A primer on Seshadri constants

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This paper is dedicated to Andrew J. Sommese.

Abstract. Seshadri constants express the so called local positivity of a line bundle on a projective variety. They were introduced in [**Dem92**] by Demailly. The original hope of using them towards a proof of the Fujita conjecture was too optimistic, but it soon became clear that they are interesting invariants quite in their own right. Lazarsfeld's book [**PAG**] contains a whole chapter devoted to local positivity and serves as a very enjoyable introduction to Seshadri constants. Since this book has appeared, the subject witnessed quite a bit of development. It is the aim of these notes to give an account of recent progress as well as to discuss many open questions and provide some examples. The idea of writing these notes occurred during the workshop on Seshadri constants held in Essen 12-15 February 2008.

Contents

1. Definitions

We begin by recalling the Seshadri criterion for ampleness [**Har70**, Theorem 1.7], as this is where the whole story begins.

Theorem 1.1 (Seshadri criterion). *Let X be a smooth projective variety and L be a line bundle on X. Then L is ample if and only if there exists a positive number ε such that for all points x on X and all (irreducible) curves C passing through x one has*

$$L \cdot C \geqslant \varepsilon \cdot \mathrm{mult}_x C.$$

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REMARK 1.2 (Insufficiency of positive intersections with curves). It is not enough to assume merely that the intersection of L with every curve is positive. In other words it is not enough to assume that L restricts to an ample line bundle on every curve $C \subset X$. Counterexamples were constructed by Mumford and Ramanujam [**Har70**, Examples 10.6 and 10.8].

It is natural to ask for optimal numbers ε in Theorem 1.1. This leads to the following definition due to Demailly [**Dem92**].

DEFINITION 1.3 (Seshadri constant at a point). Let X be a smooth projective variety and L a nef line bundle on X. For a fixed point $x \in X$ the real number

$$\varepsilon(X, L; x) := \inf \frac{L \cdot C}{\mathrm{mult}_x C}$$

is the *Seshadri constant of* L *at* x (the infimum being taken over all irreducible curves C passing through x).

DEFINITION 1.4 (Seshadri curve). We say that a curve C is a *Seshadri curve of* L *at* x if C computes $\varepsilon(X, L; x)$, i.e., if

$$\varepsilon(X, L; x) = \frac{L \cdot C}{\mathrm{mult}_x C}.$$

It is not known if Seshadri curves exist in general.

Definition 1.3 extends naturally so that we can define Seshadri constants for an arbitrary subscheme $Z \subset X$. To this end let $f : Y \longrightarrow X$ be the blowup of X along Z with the exceptional divisor E.

DEFINITION 1.5 (Seshadri constant at a subscheme). The *Seshadri constant* of L at Z is the real number

$$\varepsilon(X, L; Z) := \sup \{\lambda \,:\, f^*L - \lambda E \text{ is ample on } Y\}. \qquad \text{(Eqn 1.1)}$$

REMARK 1.6. If Z is a point, then both definitions agree. The argument is given in [**PAG**, Proposition 5.1.5].

REMARK 1.7 (Relation to the s-invariant). Note that $\varepsilon(X, L; Z)$ is the reciprocal of the s-invariant $s_L(\mathcal{I}_Z)$ of the ideal sheaf $\mathcal{I}_Z$ of Z with respect to L as defined in [**PAG**, Definition 5.4.1]

DEFINITION 1.8 (Multi-point Seshadri constant). If Z is a reduced subscheme supported at r distinct points $x_1, \ldots, x_r$ of X, then the number $\varepsilon(X, L; x_1, \ldots, x_r)$ is called the *multi-point Seshadri constant* of L at the r-tuple of points $x_1, \ldots, x_r$.

There is yet another variant of Definition 1.3 which instead of curves takes into account higher dimensional subvarieties of X passing through a given point $x \in X$.

DEFINITION 1.9 (Seshadri constants via higher dimensional subvarieties). Let X be a smooth projective variety, L a nef line bundle on X and $x \in X$ a point. The real number

$$\varepsilon_d(X, L; x) := \inf \left(\frac{L^d \cdot V}{\mathrm{mult}_x V} \right)^{\frac{1}{d}}$$

is the *d-dimensional Seshadri constant* of L at x (the infimum being taken over all subvarieties $V \subset X$ of dimension d such that $x \in V$).

REMARK 1.10. Note that the above definition agrees for $d = 1$ with Definition 1.3, so that $\varepsilon(X, L; x) = \varepsilon_1(X, L; x)$.

In the above definitions we suppress the variety X if it is clear from the context where the Seshadri constant is computed, i.e., we write $\varepsilon(L; x) = \varepsilon(X, L; x)$ etc.

There are another three interesting numbers which can be defined taking infimums over various spaces of parameters.

DEFINITION 1.11 (Seshadri constants of a line bundle, a point and a variety).

(a) The number
$$\varepsilon(X,L) := \inf_{x\in X} \varepsilon(X,L;x)$$
is the *Seshadri constant of the line bundle* L.

(b) The number
$$\varepsilon(X;x) := \inf_{L \text{ ample}} \varepsilon(X,L;x)$$
is the *Seshadri constant of the point* $x \in X$.

(c) The number
$$\varepsilon(X) := \inf_{L \text{ ample}} \varepsilon(X,L) = \inf_{x\in X} \varepsilon(X;x)$$
is the *Seshadri constant of the variety* X.

REMARK 1.12 (Reformulation of Seshadri criterion). Theorem 1.1 asserts now simply that a line bundle L is ample if and only if its Seshadri constant is positive: $\varepsilon(X,L) > 0$.

So far we defined Seshadri constants for ample or at least nef line bundles. Recently Ein, Lazarsfeld, Mustata, Nakamaye and Popa [**RVBLLS**] found a meaningful way to extend the notion of Seshadri constants to big line bundles.

To begin with, we recall the notion of augmented base locus. For this purpose it is convenient to pass to $\mathbb{Q}$-divisors.

DEFINITION 1.13 (Augmented base locus). Let D be a $\mathbb{Q}$-divisor. The *augmented base locus* of D is
$$B_+(D) := \bigcap_A \mathrm{SB}(D-A),$$
where the intersection is taken over all sufficiently small ample $\mathbb{Q}$-divisors A and $\mathrm{SB}(D-A)$ is the stable base locus of $D-A$, i.e., the common base locus of all linear series $|m(D-A)|$ for all sufficiently divisible m. (In fact $B_+(D) = \mathrm{SB}(D-A)$ for any sufficiently small ample A.)

REMARK 1.14 (Numerical nature of augmented base loci). Contrary to the stable base loci, the augmented base loci depend only on the numerical class of D [**AIBL06**, Proposition 1.4].

Intuitively, the augmented base locus of a line bundle L is the locus where L has no local positivity. This is reflected by the following definition.

DEFINITION 1.15 (Moving Seshadri constant). Let X be a smooth projective variety and $L = \mathcal{O}_X(D)$ a line bundle on X. The real number
$$\varepsilon_{\mathrm{mov}}(L;x) := \begin{cases} \sup_{f^*D=A+E} \varepsilon(A;x) & \text{if } x \text{ is not in } B_+(L), \\ 0 & \text{otherwise.} \end{cases}$$
is the *moving Seshadri constant* of L at x. The supremum in the definition is taken over all projective morphisms $f : X' \mapsto X$, with X' smooth, which are isomorphism over a neighborhood of x and all decompositions $f^*(D) = A + E$ such that E is an effective $\mathbb{Q}$-divisors and $A = f^*(D) - E$ is ample.

Note that if L is not big, then $\varepsilon_{\mathrm{mov}}(L;x) = 0$ for every point $x \in X$, so the moving Seshadri constants are meaningful for big divisors only.

REMARK 1.16 (Consistency of definitions). If L is nef, then the above definition agrees with Definition 1.3. One can also state the other definitions of this section in the moving context. This is left to the reader.

We conclude with yet another remark relating moving Seshadri constants to Zariski decompositions on surfaces. The definition of the Zariski decomposition is provided by the following theorem, see [**Zar62**] and [**Bau08**].

THEOREM 1.17 (Zariski decomposition). *Let D be an effective $\mathbb{Q}$-divisor on a smooth projective surface X. Then there are uniquely determined effective (possibly zero) $\mathbb{Q}$-divisors P and N with $D = P + N$ such that:*

(i) *P is nef ;*
(ii) *N is zero or has negative definite intersection matrix ;*
(iii) *$P \cdot C = 0$ for every irreducible component C of N.*

REMARK 1.18 (Moving Seshadri constants and Zariski decompositions). Let $L = \mathcal{O}_X(D)$ be a big line bundle on a smooth projective surface X and let $D = P + N$ be the Zariski decomposition of D, then

$$\varepsilon_{mov}(L; x) = \varepsilon(P; x) .$$

PROOF. First of all recall that one has

$$H^0(mL) = H^0(mP) \qquad \text{(Eqn 1.2)}$$

for all m sufficiently divisible. Then (Eqn 2.2) relates $\varepsilon(P; x)$ to the number of jets generated asymptotically by P at x. The same relation holds for moving Seshadri constants by [**RVBLLS**, Proposition 6.6]. Taking (Eqn 1.2) into account we have

$$\varepsilon_{mov}(L; x) = \sup_m \frac{s(mL, x)}{m} = \sup_m \frac{s(mP, x)}{m} = \varepsilon(P; x).$$

□

2. Basic properties

2.1. Upper bounds and submaximal curves. Since Seshadri constants are in particular defined by a nefness condition, it is easy to come up with an upper bound using Kleiman's criterion [**PAG**, Theorem 1.4.9]. For 0-dimensional reduced subschemes we have the following result.

PROPOSITION 2.1.1 (Upper bounds). Let X be a smooth projective variety of dimension n and L a nef line bundle on X. Let $x_1, \ldots, x_r$ be r distinct points on X, then

$$\varepsilon(X, L; x_1, \ldots, x_r) \leqslant \sqrt[n]{\frac{L^n}{r}}.$$

In particular for a single point x we always have

$$\varepsilon(X, L; x) \leqslant \sqrt[n]{L^n}.$$

PROOF. Let $f : Y \longrightarrow X$ be the blowup $x_1, \ldots, x_r$. Then the exceptional divisor $E = E_1 + \cdots + E_r$ is the sum of disjoint exceptional divisors over each of the points. By (Eqn 1.1) we must have $(f^*L - \varepsilon(X, L; x_1, \ldots, x_r)E)^n \geqslant 0$, and the claim follows. □

The above proposition leads in a natural manner to the following definition.

DEFINITION 2.1.2 (Submaximal Seshadri constants). We say that the Seshadri constant $\varepsilon(X, L; x)$ is *submaximal* if the strict inequality holds

$$\varepsilon(X, L; x) < \sqrt[n]{L^n} .$$

The above definition is paralleled by the following one.

DEFINITION 2.1.3 (Submaximal curves). Let X be a smooth projective surface and L an ample line bundle on X. We say that $C \subset X$ is a *submaximal curve* (at $x \in X$ with respect to L) if

$$\frac{L \cdot C}{\mathrm{mult}_x C} < \sqrt{L^2} .$$

If only the weak inequality holds for C, then we call C a *weakly-submaximal* curve.

REMARK 2.1.4. For surfaces submaximal Seshadri constants are always computed by Seshadri curves, see [**BauSze08**, Proposition 1.1]. In particular they are rational numbers.

In general we have the following restriction on possible values of Seshadri constants [**Ste98**, Prop. 4], which is a direct consequence of the Nakai-Moishezon criterion for $\mathbb{R}$-divisors [**CamPet90**].

Theorem 2.1.5 (Submaximal Seshadri constants are roots). *Let X be an n-dimensional smooth projective variety, L an ample line bundle on X and x a point of X.*

If $\varepsilon(L,x)$ is submaximal, that is, $\varepsilon(L,x) < \sqrt[n]{L^n}$, then it is a d-th root of a rational number, for some d with $1 \leqslant d \leqslant n-1$.

In particular, it might happen that a Seshadri constant is computed by a higher dimensional subscheme. It is interesting to note that d-dimensional Seshadri constants are partially ordered [**PAG**, Proposition 5.1.9].

Proposition 2.1.6 (Relation between d-dimensional Seshadri constants). For a line bundle L on a smooth projective variety X of dimension n, a point $x \in X$ and an integer d with $1 \leqslant d \leqslant n$ we have

$$\varepsilon(L;x) \leqslant \varepsilon_d(L;x).$$

Note that for d we just recover the bound from Proposition 2.1.1.

Recently Ross and Roé [**RosRoe08**, Remark 1.3] have raised an interesting question if

$$\varepsilon_{d_1}(L;x) \leqslant \varepsilon_{d_2}(L;x)$$

for all $d_1 \leqslant d_2$ (and the analogous version in the multi-point setting).

2.2. Lower bounds. Now we turn our attention to lower bounds. Extrapolating on Definition 1.11, one could hope that yet another infimum can be taken: For a positive integer n define

$$\varepsilon(n) := \inf \varepsilon(X),$$

where the infimum is taken this time over all smooth projective varieties of dimension n. However the numbers $\varepsilon(n)$ always equal zero. Miranda (see [**PAG**, Example 5.2.1]) constructed a sequence of examples of smooth surfaces X_n, ample line bundles L_n on X_n and points $x_n \in X_n$ such that

$$\lim_{n\to\infty} \varepsilon(X_n, L_n; x_n) = 0.$$

Miranda's construction was generalized to arbitrary dimension by Viehweg (see [**PAG**, Example 5.2.2]). In these examples only rational varieties were used but it was quickly realized in [**Bau99**, Proposition 3.3] that the same phenomenon happens on suitable blow ups of arbitrary varieties. Note that in the above sequence it is necessary to change the underlying variety all the time. It is natural to ask if one could realize the sequence (L_n, x_n) as above on a single variety X, i.e., to raise the following problems.

Question 2.2.1 (Existence of a lower bound on a fixed variety).

(a) Can it happen that $\varepsilon(X) = 0$?
(b) If not, is it possible to compute a lower bound in terms of geometric invariants of X?

This question was asked already in the pioneering paper of Demailly [**Dem92**, Question 6.9]. Up to now, we don't know. However there is one obvious instance in which there is an affirmative answer to Question 2.2.1(a), namely if the Picard number $\rho(X)$ is equal to 1. In case of surfaces there is also a sharp answer to Question 2.2.1(b). We come back to this in Theorem 6.1.4.

Another class of varieties, where answers to Question 2.2.1 are known, is constituted by abelian varieties. First of all, since on an abelian variety one can translate divisors around without changing their numerical class, it is clear that one has the lower bound

$$\varepsilon(X,L) \geqslant 1 \qquad \text{(Eqn 2.1)}$$

for any ample line bundle L on an abelian variety X. A beautiful result of Nakamaye [**Nak96**] gives precise characterization of when there is equality in (Eqn 2.1).

Theorem 2.2.2 (Seshadri constants on abelian varieties). *Let (X,L) be a polarized abelian variety. Then $\varepsilon(L) = 1$ if and only if X splits off an elliptic curve and the polarization splits as well, i.e.,*

$$X = X' \times E \text{ and } L = \pi_1^*(L') \otimes \pi_2^*(L_E),$$

where E is an elliptic curve, X' an abelian variety, L_E, L' are ample line bundles on E and X' respectively and π_i are projections in the product.

Furthermore, a lower bound for the Seshadri constant $\varepsilon(X)$ of a variety X can always be given, provided one has good control over base point freeness or very ampleness of ample line bundles on X. Specifically we have the following fact [**PAG**, Example 5.1.18].

PROPOSITION 2.2.3 (Lower bound for spanned line bundles). Let L be an ample and spanned line bundle on a smooth projective variety X, then

$$\varepsilon(X, L; x) \geqslant 1$$

for all points $x \in X$.

This proposition generalizes easily to the case when L generates s-jets at a point, i.e., when the evaluation mapping

$$H^0(X, L) \longrightarrow H^0(X, L \otimes \mathcal{O}_X/\mathcal{I}_x^{s+1})$$

is surjective. (Here $\mathcal{I}_x$ denotes the ideal sheaf of a point $x \in X$.)

PROPOSITION 2.2.4 (Lower bound under generation of higher jets). Let L be an ample line bundle generating s-jets (for $s \geqslant 1$) at a point x of a smooth projective variety X. Then

$$\varepsilon(X, L; x) \geqslant s.$$

In particular, if L is very ample, then $\varepsilon(L; x) \geqslant 1$ for all points $x \in X$.

The above proposition is a special case of the following characterization of Seshadri constants via generation of jets. Denote for $k \geqslant 1$ by $s(kL, x)$ the maximal integer s such that the linear series $|kL|$ generates s-jets at x. Then one has for L nef,

$$\varepsilon(L; x) = \sup \frac{s(kL, x)}{k} \qquad \text{(Eqn 2.2)}$$

(see [**Dem92**, 6.3]). If L is ample, then the supremum is in fact a limit:

$$\varepsilon(L; x) = \lim_{k\to\infty} \frac{s(kL, x)}{k}.$$

Whereas Question 2.2.1 has remained unanswered for several years, one can raise a seemingly easier problem concerning the Seshadri constant at a fixed point $x \in X$.

QUESTION 2.2.5 (Existence of a lower bound at a fixed point). Can it happen that $\varepsilon(X, x) = 0$?

As of this writing we don't know the answer, not even for surfaces.

2.2.6 (Seshadri function). Definition 1.15 generalizes easily to $\mathbb{R}$-divisors and it is clear that it depends only on the numerical class of D. So, we can consider Seshadri constants for elements of the Néron-Severi space $N^1(X)_{\mathbb{R}}$. It is then reasonable to ask about regularity properties of the mapping

$$\varepsilon_{\mathrm{mov}}(X, \cdot; \cdot) : N^1(X)_{\mathbb{R}} \times X \ni (L, x) \longrightarrow \varepsilon_{\mathrm{mov}}(X, L; x) \in \mathbb{R}.$$

It turns out that this mapping is continuous with respect to the first variable [**RVBLLS**, Theorem 6.2] and lower semi-continuous with respect to the second variable (in the topology which closed sets are countable unions of Zariski closed sets) [**PAG**, Example 5.1.11].

3. Projective spaces

The case of $\mathbb{P}^2$ polarized by $\mathcal{O}_{\mathbb{P}^2}(1)$ attracts most of the attention devoted to multiple point Seshadri constants. Thanks to a good interpretation in terms of polynomials the problem of estimating Seshadri constants is well tractable by computer calculations. This, together with the motivation to handle the still open Nagata conjecture, has caused a lot of effort to find lower estimates for general multiple point Seshadri constants on $\mathbb{P}^2$ which are as precise as possible. In many cases analogous methods can also be applied in higher dimensions.

For now the best estimates are obtained by M. Dumnicki using a combination of two methods contained in [**HarRoe03b**] and [**Dum07**]. Both methods appear in a different context and complement each other. The first gives us a relatively small family of all possible divisor classes that might contain curves which compute the Seshadri constants, whereas the second enables us to check if a linear system is empty.

We need the following generalization of Definition 2.1.3.

Definition 3.1 (Multi-point weakly-submaximal curve). Let X be a smooth projective variety of dimension n and L an ample line bundle on X. Let $x_1, \ldots, x_r \in X$ be r arbitrary distinct points. We say that a curve C is *weakly-submaximal* for L with respect to these points if

$$\frac{L \cdot C}{\sum \mathrm{mult}_{x_i}(C)} \leqslant \sqrt[n]{\frac{L^n}{r}} \,.$$

In the view of Proposition 2.1.1 weakly-submaximal curves are important because they contribute substantially to the infimum in Definition 1.3. It is not known in general if weakly-submaximal curves exist. In any case if there are no weakly-submaximal curves for L with respect to the given points, then the Seshadri constant computed in these points equals $\sqrt[n]{L^n/r}$.

The following theorem [**HarRoe03b**] restricts the set of candidates for divisor classes of weakly-submaximal curves in $\mathbb{P}^2$ under the assumption that the points $x_1, \ldots, x_r$ are in general position.

Theorem 3.2 (Restrictions on weakly-submaximal curves). *Let X be obtained by blowing up $r \geqslant 10$ general points $p_1, \ldots, p_r \in \mathbb{P}^2$ and let L be the pull-back of the hyperplane bundle on $\mathbb{P}^2$. If H is the class of a proper transform to X of a weakly-submaximal curve, then there exist integers t,$m > 0$ and k such that:*

(a) $H = tL - m(E_1 + \cdots + E_r) - kE_i$;
(b) $-m < k$ and $k^2 < \frac{r}{r-1}\min\{m, m+k\}$;
(c) $\begin{cases} \mathrm{m^2r + 2mk + \max\{k^2 - m, 0)\} \leqslant t^2 \leqslant m^2r + 2mk + \frac{k^2}{r}}, \ \textit{when}\ \mathrm{k} > 0; \\ m^2r - m \leqslant t^2 < m^2r, \ \textit{when}\ k = 0; \\ m^2r + 2mk + \max\{k^2 - (m+k), 0)\} \leqslant t^2 \leqslant m^2r + 2mk + \frac{k^2}{r}, \ \textit{when}\ k < 0; \end{cases}$
(d) $t^2 - (m+k)^2 - (r-1)m^2 - 3t + mr + k \geqslant -2$.

A potential curve C from the linear system on $\mathbb{P}^2$ corresponding to numbers t, m, k would give the ratio $\frac{L \cdot C}{\sum_{i=1}^{r} \mathrm{mult}_{x_i}(C)} \leqslant \frac{t}{mr+k}$. Thus there is an infinite list of linear systems on $\mathbb{P}^2$, which might contain among their elements weakly-submaximal curves. In order to give a lower estimate α for the multi-point Seshadri constant in r general points, we need to prove that these linear systems connected with the numbers (t, m, k) are empty for $\frac{t}{mr+k} < \alpha$. Observe that for each $\alpha < \sqrt{\frac{1}{r}}$ there is only a finite set of systems to check.

The emptiness of the above systems is proved applying methods of [**Dum07**]. More precisely one uses the algorithm called NSsplit, which has proved up to date to be the most efficient for checking non-speciality (in particular emptiness) of linear systems defined on $\mathbb{P}^2$ by vanishing with given multiplicities at a number of points in very general position. As this is not directly connected with the study of Seshadri constants we omit details and refer to the original paper for a precise description of the algorithm.

Recall that for all r which are squares, the Nagata conjecture holds and thus gives the exact value of the Seshadri constant. For integers r with $10 \leqslant r \leqslant 32$ which are not squares, using the above method M. Dumnicki obtained the following table of estimates:

r	lower estimate	approximate value	non-checked system	conjectured approximate value
10	$\frac{313}{990}$	$\simeq 0.3161616162$	$L(313; 99^{10})$	$\simeq 0.3162277660$
11	$\frac{242}{803}$	$\simeq 0.3013698630$	$L(242; 73^{11})$	$\simeq 0.3015113446$
12	$\frac{277}{960}$	$\simeq 0.2885416667$	$L(277; 80^{12})$	$\simeq 0.2886751346$
13	$\frac{602}{2171}$	$\simeq 0.2772915707$	$L(602; 167^{13})$	$\simeq 0.2773500981$
14	$\frac{389}{1456}$	$\simeq 0.2671703297$	$L(389; 104^{14})$	$\simeq 0.2672612419$
15	$\frac{484}{1875}$	$\simeq 0.2581333333$	$L(484; 125^{15})$	$\simeq 0.2581988897$
17	$\frac{305}{1258}$	$\simeq 0.2424483307$	$L(305; 74^{17})$	$\simeq 0.2425356250$
18	$\frac{369}{1566}$	$\simeq 0.2356321839$	$L(369; 87^{18})$	$\simeq 0.2357022604$
19	$\frac{741}{3230}$	$\simeq 0.2294117647$	$L(741; 170^{19})$	$\simeq 0.2294157339$
20	$\frac{796}{3560}$	$\simeq 0.2235955056$	$L(796; 178^{20})$	$\simeq 0.2236067977$
21	$\frac{1865}{8547}$	$\simeq 0.2182052182$	$L(1865; 407^{21})$	$\simeq 0.2182178902$
22	$\frac{924}{4334}$	$\simeq 0.2131979695$	$L(924; 197^{22})$	$\simeq 0.2132007164$
23	$\frac{585}{2806}$	$\simeq 0.2084818247$	$L(585; 122^{23})$	$\simeq 0.2085144141$
24	$\frac{965}{4728}$	$\simeq 0.2041032149$	$L(965; 197^{24})$	$\simeq 0.2041241452$
26	$\frac{622}{3172}$	$\simeq 0.1960907945$	$L(622; 122^{26})$	$\simeq 0.1961161351$
27	$\frac{956}{4968}$	$\simeq 0.1924315620$	$L(956; 184^{27})$	$\simeq 0.1924500897$
28	$\frac{2434}{12880}$	$\simeq 0.1889751553$	$L(2434; 460^{28})$	$\simeq 0.1889822365$
29	$\frac{2364}{12731}$	$\simeq 0.1856884769$	$L(2364; 439^{29})$	$\simeq 0.1856953382$
30	$\frac{2388}{13080}$	$\simeq 0.1825688073$	$L(2388; 436^{30})$	$\simeq 0.1825741858$
31	$\frac{10729}{59737}$	$\simeq 0.1796039306$	$L(10729; 1927^{31})$	$\simeq 0.1796053020$
32	$\frac{1137}{6432}$	$\simeq 0.1767723881$	$L(1137; 201^{32})$	$\simeq 0.1767766953$

In the fourth column there is included the list of systems not yet proven to be empty. The notation $L(d, m^r)$ stands for the system of curves of degree d passing with multiplicity m through each of r general points.

4. Toric varieties

Toric varieties carry strong local constraints, due to the torus action. The behavior of Seshadri constants at a given number of points is bounded by the maximal generation of jets at that number of points. Equivalently, the Seshadri criterion of ampleness, Theorem 1.1, generalizes to a criterion on the generation of multiple higher order jets. Moreover, estimates on local positivity can be explained by properties of an associated convex integral polytope.

Some of the results reported in this section are contained in [**DiR99**] to which we refer for more details regarding proofs. Some background on toric geometry will be explained, but we refer to [**Ful93**] for more.

4.1. Toric Varieties and polytopes. Let X be a non-singular toric variety of dimension n and L be an ample line bundle on X. We identify the torus T, acting on X, with $N \otimes \mathbb{C}$, for an n-dimensional lattice $N \cong \mathbb{Z}^n$. The geometry of X is completely described by a fan $\Delta \subset N$. In particular the n-dimensional cones in the fan, $\sigma_1, ..., \sigma_l$, define affine patches:

$$X = \bigcup_{i=1}^{l} U_{\sigma_i}.$$

Since X is non-singular, every cone $\sigma \in \Delta$ is given by $\sigma_j = \sum_{i=1}^n \mathbb{R}_+ n_i$, where the $\{n_i\}$ form a lattice basis for N. Let $\Delta(s)$ denote the set of cones of Δ of dimension s. Every $n_i \in \Delta(1)$ is associated to a divisor D_i.

The Picard group of X has finite rank and it is generated by the divisors D_i:

$$Pic(X) = \bigoplus_{i=1}^{d} \mathbb{Z} < D_i > .$$

Hence we can write $L = \sum_{i=1}^d a_i D_i$.

The pair (X, L) defines a convex, n-dimensional, integral polytope in the lattice M dual to N:

$$P = P_{(X,L)} = \{v \in M \mid < v, n_i > \geqslant a_i\}.$$

We will denote by $P(s)$ the set of faces of P of dimension s. In particular $P(0)$ is the set of vertices and $P(n-1)$ is the set of facets. We denote by $|F|$ the number of lattice points on the face F. There is the following one-to-one correspondence:

$$\begin{array}{ccccc} \sigma \in \Delta(n) & \Leftrightarrow & v(\sigma) \in P(0) & \Leftrightarrow & x(\sigma) \text{ fixed point} \\ n_i \in \Delta(1) & \Leftrightarrow & F_i \in P(n-1) & \Leftrightarrow & D_i \text{ invariant divisors} \\ \rho \in \Delta(n-1) & \Leftrightarrow & e_\rho \in P(1) & \Leftrightarrow & C_\rho \text{ invariant curve} \end{array}$$

Moreover $C_\rho \cong \mathbb{P}^1$ for every $\rho \in \Delta(n-1)$.

Recall also that the toric variety X being non-singular is equivalent to the polytope P being Delzant, i.e., satisfying the following two properties:

- there are exactly n edges originating from each vertex;
- for each vertex, the first integer points on the edges form a lattice basis.

By the length of an edge e_ρ we mean $|e_\rho| - 1$.

Recall that $H^0(X, L) \cong \oplus_1^{|P \cap M|} \mathbb{C}$. A basis for $H^0(X, L)$ is denoted by $\{s(m)\}_{m \in P \cap M}$.

4.2. Torus action and Seshadri constants. Seshadri constants on non-singular toric varieties are particularly easy to estimate because of an explicit criterion for the generation of k-jets.

Proposition 2.2.4 tells us that as soon as we are able to estimate the highest degree of jets generated by all multiples of L we can compute the Seshadri constant of L at any point $x \in X$.

We begin by showing that the generation of jets at the fixed points is detected by the size of the associated polytope.

LEMMA 4.2.1 (Generation of jets on toric varieties). *Let $x(\sigma)$ be a point fixed by the torus action. A line bundle L generates k-jets and not $(k+1)$-jets at $x(\sigma)$, if and only if all the edges of P originating from $x(\sigma)$ have length at least k, and there is at least one edge of length k.*

PROOF. Let $x(\sigma)$ be a fixed point. We can choose local coordinates $(x_1, ..., x_n)$, in the affine patch $U_\sigma \cong \mathbb{C}^n$ such that $x(\sigma) = 0$. After choosing the lattice basis $(m_1, ..., m_n)$, given by he first lattice points on the edges from $x(\sigma)$ the map

$$\varphi_{x(\sigma)} : H^0(X, L) \to H^0(L \otimes \mathcal{O}_X / m_{x(\sigma)}^{k+1})$$

is defined by

$$s(m = \sum b_i m_i) \mapsto (\Pi x_i^{b_i}|_{x=0}, \ldots, \frac{\partial \Pi x_i^{b_i}}{\partial x_i}|_{x=0}, \ldots, \frac{\partial^k \Pi x_i^{b_i}}{\partial^{k_1} x_{i_1} \ldots \partial^{k_j} x_{i_j}}|_{x=0}, \ldots).$$

This map is indeed surjective if and only if, in the given basis, $(b_1, ..., b_n) \in P \cap M$ for $\sum b_i = k$. By convexity this is equivalent to the length of the edges of P originating from $x(\sigma)$ being at least k. $\square$

Observing that $P_{tL} = tP$, the above criterion gives the exact value of Seshadri constants at the fixed points. Let

$$s(P, \sigma) = min_{v(\sigma) \in e_\rho} \{|e_\rho| - 1\}.$$

Corollary 4.2.2 (Seshadri constants at torus fixed points).

$$\varepsilon(L, x(\sigma)) = s(P, \sigma).$$

Proof. Theorem 4.2.1 gives that kL generates exactly $ks(P, \sigma)$-jets at $x(\sigma)$. Proposition 2.2.4 gives then that $\varepsilon(L, x(\sigma)) = s(P, \sigma)$. □

Using this criterion we cannot give an exact estimate at every point in X, but we can conclude that toric varieties admit a converse of Proposition 2.2.4, which can be interpreted combinatorially via the associated polytope.

Theorem 4.2.3 (A jet generation criterion). *A line bundle L generates k-jets at every point $x \in X$ if and only if all the edges of P originating from $v(\sigma)$ have length at least k, for all vertices $v(\sigma) \in P(0)$.*

Proof. Since the map $\varphi_x : H^0(X, L) \to H^0(L/m_x^{k+1})$ is equivariant, the subset

$$C = \{x \in X \,|\, Coker(\varphi_x) \neq \varnothing\}$$

is an invariant closed subset of X, hence it is proper.

A line bundle L fails to generate k-jets on X if and only if there is an $x \in X$ such that $Coker(\varphi_x) \neq \varnothing$. In this case $C \neq \varnothing$ and thus, by the Borel fixed point theorem $C^T \neq \varnothing$, where C^T denotes the set of fixed points in C. We conclude that L fails to generate k-jets on X if and only L fails to generate k-jets at some fixed point $x(\sigma) \in X$. Lemma 4.2.1 implies the assertion. □

Corollary 4.2.4 (Higher order Seshadri criterion). The Seshadri constant satisfies $\varepsilon(L) \geqslant s$ if and only if all the edges of P originating from $v(\sigma)$ have length at least s, for all vertices $v(\sigma) \in P(0)$.

Proof. If the edges of P originating from $v(\sigma)$ have length at least k, for all vertices $v(\sigma) \in P(0)$, then the line bundle kL is ks-jet ample for all $s \geqslant 1$, at all points $x \in X$. Proposition 2.2.4 gives then $\varepsilon(L) \geqslant s$.

If $\varepsilon(L) \geqslant s$, then $\varepsilon(L, x(\sigma)) \geqslant s$, for each fixed point $x(\sigma)$. It follows that, for all $(n-1)$ dimensional cones ρ in σ,

$$L \cdot C_\rho \geqslant s \cdot m(C_\rho) \geqslant s,$$

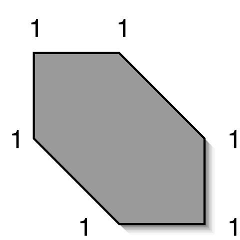
because $m_{x(\sigma)}(C_\rho) = 1$.

The property $L \cdot C_\rho \geqslant s$ for every $\rho \subset \sigma$ and for all $\sigma \in \Delta(n)$ is equivalent to the edges of P originating from $v(\sigma)$ having length at least s, for all vertices $v(\sigma) \in P(0)$, see [**DiR99**, 3.5]. □

We easily conclude that:

Corollary 4.2.5 (Global Seshadri constants are integers). $\varepsilon(L) = min_{\sigma \in \Delta} s(P, \sigma)$. In particular $\varepsilon(L)$ is always an integer.

Example 4.2.6. The polarized variety associated to the polytope



[Fig. 1]

is $(X, L) = (Bl_{P_1,P_2,P_3}(\mathbb{P}^2), \pi^*(\mathcal{O}_{\mathbb{P}^2}(3) - E_1 - E_2 - E_3))$, where π is the blow up of $\mathbb{P}^2$ at the three points fixed by the torus action and E_i are the corresponding exceptional divisors. We see that $s(P,\sigma)=1$ at all vertices, which shows that $\varepsilon(L,x(\sigma))=1$ at the six fixed points. A local calculation shows that

$$\begin{aligned}\varepsilon(L,x)&=2 \quad \text{for all } x\in X\setminus \cup_{\rho\in\Delta(1)}C_\rho\ ;\\ \varepsilon(L,x)&=1 \quad \text{for all } x\in \cup_{\rho\in\Delta(1)}C_\rho\ ;\\ \varepsilon(L)&=1\ .\end{aligned}$$

5. Slope stability and Seshadri constants

In [**RosTho07**], Ross and Thomas studied various notions of stability for polarized varieties, each of which leads to a concept of slope for varieties and subschemes. Our purpose in this section is to briefly touch upon this circle of ideas, and to see how Seshadri constants enter the picture. In order to be as specific as possible, we restrict attention to the concept of *slope stability*; by way of example we present a result on exceptional divisors of high genus from [**PanRos07**].

Slope of a polarized variety. Let X be a smooth projective variety and let L be an ample line bundle on X. We consider the Hilbert polynomial

$$P(k)=\chi(kL)=a_0k^n+a_1k^{n-1}+O(k^{n-2})$$

and define the *slope* of (X,L) to be the rational number

$$\mu(X,L)=\frac{a_1}{a_0}\ .$$

In terms of intersection numbers, we have by Riemann-Roch $a_0=\frac{1}{n!}L^n$ and $a_1=-\frac{1}{2(n-1)!}K_X\cdot L^{n-1}$, and therefore

$$\mu(X,L)=-\frac{nK_X\cdot L^{n-1}}{2L^n}\ . \tag{Eqn 5.1}$$

Slope of a subscheme. Consider next a proper closed subscheme $Z\subset X$. On the blowup $f:Y\to X$ along Z with the exceptional divisor E, the $\mathbb{Q}$-divisor f^*L-xE is ample for $0<x<\varepsilon(L,Z)$. Here $\varepsilon(L,Z)$ is the Seshadri constant of L along Z (see Definition 1.5). There are polynomials $b_i(x)$ such that

$$\chi(k(f^*L-xE))=b_0(x)k^n+b_1(x)k^{n-1}+O(k^{n-2}) \qquad \text{for } k\gg 0 \text{ with } kx\in\mathbb{N}.$$

One now sets $\widetilde{a}_i(x)=a_i-b_i(x)$ and defines the *slope of* Z with respect to a given real number c (and with respect to the polarization L) to be

$$\mu_c(\mathcal{O}_Z,L)=\frac{\int_0^c[\widetilde{a}_1(x)+\frac{1}{2}\frac{d}{dx}\widetilde{a}_0(x)]\,dx}{\int_0^c\widetilde{a}_0(x)\,dx}\ .$$

When Z is a divisor on a surface, then by Riemann-Roch one has

$$\mu_c(\mathcal{O}_Z,L)=\frac{3(2L\cdot Z-c(K_X\cdot Z+Z^2))}{2c(3L\cdot Z-cZ^2)}\ . \tag{Eqn 5.2}$$

Slope stability. One says that (X,L) is *slope semistable with respect to* Z, if

$$\mu(X,L)\leqslant\mu_c(\mathcal{O}_Z,L) \qquad \text{for } 0<c\leqslant\varepsilon(L,Z).$$

In the alternative case, one says that Z *destabilizes* (X,L). (We will see below that in order to show that a certain subscheme is destabilizing, the crucial point is to find an appropriate c in the range that is determined by the Seshadri constant of Z.) One checks that if the condition of semistability is satisfied, then it is also satisfied for mL instead of L. So the notion extends to $\mathbb{Q}$-divisors.

Remark 5.1. The condition that a certain subscheme Z destabilizes (X,L) may be seen as a bound on the Seshadri constant $\varepsilon(L,Z)$: For instance, when X is a surface, then by (Eqn 5.1) and (Eqn 5.2) a divisor Z destabilizes (X,L) iff the inequality

$$\frac{-K_X\cdot L}{L^2}>\frac{3(2L\cdot Z-c(K_X\cdot Z+Z^2))}{2c(3L\cdot Z-cZ^2)}$$

holds for some number c with $0 < c < \varepsilon(L, Z)$. And the latter condition means that $\varepsilon(L, Z)$ is bigger than the smallest root of a quadratic polynomial in c.

Interest in slope stability stems in part from the fact that it gives a concrete obstruction to other geometric conditions – for instance it is implied by the existence of constant scalar curvature Kähler metrics (see [**RosTho06**]). It is therefore natural to ask which varieties are slope stable, and to study the geometry of destabilizing subschemes. For the surface case, Panov and Ross have addressed this problem in [**PanRos07**]. They show that if a polarized surface (X, L) is slope unstable, then

- there is a *divisor* D on X such that D destabilizes (X, L), and
- if a divisor D destabilizes (X, L), then D is not nef. If in addition X has non-negative Kodaira dimension, then $D^2 < 0$.

In the other direction, they show

THEOREM 5.2. *Let X be a smooth projective surface containing an effective divisor D with $p_a(D) \geqslant 2$ whose intersection matrix is negative definite. Then there is a polarization L on X such that (X, L) is slope unstable.*

Note that the theorem does not claim that the *given* divisor destabilizes. As the proof below shows, it is rather the *numerical cycle of* D that is claimed to destabilize. (Recall that the numerical cycle of a divisor $D = \sum_i d_i D_i$ – also called *fundamental cycle* in the literature – is the smallest non-zero effective (integral) divisor $D' = \sum_i d_i' D_i$ such that $D' \cdot D_i \leqslant 0$ for all i. For its existence and uniqueness see [**Rei97**, Sect. 4.5].)

PROOF. Write $D = \sum_{i=1}^m d_i D_i$ with irreducible divisors D_i and integers $d_i > 0$. One reduces first to the case where

(Eqn 5.3) $$D \cdot D_i \leqslant 0 \qquad \text{for } i = 1, \dots, m.$$

To get (Eqn 5.3), replace D by its numerical cycle D'. Then work by Artin [**Art66**], Laufer [**Lau77**], and Némethi [**Nem99**] implies that the inequality $p_a(D') \geqslant 2$ follows from the hypothesis $p_a(D) \geqslant 2$.

Assuming now (Eqn 5.3), we fix an ample divisor H and we construct a divisor

$$L_0 := H + \sum_i q_i D_i$$

with rational coefficients q_i such that $L_0 \cdot D_i = 0$ for all j. Such a divisor exists uniquely thanks to the negative definiteness of the intersection matrix of D. As the inverse of this intersection matrix has all entries $\leqslant 0$ (cf. [**BaKuSz04**, Lemma 4.1]), it follows that $q_i \geqslant 0$ for all i. Since H is ample, we actually have $q_i > 0$ for all i. Letting now $\varepsilon = \min_i\{q_i/d_i\}$, we claim that

(Eqn 5.4) $$L_0 - cD \text{ is nef for } 0 \leqslant c \leqslant \varepsilon.$$

In fact, we have $(L_0 - cD) \cdot D_i \geqslant 0$ thanks to (Eqn 5.3), and for curves C different from the D_i we have $(L_0 - cD) \cdot C = (H + \sum_i (q_i - cd_i)D_i) \cdot C > 0$. The proof is now completed by showing that

(Eqn 5.5) $$D \text{ destabilizes } L_s := L_0 + sH \text{ for small } s > 0.$$

To see (Eqn 5.5), note first that $L_s - cD$ is clearly ample for $0 \leqslant c \leqslant \varepsilon$ and for every $s > 0$, hence $\varepsilon(L_s, D) \geqslant \varepsilon$. We have[1]

$$\mu(X, L_0) = \frac{-K_X \cdot L_0}{L_0^2},$$

which is finite because $L_0^2 = H \cdot L_0 \geqslant H^2 > 0$, and we have

$$\mu_c(\mathcal{O}_D, L_0) = \frac{3(2L_0 \cdot D - c(K_X \cdot D + D^2))}{2c(3L_0 \cdot D - cD^2)} = \frac{3(2p_a(D) - 2)}{2cD^2}.$$

[1]In the two subsequent displayed equations the expressions for $\mu(X, L_0)$ and $\mu_c(\mathcal{O}_D, L_0)$ from (Eqn 5.1) and (Eqn 5.2) are used formally even though L_0 is not ample. The formulas (Eqn 5.1) and (Eqn 5.2) may be viewed as the definitions of μ and μ_c in this case. From this perspective, the point is only that $\mu(X, L_s)$ tends to $\mu(X, L_0)$ when $s \to 0$, and similarly for μ_c.

As $D^2 < 0$, and thanks to the hypotheses on $p_a(D)$, the latter tends to $-\infty$ for $c \to 0$. We can therefore choose a c with $0 < c < \varepsilon$ such that $\mu_c(\mathcal{O}_D, L_0) < \mu(X, L_0)$. Choosing now $s > 0$ small enough, we still have $\mu_c(\mathcal{O}_D, L_s) < \mu(X, L_s)$ while $c < \varepsilon(L_s, D)$, and this proves (Eqn 5.5). □

6. Seshadri constants on surfaces

6.1. Bounds on arbitrary surfaces. Not surprisingly, the case of surfaces is the case that has been studied the most. We will in this section present some of the known results. So let S be a smooth projective surface, L an ample line bundle on S and x any point on S.

First of all note that $\varepsilon(L, x) \leqslant \sqrt{L^2}$ by Proposition 2.1.1 and that $\varepsilon(L, x)$ is rational if strict inequality holds, by Theorem 2.1.5. In fact one has the following improvement due to Oguiso [**Ogu02**, Cor. 2] (see also [**Sze01**, Lemma 3.1]):

THEOREM 6.1.1 (Submaximal global Seshadri constants). *Let (S, L) be a smooth polarized surface. If $\varepsilon(L) < \sqrt{L^2}$, then there is a point $x \in S$ and a curve $x \in C \subset S$ such that $\varepsilon(L) = \varepsilon(L, x) = \frac{L\cdot C}{\mathrm{mult}_x C}$.*

In particular, $\varepsilon(L)$ is rational unless $\varepsilon(L) = \sqrt{L^2}$ and $\sqrt{L^2}$ is irrational.

In fact, in [**Ogu02**], Oguiso studies Seshadri constants of a family of surfaces $\{f : \mathcal{S} \to B, \mathcal{L}\}$, where f is a surjective morphism onto a non-empty Noetherian scheme B, $\mathcal{L}$ is an f-ample line bundle and the fibers (S_t, L_t) are polarized surfaces of degree L_t^2 over an arbitrary closed field k. He proves [**Ogu02**, Cor.5]

THEOREM 6.1.2 (Lower semi-continuity of Seshadri constants). *(1) For each fixed $t \in B$, the function $y = \varepsilon(x) := \varepsilon(L_t, x)$ of $x \in S_t$ is lower semi-continuous with respect to the Zariski topology of S_t.*

(2) The function $y = \varepsilon(t) := \varepsilon(S_t, L_t)$ of $t \in B$ is lower semi-continuous with respect to the Zariski topology of B.

A nice visualization of this result is provided by the global Seshadri constants of quartic surfaces in Theorem 6.6.1 below: They are mostly constant but jump down along special loci in the moduli.

Much attention has been devoted to the study of (the existence of) submaximal curves (cf. Definition 3.1), that is, curves C for which $\frac{L\cdot C}{\mathrm{mult}_x C} < \sqrt{L^2}$ and to possible values of $\varepsilon_{C,x} := \frac{L\cdot C}{\mathrm{mult}_x C}$. In [**Bau99**, Thm. 4.1], the degree of submaximal curves at a *very general point* x is bounded by showing that

$$L \cdot C < \frac{L^2}{\sqrt{L^2} - \varepsilon_{C,x}} .$$

Moreover, [**Bau99**, Prop. 5.1] provides also bounds on the number of curves satisfying $\frac{L\cdot C}{\mathrm{mult}_x C} < a$ for any $a \in \mathbb{R}^+$. These results have been generalized to multi-point Seshadri constants by Roé and the third named author in [**HarRoe08**, Lemma 2.1.4 and Thm. 2.1.5]. The main result of [**HarRoe08**] implies that when the Seshadri constant is submaximal, then the set of potential Seshadri curves is finite.

As for lower bounds, we recall the following result obtained in [**Bau99**, Thm. 3.1] in terms of the quantity $\sigma(L)$, which is defined as

$$\sigma(L) := \frac{1}{\varepsilon(L, K_S)} = \min \{s \in \mathbb{R} \mid \mathcal{O}_S(sL - K_S) \text{ is nef}\} .$$

THEOREM 6.1.3 (Lower bound in terms of canonical slope). *Let (S, L) be a smooth polarized surface. Then*

$$\varepsilon(L) \geqslant \frac{2}{1 + \sqrt{4\sigma(L) + 13}}.$$

Note that for $(S, L) = (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ equality holds, as $\sigma(L) = -3$ and $\varepsilon(L) = 1$. Also note that for surfaces of Kodaira dimension zero, $\sigma(L) = 0$ and the theorem yields $\varepsilon(L) \geqslant 0,434\ldots$, whereas the optimal bound is $\varepsilon(L) \geqslant \frac{1}{2}$ on an Enriques or $K3$ surface (see Theorem 6.5.2 and the beginning of §6.6) and $\varepsilon(L) \geqslant \frac{4}{3}$ on a simple abelian surface (see Theorem

6.4.3(a)). Moreover, in the case of a smooth quartic in $\mathbb{P}^3$, the value of $\varepsilon(L)$ strongly depends on the geometry of S (see Theorem 6.6.1 below), so that one cannot expect that $\sigma(L)$ alone fully accounts for the behaviour of the Seshadri constant.

When the Picard number $\rho(S)$ of the surface is one, we have the following optimal result [**Sze08**, Theorem 7], yielding an answer to Question 2.2.1.

THEOREM 6.1.4 (Effective lower bound on surfaces with $\rho(S) = 1$). *Let S be a smooth projective surface with $\rho(S) = 1$ and let L be an ample line bundle on S. Then for any point $x \in S$*

(S) $\varepsilon(L, x) \geqslant 1$ *if S is not of general type and*
(G) $\varepsilon(L, x) \geqslant \frac{1}{1+\sqrt[4]{K_S^2}}$ *if S is of general type.*

Moreover both bounds are sharp.

Equality in (S) is for example attained for $S = \mathbb{P}^2$ and $L = \mathcal{O}_{\mathbb{P}^2}(1)$. Equality in (b) is attained in the following example (see [**Sze08**] or [**BauSze08**, Example 1.2]):

EXAMPLE 6.1.5. Let S be a smooth surface of general type with $K_S^2 = 1$, $p_g(S) = 2$ and $\rho(S) = 1$. An example of such a surface is a general surface of degree 10 in the weighted projective space $\mathbb{P}(1, 1, 2, 5)$. Then, $\rho(S) = 1$ by a result of Steenbrink [**Ste82**]. Moreover, by adjunction $K_S^2 = 1$ and sections of K_S correspond to polynomials of degree one in the weighted polynomial ring on 4 variables. Thus $p_g(S) = 2$, cf. also [**Ste82**].

We now claim that there exists an $x \in S$ such that $\varepsilon(K_S, x) = \frac{1}{2}$. Indeed, the curves in the pencil $|K_S|$ cannot carry points of multiplicity > 2 since they have arithmetic genus two and cannot all be smooth, which can be seen directly computing the topological Euler characteristic of S.

Looking back at the examples of Miranda mentioned in §2.2, we see that the lower bound $\varepsilon(L, x) \geqslant \frac{1}{1+\sqrt[4]{K_S^2}}$ holds. One could therefore hope that this (or some "nearby" number) would serve as a lower bound on arbitrary surfaces. In fact, there is a conjectural effective lower bound for all minimal surfaces [**Sze08**, Question]:

QUESTION 6.1.6 (Conjectural effective lower bound on surfaces). For any minimal surface S, an ample line bundle L and $x \in S$ is it true that

$$\varepsilon(L, x) \geqslant \frac{1}{2 + \sqrt[4]{|K_S^2|}} \ ?$$

The appearance of 2 in the denominator is in fact necessary due to Enriques and $K3$ surfaces carrying ample line bundles with $\varepsilon(L, x) = \frac{1}{2}$, see §6.5 and §6.6.

Better lower bounds are known if x is a (very) general point. We observed already in 2.2.6 that for x away of a countable union of Zariski closed subsets $\varepsilon(L; x)$ is constant. We denote its value by $\varepsilon(L; 1)$. A fundamental result of Ein and Lazarsfeld, which we recall in Theorem 7.1 states that on surfaces

$$\varepsilon(L; 1) \geqslant 1 \ .$$

In fact, if $L^2 > 1$, they proved, cf. [**EinLaz93**, Theorem] that $\varepsilon(L, x) \geqslant 1$ for all but *finitely many* points on S. This result was improved by Xu [**Xu95**, Thm. 1]:

THEOREM 6.1.7 (Xu's lower bound on surfaces). *Let (S, L) be a smooth polarized surface. Assume that, for a given integer $a > 1$, we have $L^2 \geqslant \frac{1}{3}(4a^2 - 4a + 5)$ and $L \cdot C \geqslant a$ for every irreducible curve $C \subset S$.*

Then $\varepsilon(L, x) \geqslant a$ for all $x \in S$ outside of finitely many curves on S.

(Note that in fact $\varepsilon(L, x) \geqslant a$ outside finitely many points on S if there is no curve C such that $L \cdot C = a$.)

In the case of Picard number one, Steffens [**Ste98**, Prop. 1] proved:

THEOREM 6.1.8 (Steffens' lower bound for $\rho(S) = 1$). *Let S be a smooth surface with $NS(S) \simeq \mathbb{Z}[L]$. Then*

$$\varepsilon(L; 1) \geqslant \lfloor \sqrt{L^2} \rfloor \ .$$

In particular, if $\sqrt{L^2}$ is an integer, then $\varepsilon(L; 1) = \sqrt{L^2}$.

In the case of very ample line bundles, these results have been generalized to the case of multi-point Seshadri constants at general points in [**Har03**, Thm. I.1]. Recall that $\varepsilon(L; x_1, \ldots, x_r) \leqslant \sqrt{\frac{L^2}{r}}$ by Proposition 2.1.1. Given any $c \in \mathbb{R}$, we write $\varepsilon(L; r) \geqslant c$ if $\varepsilon(L, x_1, \ldots, x_r) \geqslant c$ holds on a *Zariski-open set* of r-tuples of points x_i of X. Moreover, let $\varepsilon_{r,l}$ be the maximum element in the finite set

$$\left\{ \frac{\lfloor d\sqrt{rl} \rfloor}{dr} \,\Big|\, 1 \leqslant d \leqslant \sqrt{\frac{r}{l}} \right\} \cup \left\{ \frac{1}{\lceil \sqrt{\frac{r}{l}} \rceil} \right\} \cup \left\{ \frac{dl}{\lceil d\sqrt{rl} \rceil} \,\Big|\, 1 \leqslant d \leqslant \sqrt{\frac{r}{l}} \right\}.$$

(Note that $\varepsilon_{r,l} = \sqrt{lr}$ if $l < r$ and rl is a square, cf. [**Har03**, Prop. III.1(b)(i)].)

Then, we have the following result:

Theorem 6.1.9 (Lower bound for multi-point Seshadri constants). *Let S be a smooth surface and L a very ample line bundle on S. Set $l := L^2$.*

Then $\sqrt{l/r} \geqslant \varepsilon(L, r)$, and in addition, $\varepsilon(L, r) \geqslant \varepsilon_{r,l}$ unless $l \leqslant r$ and rl is a square, in which case $\sqrt{l/r} = \varepsilon_{r,l}$ and $\varepsilon(L, r) \geqslant \sqrt{l/r} - \delta$ for every positive rational δ.

The somewhat awkward statement in the case rl is a square is due to the possibility of there being no *open* set of points such that $\varepsilon(L, r) = \varepsilon_{r,l}$ in that case. Also note that the result holds over an algebraically closed field of any characteristic. Over the complex numbers, one obtains a generalization of the last statement in Theorem 6.1.8:

Theorem 6.1.10 (Maximality of multi-point Seshadri constants). *Let S be a smooth surface and L a very ample line bundle on S. Let $r \in \mathbb{Z}$ be such that $r \geqslant L^2$ and $\sqrt{rL^2}$ is an integer.*

Then, for a Zariski-open set of points $(x_1, \ldots, x_r) \in S^r$, we have

$$\varepsilon(L; x_1, \ldots, x_r) = \sqrt{\frac{L^2}{r}}.$$

More specific results are known when one restricts the attention to surfaces or line bundles of particular types. In the remainder of this section, we will present some of these results.

6.2. Very ample line bundles. Consider a smooth projective surface S and a very ample line bundle L on S. By Proposition 2.2.4 we have $\varepsilon(L, x) \geqslant 1$ for any $x \in S$. Moreover, equality is obviously attained if S contains a line (when embedded by the linear series $|L|$). It is then natural to ask whether this is the only case where $\varepsilon(L) = 1$ occurs, and what the next possible values of $\varepsilon(L)$ for a very ample line bundle are. Both of these questions were answered in [**Bau99**, Theorem 2.1].

Theorem 6.2.1 (Seshadri constants on embedded surfaces). *(a) Let $S \subset \mathbb{P}^N$ be a smooth surface. Then $\varepsilon(\mathcal{O}_S(1)) = 1$ if and only if S contains a line.*

(b) For $d \geqslant 4$ let $\mathcal{S}_{d,N}$ denote the space of smooth irreducible surfaces of degree d in $\mathbb{P}^N$ that do not contain any lines. Then

$$\min \left\{ \varepsilon(\mathcal{O}_S(1)) \mid S \in \mathcal{S}_{d,N} \right\} = \frac{d}{d-1}.$$

(c) If S is a surface in $\mathcal{S}_{d,N}$ and $x \in S$ is a point such that the Seshadri constant $\varepsilon(\mathcal{O}_S(1), x)$ satisfies the inequalities $1 < \varepsilon(\mathcal{O}_S(1), x) < 2$, then it is of the form

$$\varepsilon(\mathcal{O}_S(1), x) = \frac{a}{b} \ ,$$

where a, b are integers with $3 \leqslant a \leqslant d$ and $a/2 < b < a$.

(d) All rational numbers a/b with $3 \leqslant a \leqslant d$ and $a/2 < b < a$ occur as local Seshadri constants of smooth irreducible surfaces in $\mathbb{P}^3$ of degree d.

The examples in (d) are constructed in the following way: given a and b, one can choose an irreducible curve $C_0 \subset \mathbb{P}^2$ of degree a with a point x of multiplicity b. Further, take a smooth curve $C_1 \subset \mathbb{P}^2$ of degree $d - a$ not passing through x. Then there is a smooth surface $S \subset \mathbb{P}^3$ such that the divisor $C_0 + C_1$ is a hyperplane section of X and the curve C

computing $\varepsilon(\mathcal{O}_S(1), x)$ is a component of the intersection $S \cap T_xX$, and therefore $C = C_0$. So one can conclude

$$\varepsilon(\mathcal{O}_S(1), x) = \frac{L \cdot C_0}{\mathrm{mult}_x C_0} = \frac{a}{b}.$$

Note that in the case of quartic surfaces, exact values have been computed in [**Bau97**], see Theorem 6.6.1 below.

6.3. Surfaces of negative Kodaira dimension. The projective plane is discussed in §3. The case of ruled surfaces has been studied by Fuentes García in [**Fue06**]. He explicitly computes the Seshadri constants in the case of the invariant $e > 0$, cf. [**Fue06**, Theorem 4.1]. In the following we let σ and f denote the numerical class of a section and a fiber, respectively.

THEOREM 6.3.1 (Seshadri constants on ruled surfaces with $e > 0$). *Let S be a ruled surface with the invariant $e > 0$ and $A \equiv a\sigma + bf$ be a nef linear system on S. Let x be a point of S. Then*

$$\varepsilon(A; x) = \begin{cases} \min\{a, b - ae\} & \text{if} \quad x \in \sigma, \\ a & \text{if} \quad x \notin \sigma. \end{cases}$$

In particular, note that $\varepsilon(A; x)$ reaches the maximal value $\sqrt{A^2}$ only for $e = 1$ and $b = a$, at points $x \notin \sigma$, or when $A \equiv bf$, at any point $x \in S$.

Furthermore, Fuentes García gives the following bounds when $e \leqslant 0$, cf. [**Fue06**, Thm. 4.2]:

THEOREM 6.3.2 (Seshadri constants on ruled surfaces with $e \leqslant 0$). *Let S be a ruled surface with the invariant $e \leqslant 0$ and $A \equiv a\sigma + bf$ be a nef linear system on S. Let x be a point of S.*

(1) *If $e = 0$ and x lies on a curve numerically equivalent to σ, then $\varepsilon(A, x) = min\{a, b\}$.*
(2) *In all other cases $\varepsilon(A, x) = a$ if $b - \frac{1}{2}ae \geqslant \frac{1}{2}a$ and*

$$2(b - \frac{1}{2}ae) \leqslant \varepsilon(A, x) \leqslant \sqrt{A^2} = \sqrt{2a(b - \frac{1}{2}ae)}.$$

if $0 \leqslant b - \frac{1}{2}ae \leqslant \frac{1}{2}a$.

Of course, Theorem 6.3.1 and case 1. of Theorem 6.3.2 completely determine the Seshadri constants on rational ruled surfaces (as $e \geqslant 0$, and in the case $e = 0$ there is always a section passing through a given point $x \in S$). From these two theorems and some more work in the cases $e = -1$ and $e = 0$, Fuentes García is also able to explicitly compute all Seshadri constants on elliptic ruled surfaces, cf. [**Fue06**, Thms. 1.2 and 6.6]. Furthermore, he also constructs ruled surfaces and linear systems where the Seshadri constant does not reach the upper bound, but is as close as we wish:

THEOREM 6.3.3. *Given any $\delta \in \mathbb{R}^+$ and a smooth curve C of genus > 0, there is a stable ruled surface S, an ample divisor A on S and a point $x \in S$ such that*

$$\sqrt{A^2} - \delta < \varepsilon(A, x) < \sqrt{A^2}.$$

As for del Pezzo surfaces, Broustet proves the following result, cf. [**Bro06**, Thm. 1.3]. Here, S_r for $r \leqslant 8$, denotes the blow up of the plane in r general points $\{p_1, \ldots, p_r\}$. We say that $x \in S_r$ is in general position if its image point $p \in \mathbb{P}^2$ is such that the points in the set $\{p_1, \ldots, p_r, p\}$ are in general position.

THEOREM 6.3.4 (Seshadri constants of $-K_S$ on del Pezzo surfaces). *If $r \leqslant 5$, then $\varepsilon(-K_{S_r}, x) = 2$ if x is in general position and $\varepsilon(-K_{S_r}, x) = 1$ otherwise.*

If $r = 6$, then $\varepsilon(-K_{S_6}, x) = 3/2$ if x is in general position and $\varepsilon(-K_{S_6}, x) = 1$ otherwise.

If $r = 7$, then $\varepsilon(-K_{S_7}, x) = 4/3$ if x is in general position and $\varepsilon(-K_{S_7}, x) = 1$ otherwise.

If $r = 8$, then $\varepsilon(-K_{S_8}, x) = \frac{1}{2}$ in at most 12 points lying outside the exceptional divisor, and $\varepsilon(-K_{S_8}, x) = 1$ everywhere else.

6.4. Abelian surfaces. Let S be an abelian surface and L an ample line bundle on S. By homogeneity, $\varepsilon(L,x)$ does not depend on the x chosen. In particular $\varepsilon(L) = \varepsilon(L,x)$ for any $x \in S$ and one can compute this number for x being one of the half-periods of S, which is the idea in both [**BauSze98**] and [**Bau99**]. Furthermore, since $\varepsilon(kL) = k\varepsilon(L)$ for any integer $k > 0$, one may assume that L is primitive, that is, of type $(1,d)$ for some integer $d \geqslant 1$. The elementary bounds for single point Seshadri constants one has from Proposition 2.1.1 and (Eqn 2.1) are

(Eqn 6.1) $$1 \leqslant \varepsilon(L) \leqslant \sqrt{2d}$$

In the case of Picard number one, exact values for one-point Seshadri constants were computed in [**Bau99**, Thm. 6.1]. To state the result, we will need:

Notation 6.4.1 (Solution to Pell's equation). In the rest of this subsection we let (ℓ_0, k_0) denote the primitive solution of the diophantine equation

$$\ell^2 - 2dk^2 = 1,$$

known as *Pell's equation.*

Theorem 6.4.2 (Exact values on abelian surfaces with $\rho(S) = 1$). *Let (S,L) be a polarized abelian surface of type $(1,d)$ with $\rho(S) = 1$.*

If $\sqrt{2d}$ is rational, then $\varepsilon(L) = \sqrt{2d}$.

If $\sqrt{2d}$ is irrational, then

$$\varepsilon(L) = 2d \cdot \frac{k_0}{\ell_0} = \frac{2d}{\sqrt{2d + \frac{1}{k_0^2}}} \quad \Big(< \sqrt{2d} \Big).$$

In the general case, the lower bound in (Eqn 6.1) has been improved as follows:

Theorem 6.4.3 (Lower bounds on abelian surfaces). *Let (S,L) be a polarized abelian surface of type $(1,d)$.*

(a) $\varepsilon(L) \geqslant \frac{4}{3}$ unless S is non-simple a product of elliptic curves).

(b) $\varepsilon(L) \geqslant \{\varepsilon_1(L), \frac{\sqrt{7d}}{2}\}$, where $\varepsilon_1(L)$ is the minimal degree with respect to L of an elliptic curve on S.

Here, statement (a) is due to Nakamaye [**Nak96**, Thm. 1.2] and (b) was proved in [**BauSze98**, Thm. A.1(b)].

Note that (b) yields a better bound than (a) if $d > 2$. However, for $d = 2$, (a) is sharp, as equality is attained for S the Jacobian of a hyperelliptic curve and L the theta divisor on S, by [**Ste98**, Prop. 2].

Furthermore, note that it is inevitable that small values of $\varepsilon(L)$ occur for non-simple abelian surfaces regardless of d, since for any integer $e \geqslant 1$, there are non-simple polarized abelian surfaces (S,L) of arbitrarily high degree L^2 containing an elliptic curve of degree e.

The upper bound in (Eqn 6.1) can be improved in the case of $\sqrt{2d}$ being irrational, by the following result, see [**BauSze98**, Theorem A.1(a)]:

Theorem 6.4.4 (Upper bounds on abelian surfaces). *Let (S,L) be a polarized abelian surface of type $(1,d)$. If $\sqrt{2d}$ is irrational, then*

$$\varepsilon(L) \leqslant 2d \cdot \frac{k_0}{\ell_0} = \frac{2d}{\sqrt{2d + \frac{1}{k_0^2}}} \quad \Big(< \sqrt{2d} \Big).$$

In particular, together with Theorem 6.1.1, this implies

Theorem 6.4.5 (Rationality on abelian surfaces). *Seshadri constants of ample line bundles on abelian surfaces are rational.*

For the estimates obtained by combining the upper and lower bounds above for low values of d, we refer to [**BauSze98**, Rmk. A.3]. For more precise results on submaximal curves, we refer to [**Bau99**, Sec. 6]. Also note that results for non-simple abelian surfaces have been obtained in [**BauSch08**].

The case of multi-point Seshadri constants is much harder. In fact, if one wants to compute the Seshadri constant in r points for $r > 1$, one can non longer assume that the points are general.

By homogeneity, the number $\varepsilon(L; x_1, \dots, x_r)$ depends only on the differerences $x_i - x_1$ and we have

$$\varepsilon(L; x_1, \dots, x_r) \geqslant \frac{1}{r} \min \varepsilon(L; x_i) = \frac{1}{r}\varepsilon(L) \qquad \text{(Eqn 6.2)}$$

(Note that the inequality (Eqn 6.2) holds on any variety). In [**Bau99**, Proposition 8.2] it is shown that if the equality is attained, then S contains an elliptic curve E containing $x_1, \dots, x_r$ and such that $L \cdot E = r\varepsilon(L; x_1, \dots, x_r)$.

Tutaj-Gasińska gave bounds for Seshadri constants in half-period points in [**Tut04**]. In [**Tut05**] she gave exact values for the case of two half-period points (with a small gap in the proof pointed out in [**Fue07**, Remark 2.10]). More precisely, in [**Tut04**, Thm. 3] she proves that if $e_1, \dots, e_r$ are among the 16 half-period points of S, then

$$\varepsilon(L; e_1, \dots, e_r) \begin{cases} = \sqrt{\frac{2d}{r}} & \text{if} \quad \sqrt{\frac{2d}{r}} \in \mathbb{Q}\,, \\ \leqslant 2d\frac{k_0}{\ell_0} & \text{if} \quad \sqrt{\frac{2d}{r}} \notin \mathbb{Q}\,. \end{cases}$$

In the case of Picard number one, the results were generalized by a different method by Fuentes García in [**Fue07**], who computes the multi-point Seshadri constants in points of a finite subgroup of an abelian surface, cf. [**Fue07**, Theorem 1.2]. One of the corollaries obtained by Fuentes García [**Fue07**, Corollary 2.6] is:

Theorem 6.4.6 (Multi-point Seshadri constants on abelian surfaces with $\rho(S) = 1$). *Let (S, L) be a polarized abelian surface of type $(1, d)$ with $\rho(S) = 1$ and $x_1, \dots, x_r$ be general points on S.*

If $\sqrt{\frac{2d}{r}} \in \mathbb{Q}$, then $\varepsilon(L; x_1, \dots, x_r) = \sqrt{\frac{2d}{r}}$.

If $\sqrt{\frac{2d}{r}} \notin \mathbb{Q}$, then $\varepsilon(L; x_1, \dots, x_r) \geqslant 2d\frac{k_0}{\ell_0}$.

Moreover, as a direct consequence of [**Fue07**, Theorem 1.2], one obtains:

Theorem 6.4.7 (Rationality of multi-point Seshadri constants at finite subgroups). *The multiple-point Seshadri constants of ample line bundles at the points of a finite subgroup of an abelian surface are rational.*

6.5. Enriques surfaces. Let S be an Enriques surface (by definition, $h^1(\mathcal{O}_S) = 0$, $K_S \neq 0$ and $2K_S = 0$) and L an ample line bundle on S. One-point Seshadri constants on Enriques surfaces have been studied in [**Sze01**]. It is well-known that there is an effective nonzero divisor E on S satisfying $E^2 = 0$ (whence E has arithmetic genus 0) and $E \cdot L \leqslant \sqrt{L^2}$, see [**CosDol89**, Prop. 2.7.1 and Cor. 2.7.1]. As a consequence, taking any point $x \in E$, combining with Theorem 2.1.5, one obtains [**Sze01**, Thm. 3.3]:

Theorem 6.5.1 (Rationality on Enriques surfaces). *Let (S, L) be a polarized Enriques surface. Then $\varepsilon(L)$ is rational.*

To state the lower bounds obtained in [**Sze01**, Thm. 3.4 and Prop. 3.5], define the *genus g Seshadri constant of L at x* by

$$\varepsilon_g(L, x) := \inf \frac{L \cdot C}{\mathrm{mult}_x C},$$

where the infimum is taken over all irreducible curves of arithmetic genus g passing through x. (Note that since an abelian surface does not contain rational curves, this definition is consistent with the definition of the number $\varepsilon_1(L)$ in Theorem 6.4.3(b)).

Theorem 6.5.2 (Lower bounds on Enriques surfaces). *Let (S, L) be a polarized Enriques surface and $x \in S$ an arbitrary point.*

Then

$$\varepsilon(L, x) \geqslant min\Big\{\varepsilon_0(L, x), \varepsilon_1(L, x), \frac{1}{4}\sqrt{L^2}\Big\}.$$

Furthermore, $\varepsilon(L,x) < 1$ *if and only if there is an irreducible curve* E *on* S *satisfying* $p_a(E) = 0$, $L \cdot E = 1$ *and* $\text{mult}_x E = 2$ *(so that* $\varepsilon(L,x) = \frac{1}{2}$*.)*

Note that in the special case of the theorem, L cannot be globally generated, by Proposition 2.2.3 or directly from a fundamental property of line bundles on Enriques surfaces [**CosDol89**, Thm. 4.4.1]. In fact, the proof exploits the characterization of non-globally generated line bundles on Enriques surfaces. Also note that the special case attains the lower bound in Question 6.1.6.

6.6. $K3$ **surfaces.** Let S be an $K3$ surface (by definition, $h^1(\mathcal{O}_S) = 0$ and $K_S = 0$) and L an ample line bundle on S. Despite the fact that these surfaces have been studied extensively and very much is known about them, remarkably little is known about Seshadri constants on $K3$ surfaces.

Of course if L is globally generated then $\varepsilon(L,x) \geqslant 1$ for all $x \in S$ by Proposition 2.2.3. Non-globally generated ample line bundles on $K3$ surfaces have been characterized in [**S-D74**]: In this case $L = kE + R$, where $k \geqslant 3$, E is a smooth elliptic curve and R a smooth rational curve such that $E.R = 1$. In particular $|E|$ is an elliptic pencil on S such that $E \cdot L = 1$. It follows that $\varepsilon(L,x) = 1$, unless x is a singular point of one of the (finitely many) singular fibers of $|E|$, in which case $\varepsilon(L,x) = \frac{1}{2}$ [**BaDRSz00**, Prop. 3.1]. Again this is a case where the lower bound in Question 6.1.6 is reached, and the $K3$ surface is forced to have Picard number $\geqslant 2$.

Exact values for Seshadri constants in the special case of smooth quartic surfaces in $\mathbb{P}^3$ have been computed in [**Bau97**].

Theorem 6.6.1 (Quartic surfaces). *Let* $S \subset \mathbb{P}^3$ *be a smooth quartic surface. Then:*

(a) $\varepsilon(\mathcal{O}_S(1)) = 1$ *if and only if* S *contains a line.*

(b) $\varepsilon(\mathcal{O}_S(1)) = \frac{4}{3}$ *if and only if there is a point* $x \in S$ *such that the Hesse form vanishes at* x *and* S *does not contain any lines.*

(c) $\varepsilon(\mathcal{O}_S(1)) = 2$ *otherwise.*

Moreover, the cases (a) and (b) occur on sets of codimension one in the moduli space of quartic surfaces.

(The Hesse form of a smooth surface in $\mathbb{P}^3$ is a quadratic form on the tangent bundle of S, cf. [**Bau97**, Sect. 1].) In particular $\varepsilon(\mathcal{O}_S(1)) = 2$ on a general quartic surface. Since the proof very strongly uses the fact that the surface lies in $\mathbb{P}^3$, it seems very difficult to generalize it to $K3$ surfaces of higher degrees. Nevertheless, a generalization holds in the case of Picard number one, by the following result [**Knu08**, Thm.]:

Theorem 6.6.2 ($K3$ surfaces with $\rho(S) = 1$). *Let* S *be a* $K3$ *surface with Pic* $S \simeq \mathbb{Z}[L]$ *such that* L^2 *is a square. Then* $\varepsilon(L) = \sqrt{L^2}$*.*

This result is a corollary of the following more general lower bound proved in [**Knu08**, Corollary], which can be seen as an extension of Theorem 6.1.8 to *all* points on the surface:

Theorem 6.6.3 (Lower bounds on $K3$ surfaces with $\rho(S) = 1$). *Let* S *be a* $K3$ *surface with Pic* $S \simeq \mathbb{Z}[L]$*.*

Then either

$$\varepsilon(L) \geqslant \lfloor \sqrt{L^2} \rfloor,$$

or

$$(L^2, \varepsilon(L)) \in \Big\{ (\alpha^2 + \alpha - 2, \alpha - \frac{2}{\alpha+1}), (\alpha^2 + \frac{1}{2}\alpha - \frac{1}{2}, \alpha - \frac{1}{2\alpha+1}) \Big\} \qquad \text{(Eqn 6.3)}$$

for some $\alpha \in \mathbb{N}$*. (Note that in fact* $\alpha = \lfloor \sqrt{L^2} \rfloor$*.)*

In the two exceptional cases (Eqn 6.3) of the theorem, the proof shows that there has to exist a point $x \in S$ and an irreducible rational curve $C \in |L|$ (resp. $C \in |2L|$) such that C has an ordinary singular point of multiplicity $\alpha + 1$ (resp. $2\alpha + 1$) at x and is smooth outside x, and $\varepsilon(L) = L \cdot C / \text{mult}_x C$.

By a well-known result of Chen [**Che02**], rational curves in the primitive class of a *general* $K3$ surface in the moduli space are nodal. *Hence the first exceptional case in (Eqn*

6.3) cannot occur on a general $K3$ surface in the moduli space (as $\alpha \geqslant 2$). If $\alpha = 2$, so that $L^2 = 4$, this special case is case (b) in Theorem 6.6.1 above. As one also expects that rational curves in any multiple of the primitive class on a *general* $K3$ surface are always nodal (cf. [**Che99**, Conj. 1.2]), one may expect that also the second exceptional case in (Eqn 6.3) cannot occur on a *general* $K3$ surface.

6.7. Surfaces of general type. Concrete bounds at single points for the canonical divisor have been found recently, see [**BauSze08**, Theorem 1]:

THEOREM 6.7.1 (Bounds for the canonical divisor a arbitrary point). *Let S be a smooth projective surface such that the canonical divisor K_S is big and nef and let x be any point on S.*

(a) *One has $\varepsilon(K_S, x) = 0$ if and only if x lies on one of finitely many (-2)-curves on X.*

(b) *If $0 < \varepsilon(K_S, x) < 1$, then there is an integer $m \geqslant 2$ such that*

$$\varepsilon(K_S, x) = \frac{m-1}{m} \ ,$$

and there is a Seshadri curve $C \subset S$ such that $\mathrm{mult}_x(C) = m$ *and $K_S \cdot C = m - 1$.*

(c) *If $0 < \varepsilon(K_S, x) < 1$ and $K_S^2 \geqslant 2$, then either*
   (i) *$\varepsilon(K_S, x) = \frac{1}{2}$ and x is the double point of an irreducible curve C with arithmetic genus $p_a(C) = 1$ and $K_S \cdot C = 1$, or*
   (ii) *$\varepsilon(K_S, x) = \frac{2}{3}$ and x is a triple point of an irreducible curve C with arithmetic genus $p_a(C) = 3$ and $K_S \cdot C = 2$.*

(d) *If $0 < \varepsilon(K_S, x) < 1$ and $K_S^2 \geqslant 3$, then only case* (c)(i) *is possible.*

It is well known that the bicanonical system $|2K_S|$ is base point free on almost all surfaces of general type. For such surfaces one easily gets the lower bound $\varepsilon(K_S, x) \geqslant 1/2$ for all x outside the contracted locus. However, in general one only knows that $|4K_S|$ is base point free, which gives a lower bound of $1/4$. The theorem shows in particular that one has $\varepsilon(K_S, x) \geqslant 1/2$ in all cases. Moreover, by Example 6.1.5, the bound is sharp. It is not known whether all values $(m-1)/m$ for arbitrary $m \geqslant 2$ actually occur. As part (c) of Theorem 6.7.1 shows, however, values $(m-1)/m$ with $m \geqslant 4$ can occur only in the case $K_S^2 = 1$. It is shown in [**BauSze08**, Example 1.3] that curves as in (c)(i) actually exist on surfaces with arbitrarily large degree of the canonical bundle. In other words, one cannot strengthen the result by imposing higher bounds on K_S^2. It is not known whether curves as in (c)(ii) exist.

As for values at very general points we have the following bound (cf. [**BauSze08**, Thms. 2 and 3]).

THEOREM 6.7.2 (Positivity of the canonical divisor at very general points). *Let S be a smooth projective surface such that K_S is big and nef.*

If $K_S^2 \geqslant 2$, then $\varepsilon(K_S, 1) > 1$.

If $K_S^2 \geqslant 6$, then $\varepsilon(K_S, 1) \geqslant 2$ with equality occurring if and only if X admits a genus 2 *fibration $X \to B$ over a smooth curve B.*

A somewhat more general statement is given in [**BauSze08**, Props. 2.4 and 2.5].

7. S-slope and fibrations by Seshadri curves

As already observed in 2.2.6, the Seshadri constant is a lower semi-continuous function of the point. In particular there is a number, which we denote by $\varepsilon(X, L; 1)$, such that it is the maximal value of the Seshadri function. This maximum is attained for a *very general* point x. Whereas there is no general lower bound on values of Seshadri constants at arbitrary points of X, the numbers $\varepsilon(X, L; 1)$ behave much better. It was first observed by Ein and Lazarsfeld [**EinLaz93**] that there is the following universal lower bound on surfaces.

THEOREM 7.1 (Ein-Lazarsfeld lower bound on surfaces). *Let X be a smooth projective surface and L a nef and big line bundle on X. Then*

$$\varepsilon(X, L; 1) \geqslant 1.$$

It is quite natural to expect that the same bound is valid in arbitrary dimension. However up to now the best result in this direction is the following result proved by Ein, Küchle and Lazarsfeld [**EiKuLa95**].

THEOREM 7.2 (Lower bound in arbitrary dimension). *Let X be a smooth projective variety of dimension n and L a nef and big line bundle on X. Then*

$$\varepsilon(X, L; 1) \geqslant \frac{1}{n}.$$

There has been recently considerable interest in bounds of this type and there emerged several interesting improvements in certain special cases. Most notably, if X is a threefold, then Nakamaye [**Nak05**] shows $\varepsilon(X, L; 1) \geqslant \frac{1}{2}$ for L an ample line bundle on X. Under the additional assumption that the anticanonical divisor $-K_X$ is nef the inequality of Theorem 7.2 is further improved by Broustet [**Bro07**] who shows that $\varepsilon(X, L; 1) \geqslant 1$ holds in this case.

The simple example of the projective plane X with $L = \mathcal{O}_{\mathbb{P}^2}(1)$ shows that one cannot improve the bound in Theorem 7.1. One could hope however that this bound could be influenced by the degree of L. The following example shows that this is not the case.

EXAMPLE 7.3 (Polarizations of large degree and low Seshadri constants). There exist ample line bundles L on smooth projective surfaces such that $\varepsilon(X, L; 1) = 1$, with L^2 arbitrarily large.

Consider for instance the product $X = C \times D$ of two smooth irreducible curves, and denote by a slight abuse of notation the fibers of both projections again by D and C. The line bundles $L_m = mC + D$ are ample and we have $L_m \cdot C = 1$, so that in any event $\varepsilon(L_m, x) \leqslant 1$ for every point $x \in X$. One has in fact $\varepsilon(L_m, x) = 1$, which can be seen as follows: If F is any irreducible curve different from the fibers of the projections with $x \in F$, then we may take a fiber D' of the first projection with $x \in D'$, and we have

$$L_m \cdot F \geqslant D' \cdot F \geqslant \mathrm{mult}_x(D') \cdot \mathrm{mult}_x(F) \geqslant \mathrm{mult}_x(F)$$

which implies $\varepsilon(L_m, x) \geqslant 1$. So $\varepsilon(L_m, x) = 1$, but on the other hand $L_m^2 = 2m$ is unbounded.

This kind of behavior is of course not specific for dimension 2, one can easily generalize it to arbitrary dimension. Interestingly enough Nakamaye [**Nak03**] observed that the above example is in a sense a unique way to produce low Seshadri constants in every point. His result was strengthened and clarified considerably in a series of papers [**SzeTut04**], [**SyzSze07**], [**SyzSze08**], [**KnSySz**]. We summarize below what is known up to now. To this end we introduce first the following quantity.

DEFINITION 7.4 (S-slope). Let X be a smooth projective variety and L a big and nef line bundle on X. We define the *S-slope* of L as

$$\sigma(X, L) := \frac{\varepsilon(X, L; 1)}{\sqrt[n]{L^n}} \ .$$

Note that by Proposition 2.1.1 the number in the denominator is the upper bound on $\varepsilon(X, L; 1)$ (and hence on $\varepsilon(X, L; x)$ for any $x \in X$).

DEFINITION 7.5 (Seshadri fibration). We say that a surface X is *fibred by Seshadri curves* of L if there exists a surjective morphism $f : X \longrightarrow B$ onto a complete curve B such that for $b \in B$ general the fiber $F_b = f^{-1}(b)$ computes $\varepsilon(X, L; x)$ for a general $x \in F_b$.
In case of multi-point Seshadri constants we say that X is fibred by Seshadri curves of L if there exists a surjective morphism $f : X \longrightarrow B$ onto a complete curve B such that for $b \in B$ general, the fiber $F_b = f^{-1}(b)$ computes $\varepsilon(X, L; P_1, \ldots, P_r)$ for a general r-tuple $P_1, \ldots, P_r \in X$ such that $\{P_1, \ldots, P_r\} \cap F_b \neq \varnothing$.

On surfaces we have the following classification.

THEOREM 7.6 (S-slope on surfaces). *Let X be a smooth surface and L an ample line bundle on X. If*

$$\sigma(X, L) < \frac{\sqrt{7}}{\sqrt{8}} \ ,$$

then

(a) *either X is fibred by Seshadri curves or*

(b) *X is a smooth cubic surface in $\mathbb{P}^3$ with $L = \mathcal{O}_X(1)$ and $\sigma(X,L) = \frac{\sqrt{3}}{2}$ in this case, or*

(c) *X is a smooth rational surface such that for a general point $x \in X$ there is a curve C_x of arithmetic genus 3 having multiplicity 3 at x and $C_x^2 = 7$. In this case $\sigma(X,L) = \frac{\sqrt{7}}{3}$.*

Remark 7.7. We don't know if surfaces as in Theorem 7.6(c) exist.

The strategy to prove Theorem 7.6 is to consider classes of Seshadri curves of L in the Hilbert scheme. In one of its components there must be an algebraic family of such curves. Then one invokes a bound on the self-intersection of these curves in the spirit of [**Xu95**]. This either leads to the case when $C_x^2 = 0$, hence a multiple of C_x gives a morphism onto a curve and we can take the Stein factorization of this morphism, or gives restrictions on curves C_x strong enough in order to characterize exceptional cases.

Definition 7.4 generalizes easily to the multi-point case.

Definition 7.8 (Multi-point S-slope). Let X be a smooth projective variety and L a big and nef line bundle on X. We define the *multi-point S-slope* of L as

$$\sigma(X,L;r) := \frac{\varepsilon(X,L;r)}{\sqrt[n]{L^n/r}} .$$

The results presented in [**SyzSze07**] and [**SyzSze08**] may be summarized in the following multi-point counterpart of Theorem 7.6.

Theorem 7.9. *Let X be a smooth surface and L an ample line bundle on X. Let $r \geqslant 2$ be an integer. If*

$$\sigma(X,L;r) < \sqrt{\frac{2r-1}{2r}} ,$$

then

(a) *either X is fibred by Seshadri curves or*

(b) *X is a surface of minimal degree in $\mathbb{P}^r$ with $L = \mathcal{O}_X(1)$ and $\sigma(X,L;r) = \sqrt{\frac{r-1}{r}}$ in this case.*

8. Algebraic manifestation of Seshadri constants

In this section we apply results on Seshadri constants to a problem of commutative algebra concerning comparisons of powers of a homogeneous ideal in a polynomial ring with symbolic powers of the same ideal.

To begin, let $R = k[x_0, \ldots, x_N]$ be a polynomial ring in $N+1$ indeterminates x_i over an algebraically closed field k of arbitrary characteristic. We will often regard R as the homogeneous coordinate ring $R = k[\mathbb{P}^N]$ of projective N-space over k.

8.1. Symbolic powers, ordinary powers and the containment problem. Let $I \subseteq R$ be a *homogeneous* ideal, meaning $I = \oplus_i I_i$, where the homogeneous component I_i of I of degree i is the k-vector space span of all forms $F \in I$ of degree i.

Definition 8.1.1 (Symbolic power). Given an integer $m \geqslant 1$, the mth *symbolic power* $I^{(m)}$ of I is the ideal

$$I^{(m)} = \cap_{P \in \mathrm{Ass}(I)} (R \cap I^m R_P) .$$

Equivalently,

$$I^{(m)} = R \cap I^m R_U ,$$

where R_U is the localization with respect to the set $U = R - \cup_{P \in \mathrm{Ass}(I)} P$.

REMARK 8.1.2 (Homogeneous primary decomposition). All associated primes of a homogeneous ideal are themselves homogeneous, and the primary components of a homogeneous ideal, meaning the ideals in a primary decomposition, can always be taken to be homogeneous (see p. 212, [**Abh06**]). Such a primary decomposition is said to be a *homogeneous primary decomposition*; when we refer to a primary decomposition of a homogeneous ideal, we will always mean a homogeneous primary decomposition. With this convention, given a primary decomposition $I^m = \cap_{P \in \mathrm{Ass}(I^m)} Q_P$, where for each associated prime P of I^m, Q_P denotes the primary component of I^m corresponding to P, the symbolic power $I^{(m)}$ is just $\cap_{P \in S} Q_P$, where S is the set of $P \in \mathrm{Ass}(I^m)$ such that Q_P is contained in some associated prime of I.

EXAMPLES 8.1.3 (Some symbolic power examples). Let $I \subseteq R$ be a homogeneous ideal. By the definition it follows that $I^m \subseteq I^{(m)}$ for all $m \geqslant 1$, and by [**AtiMac69**, Proposition 4.9] we have $I = I^{(1)}$, but it can happen that $I^m \subsetneq I^{(m)}$ when $m > 1$; see Example 8.1.8. However, by a result of Macaulay, if I is a complete intersection, then $I^{(m)} = I^m$ for all $m \geqslant 1$ (see the proof of Theorem 32 (2), p. 110, [**Mat70**]). If I is a radical homogeneous ideal with associated primes $P_1, \dots, P_j$, then $I = P_1 \cap \dots \cap P_j$ and $I^{(m)} = P_1^{(m)} \cap \dots \cap P_j^{(m)}$, where $P_i^{(m)}$ is the smallest primary ideal containing P_i^m. Thus for an ideal $I = \cap_i P_i$ of a finite set of points $p_1, \dots, p_j \in \mathbb{P}^N$, where P_i is the ideal generated by all forms vanishing at p_i, we have $I^{(m)} = P_1^m \cap \dots \cap P_j^m$.

The problem we wish to address here is that of comparing powers of an ideal I with symbolic powers of I. The question of when $I^{(m)}$ contains I^r has an easy complete answer.

LEMMA 8.1.4 (Containment condition). *Let $0 \neq I \subsetneq R$ be a homogeneous ideal. Then $I^r \subseteq I^{(m)}$ if and only if $r \geqslant m$.*

PROOF. Clearly, $r \geqslant m$ implies $I^r \subseteq I^m \subseteq I^{(m)}$.

Conversely, say $r < m$ but $I^r \subseteq I^{(m)}$. Since $I^r \subseteq I^{(m)}$, we have $I^{(r)} \subseteq I^{(m)}$, and since $r < m$ we have $I^m \subseteq I^r$, so $I^{(m)} \subseteq I^{(r)}$ and hence $I^{(r)} = I^{(m)}$. Thus there is an associated prime P of I such that $I^r R_P = I^m R_P \neq (1)$ and so $I^r R_P = I^m R_P = (I^r R_P)(I^s R_P)$, where $s + r = m$. By Nakayama's lemma, this implies $I^r R_P = 0$, contradicting $0 \neq I$. □

The question, on the other hand, of when I^r contains $I^{(m)}$ turns out to be very delicate. This is the main problem we will consider here.

PROBLEM 8.1.5 (Open Problem). Let $I \subseteq R$ be a homogeneous ideal. Determine for which r and m we have $I^{(m)} \subseteq I^r$

In order to make a connection of this problem to computing Seshadri constants we will need the following definition. Let $M = (x_0, \dots, x_N)$ be the maximal homogeneous ideal of R.

DEFINITION 8.1.6 (M-adic order of an ideal). Given a homogeneous ideal $0 \neq I \subseteq R$, let $\alpha(I)$ be the M-adic order of I; i.e., the least t such that I contains a nonzero homogeneous element of degree t; equivalently, $\alpha(I)$ is the least t such that $I_t \neq 0$.

For any homogeneous ideal $0 \neq I \subseteq R$, it is easy to see that $\alpha(I^m) = m\alpha(I)$, but for symbolic powers we have just $\alpha(I^{(m)}) \leqslant m\alpha(I)$; as Example 8.1.8 shows, this inequality can be strict. First a definition.

DEFINITION 8.1.7 (Fat point subscheme). Given distinct points $p_1, \dots, p_j \in \mathbb{P}^N$, let $I(p_i)$ be the maximal ideal of the point p_i. Given a 0-cycle $Z = m_1 p_1 + \dots + m_j p_j$ with positive integers m_i, let $I(Z)$ denote the ideal $\cap_i I(p_i)^{m_i}$. We also write $Z = m_1 p_1 + \dots + m_j p_j$ to denote the subscheme defined by $I(Z)$. Such a subscheme is called a *fat point* subscheme.

Now we consider an easy example of a fat point subscheme of $\mathbb{P}^2$.

EXAMPLE 8.1.8 (The power and symbolic power can differ). Given $Z = p_1 + \dots + p_j$ and $m \geqslant 1$, mZ is the subscheme $mp_1 + \dots + mp_j$, and we have $I(mZ) = I(Z)^{(m)}$. The ideal $I(mZ)$ is generated by all forms that vanish to order at least m at each point p_i. If

$N = 2$ and $I = I(p_1 + p_2 + p_3)$, where $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$ and $p_3 = (0 : 0 : 1)$, then $\alpha(I) = 2$ so $\alpha(I^2) = 4$ but, since $x_0x_1x_2 \in I^{(2)}$, we have $\alpha(I^{(2)}) \leqslant 3$ (and in fact this is an equality), and thus $\alpha(I^{(2)}) < 2\alpha(I) = 4$, hence $I^2 \subsetneq I^{(2)}$.

8.2. Measurement of growth and Seshadri constants. An interesting problem, pursued in [**ArsVat03**] and [**CHST05**], is to determine how much bigger $I^{(m)}$ is than I^m. Whereas [**CHST05**] uses local cohomology to obtain an asymptotic measure of $I^{(m)}/I^m$, [**ArsVat03**] uses the regularity of I to estimate how big $I^{(m)}/I^m$ is. An alternative approach is to use an asymptotic version of α [**BocHar07**].

Definition 8.2.1 (Asymptotic M-adic order). Given a homogeneous ideal $0 \neq I \subseteq R$, then $\alpha(I^{(m)})$ is defined for all $m \geqslant 1$ and we define

$$\gamma(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m} .$$

Because α is subadditive (i.e., $\alpha(I^{(m_1+m_2)}) \leqslant \alpha(I^{(m_1)}) + \alpha(I^{(m_2)})$), this limit exists (see [**BocHar07**, Lemma 2.3.1], or [**HarRoe03a**, Remark III.7]).

Lemma 8.2.2 (Positivity of γ). *Given a homogeneous ideal $0 \neq I \subsetneq R$, then $\gamma(I) \geqslant 1$.*

Proof. To see this, consider $M = (x_0, \dots, x_N)$. Let $P \in \operatorname{Ass}(I)$. Then $I^{(m)} \subseteq P^{(m)}$. But P is homogeneous, so $P \subseteq M$, hence $P^{(m)} \subseteq M^m$ by Corollary 1 of [**EisHoc79**]. Thus $m = \alpha(M^m) \leqslant \alpha(I^{(m)})$, hence $1 \leqslant \gamma(I)$. □

Remark 8.2.3 (γ, the containment problem and Seshadri constants). We note that $0 \neq I \subsetneq R$ guarantees that $\alpha(I)$ is defined, and that $\gamma(I)$ is defined and nonzero. The quantity $\gamma(I)$ is useful not only for studying when $I^m \subsetneq I^{(m)}$ but, as we will see in Lemma 8.3.3, also for studying when $I^{(m)} \subseteq I^r$. We also will relate $\gamma(I)$ to Seshadri constants.

First we see how $\alpha(I)/\gamma(I)$ gives an asymptotic indication of when $I^m \subsetneq I^{(m)}$ in case $0 \neq I \subsetneq R$. Note by subadditivity we have for all m that

$$\gamma(I) = \lim_{t\to\infty} \frac{\alpha(I^{(tm)})}{tm} \leqslant \frac{\alpha(I^{(m)t})}{mt} = \frac{\alpha(I^{(m)})}{m} \leqslant \frac{\alpha(I^m)}{m} = \alpha(I) .$$

Thus, for example, $\alpha(I)/\gamma(I) > 1$ if and only if $\alpha(I^m) > \alpha(I^{(m)})$ for some (equivalently, infinitely many) $m > 1$, and hence $\alpha(I)/\gamma(I) > 1$ implies $I^m \subsetneq I^{(m)}$ for some (equivalently, infinitely many) $m > 1$.

As pointed out in [**BocHar07**], $\gamma(I)$ is in some cases related to a suitable Seshadri constant. In particular, if $Z = p_1 + \cdots + p_j \subset \mathbf{P}^N$, then one defines the Seshadri constant (cf. Definition 1.9)

$$\varepsilon(N, Z) := \varepsilon_{N-1}(\mathbb{P}^N, \mathcal{O}(1); Z) = \sqrt[N-1]{\inf\left\{\frac{\deg(H)}{\Sigma_{i=1}^{j}\operatorname{mult}_{p_i} H}\right\}} ,$$

where the infimum is taken with respect to all hypersurfaces H through at least one of the points p_i. It is clear from the definitions that

$$\gamma(I(Z)) \geqslant j \cdot (\varepsilon(N, Z))^{N-1} .$$

If the points p_i are generic, then equality holds (see [**BocHar07**, Lemma 2.3.1], or [**HarRoe03a**, Remark III.7]; the idea of the proof is to use the fact that the points are generic to show that one can assume that H has the same multiplicity at each point p_i).

8.3. Background for the containment problem. It is not so easy to determine for which r we have $I^{(m)} \subseteq I^r$. It is this problem that is the motivation for [**BocHar07**], which develops an asymptotic approach to this problem. If I is nontrivial (i.e., $0 \neq I \subsetneq R$), then the set $\{m/r : I^{(m)} \not\subseteq I^r\}$ is nonempty, and we define $\rho(I) = \sup\{m/r : I^{(m)} \not\subseteq I^r\}$; *a priori* $\rho(I)$ can be infinite. When an upper bound does exist, we see that $I^{(m)} \subseteq I^r$ whenever $m/r > \rho(I)$.

Swanson [**Swa00**] showed that an upper bound exists for many ideals I. This was the inspiration for the papers [**EiLaSm01**] and [**HocHun02**], whose results imply that

$\rho(I) \leqslant N$ for any nontrivial homogeneous ideal $I \subset \mathbb{P}^N$. In fact, if we define $\operatorname{codim}(I)$ to be the maximum height among associated primes of I other than M, then it follows from [**EiLaSm01**] and [**HocHun02**] that $\rho(I) \leqslant \min\{N, \operatorname{codim}(I)\}$. (If M is an associated prime of I, as happens if I is not saturated, then $I^m = I^{(m)}$ for all $m \geqslant 1$, since every homogeneous primary ideal in R is contained in M.)

This raises the question of whether the bound $\rho(I) \leqslant N$ can be improved. Results of [**BocHar07**] show that this bound and the bounds $\rho(I) \leqslant \operatorname{codim}(I)$ are optimal, in the sense that $\sup\{\rho(I) : 0 \neq I \subsetneq R \text{ homogeneous}\} = N$, and, when $e \leqslant N$,

(Eqn 8.1) $$\sup\{\rho(I) : 0 \neq I \subsetneq R \text{ homogeneous of } \operatorname{codim}(I) = e\} = e\ .$$

To justify this we introduce the following arrangements of linear subspaces.

NOTATION 8.3.2 (Generic arrangements of linear subspaces). Let $H_1, \dots, H_s$ be $s > N$ generic hyperplanes in $\mathbb{P}^N$. Let $1 \leqslant e \leqslant N$ and let $S \subset \{1, 2, \dots, s\}$ with $|S| = e$. We define the scheme $Z_S(N, s, e)$ to be $\cap_{i \in S} H_i$, so $Z_S(N, s, e)$ is a linear subspace of $\mathbf{P}^N$ of codimension e. We also let $Z(N, s, e)$ be the union of all $Z_S(N, s, e)$ with $|S| = e$.

The following result, [**BocHar07**, Lemma 2.3.2(b)], as applied in Example 8.3.4, justifies (Eqn 8.1):

LEMMA 8.3.3 (Asymptotic noncontainment). *Given a homogeneous ideal* $0 \neq I \subsetneq R$ *and* $m/r < \frac{\alpha(I)}{\gamma(I)}$, *then* $I^{(mt)} \not\subseteq I^{rt}$ *for all* $t \gg 0$*; in particular,* $\frac{\alpha(I)}{\gamma(I)} \leqslant \rho(I)$.

EXAMPLE 8.3.4 (Sharp examples of [**BocHar07**]). We write $I(mZ(N, s, e))$ to denote $I(Z(N, s, e))^{(m)}$. Then $\alpha(I(mZ(N, s, e))) = ms/e$ if $e|m$ and $\alpha(I(Z(N, s, e))) = s-e+1$ (see Lemma 8.4.7). Thus $\gamma(I(Z(N, s, e))) = s/e$ and $\rho(I(Z(N, s, e))) \geqslant e(s - e + 1)/s$. Keeping in mind $\rho(I(Z(N, s, e))) \leqslant e$ (which holds by [**HocHun02**] since $\operatorname{codim}(I(Z(N, s, e))) = e$), we now see $\lim_{s\to\infty} \rho(I(Z(N, s, e))) = e$, so the bounds of [**EiLaSm01**] and [**HocHun02**] are sharp.

REMARK 8.3.5 (A Seshadri constant computation). Let $I = I(Z(N, s, N))$. It is interesting to note, by [**BocHar07**, Theorem 2.4.3(a)], that $\rho(I) = \alpha(I)/\gamma(I)$, and hence $\rho(I) = N(s - N + 1)/s$. By an argument similar to that of [**BocHar07**, Lemma 2.3.1], discussed above in Remark 8.2.3, we can express $\rho(I)$ in terms of the Seshadri constant $\varepsilon(N, Z)$. In particular, $\gamma(I) = |Z| \cdot \varepsilon(N, Z)^{N-1}$ holds, and thus we obtain

$$\varepsilon(N, Z) = \sqrt[N-1]{\frac{s}{N\binom{s}{N}}}\ .$$

8.4. Conjectural improvements. Even though the bound $\rho(I) \leqslant N$ is optimal (in the sense that for no value d smaller than N will $\rho(I) \leqslant d$ hold for all nontrivial homogeneous ideals I), we can try to do better. The bound $\rho(I) \leqslant \operatorname{codim}(I)$ can be rephrased as saying $I^{(m)} \subseteq I^r$ if $m > r\operatorname{codim}(I)$. In fact the results of [**EiLaSm01**] and [**HocHun02**] imply the slightly stronger result that $I^{(m)} \subseteq I^r$ if $m \geqslant r\operatorname{codim}(I)$. As a next step, we can ask for the largest *integer* d_e such that $I^{(m)} \subseteq I^r$ whenever $m \geqslant re - d_e$, where $e = \operatorname{codim}(I)$.

Examples of Takagi and Yoshida [**TakYos07**] support the possibility that $I^{(m)} \subseteq I^r$ holds for $m \geqslant Nr - 1$ (i.e., perhaps it is true that $d_e \geqslant 1$). On the other hand, the obvious fact that $\alpha(I^{(m)}) < \alpha(I^r)$ implies $I^{(m)} \not\subseteq I^r$ (see Theorem 8.4.6(a) for a reference), applied with $m = re - e$ for $e > 1$ and $s \gg 0$ to $I(mZ(N, s, e))$ of Example 8.3.4, shows that $d_e < e$.

For example, the fact that $I^{(2)}$ is not always contained in I^2, as we saw in Example 8.1.8, shows that $d_2 < 2$ (at least for $I \subseteq R = k[\mathbb{P}^2]$), and hence either $d_2 = 0$ or $d_2 = 1$. Proving $d_2 = 1$ for $R = k[\mathbb{P}^2]$ would provide an affirmative answer to an as-of-now still open unpublished question raised by Craig Huneke:

QUESTION 8.4.1 (Huneke). Let $I = I(Z)$ where $Z = p_1 + \cdots + p_j$ for distinct points $p_i \in \mathbb{P}^2$. Then we know $I^{(4)} \subseteq I^2$, but is it also true that $I^{(3)} \subseteq I^2$?

The following conjectures are motivated by Huneke's question, by the fact that $d_e < e$ as we saw above, and by a number of suggestive supporting examples which we will recall below:

Conjecture 8.4.2 (Harbourne). Let I be a homogeneous ideal with $0 \neq I \subsetneq R = k[\mathbb{P}^N]$. Then $I^{(m)} \subseteq I^r$ if $m \geqslant Nr - (N-1)$.

Since $Nr-(N-1) \geqslant er-(e-1)$ for any positive integers $e \leqslant N$, the previous conjecture is a consequence of the following more precise version of the conjecture:

Conjecture 8.4.3 (Harbourne). Let I be a homogeneous ideal with $0 \neq I \subsetneq R = k[\mathbb{P}^N]$ and $\operatorname{codim}(I) = e$. Then $I^{(m)} \subseteq I^r$ if $m \geqslant er - (e-1)$.

We conclude by recalling evidence in support of these conjectures.

Example 8.4.4 (Examples of Huneke). After receiving communication of these conjectures, Huneke re-examined the methods of [**HocHun02**] and noticed that they implied that Question 8.4.1 has an affirmative answer in characteristic 2. More generally, Conjecture 8.4.2 is true if $r = p^t$ for $t > 0$, where $p = \operatorname{char}(k) > 0$ and I is the radical ideal defining a set of points $p_1, \ldots, p_j \in \mathbb{P}^N$. Huneke's argument uses the fact that in characteristic p taking a Frobenius power $J^{[r]}$ of an ideal J (defined as the ideal $J^{[r]}$ generated by the rth powers of elements of J) commutes with intersection. (To see this, note that $J^{[r]} = J \otimes_R S$, where $\varphi^t : R \to R = S$ is the tth power of the Frobenius homomorphism. Tensoring $0 \to J_1 \cap J_2 \to J_1 \oplus J_2 \to J_1 + J_2 \to 0$ by S over R gives a short exact sequence. This is because of flatness of Frobenius; see, for example, [**HunSwa06**, Lemma 13.1.3, p. 247] and [**Kun69**]. Comparing the resulting short exact sequence with $0 \to J_1^{[r]} \cap J_2^{[r]} \to J_1^{[r]} \oplus J_2^{[r]} \to J_1^{[r]} + J_2^{[r]} \to 0$ gives the result.) It also uses the observation that a large enough power of any ideal J is contained in a given Frobenius power of J. More precisely, if J is generated by h elements, then $J^m \subseteq J^{[r]}$ as long as $m \geqslant rh - h + 1$. This is because J^m is generated by monomials in the h generators, but in every monomial involving a product of at least $rh - h + 1$ of the generators there occurs a factor consisting of one of the generators raised to the power r.

In particular, since ideals of points in $\mathbb{P}^N$ are generated by N elements, following the notation of Example 8.1.3 (and keeping in mind that r must be a power of p here) we have

$$I^{(rN-N+1)} = \cap_i P_i^{rN-N+1} \subseteq \cap_i P_i^{[r]} = (\cap_i P_i)^{[r]} = I^{[r]} \subseteq I^r.$$

Huneke's argument also applies more generally to show that Conjecture 8.4.3 is true for any radical ideal I when r is a power of the characteristic, using the fact that Frobenius powers commute with colons (see [**HunSwa06**, Proof of part (6) of Theorem 13.1.2, p. 247]) and using the fact that PR_P is generated by h elements, where h is the height of the prime P.

Example 8.4.5 (Monomial ideals). As another example, we now show that Conjecture 8.4.3 holds for any monomial ideal $I \subset R$ in any characteristic. We sketch the proof, leaving basic facts about monomial ideals as exercises.

Consider a monomial ideal I; let $P_1, \ldots, P_s$ be the associated primes. These primes are necessarily monomial ideals and hence are generated by subsets of the variables. Moreover, I has a primary decomposition $I = \cap_{ij} Q_{ij}$ where the P_i-primary component of I is $\cap_j Q_{ij}$ and each Q_{ij} is generated by positive powers of the variables which generate P_i. Let e be the maximum of the heights of P_i and let $m \geqslant er - r + 1$. By definition we then have $I^{(m)} = \cap_i (I^m R_{P_i} \cap R)$, but $\cap_i (I^m R_{P_i} \cap R) \subseteq \cap_i ((\cap_j Q_{ij})^m R_{P_i} \cap R)$ since

$$I^m R_{P_i} = (\cap_{\{t : P_t \subseteq P_i\}} (\cap_j Q_{tj}))^m R_{P_i}.$$

Clearly, $(\cap_j Q_{ij})^m R_{P_i} \subseteq \cap_j Q_{ij}^m R_{P_i}$ but Q_{ij}^m is P_i-primary (hence $Q_{ij}^m R_{P_i} \cap R = Q_{ij}^m$), so we have

$$\cap_i ((\cap_j Q_{ij})^m R_{P_i} \cap R) \subseteq \cap_{ij} Q_{ij}^m.$$

Now, each Q_{ij} is generated by at most e elements, so $Q_{ij}^m \subseteq Q_{ij}^{[r]}$, where $J^{[r]}$ is defined for any monomial ideal J to be the ideal generated by the rth powers of the monomials contained in J; thus $\cap_{ij} Q_{ij}^m \subseteq \cap_{ij} Q_{ij}^{[r]}$. Finally, we note that if J_1 and J_2 are monomial ideals, then $(J_1 \cap J_2)^{[r]} = J_1^{[r]} \cap J_2^{[r]}$ (since $(J_1 \cap J_2)^{[r]}$ is generated by the rth powers of the least common multiples of the generators of J_1 and J_2, while $J_1^{[r]} \cap J_2^{[r]}$ is generated by the least common multiples of the rth powers of the generators of J_1 and J_2, and taking rth powers commutes with taking least common multiples). So we have $\cap_{ij} Q_{ij}^{[r]} = (\cap_{ij} Q_{ij})^{[r]} = I^{[r]} \subseteq I^r$, and we conclude that $I^{(m)} \subseteq I^r$.

The schemes $Z(N, s, N) \subset \mathbb{P}^N$ give additional examples for which Conjecture 8.4.2 is true. These schemes are of particular interest, since, as we saw above, they are asymptotically extremal for ρ, and thus one might expect if the conjecture were false that one of these schemes would provide a counterexample. In order to see why Conjecture 8.4.2 is true for symbolic powers of $I(Z(N, s, N))$, we need the following theorem, for which we recall the notion of the *regularity* of an ideal. We need it only in a special case:

> If I defines a 0-dimensional subscheme of projective space, the regularity $\mathrm{reg}(I)$ of I equals the least t such that $(R/I)_t$ and $(R/I)_{t-1}$ have the same k-vector space dimension.

As an example, if $I = I(p_1 + \cdots + p_j)$ for distinct generic points p_i, then (since the points impose independent conditions on forms of degree i as long as $\dim R_i \geqslant j$) we have $\mathrm{reg}(I) = t + 1$, where t is the least degree such that $\dim(R_t) \geqslant j$.

THEOREM 8.4.6 (Noncontainment and Containment Criteria). *Let $0 \neq I \subseteq R = k[\mathbb{P}^N]$ be a homogeneous ideal.*

(a) *Non-containment Criterion: If $\alpha(I^{(m)}) < \alpha(I^r)$, then $I^{(m)} \not\subseteq I^r$.*
(b) *Containment Criterion: Assume $\mathrm{codim}(I) = N$. If $r\,\mathrm{reg}(I) \leqslant \alpha(I^{(m)})$, then $I^{(m)} \subseteq I^r$.*

PROOF. (a) This is [**BocHar07**, Lemma 2.3.2(a)].
(b) This is [**BocHar07**, Lemma 2.3.4]. □

In order to verify that Question 8.4.1 has an affirmative answer for $I = I(Z(2, s, 2))$, and that $I^{(m)} \subseteq I^r$ holds whenever $m \geqslant Nr - N + 1$ for $I = I(Z(N, s, N))$, we will apply Theorem 8.4.6. To do so, we need the following numerical results.

LEMMA 8.4.7 (Some numerics). *Let $I = I(Z(N, s, e)) \subset R = k[\mathbb{P}^N]$.*

(a) *Then $\alpha(I) = s - e + 1$; if $e = N$, then $\alpha(I) = \mathrm{reg}(I) = s - N + 1$.*
(b) *If $e|m$, then $\alpha(I^{(m)}) = ms/e$.*
(c) *Let $m = iN + j$, where $i \geqslant 0$ and $0 < j \leqslant N$, and let $I = I(Z(N, s, N))$ where $s > N \geqslant 1$. Then $\alpha(I^{(m)}) = (i + 1)s - N + j$.*

PROOF. (a) This holds by [**BocHar07**, Lemma 2.4.2].
(b) This holds by [**BocHar07**, Lemma 2.4.1].
(c) See Lemma 8.4.5 and Proposition 8.5.3 of version 1 of ArKiv0810.0728 for detailed proofs. (We note that the case $N = 2$ follows easily by using induction and Bézout's theorem.) □

EXAMPLE 8.4.8 (Additional supporting evidence). Let $I = I(Z(N, s, N))$. By Lemma 8.4.7(a), $\mathrm{reg}(I) = s - N + 1$ and by Lemma 8.4.7(c), $\alpha(I^{(m)}) = (i + 1)s - N + j$, where $m = iN + j$, $i \geqslant 0$ and $0 < j \leqslant N$. Thus, if $m = rN - (N - 1) = (r - 1)N + 1$, then $\alpha(I^{(m)}) = rs - N + 1 \geqslant r(s - N + 1) = r\,\mathrm{reg}(I)$, and hence $I^{(Nr-(N-1))} \subseteq I^r$ by Theorem 8.4.6(b). Thus Conjecture 8.4.2 holds for $I = I(Z(N, s, N))$. Moreover, when $r = N = 2$, we have $I^{(3)} \subseteq I^2$, which shows that Question 8.4.1 has an affirmative answer for $I = I(Z(2, s, 2))$.

In our remaining two examples, concerning generic points in projective space, Seshadri constants play a key role.

EXAMPLE 8.4.9 (Generic points in $\mathbb{P}^2$). By [**BocHar07**, Theorem 4.1], Huneke's question again has an affirmative answer if I is the ideal of generic points $p_1, \ldots, p_j \in \mathbb{P}^2$. More generally, by [**BocHar07**, Remark 4.3] we have $\rho(I) < 3/2$ when I is the ideal of a finite set of generic points in $\mathbb{P}^2$. It follows that $I^{(m)} \subseteq I^r$ whenever $m/r \geqslant 3/2$. Since $m \geqslant 2r - 1$ implies that either $m/r \geqslant 3/2$ or $r = 1$, Conjecture 8.4.2 is true in the case $N = 2$ and I is the ideal of generic points in the plane.

The proof that $\rho(I) < 3/2$ depends on using estimates for multipoint Seshadri constants to handle the case that j is large. The few remaining cases of small j are then handled *ad hoc*. We now describe this argument for large j in more detail. Let $I = I(Z)$, where $Z = p_1 + \cdots + p_j$ for distinct generic points $p_i \in \mathbb{P}^2$. By [**BocHar07**, Corollary 2.3.5] we have $\rho(I) \leqslant \operatorname{reg}(I)/\gamma(I)$. If $j \gg 0$, we wish to show that $I^{(m)} \subseteq I^r$ for all $m \geqslant 2r-1$. The proof depends on estimating $\varepsilon(2,Z)$ and $\operatorname{reg}(I)$, and then using $\gamma(I) = j\cdot\varepsilon(2,Z)$ from Remark 8.2.3 and $\rho(I) \leqslant \operatorname{reg}(I)/\gamma(I)$.

To estimate $\operatorname{reg}(I)$, given that $\operatorname{reg}(I) = t+1$ where t is the least degree such that $\dim(R_t) \geqslant j$, use the fact that $\dim(R_t) = \binom{t+2}{2}$. It is now not hard to check that $\operatorname{reg}(I) \leqslant \sqrt{2j+(1/4)}+(1/2)$ for $j \gg 0$. We also have $j\cdot\varepsilon(2,Z) \geqslant \sqrt{j-1}$ for $j \geqslant 10$ (for characteristic 0, see [**Xu94**]; see the proof of [**BocHar07**, Theorem 4.2] in general).

Thus for $j \gg 0$ we have

$$\rho(I) \leqslant \operatorname{reg}(I)/\gamma(I) \leqslant (\sqrt{2j+(1/4)}+(1/2))/\sqrt{j-1}\ ;$$

for large j this is close to $\sqrt{2}$ and thus less than $3/2$. But $m \geqslant 2r-1$ implies $m/r \geqslant 3/2 > \rho(I)$ for all $r > 1$. Thus $I^{(m)} \subseteq I^r$ for $r > 1$. If $r = 1$, then we also have $I^{(m)} \subseteq I^{(1)} = I = I^r$.

Finally, we show that $I^{(Nr-(N-1))} \subseteq I^r$ holds for $j \gg 0$ if $I = I(Z)$, where $Z = p_1 + \cdots + p_j$ for distinct generic points $p_i \in \mathbb{P}^N$. The argument is modeled on that used in Example 8.4.9.

Example 8.4.10 (Generic points in $\mathbb{P}^N$). Let $I = I(Z)$, where $Z = p_1 + \cdots + p_j$ for distinct generic points $p_i \in \mathbf{P}^N$. To show $I^{(Nr-(N-1))} \subseteq I^r$, since the case $r = 1$ is clear, it is enough to consider $r \geqslant 2$. As in Example 8.4.9 $\rho(I) \leqslant \operatorname{reg}(I)/\gamma(I)$, so it suffices to show $\operatorname{reg}(I)/(j(\varepsilon(N,Z))^{N-1}) < (rN-(N-1))/r$ for $j \gg 0$, and since $(rN-(N-1))/r$ is least for $r = 2$, it is enough to verify this for $r = 2$. To estimate $\operatorname{reg}(I)$, use the facts that $\dim(R_t) = \binom{t+N}{N}$ and $\operatorname{reg}(I) = t+1$, where t is the least nonnegative integer such that $j \leqslant \binom{t+N}{N}$. Since $j = (\sqrt[N]{N!j})^N/(N!) \leqslant (x+N)\cdots(x+1)/(N!)$ for $x = \sqrt[N]{N!j}-1$, we see $t \leqslant \lceil \sqrt[N]{N!j}-1 \rceil \leqslant \sqrt[N]{N!j}$ and hence $\operatorname{reg}(I) \leqslant \sqrt[N]{N!j}+1$. Next, for $j \gg 0$, we have

$$\frac{j-1}{j}\frac{1}{\sqrt[N]{j-1}} = \frac{\sqrt[N]{(j-1)^{N-1}}}{j} \leqslant \varepsilon_1(\mathbb{P}^N, \mathcal{O}(1); Z)$$

by Theorem 1.1 [**Kue96b**] and

$$\varepsilon_1(\mathbb{P}^N, \mathcal{O}(1); Z) \leqslant \varepsilon(N,Z)$$

by Proposition 2.1.6 (although Proposition 2.1.6 is stated only for the case $j = 1$ of a single point, the proof (see Proposition 5.1.9 [**PAG**]) carries over for any j). Thus

$$\frac{j-1}{j}\frac{1}{\sqrt[N]{j-1}} \leqslant \varepsilon(N,Z)$$

and hence

$$\Big(\frac{j-1}{j}\Big)^{N-2}\sqrt[N]{j-1} = \Big(\frac{j-1}{j}\Big)^{N-1}\frac{j}{\sqrt[N]{j-1}^{N-1}} \leqslant j(\varepsilon(N,Z))^{N-1}$$

so

$$\frac{\operatorname{reg}(I)}{j(\varepsilon(N,Z))^{N-1}} \leqslant \frac{\sqrt[N]{N!j}+1}{\Big(\frac{j-1}{j}\Big)^{N-2}\sqrt[N]{j-1}}$$

for $j \gg 0$. But

$$\lim_{j\to\infty}\frac{\sqrt[N]{N!j}+1}{\Big(\frac{j-1}{j}\Big)^{N-2}\sqrt[N]{j-1}} = \sqrt[N]{N!}$$

so

$$\frac{\operatorname{reg}(I)}{j(\varepsilon(N,Z))^{N-1}} < \frac{N+1}{2}$$

follows for $j \gg 0$ if we have

$$\sqrt[N]{N!} < \frac{N+1}{2},$$

and this is equivalent to $2^N N! < (N+1)^N$. This last is true for $N = 2$, and if it is true for some $N \geqslant 2$, then it holds for $N+1$ (and hence for all $N \geqslant 2$ by induction) if

$$2(N+1) \leqslant (N+2)((N+2)/(N+1))^N,$$

since then

$$2^{N+1}(N+1)! = 2(N+1)2^N N! < (N+2)((N+2)/(N+1))^N(N+1)^N = (N+2)^{N+1}.$$

But taking $n = N+1$, we can rewrite $2(N+1) \leqslant (N+2)((N+2)/(N+1))^N$ as $2 \leqslant ((N+2)/(N+1))^{N+1} = (1+\frac{1}{n})^n = 1^n + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots$, which is obvious.

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