As an example which we make use of later, when $X = \mathbf{n}$ and $\varepsilon = (1, 2\,|\,3\,|\,\cdots\,|\,n)$, we obtain an isomorphism $\Upsilon : \varepsilon\mathscr{I}_n^*\varepsilon \to \mathscr{I}_{n-1}^*$. Diagrammatically, we obtain a graph of $\beta\Upsilon \in \mathscr{I}_{n-1}^*$ from a graph of $\beta \in \varepsilon\mathscr{I}_n^*\varepsilon$ by identifying vertices $1 \equiv 2$ and $1' \equiv 2'$, relabelling the vertices, and adjusting the edges accordingly; an example is given in Fig. 3.

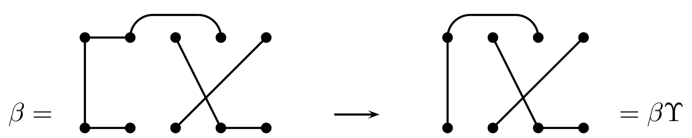
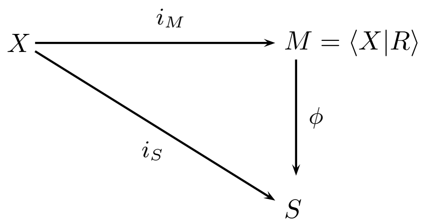


Figure 3: The action of the map $\Upsilon : \varepsilon\mathscr{I}_n^*\varepsilon \to \mathscr{I}_{n-1}^*$ in the case $n = 5$.

1.2 Presentations

Let X be an alphabet (a set whose elements are called *letters*), and denote by X^* the free monoid on X. For $R \subseteq X^* \times X^*$ we denote by $R^\sharp$ the congruence on X^* generated by R, and we define $\langle X|R\rangle = X^*/R^\sharp$. We say that a monoid M *has presentation* $\langle X|R\rangle$ if $M \cong \langle X|R\rangle$. Elements of X and R are called *generators* and *relations* (respectively), and a relation $(w_1, w_2) \in R$ is conventionally displayed as an equation: $w_1 = w_2$. We will often make use of the following universal property of $\langle X|R\rangle$. We say that a monoid S *satisfies* R (or that R *holds in* S) via a map $i_S : X \to S$ if for all $(w_1, w_2) \in R$ we have $w_1 i_S^* = w_2 i_S^*$ (where $i_S^* : X^* \to S$ is the natural extension of i_S to X^*). Then $M = \langle X \mid R\rangle$ is the monoid, unique up to isomorphism, which is universal with respect to the property that it satisfies R (via $i_M : x \mapsto xR^\sharp$); that is, if a monoid S satisfies R via i_S, there is a *unique* homomorphism $\phi : M \to S$ such that $i_M\phi = i_S$:



This map ϕ is called the *canonical homomorphism*. If X generates S via i_S, then ϕ is surjective since i_S^* is.

2 Inverse Monoids Generated by Completely Regular Elements

In this section we present two general results which give necessary and sufficient conditions for a monoid generated by completely regular elements to be inverse, with a semilattice of idempotents specified by the generators.

For a monoid S, we write $E(S)$ and $G(S)$ for the set of idempotents and group of units of S (respectively). Suppose now that S is an inverse monoid (so that $E(S)$ is in fact a semilattice). The *factorizable part* of S is $F(S) = E(S)G(S) = G(S)E(S)$, and S is *factorizable* if $S = F(S)$; in general, $F(S)$ is the largest factorizable inverse submonoid of S.

Recall that an element x of a monoid S is said to be *completely regular* if its $\mathscr{H}$-class H_x is a group. For a completely regular element $x \in S$, we write x^{-1} for the inverse of x in H_x, and x^0 for the identity element of H_x. Thus, $xx^{-1} = x^{-1}x = x^0$ and, of course, $x^0 \in E(S)$. If $X \subseteq S$, we write $X^0 = \{x^0 \mid x \in X\}$.

Proposition 1 *Let S be a monoid, and suppose that $S = \langle X \rangle$ with each $x \in X$ completely regular. Then S is inverse with $E(S) = \langle X^0 \rangle$ if and only if, for all $x, y \in X$,*

(i) $x^0 y^0 = y^0 x^0$, *and*

(ii) $y^{-1} x^0 y \in \langle X^0 \rangle$.

Proof If S is inverse, then (i) holds. Also, for $x, y \in X$, we have

$$(y^{-1}x^0y)^2 = y^{-1}x^0y^0x^0y = y^{-1}x^0x^0y^0y = y^{-1}x^0y,$$

so that $y^{-1}x^0y \in E(S)$. So if $E(S) = \langle X^0 \rangle$, then (ii) holds.

Conversely, suppose that (i) and (ii) hold. From (i) we see that $\langle X^0 \rangle \subseteq E(S)$ and that $\langle X^0 \rangle$ is a semilattice. Now let $y \in X$. Next we demonstrate, by induction on n, that

$$y^{-1}(X^0)^n y \subseteq \langle X^0 \rangle, \tag{A}$$

for all $n \in \mathbb{N}$. Clearly (A) holds for $n = 0$. Suppose next that (A) holds for some $n \in \mathbb{N}$, and that $w \in (X^0)^{n+1}$. So $w = x^0 v$ for some $x \in X$ and $v \in (X^0)^n$. But then by (i) we have

$$y^{-1}wy = y^{-1}x^0vy = y^{-1}y^0x^0vy = y^{-1}x^0y^0vy = (y^{-1}x^0y)(y^{-1}vy).$$

Condition (ii) and an inductive hypothesis that $y^{-1}vy \in \langle X^0 \rangle$ then imply $y^{-1}wy \in \langle X^0 \rangle$. So (A) holds.

Next we claim that for each $w \in S$ there exists $w' \in S$ such that

$$w'\langle X^0 \rangle w \subseteq \langle X^0 \rangle, \tag{B1}$$

$$ww', w'w \in \langle X^0 \rangle, \tag{B2}$$

$$ww'w = w, \; w'ww' = w'. \tag{B3}$$

We prove the claim by induction on the *length* of w (that is, the minimal value of $n \in \mathbb{N}$ for which $w \in X^n$). The case $n = 0$ is trivial since then $w = 1$ and we may take $w' = 1$. Next suppose that $n \in \mathbb{N}$ and that the claim is true for elements of length n. Suppose that $w \in S$ has length $n + 1$, so that $w = xv$ for some $x \in X$ and $v \in S$ of length n. Put $w' = v'x^{-1}$. Then

$$w'\langle X^0 \rangle w = v'x^{-1}\langle X^0 \rangle xv \subseteq v'\langle X^0 \rangle v \subseteq \langle X^0 \rangle,$$

the first inclusion holding by (A) above, and the second by inductive hypothesis. Thus (B1) holds. Also,

$$ww' = xvv'x^{-1} \in x\langle X^0 \rangle x^{-1} \subseteq \langle X^0 \rangle$$

and

$$w'w = v'x^{-1}xv = v'x^0v \in v'\langle X^0 \rangle v \subseteq \langle X^0 \rangle$$

by (A), (B1), and the induction hypothesis, establishing (B2). For (B3), we have

$$ww'w = xvv'x^{-1}xv = xvv'x^0v = xx^0vv'v = xv = w,$$

using (B2), (i), and the inductive hypothesis. Similarly we have

$$w'ww' = v'x^0vv'x^{-1} = v'vv'x^0x^{-1} = v'x^{-1} = w',$$

completing the proof of (B3).

Since S is regular, by (B3), the proof will be complete if we can show that $E(S) \subseteq \langle X^0 \rangle$. So suppose that $w \in E(S)$, and choose $w' \in S$ for which (B1—B3) hold. Then

$$w' = w'ww' = (w'w)(ww') \in \langle X^0 \rangle$$

by (B2), whence $w' \in E(S)$. But then

$$w = ww'w = (ww')(w'w) \in \langle X^0 \rangle,$$

again by (B2). This completes the proof. □

Proposition 2 *Suppose that S is a monoid and that $S = \langle G \cup \{z\} \rangle$ where $G = G(S)$ and $z^3 = z$. Then S is inverse with*

$$E(S) = \langle g^{-1}z^2g \mid g \in G \rangle \quad \textit{and} \quad F(S) = \langle G \cup \{z^2\} \rangle$$

if and only if, for all $g \in G$,

$$g^{-1}z^2gz^2 = z^2g^{-1}z^2g \tag{C1}$$

$$zg^{-1}z^2gz \in \langle G \cup \{z^2\} \rangle. \tag{C2}$$

Proof First observe that z is completely regular, with $z = z^{-1}$ and $z^0 = z^2$. Now put

$$X = G \cup \{g^{-1}zg \mid g \in G\}.$$

Then $S = \langle X \rangle$, and each $x \in X$ is completely regular. Further, if $y = g^{-1}zg$ (with $g \in G$), then $y^{-1} = y$ and $y^0 = y^2 = g^{-1}z^2g$. Thus, $X^0 = \{1\} \cup \{g^{-1}z^2g \mid g \in G\}$.

Now if S is inverse, then (C1) holds. Also, $zg^{-1}z^2gz \in E(S)$ for all $g \in G$ so that (C2) holds if $F(S) = \langle G \cup \{z^2\} \rangle$.

Conversely, suppose now that (C1) and (C2) hold. We wish to verify Conditions (i) and (ii) of Proposition 1, so let $x, y \in X$. If $x^0 = 1$ or $y^0 = 1$, then (i) is immediate, so suppose $x^0 = g^{-1}z^2g$ and $y^0 = h^{-1}z^2h$ (where $g, h \in G$). By (C1) we have

$$x^0y^0 = h^{-1}(hg^{-1}z^2gh^{-1})z^2h = h^{-1}z^2(hg^{-1}z^2gh^{-1})h = y^0x^0,$$

and (i) holds.

If $x^0 = 1$ or $y \in G$, then (ii) is immediate, so suppose $x^0 = g^{-1}z^2g$ and $y = h^{-1}zh$ (where $g, h \in G$). Then $y = y^{-1}$ and, by (C2),

$$y^{-1}x^0y = h^{-1}(zhg^{-1}z^2gh^{-1}z)h \in h^{-1}\langle G \cup \{z^2\}\rangle h \subseteq \langle G \cup \{z^2\}\rangle.$$

But by [1, Lemma 2] and (C1), $\langle G \cup \{z^2\}\rangle$ is a factorizable inverse submonoid of S with $E\big(\langle G \cup \{z^2\}\rangle\big) = \langle X^0\rangle$. Since

$$(y^{-1}x^0y)^2 = y^{-1}x^0y^0x^0y = y^{-1}x^0y \in E\big(\langle G \cup \{z^2\}\rangle\big),$$

it follows that $y^{-1}x^0y \in \langle X^0\rangle$, so that (ii) holds. So, by Proposition 1, S is inverse with $E(S) = \big\langle X^0 \big\rangle = \big\langle g^{-1}z^2g \mid g \in G \big\rangle$ and, moreover, its factorizable part satisfies

$$F(S) = E(S)G \subseteq \langle G \cup X^0\rangle \subseteq \langle G \cup \{z^2\}\rangle \subseteq F(S).$$

Hence $F(S) = \langle G \cup \{z^2\}\rangle$, and the proof is complete. □

3 A Presentation of $\mathscr{I}_n^*$

If $n \leq 2$, then $\mathscr{I}_n^* = \mathscr{F}_n$ is equal to its factorizable part. A presentation of $\mathscr{F}_n$ (for any n) may be found in [1] so, without loss of generality, we will assume for the remainder of the article that $n \geq 3$. We first fix an alphabet

$$\mathscr{X} = \mathscr{X}_n = \{x, s_1, \dots, s_{n-1}\}.$$

Several notational conventions will prove helpful, and we note them here. The empty word will be denoted by 1. A word $s_i \cdots s_j$ is assumed to be empty if either (i) $i > j$ and the subscripts are understood to be ascending, or (ii) if $i < j$ and the subscripts are understood to be descending.

For $1 \leq i, j \leq n-1$, we define integers

$$m_{ij} = \begin{cases} 1 & \text{if } i = j \\ 3 & \text{if } |i-j| = 1 \\ 2 & \text{if } |i-j| > 1. \end{cases}$$

It will be convenient to use abbreviations for certain words in the generators which will occur frequently in relations and proofs. Namely, we write

$$\sigma = s_2s_3s_1s_2,$$

and inductively we define words $l_2, \dots, l_{n-1}$ and $y_3, \dots, y_n$ by

$$l_2 = xs_2s_1 \qquad \text{and} \qquad l_{i+1} = s_{i+1}l_is_{i+1}s_i \quad \text{for } 2 \leq i \leq n-2,$$

and

$$y_3 = x \qquad \text{and} \qquad y_{i+1} = l_i y_i s_i \qquad\qquad \text{for } 3 \leq i \leq n-1.$$

Consider now the set $\mathscr{R} = \mathscr{R}_n$ of relations

$$\begin{aligned}
(s_i s_j)^{m_{ij}} &= 1 && \text{for } 1 \leq i \leq j \leq n-1 && \text{(R1)}\\
x^3 &= x && && \text{(R2)}\\
x s_1 = s_1 x &= x && && \text{(R3)}\\
x s_2 x = x s_2 x s_2 &= s_2 x s_2 x && &&\\
&= x s_2 x^2 = x^2 s_2 x && && \text{(R4)}\\
x^2 \sigma x^2 \sigma = \sigma x^2 \sigma x^2 &= x s_2 s_3 s_2 x && && \text{(R5)}\\
y_i s_i y_i &= s_i y_i s_i && \text{for } 3 \leq i \leq n-1 && \text{(R6)}\\
x s_i &= s_i x && \text{for } 4 \leq i \leq n-1. && \text{(R7)}
\end{aligned}$$

Before we proceed, some words of clarification are in order. We say a relation belongs to $\mathscr{R}_n$ *vacuously* if it involves a generator s_i which does not belong to $\mathscr{X}_n$; for example, (R5) is vacuously present if $n = 3$ because $\mathscr{X}_3$ does not contain the generator s_3. So the reader might like to think of $\mathscr{R}_n$ as the set of relations (R1—R4) if $n = 3$, (R1—R6) if $n = 4$, and (R1—R7) if $n \geq 5$. We also note that we will mostly refer only to the $i = 3$ case of relation (R6), which simply says $x s_3 x = s_3 x s_3$.

We aim to show that $\mathscr{I}_n^*$ has presentation $\langle \mathscr{X} \mid \mathscr{R} \rangle$, so put $M = M_n = \langle \mathscr{X} \mid \mathscr{R} \rangle = \mathscr{X}^* / \mathscr{R}^\sharp$. Elements of M are $\mathscr{R}^\sharp$-classes of words over $\mathscr{X}$. However, in order to avoid cumbersome notation, we will think of elements of M simply as words over $\mathscr{X}$, identifying two words if they are equivalent under the relations $\mathscr{R}$. Thus, the reader should be aware of this when reading statements such as "*Let* $w \in M$" and so on.

With our goal in mind, consider the map

$$\Phi = \Phi_n : \mathscr{X} \to \mathscr{I}_n^*$$

defined by

$$x\Phi = \left(\begin{array}{c|c|c|c|c} 1,2 & 3 & 4 & \cdots & n \\ 3 & 1,2 & 4 & \cdots & n \end{array} \right) \quad \text{and} \quad s_i\Phi = \left(\begin{array}{c|c|c|c|c|c|c|c} 1 & \cdots & i-1 & i & i+1 & i+2 & \cdots & n \\ 1 & \cdots & i-1 & i+1 & i & i+2 & \cdots & n \end{array} \right) \text{ for } 1 \leq i \leq n-1.$$

See also Fig. 4 for illustrations.

Lemma 1 *The monoid $\mathscr{I}_n^*$ satisfies $\mathscr{R}$ via Φ.*

Proof This lemma may be proved by considering the relations one-by-one and diagrammatically verifying that they each hold. This is straightforward in most

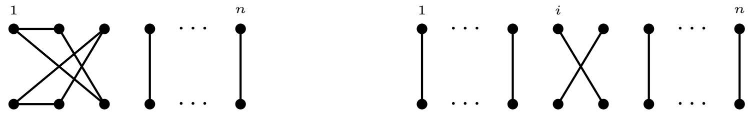


Figure 4: The block bijections $x\Phi$ (left) and $s_i\Phi$ (right) in $\mathscr{I}_n^*$.

cases, but we include a proof for the more technical relation (R6). First, one may check that $l_i\Phi$ $(2 \le i \le n-1)$ and $y_j\Phi$ $(3 \le j \le n)$ have graphical representations as pictured in Fig. 5.

Figure 5: The block bijections $l_i\Phi$ (left) and $y_j\Phi$ (right) in $\mathscr{I}_n^*$.

Using this, we demonstrate in Fig. 6 that relation (R6) holds. □

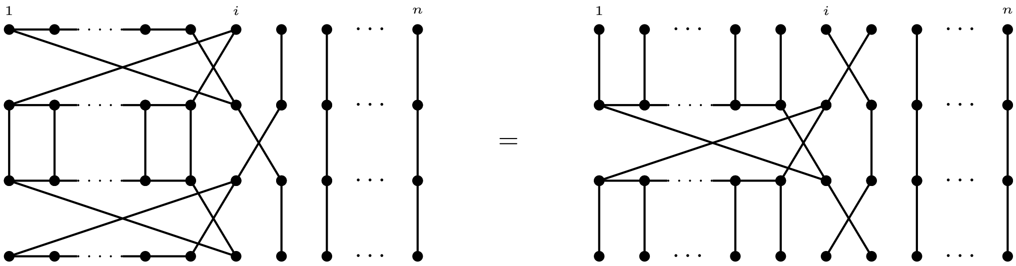


Figure 6: A diagrammatic verification that relation (R6) is satisfied in $\mathscr{I}_n^*$ via Φ.

By Lemma 1, Φ extends to a homomorphism from $M = \langle \mathscr{X} \mid \mathscr{R} \rangle$ to $\mathscr{I}_n^*$ which, without causing confusion, we will also denote by $\Phi = \Phi_n$. By [5, Proposition 16], $\mathscr{I}_n^*$ is generated by $\mathscr{X}\Phi$, so that Φ is in fact an epimorphism. Thus, it remains to show that Φ is injective, and the remainder of the paper is devoted to this task. The proof we offer is perhaps unusual in the sense that it uses, in the general case, not a normal form for elements of M, but rather structural information about M and an inductive argument. The induction is founded on the case $n = 3$, for which a normal form is given in the next proposition.

Proposition 3 *The map Φ_3 is injective.*

Proof Consider the following list of 25 words in M_3:

- the 6 units $\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$,
- the 18 products in $\{1, s_2, s_1s_2\}\{x, x^2\}\{1, s_2, s_2s_1\}$, and
- the zero element xs_2x.

This list contains the generators, and is easily checked to be closed under multiplication on the right by the generators. Thus, $|M_3| \leq 25$. But Φ_3 is a surjective map from M_3 onto $\mathscr{I}_3^*$, which has cardinality 25. It follows that $|M_3| = 25$, and that Φ_3 is injective. □

From this point forward, we assume that $n \geq 4$. The inductive step in our argument relies on Proposition 4 below, which provides a sufficient condition for a homomorphism of inverse monoids to be injective. Let S be an inverse monoid and, for $s, t \in S$, write $s^t = t^{-1}st$. We say that a non-identity idempotent $e \in E(S)$ has property (P) if, for all non-identity idempotents $f \in E(S)$, there exists $g \in G(S)$ such that $f^g \in eSe$.

Proposition 4 *Let S be an inverse monoid with $E = E(S)$ and $G = G(S)$. Suppose that $1 \neq e \in E$ has property (P). Let $\phi : S \to T$ be a homomorphism of inverse monoids for which $\phi|_E$, $\phi|_G$, and $\phi|_{eSe}$ are injective. Then ϕ is injective.*

Proof By the kernel-and-trace description of congruences on S [4, Section 5.1], and the injectivity of $\phi|_G$, it is enough to show that $x\phi = f\phi$ (with $x \in S$ and $1 \neq f \in E$) implies $x = f$, so suppose that $x\phi = f\phi$. Choose $g \in G$ such that $f^g \in eSe$. Now $(xx^{-1})\phi = f\phi = (x^{-1}x)\phi$, so that $f = xx^{-1} = x^{-1}x$, since $\phi|_E$ is injective. Thus $f\mathscr{H}x$ and it follows that $f^g\mathscr{H}x^g$, so that $x^g \in eSe$. Now $x\phi = f\phi$ also implies $x^g\phi = f^g\phi$ and so, by the injectivity of $\phi|_{eSe}$, we have $x^g = f^g$, whence $x = f$. □

It is our aim to apply Proposition 4 to the map $\Phi : M \to \mathscr{I}_n^*$. In order to do this, we first use Proposition 2 to show (in Section 3.2) that M is inverse, and we also deduce information about its factorizable part $F(M)$, including the fact that $\Phi|_{F(M)}$ is injective; this then implies that both $\Phi|_{E(M)}$ and $\Phi|_{G(M)}$ are injective too. Finally, in Section 3.3, we locate a non-identity idempotent $e \in M$ which has property (P). We then show that the injectivity of $\Phi|_{eMe}$ is equivalent to the injectivity of Φ_{n-1} which we assume, inductively. We first pause to make some observations concerning the factorizable part $\mathscr{F}_n$ of $\mathscr{I}_n^*$.

3.1 The factorizable part of $\mathscr{I}_n^*$

Define an alphabet $\mathscr{X}_F = \{t, s_1, \ldots, s_{n-1}\}$, and consider the set $\mathscr{R}_F$ of relations

$$(s_i s_j)^{m_{ij}} = 1 \qquad \text{for } 1 \le i \le j \le n-1 \tag{F1}$$

$$t^2 = t \tag{F2}$$

$$ts_1 = s_1 t = t \tag{F3}$$

$$ts_i = s_i t \qquad \text{for } 3 \le i \le n-1 \tag{F4}$$

$$ts_2ts_2 = s_2ts_2t \tag{F5}$$

$$t\sigma t\sigma = \sigma t\sigma t. \tag{F6}$$

(Recall that σ denotes the word $s_2s_3s_1s_2$.) The following result was proved in [1].

Theorem 2 *The monoid $\mathscr{F}_n = F(\mathscr{I}_n^*)$ has presentation $\langle \mathscr{X}_F \mid \mathscr{R}_F \rangle$ via*

$$s_i \mapsto s_i\Phi,\ t \mapsto (1, 2 \,|\, 3 \,|\, \cdots \,|\, n).\Box$$

Lemma 3 *The relations $\mathscr{R}_F$ hold in M via the map $\Theta : t \mapsto x^2$, $s_i \mapsto s_i$.*

Proof Relations (F1—F3) are immediate from (R1—R3); (F5) follows from several applications of (R4); and (F6) forms part of (R5). The $i \ge 4$ case of (F4) follows from (R7), and the $i = 3$ case follows from (R1) and (R6), since

$$x^2 s_3 = xs_3s_3xs_3 = xs_3xs_3x = s_3xs_3s_3x = s_3x^2.$$

$\Box$

It follows that $\Theta \circ \Phi$ extends to a homomorphism of $\langle \mathscr{X}_F \mid \mathscr{R}_F \rangle$ to $\mathscr{F}_n$, which is an isomorphism by Theorem 2. We conclude that $\Phi|_{\langle x^2, s_1, \ldots, s_{n-1} \rangle} = \Phi|_{\mathrm{im}(\Theta)}$ is injective (and therefore an isomorphism).

3.2 The structure of M

It is easy to see that the group of units $G(M)$ is the subgroup generated by $\{s_1, \ldots, s_{n-1}\}$. The reason for this is that relations (R2—R7) contain at least one occurrence of x on both sides. Now (R1) forms the set of defining relations in Moore's famous presentation [6] of the symmetric group $\mathcal{S}_n$. Thus we may identify $G(M)$ with $\mathcal{S}_n$ in the obvious way. Part (i) of the following well-known result (Lemma 8) gives a normal form for the elements of $\mathcal{S}_n$ (and is probably due to Burnside; a proof is also sketched in [1]). The second part follows immediately from the first, and is expressed in terms of a convenient contracted notation which is defined as follows. Let $1 \le i \le n-1$, and $0 \le k \le n-1$. We write

$$s_i^k = \begin{cases} s_i & \text{if } i \le k \\ 1 & \text{otherwise.} \end{cases}$$

The reader might like to think of this as abbreviating $s_i^{k \ge i}$, where $k \ge i$ is a boolean value, equal to 1 if $k \ge i$ holds and 0 otherwise.

Lemma 4 *Let $g \in G(M) = \langle s_1, \ldots, s_{n-1} \rangle$. Then*

(i) $g = (s_{i_1} \cdots s_{j_1}) \cdots (s_{i_k} \cdots s_{j_k})$ *for some $k \ge 0$ and some $i_1 \le j_1, \ldots, i_k \le j_k$ with $1 \le i_k < \cdots < i_1 \le n-1$, and*

(ii) $g = hs_2^k s_3^k s_4^k (s_5 \cdots s_k) s_1^\ell s_2^\ell s_3^\ell (s_4 \cdots s_\ell) = hs_2^k s_3^k s_4^k s_1^\ell s_2^\ell s_3^\ell (s_5 \cdots s_k)(s_4 \cdots s_\ell)$ *for some $h \in \langle s_3, \ldots, s_{n-1} \rangle$, $k \ge 1$ and $\ell \ge 0$.* □

We are now ready to prove the main result of this section.

Proposition 5 *The monoid $M = \langle \mathscr{X} \mid \mathscr{R} \rangle$ is inverse, and we have*

$$E(M) = \langle g^{-1}x^2g \mid g \in G(M) \rangle \quad \textit{and} \quad F(M) = \langle x^2, s_1, \ldots, s_{n-1} \rangle.$$

Proof Put $G = G(M) = \langle s_1, \ldots, s_{n-1} \rangle$. So $M = \langle G \cup \{x\} \rangle$ and $x = x^3$. We will now verify conditions (C1) and (C2) of Proposition 2. By Lemma 3, $\langle G \cup \{x^2\} \rangle$ is a homomorphic (in fact isomorphic) image of $\mathscr{F}_n$, so $g^{-1}x^2g$ commutes with x^2 for all $g \in G$, and condition (C1) is verified.

To prove (C2), let $g \in G$. By Lemma 4, we have

$$g = hs_2^k s_3^k s_4^k s_1^\ell s_2^\ell s_3^\ell (s_5 \cdots s_k)(s_4 \cdots s_\ell)$$

for some $h \in \langle s_3, \ldots, s_{n-1} \rangle$, $k \ge 1$ and $\ell \ge 0$. Now

$$\begin{aligned} &xg^{-1}x^2gx \\ &= x(s_\ell \cdots s_4)(s_k \cdots s_5) s_3^\ell s_2^\ell s_1^\ell s_4^k s_3^k s_2^k (h^{-1}x^2h) s_2^k s_3^k s_4^k s_1^\ell s_2^\ell s_3^\ell (s_5 \cdots s_k)(s_4 \cdots s_\ell)x \\ &= (s_\ell \cdots s_4)(s_k \cdots s_5) x s_3^\ell s_2^\ell s_1^\ell s_4^k s_3^k s_2^k x^2 s_2^k s_3^k s_4^k s_1^\ell s_2^\ell s_3^\ell x (s_5 \cdots s_k)(s_4 \cdots s_\ell), \end{aligned}$$

by (R7), (F4), and (R1). Thus it suffices to show that $x(x^2)^\pi x \in \langle G \cup \{x^2\} \rangle$, where we have written $\pi = s_2^k s_3^k s_4^k s_1^\ell s_2^\ell s_3^\ell$. Altogether there are 16 cases to consider for all pairs (k, ℓ) with $k = 1, 2, 3, \ge 4$ and $\ell = 0, 1, 2, \ge 3$. Table 1 below contains an equivalent form of $x(x^2)^\pi x$ as a word over $\{x^2, s_1, \ldots, s_{n-1}\}$ for each (k, ℓ), as well as a list of the relations used in deriving the expression. We performed the calculations in the order determined by going along the first row

	$\ell = 0$	$\ell = 1$	$\ell = 2$	$\ell \geq 3$
$k = 1$	x^2 (R2)	x^2 (R2,3)	$x^2 s_2 x^2$ (R3,4)	$x^2\sigma x^2\sigma$ (R1,2,3,5) and (F4)
$k = 2$	$x^2 s_2 x^2$ (R4)	$x^2 s_2 x^2$ (R3,4)	$x^2 s_2 x^2$ (R1,3,4)	$x^2\sigma x^2\sigma$ (R1,3) and $(k,\ell) = (1, \geq 3)$
$k = 3$	$x^2\sigma x^2\sigma$ $(k,\ell) = (1, \geq 3)$	$x^2\sigma x^2\sigma$ (R3) and $(k,\ell) = (1, \geq 3)$	$x^2 s_2 s_3 s_2 x^2$ (R1,2,5)	$x^2 s_2 s_3 s_2 x^2$ (R1,3) and $(k,\ell) = (3,2)$
$k \geq 4$	$s_4 x^2\sigma x^2\sigma s_4$ (R7) and $(k,\ell) = (1, \geq 3)$	$s_4 x^2\sigma x^2\sigma s_4$ (R3) and $(k,\ell) = (\geq 4, 0)$	$s_4 x^2 s_2 s_3 s_2 x^2 s_4$ (R1,7) and $(k,\ell) = (3,2)$	$s_3 s_4 x^2\sigma x^2\sigma s_4 s_3$ (R1,2,5,6,7) and (F4)

Table 1: Expressions for $x(x^2)^\pi x$ and the relations used. See text for further explanation.

from left to right, then the second, third, and fourth rows. Thus, as for example in the case $(k,\ell) = (2,1)$, we have used expressions from previously considered cases.
In order that readers need not perform all the calculations themselves, we provide a small number of sample derivations. The first case we consider is that in which $(k,\ell) = (1,2)$. In this case we have $\pi = s_1 s_2$ and, by (R3) and several applications of (R4), we calculate

$$x(x^2)^\pi x = x s_2 s_1 x^2 s_1 s_2 x = x s_2 x^2 s_2 x = x^2 s_2 x^2.$$

Next suppose $(k,\ell) = (1, \geq 3)$. (Here we mean that $k = 1$ and $\ell \geq 3$.) Then $\pi = s_1 s_2 s_3$ and

$$\begin{aligned} x(x^2)^\pi x &= x s_3 s_2 s_1 x^2 s_1 s_2 s_3 x \\ &= x s_3 s_2 x^2 s_2 s_3 x && \text{by (R3)} \\ &= x s_3 s_2 s_3 x^2 s_3 s_2 s_3 x && \text{by (R1) and (F4)} \\ &= (x^2\sigma x^2\sigma)(\sigma x^2\sigma x^2) && \text{by (R1) and (R5)} \\ &= x^2\sigma x^2\sigma x^2 && \text{by (R1) and (R2)} \\ &= x^2\sigma x^2\sigma && \text{by (R5) and (R2).} \end{aligned}$$

If $(k,\ell) = (3,2)$, then $\pi = \pi^{-1} = \sigma$ by (R1) and, by (R2) and (R5), we have

$$x(x^2)^\pi x = x\sigma x^2\sigma x = x(\sigma x^2\sigma x^2)x = x(x s_2 s_3 s_2 x)x.$$

If $(k,\ell) = (\geq 4, 2)$ then $\pi = s_2 s_3 s_4 s_1 s_2 = \sigma s_4$ by (R1), and so, using (R7) and the $(k,\ell) = (3,2)$ case, we have

$$x(x^2)^\pi x = x s_4\sigma x^2\sigma s_4 x = s_4 x\sigma x^2\sigma x s_4 = s_4(x^2 s_2 s_3 s_2 x^2)s_4.$$

Finally, we consider the $(k, \ell) = (\geq 4, \geq 3)$ case. Here we have $\pi = s_2s_3s_4s_1s_2s_3 = \sigma s_4s_3$ by (R1), and so

$$\begin{aligned}
x(x^2)^\pi x &= xs_3s_4\sigma x^2\sigma s_4s_3x \\
&= xx^2s_3s_4\sigma x^2\sigma s_4s_3x && \text{by (R2)} \\
&= xs_3s_4x^2\sigma x^2\sigma s_4s_3x && \text{by (F4)} \\
&= xs_3s_4xs_2s_3s_2xs_4s_3x && \text{by (R5)} \\
&= xs_3xs_4s_2s_3s_2s_4xs_3x && \text{by (R7)} \\
&= s_3xs_3s_4s_3s_2s_3s_4s_3xs_3 && \text{by (R1) and (R6)} \\
&= s_3xs_4s_3s_4s_2s_4s_3s_4xs_3 && \text{by (R1)} \\
&= s_3s_4xs_3s_2s_3xs_4s_3 && \text{by (R1) and (R7)} \\
&= s_3s_4(x^2\sigma x^2\sigma)s_4s_3 && \text{by (R1) and (R5).}
\end{aligned}$$

After checking the other cases, the proof is complete. □

After the proof of Lemma 3, we observed that $\Phi|_{\langle x^2,s_1,\ldots,s_{n-1}\rangle}$ is injective. By Proposition 5, we conclude that $\Phi|_{F(M)}$ is injective. In particular, both $\Phi|_{E(M)}$ and $\Phi|_{G(M)}$ are injective.

3.3 A local submonoid

Now put $e = x^2 \in M$. So clearly e is a non-identity idempotent of M. Our goal in this section is to show that e has property (P), and that eMe is a homomorphic image of M_{n-1}.

Lemma 5 *The non-identity idempotent $e = x^2 \in M$ has property (P).*

Proof Let $1 \neq f \in E(M)$. By Proposition 5 we have $f = e^{g_1}e^{g_2}\cdots e^{g_k}$ for some $k \geq 1$ and $g_1, g_2, \ldots, g_k \in G(M)$. But then

$$f^{g_1^{-1}} = e\, e^{g_2g_1^{-1}} \cdots e^{g_kg_1^{-1}} e \in eMe.$$

□

We now define words

$$X = s_3x\sigma xs_3,\ S_1 = x, \quad \text{and} \quad S_j = es_{j+1} \quad \text{for } j = 2, \ldots, n-2,$$

and write $\mathscr{Y} = \mathscr{Y}_{n-1} = \{X, S_1, \ldots, S_{n-2}\}$. We note that e is a left and right identity for the elements of $\mathscr{Y}$ so that $\mathscr{Y} \subseteq eMe$, and that $X = y_4$ (by definition).

Proposition 6 *The submonoid eMe is generated (as a monoid with identity e) by* $\mathscr{Y}$.

Proof We take $w \in M$ with the intention of showing that $u = ewe \in eMe$ belongs to $\langle \mathscr{Y} \rangle$. We do this by induction on the (minimum) number $d = d(w)$ of occurrences of x^δ ($\delta \in \{1,2\}$) in w. Suppose first that $d = 0$, so that $u = ege$ where $g \in G(M)$. By Lemma 4 we have

$$g = h(s_2^j s_3^j s_1^i s_2^i)(s_4 \cdots s_j)(s_3 \cdots s_i)$$

for some $h \in \langle s_3, \ldots, s_{n-1} \rangle$ and $j \geq 1$, $i \geq 0$. Put $h' = (s_4 \cdots s_j)(s_3 \cdots s_i) \in \langle s_3, \ldots, s_{n-1} \rangle$. Now by (F2) and (F4) we have

$$u = ege = eh \cdot e(s_2^j s_3^j s_1^i s_2^i)e \cdot eh'.$$

By (F2) and (F4) again, we see that $eh, eh' \in \langle S_2, \ldots, S_{n-1} \rangle$, so it is sufficient to show that the word $e\pi e$ belongs to $\langle \mathscr{Y} \rangle$, where we have written $\pi = s_2^j s_3^j s_1^i s_2^i$. Table 2 below contains an equivalent form of $e\pi e$ as a word over $\mathscr{Y}$ for each (i,j), as well as a list of the relations used in deriving the expression.

	$j=1$	$j=2$	$j\geq 3$
$i=0$	e (R2)	X^2 (R1,2,5) and (F4)	X^2S_2 (R2), (F4), and $(i,j)=(0,2)$
$i=1$	e (R2,3)	X^2 (R3) and $(i,j)=(0,2)$	X^2S_2 (R3) and $(i,j)=(0,\geq 3)$
$i\geq 2$	X^2 (R3) and $(i,j)=(0,2)$	X^2 (R1,3) and $(i,j)=(0,2)$	$S_1S_2XS_2S_1$ (R1,2) and (F4)

Table 2: Expressions for $e\pi e$ and the relations used. See text for further explanation.

Most of these derivations are rather straightforward, but we include two example calculations. For the $(i,j) = (0,2)$ case, note that

$$\begin{aligned} X^2 = s_3x\sigma xs_3s_3x\sigma xs_3 &= s_3x(x^2\sigma x^2\sigma)xs_3 && \text{by (R1) and (R2)} \\ &= s_3x(xs_2s_3s_2x)xs_3 && \text{by (R5)} \\ &= s_3x^2s_3s_2s_3x^2s_3 && \text{by (R1)} \\ &= x^2s_2x^2 && \text{by (F4) and (R1)} \\ &= e\pi e. \end{aligned}$$

For the $(i,j) = (\geq 2, \geq 3)$ case, we have $\pi = \sigma$ and

$$\begin{aligned} e\pi e = x^2\sigma x^2 &= xs_3s_3x\sigma xs_3s_3x && \text{by (R1)} \\ &= x(x^2s_3)s_3x\sigma xs_3(s_3x^2)x && \text{by (R2)} \\ &= x(x^2s_3)s_3x\sigma xs_3(x^2s_3)x && \text{by (F4)} \\ &= S_1S_2XS_2S_1. \end{aligned}$$

This establishes the $d = 0$ case. Now suppose $d \geq 1$, so that $w = vx^\delta g$ for some $\delta \in \{1, 2\}$, $v \in M$ with $d(v) = d(w) - 1$, and $g \in G(M)$. Then by (R2),

$$u = ewe = (eve)x^\delta(ege).$$

Now x^δ belongs to $\langle \mathscr{Y} \rangle$ since x^δ is equal to S_1 (if $\delta = 1$) or e (if $\delta = 2$). By an induction hypothesis we have $eve \in \langle \mathscr{Y} \rangle$ and, by the $d = 0$ case considered above, we also have $ege \in \langle \mathscr{Y} \rangle$. □

The next step in our argument is to prove (in Proposition 7 below) that the elements of $\mathscr{Y}_{n-1}$ satisfy the relations $\mathscr{R}_{n-1}$ via the obviously defined map. Before we do this, however, it will be convenient to prove the following basic lemma. If $w \in M$, we write $\text{rev}(w)$ for the word obtained by writing the letters of w in reverse order. We say that w is *symmetric* if $w = \text{rev}(w)$.

Lemma 6 *If $w \in M$ is symmetric, then $w = w^3$ and $w^2 \in E(M)$.*

Proof Now $z = z^{-1}$ for all $x \in \mathscr{X}$ and it follows that $w^{-1} = \text{rev}(w)$ for all $w \in M$. So, if w is symmetric, then $w = w^{-1}$, in which case $w = ww^{-1}w = w^3$. □

Proposition 7 *The elements of $\mathscr{Y}_{n-1}$ satisfy the relations $\mathscr{R}_{n-1}$ via the map*

$$\Psi : \mathscr{X}_{n-1} \to eMe : x \mapsto X,\ s_i \mapsto S_i.$$

Proof We consider the relations from $\mathscr{R}_{n-1}$ one at a time. In order to avoid confusion, we will refer to the relations from $\mathscr{R}_{n-1}$ as (R1)′, (R2)′, etc. We also extend the use of upper case symbols for the element $\Sigma = S_2S_3S_1S_2$ as well as the words L_i (for $i = 2, \ldots, n-2$) and Y_j (for $j = 3, \ldots, n-1$). It will also prove convenient to refer to the idempotents

$$e_i = e^{(s_2 \cdots s_i)(s_1 \cdots s_{i-1})} \in E(M),$$

defined for each $1 \leq i \leq n$. Note that $e_1 = e$, and that $e_i\Phi \in \mathscr{I}_n^*$ is the idempotent with domain $(1 \,|\, \cdots \,|\, i-1 \,|\, i, i+1 \,|\, i+2 \,|\, \cdots \,|\, n)$.

We first consider relation (R1)′. We must show that $(S_iS_j)^{m_{ij}} = e$ for all $1 \leq i \leq j \leq n-2$. Suppose first that $i = j$. Now $S_1^2 = e$ by definition and if $2 \leq i \leq n-2$ then, by (R1), (R2), (R7), and (F4), we have $S_i^2 = es_{i+1}es_{i+1} = e^2s_{i+1}^2 = e$. Next, if $2 \leq j \leq n-2$, then

$$\begin{aligned}(S_1S_j)^{m_{1j}} &= (xes_{j+1})^{m_{1j}} = (xs_{j+1})^{m_{1j}} \\ &= \begin{cases} xs_3xs_3xs_3 = s_3xs_3s_3xs_3 = s_3x^2s_3 = x^2 = e & \text{if } j = 2 \\ xs_{j+1}xs_{j+1} = x^2s_{j+1}^2 = x^2 = e & \text{if } j \geq 3, \end{cases}\end{aligned}$$

by (R1), (R2), (R6), (R7), and (F4). Finally, if $2 \le i \le j \le n-2$, then

$$(S_iS_j)^{m_{ij}} = (es_{i+1}es_{j+1})^{m_{i+1,j+1}} = e(s_{i+1}s_{j+1})^{m_{i+1,j+1}} = e,$$

by (R1), (R2), and (F4). This completes the proof for (R1)′.

For (R2)′, we have

$$\begin{aligned} X^3 &= (s_3x\sigma xs_3)(s_3x\sigma xs_3)(s_3x\sigma xs_3) \\ &= s_3x\sigma x^2\sigma x^2\sigma xs_3 && \text{by (R1)} \\ &= s_3xx^2\sigma x^2\sigma\sigma xs_3 && \text{by (R5)} \\ &= s_3x\sigma xs_3 && \text{by (R1) and (R2)} \\ &= X. \end{aligned}$$

For (R3)′, first note that

$$\begin{aligned} XS_1 &= s_3x\sigma xs_3x \\ &= s_3x\sigma s_3xs_3 && \text{by (R6)} \\ &= s_3xs_1\sigma xs_3 && \text{by (R1)} \\ &= s_3x\sigma xs_3 && \text{by (R3)} \\ &= X. \end{aligned}$$

(Here, and later, we use the fact that $\sigma s_3 = s_1\sigma$, and so also $\sigma s_1 = s_3\sigma$. These are easily checked using (R1) or by drawing pictures.) The relation $S_1X = X$ is proved by a symmetrical argument.

To prove (R4)′, we need to show that XS_2X is a (left and right) zero for X and S_2. Since $X, S_2 \in \langle x, s_1, s_2, s_3\rangle$, it suffices to show that XS_2X is a zero for each of x, s_1, s_2, s_3. In order to contract the proof, it will be convenient to use the following "arrow notation". If a and b are elements of a semigroup, we write $a \rightarrowtail b$ and $a \twoheadrightarrow b$ to denote the relations $ab = a$ (a is a left zero for b) and $ba = a$ (a is a right zero for b), respectively. The arrows may be superimposed, so that $a \rightarrowtail\!\!\!\twoheadrightarrow b$ indicates the presence of both relations. We first calculate

$$\begin{aligned} XS_2X &= (s_3x\sigma xs_3)es_3(s_3x\sigma xs_3) \\ &= s_3x\sigma xs_3x\sigma xs_3 && \text{by (R1) and (R2)} \\ &= s_3x\sigma s_3xs_3\sigma xs_3 && \text{by (R6)} \\ &= s_3xs_1\sigma x\sigma s_1xs_3 && \text{by (R1)} \\ &= s_3x\sigma x\sigma xs_3 && \text{by (R3).} \end{aligned}$$

Put $w = s_3x\sigma x\sigma xs_3$. We see immediately that $w \rightarrowtail x$ since

$$wx = s_3x\sigma x\sigma xs_3x = s_3x\sigma x\sigma s_3xs_3 = s_3x\sigma xs_1\sigma xs_3 = s_3x\sigma x\sigma xs_3 = w,$$

by (R1), (R3), and (R6), and a symmetrical argument shows that $w \twoheadrightarrow x$. Next, note that w is symmetric so that $w = w^3$ and $w \rightarrowtail\!\!\!\twoheadrightarrow w^2$, by Lemma

6. Since $\ggg$ is transitive, the proof of (R4)$'$ will be complete if we can show that $w^2 \ggg s_1, s_2, s_3$. Now by Lemma 6 again we have $w^2 \in E(M)$ and, since $w^2\Phi = (e_1e_2e_3)\Phi$ as may easily be checked diagrammatically, we have $w^2 = e_1e_2e_3$ by the injectivity of $\Phi|_{E(M)}$. But $(e_1e_2e_3)\Phi \ggg s_1\Phi, s_2\Phi, s_3\Phi$ in $\mathscr{F}_n$ and so, by the injectivity of $\Phi|_{F(M)}$, it follows that $w^2 = e_1e_2e_3 \ggg s_1, s_2, s_3$.

Relations (R5—R7)$'$ all hold vacuously if $n = 4$, so for the remainder of the proof we assume that $n \geq 5$.

Next we consider (R5)$'$. Now, by (R2) and (F4), we see that

$$\Sigma = S_2S_3S_1S_2 = (es_3)(es_4)x(es_3) = s_3s_4xs_3.$$

In particular, Σ is symmetric, by (R7), and $\Sigma^{-1} = \Sigma$. Also, $X = s_3x\sigma xs_3$ is symmetric and so X^2 is idempotent by Lemma 6. It follows that $X^2\Sigma X^2\Sigma$ and $\Sigma X^2\Sigma X^2$ are both idempotent. Since $(X^2\Sigma X^2\Sigma)\Phi = (\Sigma X^2\Sigma X^2)\Phi$, as may easily be checked diagrammatically, we conclude that $X^2\Sigma X^2\Sigma = \Sigma X^2\Sigma X^2$, by the injectivity of $\Phi|_{E(M)}$. It remains to check that $XS_2S_3S_2X = \Sigma X^2\Sigma X^2$. Since $(XS_2S_3S_2X)\Phi = (\Sigma X^2\Sigma X^2)\Phi$, it suffices to show that $XS_2S_3S_2X \in E(M)$. By (R1), (R2), and (F4),

$$XS_2S_3S_2X = (s_3x\sigma xs_3)es_3es_4es_3(s_3x\sigma xs_3) = s_3(x\sigma x)s_4(x\sigma x)s_3,$$

so it is enough to show that $v = (x\sigma x)s_4(x\sigma x)$ is idempotent. We see that

$$\begin{aligned} v^2 &= x\sigma xs_4x\sigma x^2\sigma xs_4x\sigma x \\ &= x\sigma xs_4x(x^2\sigma x^2\sigma)xs_4x\sigma x && \text{by (R2)} \\ &= x\sigma xs_4x(xs_2s_3s_2x)xs_4x\sigma x && \text{by (R5)} \\ &= x\sigma xs_4s_2s_3s_2s_4x\sigma x && \text{by (R2) and (R7).} \end{aligned}$$

Put $u = x\sigma x$. Since u is symmetric, Lemma 6 says that $u^2 \in E(M)$. One verifies easily that $(u^2s_4s_2s_3s_2s_4u^2)\Phi = (u^2s_4u^2)\Phi$ in $\mathscr{F}_n$ and it follows, by the injectivity of $\Phi|_{F(M)}$, that $u^2s_4s_2s_3s_2s_4u^2 = u^2s_4u^2$. By Lemma 6 again, we also have $u = u^3$ so that

$$v^2 = us_4s_2s_3s_2s_4u = uu^2s_4s_2s_3s_2s_4u^2u = uu^2s_4u^2u = us_4u = v,$$

and (R5)$'$ holds.

Now we consider (R6)$'$, which says $Y_iS_iY_i = S_iY_iS_i$ for $i \geq 3$. So we must calculate the words Y_i which, in turn, are defined in terms of the words L_i. Now $L_2 = XS_2S_1$ and $L_{i+1} = S_{i+1}L_iS_{i+1}S_i$ for $i \geq 2$. A straightforward induction shows that $L_i = l_{i+1}e$ for all $i \geq 2$. This, together with the definition of the words Y_i (as $Y_3 = X$, and $Y_{i+1} = L_iY_iS_i$ for $i \geq 3$) and a simple induction, shows that $Y_i = y_{i+1}$ for all $i \geq 3$. But then for $3 \leq i \leq n-2$, we have

$$Y_iS_iY_i = y_{i+1}es_{i+1}y_{i+1} = y_{i+1}s_{i+1}y_{i+1} = s_{i+1}y_{i+1}s_{i+1} = s_{i+1}ey_{i+1}es_{i+1} = S_iY_iS_i.$$

Here we have used (R6) and (F4), and the fact, verifiable by a simple induction, that $y_j e = e y_j = y_j$ for all j.

Relation (R7)$'$ holds vacuously when $n = 5$ so, to complete the proof, suppose $n \geq 6$ and $i \geq 4$. Now $Xe = eX = X$ as we have already observed, and $X s_{i+1} = s_{i+1} X$ by (R1) and (R7). Thus X commutes with $e s_{i+1} = S_i$. This completes the proof. □

3.4 Conclusion

We are now ready to tie together all the loose ends.

Theorem 7 *The dual symmetric inverse monoid* $\mathscr{I}_n^*$ *has presentation* $\langle \mathscr{X} \mid \mathscr{R} \rangle$ *via* Φ.

Proof All that remains is to show that $\Phi = \Phi_n$ is injective. In Proposition 3 we saw that this was true for $n = 3$, so suppose that $n \geq 4$ and that Φ_{n-1} is injective. By checking that both maps agree on the elements of $\mathscr{X}_{n-1}$, it is easy to see that $\Psi \circ \Phi|_{eMe} \circ \Upsilon = \Phi_{n-1}$. (The map Υ was defined at the end of Section 1.1., and Ψ in Proposition 13.) Now Ψ is surjective (by Proposition 6) and Φ_{n-1} is injective (by assumption), so it follows that $\Phi|_{eMe}$ is injective. After the proof of Proposition 5, we observed that $\Phi|_{E(M)}$ and $\Phi|_{G(M)}$ are injective. By Lemma 5, e has property (P) and it follows, by Proposition 4, that Φ is injective. □

We remark that the method of Propositions 5 and 13 may be used to provide a concise proof of the presentation of $\mathscr{I}_n$ originally found by Popova [7].

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