each two nonparallel elements of G cross each other. Obviously the conclusions of the theorem do not hold.

The following example will show that the condition that no two elements of the collection G shall have a complementary domain in common is also necessary. In the cartesian plane let M be a circle of radius 1 and center at the origin, and N a circle of radius 1 and center at the point (5, 5). Let G_1 be a collection which contains each continuum which is the sum of M and a horizontal straight line interval of length 10 whose left-hand end point is on the circle M and which contains no point within M. Let G_2 be a collection which contains each continuum which is the sum of N and a vertical straight line interval of length 10 whose upper end point is on the circle N and which contains no point within N. Let $G=G_1+G_2$. No element of G crosses any other element of G, but uncountably many have a complementary domain in common with some other element of the collection. However, it is evident that no countable subcollection of G covers the set of points each of which is common to two continua of the collection G.

It is not known whether or not the condition that each element of G shall separate some complementary domain of every other one can be omitted.

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A PRINCIPAL AXIS TRANSFORMATION FOR NON-HERMITIAN MATRICES

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The availability of the principal axis transformation for hermitian matrices often simplifies the proof of theorems concerning them. In working with non-hermitian matrices (square or rectangular) it was found that a generalization of this transformation has a similar use for them.* A special case of this generalization has been investigated by Sylvester† who proved Theorem 1 (below) for square matrices with real elements. The unitary matrices U and V are in that case orthogonal matrices with real elements. Special cases had also been

* C. Eckart, *The kinetic energy of polyatomic molecules*, Physical Review, vol. 46 (1934), p. 383; C. Eckart and G. Young, *The approximation of one matrix by another of lower rank*, Psychometrika, vol. 1 (1936), p. 211; A. S. Householder and G. Young, *Matrix approximation and latent roots*, American Mathematical Monthly, vol. 45 (1938), p. 302.

† Sylvester, Messenger of Mathematics, vol. 19 (1889), p. 42.

discussed earlier by Beltrami and Jordan, and more recently Autonne and E. T. Browne have proved the theorem for square matrices with complex elements.†

The following definitions will be convenient for the present purpose. An (r, s) matrix is one having r rows and s columns; its elements may be complex numbers. The hermitian transpose of an (r, s) matrix A, whose elements are a_{ij}, is the (s, r) matrix A^* whose elements are $(a^*)_{ji} = \bar{a}_{ij}$. An (r, s) matrix is diagonal if its elements a_{ij} are all zero unless $i = j$.

THEOREM 1. *For every (r, s) matrix A, there are two unitary matrices U and V, such that*

$$D = U^*AV$$

is a diagonal matrix with real elements, none of which are negative.

The proof of this theorem may be based on the observation that AA^* is a non-negative definite hermitian (r, r) matrix; for it is the Gram matrix of the rows of A, considered as vectors. Consequently there are r vectors (that is, r $(r, 1)$ matrices) X_i such that

$$AA^*X_i = d_i^2 X_i \tag{1}$$

and

$$X_i^*X_k = \delta_{ik}, \qquad i, k = 1, \cdots, r. \tag{2}$$

The numbers d_i^2 are the characteristic values of AA^*, and the X_i are unit vectors along its principal axes. The numbers d_i are real and may be defined to be nonnegative. It is convenient to arrange the numbering of these vectors so that

$$d_1 \geqq d_2 \geqq \cdots \geqq d_n > 0, \qquad d_{n+1} = \cdots = d_r = 0. \tag{3}$$

In the same way, there are s vectors (that is, s $(s, 1)$ matrices) Y_j such that

$$A^*AY_j = e_j^2 Y_j \tag{4}$$

and

$$Y_j^*Y_l = \delta_{jl}. \tag{5}$$

If

$$e_1 \geqq e_2 \geqq \cdots \geqq e_m > 0, \qquad e_{m+1} = \cdots = e_s = 0,$$

† Autonne, *Sur les matrices hypohermitiennes et les unitaires*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 156 (1913), pp. 858–860; E. T. Browne, this Bulletin, vol. 36 (1930), p. 707.

it can be shown that $m=n$ and $e_i=d_i$ whenever $i\leqq n$. For if X_i are the vectors of (1), (2), and (3), then the n vectors defined by

$$Y_i = A^*X_i/d_i, \qquad i \leqq n, \tag{6}$$

will satisfy (4) and (5) with $e_i=d_i$. Since the characteristic values are unique, it follows that m cannot be less than n; inverting the argument, we see that n cannot be less than m.

Any set of vectors X_i for which (1), (2), and (3) hold may be considered as the columns of a unitary (r, r) matrix U. Then let (6) define the first n columns of an (s, s) matrix V, and fill in the remaining columns to make V unitary. These matrices U and V then satisfy the requirements of the theorem.

To prove this we may first observe that if D is the matrix U^*AV, then $DD^*=U^*AA^*U$ is a diagonal matrix with diagonal elements d_i^2, and $D^*D=V^*A^*AV$ is a diagonal matrix with diagonal elements e_i^2. Furthermore, if the matrix D is written as

$$D = \begin{Vmatrix} D_1 & D_2 \\ D_3 & D_4 \end{Vmatrix},$$

where D_1, D_2, D_3, and D_4 are (n, n), $(n, s-n)$, $(r-n, n)$, and $(r-n, s-n)$ matrices, respectively, then these properties of DD^* and D^*D imply

$$D_3D_3^* + D_4D_4^* = 0, \qquad D_2^*D_2 + D_4^*D_4 = 0$$

(among other equations). Since $D_2^*D_2$, $D_3D_3^*$, and $D_4D_4^*$ are all non-negative definite hermitian matrices, it follows that they are all null matrices, and from this, that D_2, D_3, and D_4 are all null matrices. It remains to be shown that D_1 is diagonal with no negative elements. Its ij-element may be written $X_i^*AY_j$; and from (6), (1), and (2) it readily follows that

$$X_i^*AY_j = d_j\delta_{ij}, \qquad i, j \leqq n,$$

which completes the proof of Theorem 1.

The above also proves the following result:

COROLLARY. *In Theorem* 1, U^* *may be any unitary matrix which diagonalizes* AA^*, *and there then exists a unitary matrix* V *such that the theorem is true. Similarly,* V^* *may be taken as any unitary matrix which diagonalizes* A^*A, *and there then exists a matrix* U *satisfying the requirements of the theorem.*

The theorem on the simultaneous transformation of two hermitian matrices to principal axes also generalizes.

THEOREM 2. *If A and B are both (r, s) matrices, then there are two unitary matrices U and V such that $E = U^*A V$, and $F = U^*B V$ are both diagonal matrices with real elements and such that E has no negative elements, if and only if AB^* and B^*A are both hermitian matrices.*

The necessity of the condition is an immediate consequence of the invariance of the class of hermitian matrices under transformations of the form UCU^* when U is unitary, for EF^* is hermitian and $AB^* = UEF^*U^*$, and so on. The sufficiency may be proved as follows. Because of Theorem 1, it is no loss of generality to suppose that A has already been transformed to the form

$$\begin{Vmatrix} D & O_2 \\ O_3 & O_4 \end{Vmatrix},$$

where D is a real diagonal (n, n) matrix of rank n, having no negative elements, and O_2, O_3, O_4 are null matrices. The matrix B may be divided into corresponding submatrices:

$$\begin{Vmatrix} G & K \\ L & H \end{Vmatrix}.$$

Then the condition of the theorem leads to

$$K = O_2, \qquad L = O_3, \qquad DG^* = GD, \qquad G^*D = DG.$$

In element notation, the last two equations are

$$d_i\bar{g}_{ji} = d_jg_{ij}, \qquad d_j\bar{g}_{ji} = d_ig_{ij}.$$

Since d_i, $d_j > 0$, it readily follows that G is hermitian and that $DG = GD$. From the theorem on the simultaneous transformation of two hermitian matrices to principal axes, it follows that there is a unitary (n, n) matrix P such that $P^*DP = D$ and P^*GP is diagonal with real elements. From Theorem 1, it follows that there are also two unitary matrices Q and R such that Q^*HR is a diagonal matrix with real elements. It is then readily seen that the matrices

$$U = \begin{Vmatrix} P & O_3^* \\ O_3 & Q \end{Vmatrix}, \qquad V = \begin{Vmatrix} P & O_2 \\ O_2^* & R \end{Vmatrix}$$

will satisfy Theorem 2.

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