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DRAFT: A PRINCIPLE OF SIMILARITY FOR NONLINEAR VIBRATION ABSORBERS

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ABSTRACT

With continual interest in expanding the performance envelope of engineering systems, nonlinear components are increasingly utilized in real-world applications. This causes the failure of well-established techniques to mitigate resonant vibrations. In particular, this holds for the linear tuned vibration absorber (LTVA), which requires an accurate tuning of its natural frequency to the resonant vibration frequency of interest. This is why the nonlinear tuned vibration absorber (NLTVA), the nonlinear counterpart of the LTVA, has been recently developed. An unconventional aspect of this absorber is that its restoring force is tailored according to the nonlinear restoring force of the primary system. This allows the NLTVA to extend the so-called Den Hartog's equal-peak rule to the nonlinear range.

In this work, a fully analytical procedure, exploiting harmonic balance and perturbation techniques, is developed to define the optimal value of the nonlinear terms of the NLTVA. The developments are such that they can deal with any polynomial nonlinearity in the host structure. Another interesting feature of the NLTVA, discussed in the paper, is that nonlinear terms of different orders do not interact with each other in first approximation, thus they can be treated separately. Numerical results obtained through the shooting method coupled with pseudo-arclength continuation validate the analytical developments.

Keywords: principle of similarity, nonlinear resonances, equal-peak method, vibration absorber.

INTRODUCTION

The engineering trend of designing more and more slender lightweight structure, in order to expand the performance envelope, results in an increased appearance of nonlinear behaviors in real-world applications. Mitigating the resonant vibrations of nonlinear structures is therefore becoming a problem of great practical significance [1]. In particular, passive means for mitigating resonant vibrations generally relay on the invariance of the resonant frequency with respect to variations of the forcing amplitude, which is the case for the linear tuned vibration absorber (LTVA). If nonlinearities are involved, frequency-amplitude invariance is lost, causing the failure of these mitigation techniques.

In the last decades, several nonlinear vibration absorbers have been developed, including the autoparametric vibration absorber [2, 3], the nonlinear energy sink (NES) [4–6], impact dampers [7, 8] and other similar solutions [9, 10]. In general, these nonlinear absorbers abandon the principle of tuning according to the resonant frequency of interest of the host structure, instead they exploit a wider frequency range of effectiveness. This is the case of the NES, an essentially nonlinear absorber, which can resonate at virtually any frequency of the host structure, thanks to its increased bandwidth [11]. This makes nonlinear vibration absorbers proper candidates for vibration mitigation of nonlinear host structures. Per contra, they present a critical dependence on the input energy [5], thus, if the energy content is insufficient, there is no coupling between the primary system and the absorber, hence no energy is transferred to the ab-

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sorber, nor the dissipation mechanism is activated. Similarly, the saturation phenomenon, characteristic of autoparametric vibration absorber, occurs only when the forcing amplitude exceeds a certain threshold [2].

This paper builds upon previous works [12–15], to develop a vibration absorber for mitigating a nonlinear resonance, termed the nonlinear tuned vibration absorber (NLTVA), because its nonlinear restoring force is tuned according to the nonlinear restoring force of the host structure. Acknowledging the excellent performance of the LTVA when applied to a linear host structure [16], almost insurmountable by other passive devices except in very special conditions [17], the objective of the NLTVA is to extend the base principle of the LTVA, i.e. Den Hartog's equal-peak method [16], to the nonlinear range. The NLTVA is thus tuned to target a specific resonant frequency of the host structure and keep a 1:1 resonance condition also in the nonlinear range. To do so, the NLTVA exploit the 'principle of similarity' [18], in other words, its nonlinear restoring force is not defined a priori, but it has the same mathematical form of that of the host structure and it is tuned accordingly. In [13, 15] it was shown that the NLTVA is able to mitigate a nonlinear resonant frequency, where the nonlinearity is related to a cubic spring, for a large range of forcing amplitude, always outperforming the LTVA. Experimental validations of the numerical developments are described in [19]. In spite of these convincing results, a solid theoretical basis justifying the effectiveness of the NLTVA is still missing, most of the results having been obtained with semi-analytical, numerical or experimental methods. The development of this theoretical basis is the main focus of the present study.

The primary objective of this work is to define a fully analytical procedure to correctly tune the nonlinear coefficients of the NLTVA. The initial adimensionalization of the system will already explain the importance of the application of the 'principle of similarity'. Then, the analytical investigation, adopting the harmonic balance method and a perturbation technique, will result in a compact formula relevant for parameter tuning. In the following, analytical findings will be verified through numerical evaluation of a case study. The overall procedure will clarify the effective mitigation mechanisms obtained through the NLTVA. Another important contribution of this paper is to generalize the results, traditionally limited to cubic nonlinear restoring force function, to any polynomial function.

MECHANICAL MODEL

The primary system considered throughout this paper is a harmonically forced, lightweight undamped nonlinear oscillator (Fig. 1). The restoring force of the oscillator is considered to be polynomial. Following the results obtained in [13], an NLTVA is attached to the primary system. Addressing the 'principle of similarity', the NLTVA posses a restoring force with the same mathematical form of the primary system.

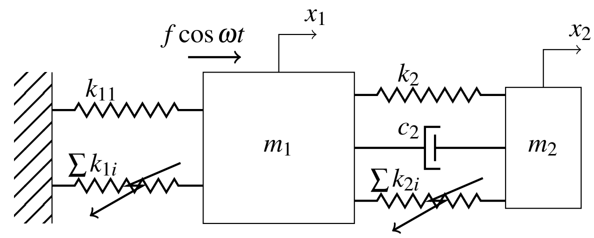

FIGURE 1. MECHANICAL MODEL.

The equations of motion of the coupled system are

$$\begin{aligned} m_1\ddot{x}_1+\sum_{i=1}^{n}k_{1i}x_1^i+c_2\left(\dot{x}_1-\dot{x}_2\right)+\sum_{i=1}^{n}k_{2i}\left(x_1-x_2\right)^i=f\cos\omega t \\ m_2\ddot{x}_2+c_2\left(\dot{x}_2-\dot{x}_1\right)+\sum_{i=1}^{n}k_{2i}\left(x_2-x_1\right)^i=0 \end{aligned} \tag{1}$$

where x_1 and x_2 are the displacement of the primary system and of the NLTVA; m_1, m_2, k_{11} and k_{21} are the masses and the linear restoring force coefficients of the primary system and of the NLTVA, respectively; c_2 is the damping coefficient of the NLTVA, k_{1i} and k_{2i}, $i=2,...,n$, are the nonlinear restoring force coefficients of the primary system and of the NLTVA, respectively; f and ω are the forcing amplitude and frequency, respectively, and n is the highest order of nonlinearity present in the primary system.

Defining the dimensionless time $\tau=t\omega_{n1}$ ($\omega_{n1}=\sqrt{k_{11}/m_1}$) and introducing the variables $q_1=x_1k_{11}/f$ and $q_2=x_2k_{11}/f$, the system is transformed into

$$\begin{aligned} q_1''+q_1+2\mu_2\lambda\varepsilon\left(q_1'-q_2'\right)+\lambda^2\varepsilon\left(q_1-q_2\right) \\ +\sum_{i=2}^{n}\alpha_iq^i+\varepsilon\sum_{i=2}^{n}b_i\alpha_i\left(q_1-q_2\right)^i=\cos\gamma\tau \\ \varepsilon q_2''+2\mu_2\lambda\varepsilon\left(q_2'-q_1'\right)+\lambda^2\varepsilon\left(q_2-q_1\right) \\ +\varepsilon\sum_{i=2}^{n}b_i\alpha_i\left(q_2-q_1\right)^i=0 \end{aligned} \tag{2}$$

where $\mu_2=c_2/\left(2\sqrt{m_2k_{21}}\right)$, $\lambda=\omega_{n2}/\omega_{n1}=\sqrt{k_{21}m_1/(k_{11}m_2)}$, $\varepsilon=m_2/m_1$, $\gamma=\omega/\omega_{n1}$, $\alpha_i=k_{1i}f^{i-1}/k_{11}^i$ and $b_i=k_{2i}/(\varepsilon k_{1i})$. The prime indicates derivation with respect to the dimensionless time τ. It should be noted that in the dimensionless system the forcing amplitude does not appear explicitly, but it is included in the coefficient of the nonlinear terms, thus it is equivalent considering a system strongly excited or strongly nonlinear.

Furthermore, we highlight that the forcing amplitude f, considering Eqn. (2), appears only in the nonlinear terms; in the coefficients of the terms of order i, f has order $i-1$. As better explained in [13], this means that, if the absorber does not have terms of the same order of the primary system, it cannot be efficient for a large range of forcing amplitude f, since its variations would affect terms of different order in a different way. Far from proving its effectiveness, this result suggests that the 'principle of similarity' can be exploited to develop an efficient nonlinear vibration absorber.

AN ANALYTICAL PROCEDURE TO EXTEND EQUAL-PEAK CONCEPT TO NONLINEAR RANGE

The objective of the present section is to define a procedure to identify the optimal values of $\mathbf{b} \equiv [b_2, ., b_n]$ such that the system satisfies the equal-peak rule also in the nonlinear range. The NLTVA is defined by the parameters ε, λ, μ_2 and $\mathbf{b}$. ε is generally given by practical constraints. The values of λ and μ_2 are chosen in order to satisfy the equal-peak rule in the underlying linear system. These were exactly defined in [20], correcting the approximate values given by Den Hartog in [16], namely

$$\begin{aligned}
\lambda &= \frac{2}{1+\varepsilon}\sqrt{\frac{2\left(16+23\varepsilon+9\varepsilon^2+2(2+\varepsilon)\sqrt{4+3\varepsilon}\right)}{3(64+80\varepsilon+27\varepsilon^2)}} \\
\mu_2 &= \frac{1}{4}\sqrt{\frac{8+9\varepsilon-4\sqrt{4+3\varepsilon}}{1+\varepsilon}}
\end{aligned} \tag{3}$$

for an undamped primary system. They depend uniquely on the mass ratio ε.

The set of values $\mathbf{b}$ should be chosen in order to extend the equal-peak rule to the nonlinear range, such that it is verified for a reasonable range of forcing amplitude f. Considering the mathematical form chosen for the absorber, this implies that a constant set of values $\mathbf{b}$ should be valid for variations of the set of parameters α_i, $i = 2, ..., n$, given specific values of the structural parameters of the host structure (k_{1i}, $i = 1, ..., n$).

Since this study considers the nonlinear system as an extension of the underlying linear one, α_i can be considered infinitesimal, significantly simplifying the procedure. This assumption will not limit the results to small nonlinearities, thanks to the adoption of the 'principle of similarity. In [13] it was shown the validity of this approach for a single nonlinear term of order three.

Approximate solution of the nonlinear problem

In order to transform the system of nonlinear differential equations into a system of nonlinear algebraic equations, the harmonic balance procedure is adopted, i.e., q_1 and q_2 are expanded in Fourier series

$$\begin{aligned}
q_1 &\approx A_0 + \sum_{i=1}^{m} A_{1i}\cos(i\gamma\tau) + \sum_{i=1}^{m} A_{2i}\sin(i\gamma\tau) \\
q_2 &\approx B_0 + \sum_{i=1}^{m} B_{1i}\cos(i\gamma\tau) + \sum_{i=1}^{m} B_{2i}\sin(i\gamma\tau),
\end{aligned} \tag{4}$$

where $m \geq n$.

Regardless of the chosen maximal harmonic m, the system can be expressed in the form

$$\mathbf{W}(\gamma)\mathbf{y} + \sum_{i=2}^{n} \alpha_i \left(\mathbf{d}_{i0}(\mathbf{y}) + b_i \mathbf{d}_{i1}(\mathbf{y})\right) = \mathbf{c}, \tag{5}$$

where $\mathbf{W}$ is related to the linear part of the system, $\mathbf{y}$ collects the amplitude of the different harmonics of the solution, $\mathbf{d}_{i0}$ and $\mathbf{d}_{i1}$ contain the nonlinear terms and $\mathbf{c}$ is related to the external forcing.

Exploiting the fact that α_i are small parameters, we expand $\mathbf{y}$ with respect to α_i at the first order, i.e.

$$\mathbf{y} \approx \mathbf{y}_0 + \sum_{i=1}^{n} \alpha_i \mathbf{y}_i. \tag{6}$$

Decomposing Eqn. (5) with respect to the different parameters α_i, the vectors $\mathbf{y}_i$ can be explicitly defined through the formulas

$$\begin{aligned}
&\mathbf{W}\mathbf{y}_0 = \mathbf{c} \rightarrow \mathbf{y}_0 = \mathbf{W}^{-1}\mathbf{c} \\
&\mathbf{W}\mathbf{y}_i + \left(\mathbf{d}_{i0} + b_i \mathbf{d}_{i1}\right)|_{\mathbf{y}=\mathbf{y}_0} = \mathbf{0} \rightarrow \mathbf{y}_i = -\mathbf{W}^{-1}\left(\mathbf{d}_{i0} + b_i \mathbf{d}_{i1}\right)_{\mathbf{y}=\mathbf{y}_0}.
\end{aligned} \tag{7}$$

The frequency response function $h(\gamma)$, describing the maximal value of q_1 for different forcing frequency, which is of interest for the application of the equal-peak rule, can be identified from the parameters contained in $\mathbf{y}$. For practical reasons, it is convenient to consider the square of the frequency response $H = h^2$ and substitute the parameter γ with its square $\Gamma = \gamma^2$. Neglecting terms of order higher than the first with respect to all the coefficients α_i, the square of the frequency response has the form

$$H = H_0 + \sum_{i=2}^{n} \alpha_i \left(H_{i0} + b_i H_{i1}\right), \tag{8}$$

where H_0, H_{i0} and H_{i1} are complicated functions of Γ.

Identification of the nonlinear resonant frequencies

The squares of the resonant frequencies of the underlying linear system, as defined in [20], are

$$\begin{aligned}\hat{\Gamma}_A, \hat{\Gamma}_B = &\frac{1}{2}\left(1+(1+\varepsilon)\lambda_{\text{opt}}^2 - 2(1+\varepsilon)^2\lambda_{\text{opt}}^2\mu_{2,\text{opt}}^2\right) \\ &\mp\sqrt{\frac{1}{4}\left(1+(1+\varepsilon)\lambda_{\text{opt}}^2 - 2(1+\varepsilon)^2\lambda_{\text{opt}}^2\mu_{2,\text{opt}}^2\right)^2 - \lambda_{\text{opt}}^2 r}\end{aligned} \tag{9}$$

where $r = 8\left((4+3\varepsilon)^{3/2}-\varepsilon\right)/\left(64+80\varepsilon+27\varepsilon^2\right)$.

Considering the full nonlinear system, the resonant frequencies Γ_A and Γ_B are given by the zeros of $H_\Gamma = 0$, where the subscript $_\Gamma$ indicates derivation with respect to Γ. Since we consider small values of α_i, $i = 2,...,n$, Γ_A and Γ_B can be approximated to small variations of $\hat{\Gamma}_A$ and $\hat{\Gamma}_B$, which is obtained linearizing H_Γ around $\hat{\Gamma}_A$ and $\hat{\Gamma}_B$, respectively.

Looking for Γ_A we have

$$H_\Gamma \approx H_\Gamma|_{\Gamma=\hat{\Gamma}_A} + H_{\Gamma\Gamma}|_{\Gamma=\hat{\Gamma}_A}\left(\Gamma-\hat{\Gamma}_A\right), \tag{10}$$

where $H_\Gamma|_{\Gamma=\hat{\Gamma}_A}$ and $H_{\Gamma\Gamma}|_{\Gamma=\hat{\Gamma}_A}$ are explicitly calculated and given by

$$\begin{aligned}H_\Gamma|_{\Gamma=\hat{\Gamma}_A} &= H_{0\Gamma}|_{\Gamma=\hat{\Gamma}_A} + \sum_{i=2}^{n}\alpha_i\left(H_{i0\Gamma}+b_iH_{i1\Gamma}\right)|_{\Gamma=\hat{\Gamma}_A} \\ H_{\Gamma\Gamma}|_{\Gamma=\hat{\Gamma}_A} &= H_{0\Gamma\Gamma}|_{\Gamma=\hat{\Gamma}_A} + \sum_{i=2}^{n}\alpha_i\left(H_{i0\Gamma\Gamma}+b_iH_{i1\Gamma\Gamma}\right)|_{\Gamma=\hat{\Gamma}_A},\end{aligned} \tag{11}$$

where $H_{0\Gamma}|_{\Gamma=\hat{\Gamma}_A} = 0$. Imposing $H_\Gamma = 0$ we have

$$\delta_A = \Gamma_A - \hat{\Gamma}_A \approx -\left.\frac{H_\Gamma}{H_{\Gamma\Gamma}}\right|_{\Gamma=\hat{\Gamma}_A}. \tag{12}$$

Linearizing δ_A with respect to $\alpha_i = 0$, $i = 2,...,n$, we have

$$\delta_A \approx -\sum_{i=2}^{n}\left.\frac{H_{i0\Gamma}+b_iH_{i1\Gamma}}{H_{0\Gamma\Gamma}}\right|_{\Gamma=\hat{\Gamma}_A}\alpha_i \tag{13}$$

and analogously for Γ_B

$$\delta_B = \Gamma_B - \hat{\Gamma}_B \approx -\sum_{i=2}^{n}\left.\frac{H_{i0\Gamma}+b_iH_{i1\Gamma}}{H_{0\Gamma\Gamma}}\right|_{\Gamma=\hat{\Gamma}_B}\alpha_i. \tag{14}$$

Definition of optimal **b**

The equal-peak condition is verified if and only if the objective function

$$F = H|_{\Gamma=\Gamma_A} - H|_{\Gamma=\Gamma_B} = 0. \tag{15}$$

The condition $F = 0$ is satisfied for the underlying linear system if λ and μ_2 are chosen according to Eqn. (3).

Expanding H in Taylor series around $\hat{\Gamma}_A$ and $\hat{\Gamma}_B$, F becomes

$$\begin{aligned}F =& H|_{\Gamma=\hat{\Gamma}_A} + H_\Gamma|_{\Gamma=\hat{\Gamma}_A}\delta_A + O(\delta_A^2) \\ &- H|_{\Gamma=\hat{\Gamma}_B} - H_\Gamma|_{\Gamma=\hat{\Gamma}_B}\delta_B + O(\delta_B^2) \approx \\ \approx& \sum_{i=2}^{n}\alpha_i\left(H_{i0}|_{\Gamma=\hat{\Gamma}_A} + b_iH_{i1}|_{\Gamma=\hat{\Gamma}_A}\right) \\ &+\left(H_{0\Gamma}|_{\Gamma=\hat{\Gamma}_A} + \sum_{i=2}^{n}\alpha_i\left(H_{i0\Gamma}|_{\Gamma=\hat{\Gamma}_A} + b_iH_{i1\Gamma}|_{\Gamma=\hat{\Gamma}_A}\right)\right)\delta_A \\ &-\sum_{i=2}^{n}\alpha_i\left(H_{i0}|_{\Gamma=\hat{\Gamma}_A} + b_iH_{i1}|_{\Gamma=\hat{\Gamma}_A}\right) - \\ &\left(H_{0\Gamma}|_{\Gamma=\hat{\Gamma}_A} + \sum_{i=2}^{n}\alpha_i\left(H_{i0\Gamma}|_{\Gamma=\hat{\Gamma}_A} + b_iH_{i1\Gamma}|_{\Gamma=\hat{\Gamma}_A}\right)\right)\delta_A = 0.\end{aligned} \tag{16}$$

Substituting δ_A and δ_B as in Eqns. (13) and (14) and limiting the analysis to terms of first order of α_i, $i = 2,...,n$, we obtain

$$\begin{aligned}F \approx \sum_{i=2}^{n}\alpha_i\Bigg(&H_{i0}|_{\Gamma=\hat{\Gamma}_A} + b_iH_{i1}|_{\Gamma=\hat{\Gamma}_A} - H_{0\Gamma}|_{\Gamma=\hat{\Gamma}_A}\left.\frac{H_{i0\Gamma}+b_iH_{i1\Gamma}}{H_{0\Gamma\Gamma}}\right|_{\Gamma=\hat{\Gamma}_A} \\ &- H_{i0}|_{\Gamma=\hat{\Gamma}_B} - b_iH_{i1}|_{\Gamma=\hat{\Gamma}_B} + H_{0\Gamma}|_{\Gamma=\hat{\Gamma}_B}\left.\frac{H_{i0\Gamma}+b_iH_{i1\Gamma}}{H_{0\Gamma\Gamma}}\right|_{\Gamma=\hat{\Gamma}_B}\Bigg) = 0.\end{aligned} \tag{17}$$

Decomposing Eqn. (17) with respect to α_i, $i = 2,...,n$, and solving with respect to **b**, we have

$$b_i = \frac{\left(H_{i0} - (H_{0\Gamma}H_{i0\Gamma}/H_{0\Gamma\Gamma})\right)|_{\Gamma=\hat{\Gamma}_A} - \left(H_{i0} - (H_{0\Gamma}H_{i0\Gamma}/H_{0\Gamma\Gamma})\right)|_{\Gamma=\hat{\Gamma}_B}}{-\left(H_{i1} - (H_{i1\Gamma}/H_{0\Gamma\Gamma})\right)|_{\Gamma=\hat{\Gamma}_A} + \left(H_{i1} - (H_{i1\Gamma}/H_{0\Gamma\Gamma})\right)|_{\Gamma=\hat{\Gamma}_B}}. \tag{18}$$

The coefficients b_i, as expressed in Eqn. (18), depend only on the order of nonlinearity under consideration i and on the linear terms. Since the coefficients of the linear terms are fully identified by ε, b_i are function of ε only. They can be defined from the knowledge of the frequency response and its derivatives in $\hat{\Gamma}_A$ and $\hat{\Gamma}_B$, without requiring any further information about the system. If $H(\Gamma)$ is kept in its analytical form, applying Eqn. (18), through computer algebra, the coefficients $\mathbf{b}$ can be defined analytically. However, even considering a system with a single cubic nonlinearity and limiting the analysis to a single harmonic, the final formula expressing b_3 is extremely long and not practically useful.

Another remarkable result of the outlined procedure is that nonlinear terms of different order do not interact with each other, in first approximation. This greatly simplify the design of the NLTVA. As it will be shown in the following sections, this circumstance is verified also numerically.

TUNING RULE

The optimal values of b_i, for $i = 3$, 5 and 7, obtained applying Eqn. (18), are depicted in Fig. 2. The values of b_i can be directly used to correctly tune the nonlinear spring of the NLTVA. It turns out that the regression

$$b_{i,\mathrm{opt}} = \frac{(2\varepsilon)^{\frac{i-1}{2}}}{1 + 3.5 \times 1.5^{\frac{i-3}{2}}\varepsilon}, \quad ,i = 3,5,7,... \tag{19}$$

provides an excellent approximation of the analytical results for b_3 and b_5, and a slight overestimate of b_7 (see dashed lines in Fig. 2). Eqn. (19), thanks to its simplicity, allows to immediately identify the optimal values of b_i, which is very practical for engineering application and for a rapid design of an operational NLTVA. Considering the dimensional parameters, Eqn. (19) becomes

$$k_{2i} = \frac{(2\varepsilon)^{\frac{i-1}{2}}\varepsilon}{1 + 3.5 \times 1.5^{\frac{i-3}{2}}\varepsilon} k_{1i}, \quad i = 3,5,7,... \tag{20}$$

Eqns. (3) and (19) (or Eqn. (20)) completely define the design of the NLTVA.

Although not shown in this work, the adopted procedure can be directly implemented for even order nonlinearities.

Numerical validation

In order to verify the analytical results, three cases, for which the LTVA is unable to mitigate the resonant vibrations, are considered, namely a primary system with third, fifth or seventh order nonlinearity. The resulting frequency response curves are shown in Fig. 3. These curves were computed using a path-following algorithm combining shooting and pseudo-arclength continuation. The algorithm is similar to that used in [21]. The red dashed curves, referring to the nonlinear primary system with an attached LTVA ($\mathbf{b} = \mathbf{0}$), clearly illustrate that the linear absorber is almost completely ineffective in all the three cases. On the other hand, the addition of a nonlinear spring of the appropriate order (black solid lines) is able to guarantee the correct operation of the absorber. Remarkably, the frequency response of the primary system with an attached NLTVA has a shape and an amplitude which remind those of the underlying linear system. In some sense, the addition of a proper nonlinear component allows to retrieve a linear behavior in a sort of compensation of the nonlinearities. The black lines of Fig. 3 are obtained tuning b_3, b_5 and b_7 according to Eqn. (19). Acknowledging that the resonant peaks have almost the same high, the analytical procedure can be considered verified.

Of course the range of safe operation of the NLTVA is limited and, if the forcing amplitude is further incremented, detrimental dynamics (quasiperiodic motions, detached resonant curves, etc.) arise, causing for certain values the failure of the NLTVA. Another aspect which should be taken into account is the stability of the solutions, which is completely overlooked in this work. For a thorough study on these matters, on the limitation of the NLTVA and the effects of detuning of the parameters, in the case of a third order nonlinearity, we address the interest readers to [15]. A comparison between the NLTVA and the LTVA can be found in [13].

CASE STUDY

We now consider a primary system having a restoring force comprising a third, a fifth and a seventh order nonlinear term. Fig. 4 (a) depicts the frequency response of the host system coupled to an LTVA. The parameter values used during the computation are $\varepsilon = 0.05$, $\alpha_3 = 0.01$, $\alpha_5 = 0.00012$, $\alpha_7 = 1.37 \times 10^{-6}$. α_3, α_5 and α_7 are chosen such that their relative terms have comparable numerical values around the resonant peaks of the underlying linear system. Also in this case, the LTVA is clearly incapable of mitigating the resonant vibrations.

In order to improve the performance of the absorber, nonlinear terms are added to the absorber. According to the procedure depicted in the previous section, the different order terms can be treated separately. Thus, the coefficients b_3, b_5 and b_7 can be tuned according to Eqn. (19).

Figure 4 (b) depicts the effect of adopting an incomplete NLTVA, i.e. either considering a third and fifth order spring (magenta dash-dotted line), a third and seventh order spring (green dashed line) or a fifth and seventh order spring (blue solid line). In all the considered cases, the incomplete NLTVA has an effect analogous to that of the LTVA, that is the maximal peak of the frequency response function is almost unchanged by the addition of nonlinear components. Instead, the three curves overlap each other for large amplitudes.

Figure 4 (c) illustrate the effectiveness of the NLTVA when the ‘principle of similarity’ is fully addressed. Adopting an NLTVA with a third, fifth and seventh order nonlinear spring, like in the primary system, the amplitude of the frequency response is dramatically decreased, resembling, once again, the frequency

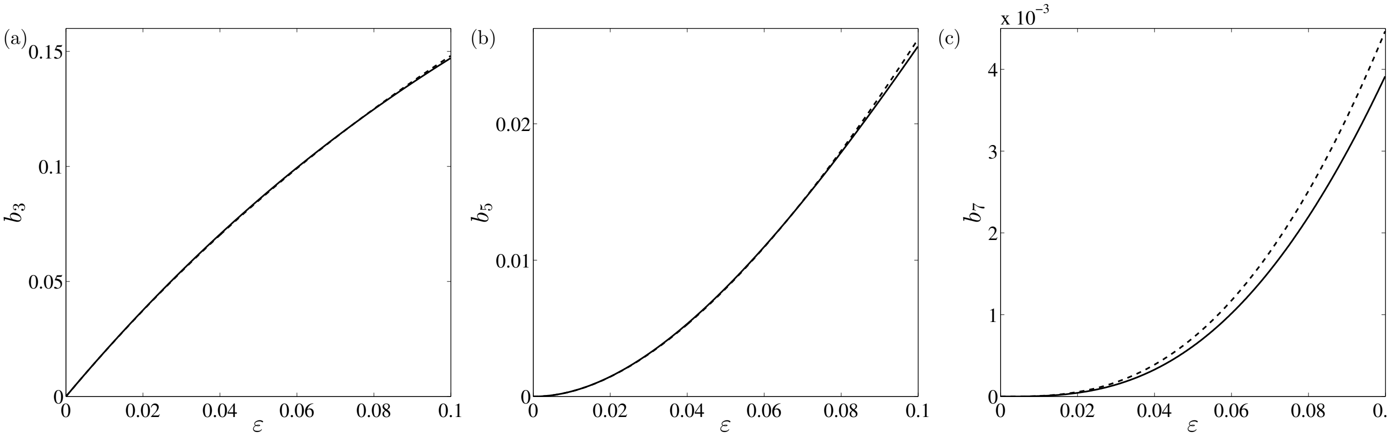

FIGURE 2. VARIATION OF THE PARAMETER b_3 (a), b_5 (b) AND b_7 (c) AS A FUNCTION OF ε. SOLID LINES: ANALYTICAL RESULTS, DASHED LINES: APPROXIMATION THROUGH THE FORMULA IN EQN. (19).

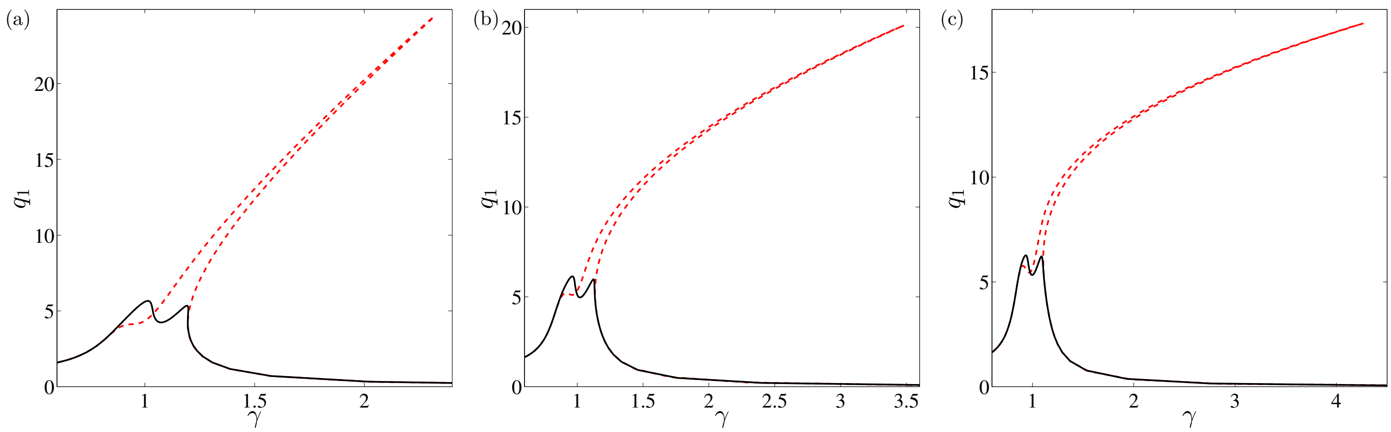

FIGURE 3. FREQUENCY RESPONSE OF A PRIMARY SYSTEM WITH A CUBIC (a), A QUINTIC (b) OR A SEVENTH ORDER (c) NONLINEARITY AND AN LTVA OR NLTVA ATTACHED. RED DASHED LINES: LTVA, BLACK SOLID LINES: NLTVA. DURING THE COMPUTATION $\varepsilon = 0.05$, $\alpha_3 = 0.01$ (a), $\alpha_5 = 0.00012$ (b), $\alpha_7 = 1.37 \times 10^{-6}$ (c), WHILE b_3, b_5 AND b_7 ARE CHOSEN ACCORDING TO EQN. (19).

response of the underlying linear system.

The successful application of the NLTVA to a system with several nonlinear terms, suggests that it can be applied to a larger assortment of nonlinear oscillators, i.e. to systems whose restoring force can be approximated by a polynomial function in the range of interest. However, the application of the NLTVA to a real structure having a complicated non polynomial restoring force is out of the scope of this paper.

CONCLUSIONS

The objective of this study was to demonstrate how the principle of similarity can be exploited to design a nonlinear vibration absorber, able to extend the Den Hartog equal-peak rule to nonlinear regimes. The procedure addressing the optimization of the nonlinear parameter, carried out completely analytically, resulted in an explicit formula (Eqn. (18)), which treats individually the different nonlinearities of the host structure, allowing a much simpler design of the absorber. Although during the analytical development the nonlinear terms are considered small, the numerical results illustrate how the NLTVA is effective also in strong nonlinear regimes. It turns out that the numerical regression in Eqn. (19) well approximates the analytical results for the cases of symmetric odd nonlinearities. This formula can be applied for immediate definition of the optimal nonlinear coefficients of the absorber.

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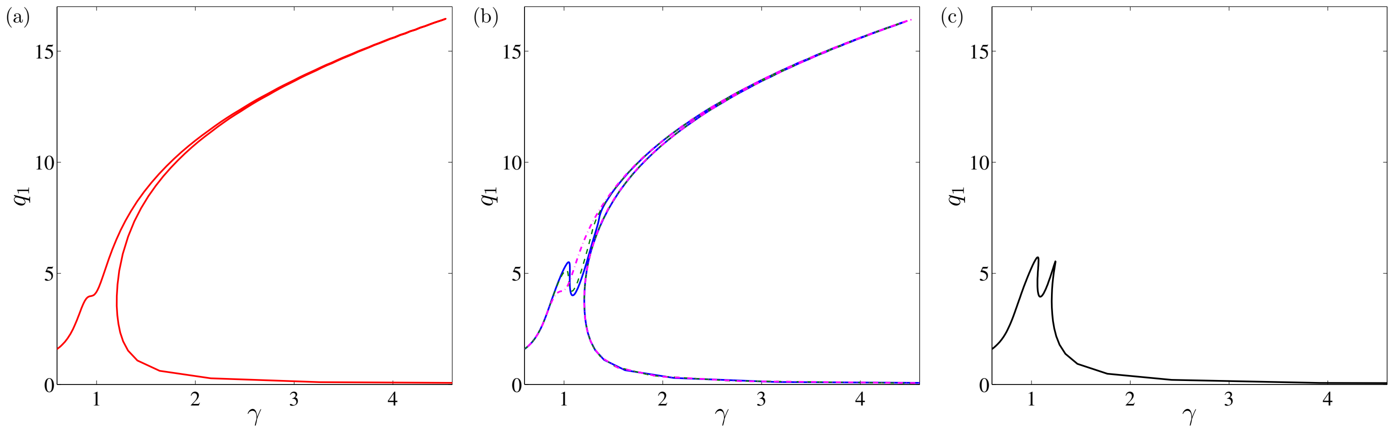

FIGURE 4. FREQUENCY RESPONSE OF A PRIMARY SYSTEM WITH A CUBIC, QUINTIC AND SEVENTH ORDER NONLINEARITY AND AN LTVA OR NLTVA ATTACHED. (a) LTVA; (b) INCOMPLETE NLTVA, MISSING SEVENTH (DASH-DOT MAGENTA LINE, $b_7 = 0$), FIFTH (DASHED GREEN LINE, $b_5 = 0$) OR THIRD (SOLID BLUE LINE, $b_3 = 0$) ORDER TERM; (c) COMPLETE NLTVA ATTACHED. DURING THE COMPUTATION $\varepsilon = 0.05$, $\alpha_3 = 0.01$, $\alpha_5 = 0.00012$, $\alpha_7 = 1.37 \times 10^{-6}$, WHILE b_3, b_5 AND b_7 ARE CHOSEN ACCORDING TO EQN. (19).

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