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A PRINCIPLE OF IMPOTENCE ALLOWING FOR NEWTONIAN COSMOLOGIES WITH A TIME-DEPENDENT GRAVITATIONAL CONSTANT

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SUMMARY

A principle of impotence is given. Subject to some additional assumptions, it leads for Newtonian cosmologies to a gravitational interaction of the form

$$\nu(r, t) = -m_1 m_2 \left[\frac{1}{r} + Dr^j\right] G(t),$$

where the gravitational 'constant' $G(t)$ is related to the Hubble constant $H(t)$ by

$$G(t) = \exp\left[-j \int^t H(t)\, dt\right],$$

and j is a constant. This allows for the normal zero-pressure Friedmann cosmologies, the Dirac cosmology and also other models not previously considered. Comparison with observation restricts the likely values of j to the range $-1 < j < 4$.

1. INTRODUCTION

If Newtonian cosmology is discussed for the simple situations in which it leads to the same models as relativistic cosmology, one finds that the Hubble law and the inverse square law of interaction are intimately related. If one adopts the principle that the radius of the Universe shall not be obtainable from a study of the local dynamics, one can in fact *deduce* the inverse square law if the Hubble law is given (Landsberg 1973a). While no doubt of some interest, this use of the principle does not shed light on possible generalizations of the inverse square law. This situation changes drastically, however, if one assumes a gravitational interaction energy $-m_1 m_2 \nu(r, t)$ which is unknown, not only as a function of inter-particle distance r, but also as a function of cosmological time t. The principle then enables one to arrive at a permissible class of interaction functions

$$\nu(r, t) = \frac{G(t)}{r} + \sum_j D_j r^j \exp\left[-j \int^t H(t)\, dt\right], \tag{1.1}$$

where G is a (possibly time-dependent) gravitational 'constant', H is Hubble's constant and the D_j are constants which may be zero. For factorizable functions $\nu(r, t)$, at most one value of j is possible and the class of new cosmological models is sufficiently restricted to make their study worthwhile. Sections 2 and 3 of this paper give the argument which leads to these results, while Section 4 gives a discussion of the various models. It is found that on theoretical grounds $j > -3$.

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The restrictions on the models which result from the approximate knowledge of the present Hubble constant H_0, of the present deceleration parameter q_0, and the present age T_0 of the Universe lead to

$$-1 \lesssim j \lesssim 4.$$

These models have the property that for a given value of q_0 the highest tolerable value of j leads to the smallest ' missing matter ' problem. The Dirac (1937, 1973) cosmology is included in principle among these models and can be specified by $j = 3$ and $q_0 = 2$. However, the resulting value of T_0 is rather low and q_0 is somewhat large; models still having $j = 3$ but with smaller values of q_0 (and hence greater values of T_0) are included and cannot be ruled out ony any of the grounds considered here. These matters are discussed in Section 5.

2. THE ENERGY EQUATION OF NEWTONIAN COSMOLOGY

We consider a standard spherical homogeneous Newtonian universe of radius $q(t)$, and with an unknown gravitational interaction $\nu(r, t)$. The Hubble parameter H and the density $\rho(t)$ are related by

$$H = \frac{\dot{q}}{q} = -\frac{\dot{\rho}}{3\rho}. \tag{2.1}$$

The kinetic energy of this system is

$$T = \frac{2\pi}{5}\rho(t)[H(t)]^2\,[q(t)]^5. \tag{2.2}$$

For a general two-body gravitational interaction which may for given r depend on time, the interaction energy is

$$-m_1 m_2 \nu(r, t). \tag{2.3}$$

The potential energy of the model is

$$V = -8\pi^2[\rho(t)]^2\,J(q, t) \tag{2.4}$$

where

$$J(q, t) = \frac{1}{48}\int_0^{2q(t)} (r^5 - 12q^2r^3 + 16q^3r^2)\,\nu(r, t)\,dr. \tag{2.5}$$

Equations (2.4), (2.5) have been given before in a slightly different form (Landsberg 1973a, equations (5) and (6)).

In general the energy equation can be written in the form

$$E = \phi(t)[q(t)]^5 - 8\pi^2[\rho(t)]^2\,J(q, t) \tag{2.6}$$

where in the simplest case

$$\phi(t) = \frac{2\pi}{5}\rho(t)[H(t)]^2 - \frac{2\pi\lambda}{15}\rho(t). \tag{2.7}$$

Here λ is the cosmological ' constant ' and a time dependence of λ will be allowed. Other terms, for instance one due to a rigid rotation, could be added in (2.7), but will not be needed here. Introducing a constant a and the scale factor $R(t)$ by

$$\rho(t) = a[R(t)]^{-3}, \tag{2.8}$$

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the energy equation (2.6) can be cast into the form

$$\frac{5}{2\pi a}[R(t)]^5\,\phi(t) - 20\pi a\,\frac{J(q,t)}{[q(t)]^5\,R(t)} = \frac{5E}{2\pi a}\left[\frac{R(t)}{q(t)}\right]^5 (\equiv -kc^2). \tag{2.9}$$

With (2.7) the energy equation (2.9) is

$$\dot{R}^2 = \frac{20\pi\rho(t)\,J(q,t)}{[q(t)]^5}[R(t)]^2 + \frac{\lambda(t)[R(t)]^2}{3} - kc^2. \tag{2.10}$$

We have used the energy equation approach rather than one based on an equation of motion of a fluid, because this procedure saves an integration; it is easy to generalize by adding additional terms, as for instance if rigid rotation is to be included, and it has a clear conceptual basis even for a finite set of arbitrarily placed galaxies (Landsberg 1973b).

3. A PRINCIPLE OF IMPOTENCE

Defining q_1 by $q(t) = q_1 R(t)$ and regarding now J as a function of q_1 and t, one finds from (2.5)

$$\left\{\frac{\partial^2}{\partial q_1{}^2}\left[q_1{}^{-1}\left(\frac{\partial J}{\partial q_1}\right)_t\right]\right\}_t = 8q_1{}^2[R(t)]^6\,\nu(2q_1R,t). \tag{3.1}$$

We now formulate a cosmological principle which leads to a functional form for J and hence implies a functional form for the gravitational interaction. The study of the dynamics of the Universe can lead in principle to a knowledge of $\rho(t)$, $\dot{\rho}(t)$, $\lambda(t)$, etc., and hence to $\dot{\rho}/\rho = -3H(t)$. Also R and $\dot{R}$ are known from ρ and the assigned value of a by means of equation (2.8). Hence if kc^2, essentially the energy of the Universe, were also known, then (2.10) could be used to compute the function $q(t)$ from a knowledge of the interaction function $\nu(r,t)$. However, we shall require that q_1 and therefore $q(t)$ *shall not be obtainable from the local dynamics of the Universe*. This principle of impotence is a constraint on the permissible form of J and leads uniquely (using (3.1)) to an inverse square law of gravitation. This is essentially the standpoint adopted previously (Landsberg 1973a).

In fact, the term kc^2 is also not known, and in order to eliminate it from (2.9), consider instead

$$[R(\tau)]^5\,\phi(\tau) - [R(\tau')]^5\,\phi(\tau') = 8\pi^2a^2\left\{\frac{J(q_1,\tau)}{q_1{}^5[R(\tau)]^6} - \frac{J(q_1,\tau')}{q_1{}^5[R(\tau')]^6}\right\}. \tag{3.2}$$

Knowledge of R, $\dot{R}$, a, λ, ρ leads to $\phi(t)$, the gravitational interaction yields the function $J(q_1,t)$ in (3.2), hence (3.2) can be solved for q_1 and one now finds the radius $q(t)$ of the Universe unless q_1 cancels out of (3.2). Thus one finds the principle of impotence to require that there exists a function $h(\tau,\tau')$, independent of q_1, such that

$$K(q_1,\tau) - K(q_1,\tau') = h(\tau,\tau') \quad \text{where} \quad K(q_1,\tau) \equiv \frac{J(q_1,\tau)}{q_1{}^5[R(\tau)]^6}.$$

It therefore follows that $K(q_1,t)$ is a function of t plus a function of q_1, i.e.

$$J = \frac{2}{15}G(t)[R(t)]^5\,q_1{}^5 + \chi(q_1)[R(t)]^6,$$

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where the coefficient of q_1^5 has been chosen for later convenience. If $\chi(q_1)$ can be expressed as a power series, one finds by (3.1) that

$$\nu(r, t) = \frac{G(t)}{r} + \sum_j d_j \left[\frac{r}{R(t)}\right]^j, \tag{3.3}$$

where the d_j are appropriate constants; (3.3) is equivalent to (1.1).

For the rest of this paper we shall confine attention to factorizable potentials $\nu(r, t)$. Then at most one power j can occur in (3.3) and

$$\nu(r, t) = \left(\frac{1}{r} + Dr^j\right) G(t). \tag{3.4}$$

Here $G(t)$ is an unknown function of t if $D = 0$, and this includes Newton's law of gravitation. However, if this law is amended for large distances by the existence of a non-zero value of D, then the time-dependence of G is, from (3.3),

$$\left.\begin{aligned} G(t) &= G_1[R(t)]^{-j}, \\ \text{i.e.} \quad G(t) &= G(t_2) \exp\left[-j \int_{t_2}^{t} H(t)\, dt\right], \end{aligned}\right\} \tag{3.5}$$

where G_1 is a constant and t_2 is a fixed time. This result holds independently of the form of the function $\phi(t)$ in (2.6). In the special case (2.10) the energy equation takes the familiar form

$$\dot{R}^2 = \frac{8}{3}\pi a \frac{G(t)}{R(t)} + \frac{1}{3}\lambda(t)[R(t)]^2 - k'c^2 \tag{3.6}$$

where k' incorporates a constant term arising from $J(q_1, t)/q_1{}^5R^6$.

4. PROPERTIES OF THE COSMOLOGICAL MODELS

For convenience of exposition three further assumptions are made. The first is to assume R rescaled so that $k' = 0$ or 1 or -1, as in normal Friedmann cosmologies. The second is that $\lambda(t)$ is in fact a constant. The third is represented by (3.5) for some value of $j > -3$. This is an additional assumption if D should vanish, but it is a consequence of what has already been assumed if D should not be zero. The reason is that if $D \neq 0$ and $j < -3$ in (3.4) then equation (2.5) shows that the potential energy of any uniform sphere diverges. The value of D will not be specified and all that follows holds if $D = 0$ and if $D \neq 0$.

Equation (3.6) can be written as

$$\dot{R}^2 = \frac{8\pi a}{3} \frac{G_1}{[R(t)]^{j+1}} + \frac{\lambda}{3}[R(t)]^2 - k'c^2 \tag{4.1}$$

and leads to a qualitative classification of models. This is given in Table I. We find that all models for which $j > -1$ have the same qualitative behaviour as the standard Friedmann models which, by virtue of (3.5), correspond to $j = 0$. Departures from these occur for $j \leqslant -1$. One observes that accelerating expansion is helped by (i) increasing λ, (ii) decreasing k, and (iii) decreasing j. Also, for

$$-3 < j < -1, \quad \lambda < 0, \quad k = 1,$$

one has either a stable-static solution or an oscillatory solution free of singularities.

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TABLE I

Qualitative properties of (4.1): $\dot{R}^2 = C/[R(t)]^{j+1} + \lambda/3[R(t)]^2 - k'c^2$ *where* $C = (8/3)\,\pi a G_1$

λ	k'	$j > -1$	$j = -1$	$-3 < j < -1$
<0	$+1$	One oscillation	One oscillation if $C > c^2$. No model if $C \leqslant c^2$	Oscillations between the two solutions of $CR^{-(j+1)} - \lvert\lambda\rvert/3R^2 - c^2 = 0$. Stable static solution if, in addition, $\lambda = \lambda_c$*
	0	One oscillation	One oscillation	One oscillation
	-1	One oscillation	One oscillation	One oscillation
0	$+1$	One oscillation	Expansion with constant $\dot{R}$ if $C > c^2$. Neutral static solution* if $C = c^2$. No model if $C < c^2$	Accelerating expansion starting from $R = [c^2/C]^{-1/(j+1)}$
	0	Expansion according to the equation $2R^{(j+3)/2} = (j+3)\,C^{1/2}t$	Expansion with constant $\dot{R}$	Accelerating expansion
	-1	Decelerating expansion	Expansion with constant $\dot{R}$	Accelerating expansion
>0	$+1$	If $0 < \lambda < \lambda_c$, either one oscillation or decelerating contraction to a value of $R > R_0$, followed by accelerating expansion. If $\lambda = \lambda_c$*, either an unstable static solution with $R = R_0 \equiv [(j+3)\,C/2c^2]^{1/(j+1)}$ or accelerating expansion starting from $R = R_0$. If $\lambda > \lambda_c$, expansion first decelerating then accelerating	Accelerating expansion starting from $R = 0$ if $C > c^2$, but from $R = [3(c^2 - C)/\lambda]^{1/2}$ if $C < c^2$	Accelerating expansion starting from a non-zero value of R given by $CR^{-(j+1)} + \lambda R^2/3 - c^2 = 0$
	0	Expansion, first decelerating, then accelerating	Accelerating expansion	Accelerating expansion
	-1	Expansion, first decelerating, then accelerating	Accelerating expansion	Accelerating expansion

* Note that a necessary condition for any of the three types of static model is $\lambda = \lambda_c \equiv 3(j+1)\,C/2R^{j+3}$.

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5. DISCUSSION

In this section it will be assumed that

$$\lambda = 0, \quad H_0 = 60 \text{ km s}^{-1} \text{ Mpc}^{-1}. \tag{5.1}$$

It is convenient to introduce the deceleration parameter q*

$$-\frac{\ddot{R}R}{\dot{R}^2} \equiv q = \frac{1}{2}\frac{\rho(t)}{\rho_c(t)}(j+1), \quad \text{where} \quad \rho_c = \frac{3[H(t)]^2}{8\pi G(t)} \tag{5.2}$$

is the critical mass density, with present value 7×10^{-30} g cm^{-3} if (5.1) is used.

5.1 *The value of j*

In the discussion of parameters we shall follow the recent reviews of Peebles (1971) and Weinberg (1972). One can use the result (Shapiro *et al.* 1971)

$$|\dot{G}/G| \lesssim 4 \times 10^{-10} \text{ yr}^{-1} \tag{5.3}$$

from which it follows, using (5.1), that $|j| \lesssim 7$. On the other hand (Sandage 1972; Sandage & Hardy 1973) it is believed that $q_0 \sim 1$, although all that can be said with certainty is

$$0 < q_0 < 2, \tag{5.4}$$

so that by (5.2) $-1 < j$. Indeed, geophysical evidence suggests that if G is changing then it is decreasing (Wesson 1973; Hoyle 1972) in which case $0 < j$. This shows that the assumption $j > -3$ made at the beginning of Section 4 is in accord with experience. As pointed out by Weinberg (1972, p. 631) a variation of G as t^{-1}, i.e. $j = +3$, or a more rapid variation of G would lead to an average temperature of the Earth's surface 10^9 yr ago which is rather high. We therefore adopt somewhat arbitrarily the constraint

$$-1 < j < 4 \tag{5.5}$$

The present mass-density ρ_0 of the Universe together with its age, if these parameters were known accurately, would also restrict the permitted values of j. The Galaxy and hence the Universe has an age $T_0 > 4.5 \times 10^9$ yr, as determined from the radioactive elements and their decay products in the crust of the Earth (Patterson 1956; Ostic, Russell & Reynolds 1963). Radioactive dating of the Galaxy appears to push this limit up to

$$T_0 > 7 \times 10^9 \text{ yr} \tag{5.6}$$

(Weinberg 1972, p. 488). This imposes an upper limit on j for a given present mass density ρ_0. It should be noted that the oldest stars are believed (Sandage 1970) to be $(10 \pm 3) \times 10^9$ yr old. This estimate would, however, be reduced if G is decreasing, by an amount estimated by Dicke (1962).

5.2 *Alleviation of the ' missing matter ' problem*

Assuming that all the matter in the Universe is in the galaxies, the smeared-out mass density is

$$\rho_G \sim 2 \times 10^{-31} \text{ g cm}^{-3} \tag{5.7}$$

(Oort 1958; van den Bergh 1961; Peebles 1971, p. 58; Weinberg 1972, p. 478). The larger value of the present ' cosmological ' mass density, ρ_0, inferred from q_0

* The radius q of the Universe no longer occurs in this section.

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by equations like (5.2) leads to the missing matter problem in the form: why is ρ_G/ρ_0 so much smaller than unity? The existence of a non-zero value of j in (5.2) eases this problem compared with the problem as it arises in simple Friedmann cosmologies (with $\lambda = 0$) by a factor

$$\frac{\rho_0\,(\text{here})}{\rho_0\,(\text{Friedmann})} = \frac{1}{j+1}.$$

Contours of constant T_0 and of constant j are plotted in the $(\rho_G/\rho_0, q_0)$ plane (Fig. 1). The shaded area of Fig. 1 contains those models which either have too small an age ($T_0 < 7 \times 10^9$ yr) or too large a value of j ($j > 4$). As is clear from our remarks on relation (5.4), a model with $q_0 \sim 1$ and $\rho_G/\rho_0 \sim 1$ would be most desirable. It is labelled A on Fig. 1 and is therefore ruled out by a good margin.

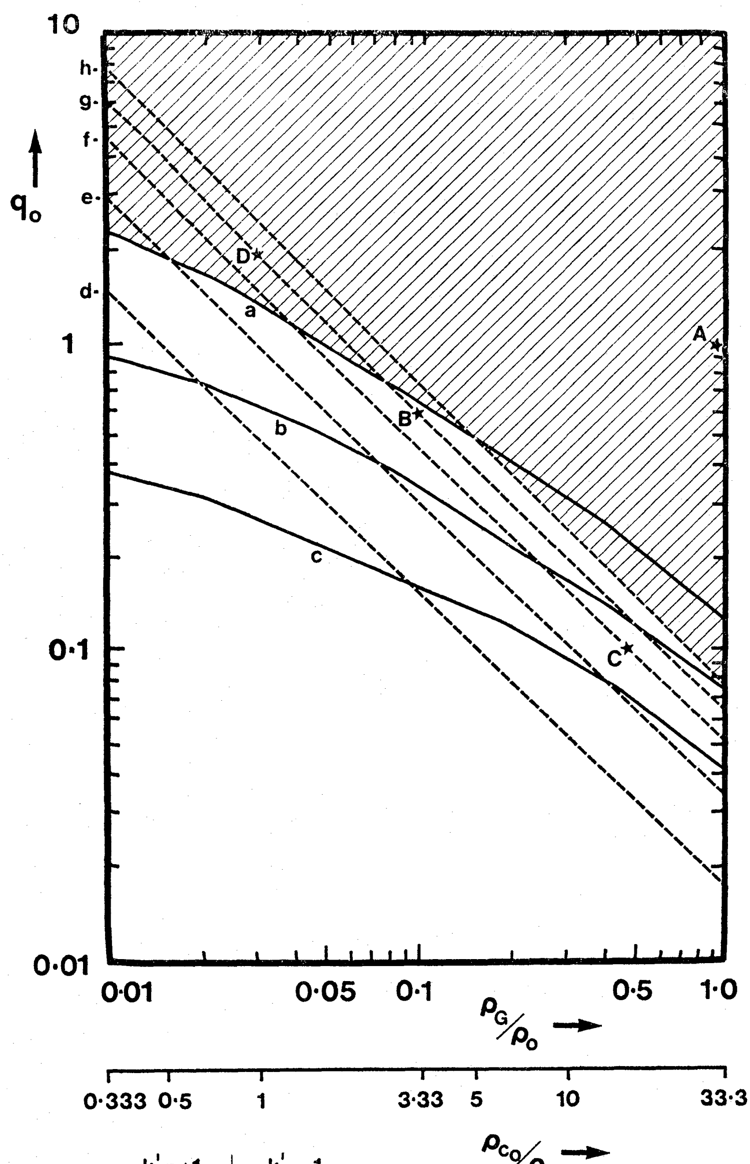

FIG. 1. *Contours of constant age T_0 and of constant j in the $(\rho_G/\rho_0, q_0)$ plane. Solid lines: (a) $T_0 = 7 \times 10^9$ yr, (b) $T_0 = 10 \times 10^9$ yr, (c) $T_0 = 13 \times 10^9$ yr. Broken lines: (d) $j = 0$, (e) $j = 1$, (f) $j = 2$, (g) $j = 3$, (h) $j = 4$. The figure is based on equations* (4.1) *and* (5.2) *subject to* (5.1) *and* (5.7).

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However, compromise models which raise a missing matter problem in a comparatively mild form are possible. Examples, indicated explicitly in Fig. 1, are

$$\text{B:}\quad j = 3,\quad q_0 \sim 0{\cdot}6,\quad \rho_G/\rho_0 \sim 0{\cdot}1,\quad \rho_{c0}/\rho_0 \sim 3{\cdot}3$$
$$\text{C:}\quad j = 2,\quad q_0 \sim 0{\cdot}1,\quad \rho_G/\rho_0 \sim 0{\cdot}5,\quad \rho_{c0}/\rho_0 \sim 16{\cdot}5.$$

5.3 *Connection with the Dirac cosmology*

Dirac (1937, 1973) has proposed the relation $H(t) = H_1 G(t)$ for all times, H_1 being a constant. Using this assumption one readily deduces that for our class of models $k' = 0, j = 3, q = 2, \rho = \rho_c, \dot{G}/G = -3H$, and the present age $T_0 = 5{\cdot}4 \times 10^9$ yr. This model is labelled D in Fig. 1, and it is seen that the rather large value of q_0 causes difficulty with the age T_0.

5.4 *Qualitative comments*

The main physical idea needed for understanding this section is to bear in mind that both an increase in the present density ρ_0 and in the past gravitational ' constant ' $G(t)$ favour independently a more rapid deceleration $q(t)$, so producing a smaller inferred age T_0. The past values of $G(t)$ are increased by increasing j. These features can all be seen in Fig. 1.

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NOTE ADDED IN PROOF

The recent result $\dot{G}/G = (-8 \pm 5) \times 10^{-11}$ years^{-1} (van Flandern 1975) leads $j = 1{\cdot}4 \pm 0{\cdot}9$ if (3.5) and (5.1) are used, and lies almost exactly in the middle of the range (5.5) adopted in this paper.

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