

A PRIORI ESTIMATES FOR THE YAMABE PROBLEM IN THE NON-LOCALLY CONFORMALLY FLAT CASE

FERNANDO CODA MARQUES

Abstract

Given a compact Riemannian manifold (M^n, g) , with positive Yamabe quotient, not conformally diffeomorphic to the standard sphere, we prove *a priori* estimates for solutions to the Yamabe problem. We restrict ourselves to the dimensions where the Positive Mass Theorem is known to be true, that is, when $n \leq 7$. We also show that, when $n \geq 6$, the Weyl tensor has to vanish at a point where solutions to the Yamabe equation blow up.

1. Introduction

Let (M^n, g) be a compact Riemannian n -manifold, $n \geq 3$, without boundary. The classical Yamabe problem consists of finding a metric \tilde{g} , conformally related to g , with constant scalar curvature on M . It can be considered as a generalization of the classical Uniformization Theorem on Riemann surfaces to the setting of higher dimensional manifolds.

In analytical terms, the problem is equivalent to show the existence of a positive solution u to the equation

$$(1.1) \quad \Delta_g u - \frac{n-2}{4(n-1)} R_g u + K u^{\frac{n+2}{n-2}} = 0 \text{ on } M,$$

where Δ_g denotes the Laplace–Beltrami operator associated with the metric g , R_g denotes the scalar curvature of the metric g , and K is a constant. The linear operator $L_g = \Delta_g - \frac{n-2}{4(n-1)} R_g$ is called the conformal Laplacian of the metric g .

The solution of the Yamabe problem was an outstanding achievement since, for the first time, it was given a very satisfactory existence theory to a non-linear partial differential equation involving a critical exponent. After the initial paper in 1960 by Yamabe [20], which contained an error, contributions made by Trudinger [19], Aubin [1], and finally by Schoen [16] in 1984, solved the problem completely in the affirmative.

In this paper, we shall be interested in the set of solutions to the Yamabe problem. When the first eigenvalue of the conformal Laplacian L_g is negative, it is not difficult to see that the solution is unique. If

this eigenvalue is zero, the equation becomes linear and then solutions are unique up to a constant. Therefore, the only interesting case left is the positive one.

We know that when the underlying manifold is the sphere (\mathbb{S}^n, g_0) , endowed with the standard metric, Obata's theorem (see [12]) classifies all solutions to the equation, and this set is non-compact in the C^2 topology. On the other hand, the standard sphere is the only compact manifold with a non-compact group of conformal diffeomorphisms, so one should expect different behavior in the other cases.

In [14], Schoen proved the compactness, in the C^2 topology, of the set of solutions to the Yamabe equation, in the positive case, for every locally conformally flat manifold not conformally diffeomorphic to the sphere. He also suggested a strategy to prove these *a priori* $C^{2,\alpha}$ estimates in the non-locally conformally flat case, based on Pohozaev-type identities. In [10], Li and Zhu followed these lines and proved the theorem in dimension 3, in which case, standard estimates on the blowing up solutions are sufficient for a Pohozaev identity to be applied. The compactness result in dimensions 4 and 5 was obtained by Druet in [4, 5]. Other compactness theorems for the Yamabe equation, in the locally conformally flat case, are proved in [6] for manifolds with boundary, and in [13] for singular solutions on the sphere.

Our main result in this paper is the following *a priori* estimates theorem in the general case:

Theorem 1.1. *Let (M^n, g) be a smooth closed Riemannian manifold with positive Yamabe quotient, not conformally equivalent to (\mathbb{S}^n, g_0) . Assume $n \leq 7$. Then, for every $\epsilon > 0$, there exists a positive constant $C = C(\epsilon, g)$ so that*

$$\begin{cases} 1/C \leq u \leq C & \text{and} \\ \|u\|_{C^{2,\alpha}(M)} \leq C \end{cases}$$

for every $u \in \cup_{1+\epsilon \leq p \leq \frac{n+2}{n-2}} \mathbb{M}_p$, where $0 < \alpha < 1$ and

$$\mathbb{M}_p = \left\{ u > 0 : \Delta_g u - \frac{n-2}{4(n-1)} R_g u + K u^p = 0 \text{ on } M \right\}.$$

These estimates clearly imply the compactness of the set of solutions to the Yamabe equation in the C^2 topology. We restrict ourselves to the dimensions covered by the Positive Mass Theorem due to Schoen and Yau [17], i.e., $n \leq 7$, since the final global argument in our proof depends essentially on this result. Due to technical problems concerning singularities of minimizing hypersurfaces, the Positive Mass Theorem is still not known to be true for dimensions greater than 7.

Our result will follow from a contradiction between a local restriction coming from a Pohozaev-type identity and a global argument provided by the Positive Mass Theorem. In order to accomplish that, we will

need a careful blowup analysis of solutions, part of it inspired by the work of Chen and Lin [3]. We introduce some new symmetry estimates which allow us to control how close the blowing up solutions get to some specific rotationally symmetric functions. These symmetric functions will be solutions to the corresponding critical ($p = \frac{n+2}{n-2}$) equation in \mathbb{R}^n . Since we are also dealing with subcritical equations, estimates on $\tau = \frac{n+2}{n-2} - p$ are also given.

We would like to point out that the symmetry estimates (Proposition 5.4), when $n \leq 5$, are just as in the conformally flat case. However, when $n \geq 6$, we can no longer expect these same estimates to hold. This is because, in general, the asymptotic expansion of the Green function for the conformal Laplacian has additional terms (see [7]).

One important difficulty we must overcome when $n \geq 6$, pointed out by Schoen in [14], is to show conformal flatness of the metric to a sufficiently high order at a blowup point. This is needed in order to apply the Positive Mass Theorem when $n = 6$ or 7 . That is the content of our next theorem, where W_g denotes the Weyl tensor of the metric g :

Theorem 1.2. *Assume $n \geq 6$ and let u_i be a sequence of positive solutions to (1.1). Suppose $x_i \rightarrow \bar{x}$ is a sequence of points such that $u_i(x_i) \rightarrow \infty$ as $i \rightarrow \infty$. Then*

$$W_g(\bar{x}) = 0.$$

In general, one should expect, as indicated by Schoen in [15], that at a blowup point \bar{x} we must have:

$$\nabla^k W_g(\bar{x}) = 0$$

for all $0 \leq k \leq \frac{n-6}{2}$. The proof of Theorem 1.2 also relies upon the symmetry estimates and a Pohozaev-type identity.

It is not difficult to check that the Theorem 1.1 implies the existence of a solution to the Yamabe problem. This is because solutions u_i to the subcritical equations, with $p_i \rightarrow \frac{n+2}{n-2}$ as $i \rightarrow \infty$, can be constructed by standard variational methods. Actually one can say more. Another consequence of the compactness theorem is the computation of the total Leray–Schauder degree of all solutions to equation (1.1):

Theorem 1.3. *Let (M^n, g) be a smooth closed Riemannian manifold with positive Yamabe quotient, not conformally equivalent to (\mathbb{S}^n, g_0) , $4 \leq n \leq 7$. Then, if Λ is sufficiently large,*

$$\deg(F, \Omega_\Lambda, 0) = -1,$$

where $F(u) = u + L_g^{-1}(E(u)u^{\frac{n+2}{n-2}})$, $E(u) = -\int_M u L_g(u) dv_g$ and

$$\Omega_\Lambda = \left\{ u \in C^{2,\alpha}(M) : \min_M u > \Lambda^{-1}, \|u\|_{2,\alpha} < \Lambda \right\}.$$

This theorem follows from Theorem 1.1 and arguments given by Schoen in [14].

The author was just recently communicated that Li and Zhang ([9]) have independently proved the same compactness result.

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2. A Pohozaev-type identity

In this section, we will establish a Pohozaev-type identity which will be very useful in the subsequent blowup analysis.

Suppose $u : B_\rho(0) \setminus \{0\} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is a positive C^2 solution to the equation

$$(2.1) \quad a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + c(x)u + K(x)u^p = 0,$$

where $p \neq -1$, $K \in C^1$ and a^{ij}, b^i, c are continuous functions, $1 \leq i, j \leq n$. Here, we are using the summation convention.

Define

$$(2.2) \quad P(r, u) = \int_{|x|=r} \left(\frac{n-2}{2} u \frac{\partial u}{\partial r} - \frac{|x|}{2} |\nabla u|^2 + |x| \left| \frac{\partial u}{\partial r} \right|^2 + \frac{1}{p+1} K(x) |x| u^{p+1} \right) d\sigma(r),$$

whenever $0 < r < \rho$.

The following lemma gives the Pohozaev-type identity we are interested in.

Lemma 2.1. *Given $0 < s \leq r < \rho$,*

$$\begin{aligned} P(r, u) - P(s, u) = & - \int_{s \leq |x| \leq r} \left(x^k \partial_k u + \frac{n-2}{2} u \right) A(u) dx \\ & + \frac{1}{p+1} \int_{s \leq |x| \leq r} (x^k \partial_k K(x)) u^{p+1} dx \\ & + \left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_{s \leq |x| \leq r} K(x) u^{p+1} dx, \end{aligned}$$

where $A(u) = (a^{ij} - \delta_{ij})\partial_{ij}u + b^i\partial_i u + cu$.

Proof. Multiplying the equation (2.1) by $x^k \partial_k u$, and integrating over the set $\{x : s \leq |x| \leq r\}$, we obtain

$$(2.3) \quad \int_{s \leq |x| \leq r} (x^k \partial_k u)(\Delta u + A(u) + K(x)u^p) dx = 0.$$

Integration by parts gives:

$$\begin{aligned}
 & \int_{s \leq |x| \leq r} (x^k \partial_k u) \partial_{ii} u dx \\
 &= - \int_{s \leq |x| \leq r} (\delta_i^k \partial_k u \partial_i u + \frac{1}{2} x^k \partial_k [(\partial_i u)^2]) dx \\
 & \quad + \frac{1}{r} \int_{|x|=r} (x^k \partial_k u)(x_i \partial_i u) - \frac{1}{s} \int_{|x|=s} (x^k \partial_k u)(x_i \partial_i u) \\
 &= - \int_{s \leq |x| \leq r} (\delta_i^k \partial_k u \partial_i u - \frac{n}{2} (\partial_i u)^2) dx - \frac{r}{2} \int_{|x|=r} (\partial_i u)^2 + \frac{s}{2} \int_{|x|=s} (\partial_i u)^2 \\
 & \quad + \frac{1}{r} \int_{|x|=r} (x^k \partial_k u)(x_i \partial_i u) - \frac{1}{s} \int_{|x|=s} (x^k \partial_k u)(x_i \partial_i u),
 \end{aligned}$$

and summing over $i = 1, \dots, n$, we obtain

$$\begin{aligned}
 (2.4) \quad & \int_{s \leq |x| \leq r} (x^k \partial_k u) \Delta u dx \\
 &= \frac{n-2}{2} \int_{s \leq |x| \leq r} |\nabla u|^2 dx - \frac{r}{2} \int_{|x|=r} |\nabla u|^2 + \frac{s}{2} \int_{|x|=s} |\nabla u|^2 \\
 & \quad + r \int_{|x|=r} \left(\frac{\partial u}{\partial r} \right)^2 - s \int_{|x|=s} \left(\frac{\partial u}{\partial r} \right)^2.
 \end{aligned}$$

Also

$$\begin{aligned}
 (2.5) \quad & \int_{s \leq |x| \leq r} (x^k \partial_k u) K(x) u^p dx \\
 &= \frac{1}{p+1} \int_{s \leq |x| \leq r} x^k \partial_k (u^{p+1}) K(x) dx \\
 &= - \frac{n}{p+1} \int_{s \leq |x| \leq r} K(x) u^{p+1} dx - \frac{1}{p+1} \int_{s \leq |x| \leq r} (x^k \partial_k K(x)) u^{p+1} dx \\
 & \quad + \frac{r}{p+1} \int_{|x|=r} K(x) u^{p+1} - \frac{s}{p+1} \int_{|x|=s} K(x) u^{p+1}.
 \end{aligned}$$

On the other hand, multiplying the equation (2.1) by u , and once again integrating by parts, we get

$$\begin{aligned}
 (2.6) \quad & \int_{s \leq |x| \leq r} |\nabla u|^2 dx \\
 &= \int_{s \leq |x| \leq r} (uA(u) + K(x)u^{p+1}) dx + \int_{|x|=r} u \frac{\partial u}{\partial r} - \int_{|x|=s} u \frac{\partial u}{\partial r}.
 \end{aligned}$$

Now, we substitute equalities (2.4), (2.5) and equality (2.6) in equality (2.3) and the Pohozaev identity follows by rearranging terms. q.e.d.

When u is a solution to the equation (2.1) in the entire ball, by taking the limit as $s \rightarrow 0$, we get

$$(2.7) \quad P(r, u) = - \int_{|x| \leq r} \left(x^k \partial_k u + \frac{n-2}{2} u \right) A(u) dx \\ + \frac{1}{p+1} \int_{|x| \leq r} (x^k \partial_k K(x)) u^{p+1} dx \\ + \left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_{|x| \leq r} K(x) u^{p+1} dx.$$

Integrating by parts once more, we can also get

$$(2.8) \quad P(r, u) = - \int_{|x| \leq r} \left(x^k \partial_k u + \frac{n-2}{2} u \right) ((a^{ij} - \delta^{ij}) \partial_{ij} u + b^i \partial_i u) dx \\ + \int_{|x| \leq r} \left(\frac{1}{2} x^k \partial_k c + c \right) u^2 dx - \frac{r}{2} \int_{|x|=r} c u^2 d\sigma(r) \\ + \frac{1}{p+1} \int_{|x| \leq r} (x^k \partial_k K(x)) u^{p+1} dx \\ + \left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_{|x| \leq r} K(x) u^{p+1} dx.$$

3. Conformal scalar curvature equation

In this section, we will introduce the partial differential equation we are interested in, and we shall discuss some of its properties related to conformal deformation of metrics.

Let $\Omega \in \mathbb{R}^n$ be an open set, and suppose g is a Riemannian metric in Ω . Suppose also f is a positive C^1 function defined in Ω .

Consider a positive C^2 function u satisfying

$$(3.1) \quad \Delta_g u - c(n) R_g u + K f^{-\tau} u^p = 0 \text{ in } \Omega,$$

where $c(n) = \frac{n-2}{4(n-1)}$, $K = n(n-2)$, $1 < p \leq \frac{n+2}{n-2}$ and $\tau = \frac{n+2}{n-2} - p$. We will use the notation R_g for the scalar curvature of g . The operator $L_g = \Delta_g - c(n) R_g$ is called the *conformal Laplacian* of the metric g .

When $p = \frac{n+2}{n-2}$, this partial differential equation is intimately related to conformal geometry, particularly when one studies conformally related metrics with constant scalar curvature. More specifically, given a positive solution u , the metric $u^{\frac{4}{n-2}} g$ has constant scalar curvature equal to $4n(n-1)$.

Now, let us describe an important feature of solutions to that type of equation. Let u be a solution to equation (3.1) and choose $\bar{x} \in \Omega$. Given $s > 0$, define the renormalized function

$$v(y) = s^{\frac{2}{p-1}} u(\exp_{\bar{x}}(sy)).$$

Then

$$L_h v + K \tilde{f}^{-\tau} v^p = 0,$$

where $\tilde{f}(y) = f(sy)$ and the components of the metric h in normal coordinates are given by $h_{kl}(y) = g_{kl}(sy)$. The important point here is that v satisfies an equation of the same type.

The equation is also conformally invariant in the following sense. Suppose $\tilde{g} = \phi^{\frac{4}{n-2}} g$ is a metric conformal to g . Let us recall

$$(3.2) \quad L_{\tilde{g}}(\phi^{-1}u) = \phi^{-\frac{n+2}{n-2}} L_g(u)$$

for any function u , and

$$(3.3) \quad R_{\tilde{g}} = -c(n)^{-1} \phi^{-\frac{n+2}{n-2}} L_g(\phi).$$

(See [7].)

Therefore, if u is a solution to (3.1), then $\phi^{-1}u$ satisfies

$$L_{\tilde{g}}(\phi^{-1}u) + K(\phi f)^{-\tau} (\phi^{-1}u)^p = 0,$$

which is again an equation of the same type.

This will have very important consequences in what follows. We will study sequences of solutions u_i to equation (3.1). When what we want to study is conformally invariant, we are allowed to replace u_i by another sequence of functions $v_i = \phi_i^{-1}u_i$, at the same time replacing the metric g_i by $\tilde{g}_i = \phi_i^{\frac{4}{n-2}} g_i$, as long as we have a uniform control on the conformal factors ϕ_i . In this paper, there will be two examples of such a procedure.

First, we can suppose the metric g_i has positive scalar curvature in a small ball centered at some fixed point x_i . To see this, fix $\sigma > 0$ small and let ϕ be the first eigenfunction of Δ_g with respect to the Dirichlet condition:

$$(3.4) \quad \begin{cases} \Delta_g \phi + \lambda_1 \phi = 0 & \text{in } B_{2\sigma}(x) \\ \phi = 0 & \text{on } \partial B_{2\sigma}(x). \end{cases}$$

Recall that the corresponding eigenspace is one-dimensional and we can choose $\phi > 0$ on $B_{2\sigma}(x)$. Now, since $\lambda_1 \rightarrow \infty$ as $\sigma \rightarrow 0$, we can choose σ small enough so that

$$\Delta_g \phi - c(n) R_g \phi < 0$$

in $B_\sigma(x)$. Defining $\tilde{g} = \phi^{\frac{4}{n-2}} g$, relation (3.3) implies $R_{\tilde{g}} > 0$ in $B_\sigma(x)$. Moreover, if we take as a conformal factor the solution ψ to:

$$(3.5) \quad \begin{cases} \Delta_g \psi - c(n) R_g \psi = 0 & \text{in } B_\sigma(x) \\ \psi = 1 & \text{on } \partial B_\sigma(x), \end{cases}$$

we can also have $R_{\tilde{g}} = 0$.

The second example is related to the so-called conformal normal coordinates. (See [7].) Given an integer $N \geq 2$, there exists a positive function ϕ (which can be constructed explicitly), such that, setting $\tilde{g} = \phi^{\frac{4}{n-2}}g$, the volume element satisfies:

$$\det(\tilde{g}_{ij}) = 1 + O(r^N),$$

in \tilde{g} -normal coordinates around x , where $r = d_{\tilde{g}}(x, \cdot)$. This allows us to simplify the local asymptotic analysis. For example, in conformal normal coordinates around x , $R_{\tilde{g}} = O(r^2)$ and $\Delta R_{\tilde{g}}(x) = -\frac{1}{6}|W_{\tilde{g}}(x)|^2$, where W stands for the Weyl tensor.

4. Isolated and isolated simple blowup points

In this section, we will define isolated and isolated simple blowup points and we shall discuss their basic properties. The results in this section are well-known in the locally conformally flat setting ([8], [18]) and in general when $n = 3$ ([10]). We will slightly modify their proofs in [10] to make them work in any dimension.

Let $\Omega \subset \mathbb{R}^n$ be an open set, and suppose g_i is a sequence of Riemannian metrics in Ω converging, in the C_{loc}^2 topology, to a metric g . Suppose also that f_i is a sequence of positive C^1 functions converging in the C_{loc}^1 topology to a positive function f .

We will consider a sequence u_i of positive C^2 functions satisfying

$$(4.1) \quad L_{g_i} u_i + K f_i^{-\tau_i} u_i^{p_i} = 0 \text{ in } \Omega,$$

where $c(n) = \frac{n-2}{4(n-1)}$, $K = n(n-2)$, $1 + \epsilon_0 < p_i \leq \frac{n+2}{n-2}$ for some $\epsilon_0 > 0$ and $\tau_i = \frac{n+2}{n-2} - p_i$.

We will sometimes omit the subscript i , for the sake of simplicity, and we will use the symbols c, C to denote various positive constants.

Definition 4.1. We say that $\bar{x} \in \Omega$ is an isolated blowup point for u_i if there exists a sequence $x_i \in \Omega$, converging to \bar{x} , so that:

- 1) x_i is a local maximum point of u_i ;
- 2) $M_i := u_i(x_i) \rightarrow \infty$ as $i \rightarrow \infty$;
- 3) there exist $r, C > 0$ such that

$$(4.2) \quad u_i(x) \leq C d_{g_i}(x, x_i)^{-\frac{2}{p_i-1}}$$

for every $x \in B_r(x_i) \subset \Omega$. Here, $B_r(x_i)$ denotes the geodesic ball of radius r , centered at x_i , with respect to the metric g_i .

Remark. In various parts of the text, we will identify x_i with the origin, that meaning we are making use of normal coordinates in a small ball around x_i . More precisely, we will sometimes write $u_i(x)$ instead of $u_i(\exp_{x_i}(x))$ and $|x|$ instead of $d_{g_i}(x, x_i)$, and those functions will be defined in balls centered at 0.

Note that the definition of isolated blowup points is invariant under renormalization, which was described in the last section. This follows from the fact that, if $v(y) = s^{\frac{2}{p_i-1}}u(sy)$, then

$$u(x) \leq C|x|^{-\frac{2}{p_i-1}} \Leftrightarrow v(y) \leq C|y|^{-\frac{2}{p_i-1}}.$$

The first result concerning isolated blowup points is the following Harnack inequality.

Proposition 4.2. *Suppose that u_i is a sequence of positive functions satisfying equation (4.1) and assume $x_i \rightarrow \bar{x}$ is an isolated blowup point. Then there exists a constant $C > 0$ such that*

$$\max_{\frac{s}{2} \leq d_{g_i}(x, x_i) \leq 2s} u_i(x) \leq C \min_{\frac{s}{2} \leq d_{g_i}(x, x_i) \leq 2s} u_i(x),$$

where $0 < s < \frac{r}{3}$.

Proof. Let (x^1, \dots, x^n) be normal coordinates with respect to the metric g_i on the ball $B_r(x_i)$. (See remark after Definition 4.1.)

Define

$$v_i(y) = s^{\frac{2}{p_i-1}}u_i(sy),$$

where $|y| < 3$.

Then, as discussed in the last section,

$$L_{h_i}v_i(y) + K\tilde{f}_i^{-\tau_i}v_i^{p_i}(y) = 0,$$

where $\tilde{f}_i(y) = f_i(sy)$ and $(h_i)_{kl}(y) = (g_i)_{kl}(sy)$, and we also know that

$$v_i(y) \leq C|y|^{-\frac{2}{p_i-1}},$$

whenever $|y| < 3$.

It follows from this last inequality that v_i is uniformly bounded in compact subsets of $B_3(0) \setminus \{0\}$. The Harnack inequality for elliptic linear equations then implies that there exists $C > 0$ such that

$$\max_{\frac{1}{2} \leq |y| \leq 2} v_i(y) \leq C \min_{\frac{1}{2} \leq |y| \leq 2} v_i(y).$$

The result now follows directly.

q.e.d.

The Proposition 4.2 clearly implies the so-called *spherical Harnack inequality* for isolated blowup points. Namely, given $0 < s \leq \frac{2}{3}r$, there exists a positive constant C , not depending on s , such that

$$(4.3) \quad \max_{d_{g_i}(x, x_i)=s} u_i(x) \leq C \min_{d_{g_i}(x, x_i)=s} u_i(x).$$

Define $U_0(y) = (1 + |y|^2)^{\frac{2-n}{2}}$. It is not difficult to check that

$$\Delta U_0(y) + KU_0^{\frac{n+2}{n-2}}(y) = 0.$$

The next proposition says that, in the case of an isolated blowup point, the functions u_i , when renormalized, converge in the C^2 topology to the rotationally symmetric function U_0 .

Proposition 4.3. *Let u_i be a sequence of positive functions satisfying the equation (4.1) and $x_i \rightarrow \bar{x}$ be an isolated blowup point. Assume that $R_i \rightarrow \infty$ and $\epsilon_i \rightarrow 0$ are given. Then $p_i \rightarrow \frac{n+2}{n-2}$ and, after possibly passing to a subsequence,*

$$(4.4) \quad \left\| M_i^{-1} u_i \left(M_i^{-\frac{p_i-1}{2}} y \right) - U_0(y) \right\|_{C^2(B_{R_i}(0))} \leq \epsilon_i$$

and

$$\frac{R_i}{\log M_i} \rightarrow 0 \text{ as } i \rightarrow \infty.$$

Here, M_i is as in Definition 4.1.

Proof. Let (x^1, \dots, x^n) be normal coordinates with respect to the metric g_i on the ball $B_r(x_i)$. (See first remark after Definition 4.1.)

Define

$$v_i(y) = M_i^{-1} u_i \left(M_i^{-\frac{p_i-1}{2}} y \right)$$

for $|y| < r M_i^{\frac{p_i-1}{2}}$. Here, r is as in Definition 4.1.

Then

$$L_{h_i} v_i(y) + K \tilde{f}_i^{-\tau_i} v_i^{p_i}(y) = 0,$$

where $\tilde{f}_i(y) = f_i(M_i^{-\frac{p_i-1}{2}} y)$ and $(h_i)_{kl}(y) = (g_i)_{kl}(M_i^{-\frac{p_i-1}{2}} y)$.

Note also that

$$(4.5) \quad \begin{cases} v_i(0) = 1, & \nabla v_i(0) = 0 \\ 0 < v_i(y) \leq C |y|^{-\frac{2}{p_i-1}} \text{ for } |y| < r M_i^{\frac{p_i-1}{2}}. \end{cases}$$

Claim. There exists $C > 0$ such that $v_i(y) \leq C$, whenever $|y| < r M_i^{\frac{p_i-1}{2}}$.

Proof of Claim. From properties (4.5), we get

$$(4.6) \quad v_i(y) \leq C,$$

if $1 \leq |y| \leq r M_i^{\frac{p_i-1}{2}}$.

Now, from what was discussed in the previous section, up to a conformal deformation, we can suppose our metrics have zero scalar curvature in small balls. In particular, their conformal Laplacians will satisfy the maximum principle. This implies there exists $C > 0$ so that

$$\min_{|y| \leq r} v_i(y) \geq C^{-1} \min_{|y|=r} v_i(y) \quad \forall i,$$

and $0 < r \leq 1$. The spherical Harnack inequality (4.3) implies

$$(4.7) \quad \begin{aligned} \max_{|y|=r} v_i(y) &\leq C \min_{|y|=r} v_i(y) \leq C \min_{|y|\leq r} v_i(y) \\ &\leq C v_i(0) = C, \end{aligned}$$

for $0 < r \leq 1$. This and inequality (4.6) imply the claim. q.e.d.

Standard elliptic estimates now imply that, after passing to a subsequence, $v_i \rightarrow v > 0$ in $C^2_{loc}(\mathbb{R}^n)$, where

$$(4.8) \quad \begin{cases} \Delta v(y) + K v^p(y) = 0, & y \in \mathbb{R}^n \\ v(0) = 1, \quad \nabla v(0) = 0, \end{cases}$$

where $p = \lim_{i \rightarrow \infty} p_i$. Here, Δ denotes the Euclidean Laplacian.

A well-known theorem by Caffarelli, Gidas and Spruck [2], states that we necessarily have $p = \frac{n+2}{n-2}$ and $v(y) = U_0(y)$.

The Proposition now follows easily. q.e.d.

Now, let us introduce the notion of an isolated simple blowup point.

Suppose u_i is a sequence of positive functions satisfying equation (4.1) and $x_i \rightarrow \bar{x}$ is an isolated blowup point. Define

$$\bar{u}_i(r) = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{\partial B_r(x_i)} u_i d\sigma(r),$$

where σ_{n-1} denotes the area of a unit sphere in \mathbb{R}^n . We are using g_i -normal coordinates and integrating with respect to the Euclidean volume form.

Definition 4.4. We say $x_i \rightarrow \bar{x}$ is an *isolated simple blowup point* if there exists a real number $0 < \rho < r$ such that the functions

$$\hat{u}_i(r) = r^{\frac{2}{p_i-1}} \bar{u}_i(r)$$

have exactly one critical point in the interval $(0, \rho)$, for i large.

It is not difficult to see that Proposition 4.3 implies that \hat{u}_i has exactly one critical point in the interval $(0, R_i u_i(x_i)^{-\frac{p_i-1}{2}})$. Moreover, its derivative is negative right after the critical point. As a result, if the blowup is isolated simple, then

$$\hat{u}'_i(r) < 0$$

for all $R_i M_i^{-\frac{p_i-1}{2}} \leq r < \rho$.

Now, we turn to the first estimate on isolated simple blowup points.

Proposition 4.5. *Let u_i be a sequence of positive functions satisfying equation (4.1) and $x_i \rightarrow \bar{x}$ be an isolated simple blowup point. Then there exists a constant $C > 0$ and $0 < \rho_1 < \rho$ such that, for each i ,*

$$(4.9) \quad M_i u_i(x) \leq C d_{g_i}(x, x_i)^{2-n},$$

whenever $d_{g_i}(x, x_i) \leq \rho_1$. Moreover, if $R_i M_i^{-\frac{p_i-1}{2}} \leq d_{g_i}(x, x_i) \leq \rho_1$, then

$$(4.10) \quad M_i u_i(x) \geq C^{-1} G_i(x_i, x),$$

where G_i is the Green function of L_{g_i} with respect to the Dirichlet boundary condition on $B_{\rho_1}(x_i)$.

Proof. We first need a slightly different estimate.

Let $\delta > 0$, and define $\lambda_i = (n - 2 - \delta) \frac{p_i-1}{2} - 1$.

Let us apply the Proposition 4.3 to some $R_i \rightarrow \infty$ and $0 < \epsilon_i < e^{-R_i}$.

Claim 1. If δ is sufficiently small, there exist constants $0 < \rho_1 < \rho$ and $C > 0$ such that

$$(4.11) \quad M_i^{\lambda_i} u_i(x) \leq C d(x, x_i)^{2-n+\delta},$$

$$(4.12) \quad M_i^{\lambda_i} |\nabla u_i(x)| \leq C d(x, x_i)^{1-n+\delta},$$

$$(4.13) \quad M_i^{\lambda_i} |\nabla^2 u_i(x)| \leq C d(x, x_i)^{-n+\delta},$$

for every x so that $R_i M_i^{-\frac{p_i-1}{2}} \leq d(x, x_i) \leq \rho_1$.

The proof of Claim 1 is analogous to the proof of the Lemma 3.3 in [10].

Remark. It is not difficult to see that the previous estimates imply

$$(4.14) \quad \begin{cases} v_i(y) \leq C M_i^{\delta \frac{p_i-1}{2}} (1 + |y|)^{2-n}, \\ |\nabla v_i(y)| \leq C M_i^{\delta \frac{p_i-1}{2}} (1 + |y|)^{1-n}, \\ |\nabla^2 v_i(y)| \leq C M_i^{\delta \frac{p_i-1}{2}} (1 + |y|)^{-n} \end{cases}$$

for any $|y| \leq \rho_1 M_i^{\frac{p_i-1}{2}}$.

Let us now estimate τ_i .

Claim 2. There exists $C > 0$ such that

$$\tau_i \leq \begin{cases} C M_i^{(-1+\delta)\frac{4}{n-2}+o(1)} & \text{if } n > 4, \\ C M_i^{(-1+\delta)2+o(1)} \log M_i & \text{if } n = 4, \end{cases}$$

and, in particular, $M_i^{\tau_i} \rightarrow 1$ as $i \rightarrow \infty$.

Proof of Claim 2. We will apply the Pohozaev identity (2.7) from Section 2 to u_i on the ball of radius $\frac{\rho_1}{2}$:

$$(4.15) \quad P\left(\frac{\rho_1}{2}, u_i\right) = - \int_{|x| \leq \frac{\rho_1}{2}} \left(x^m \partial_m u_i + \frac{n-2}{2} u_i\right) A_i(x) dx \\ + \left(\frac{n}{p_i+1} - \frac{n-2}{2}\right) \int_{|x| \leq \frac{\rho_1}{2}} K f_i^{-\tau_i} u_i^{p_i+1} dx \\ - \frac{\tau_i}{p_i+1} \int_{|x| \leq \frac{\rho_1}{2}} K f_i^{-\tau_i-1} (x^m \partial_m f_i) u_i^{p_i+1} dx,$$

where

$$A_i(x) = (g^{kl} - \delta^{kl})(x) \partial_{kl} u_i(x) \\ + (\partial_k g^{kl} + |g|^{-\frac{1}{2}} \partial_k (|g|^{\frac{1}{2}}) g^{kl})(x) \partial_l u_i(x) - c(n) R_g(x) u_i(x)$$

and recall

$$P\left(\frac{\rho_1}{2}, u_i\right) = \int_{|x| = \frac{\rho_1}{2}} \left(\frac{n-2}{2} u_i \frac{\partial u_i}{\partial r} - \frac{|x|}{2} |\nabla u_i|^2 + |x| \left|\frac{\partial u_i}{\partial r}\right|^2\right) d\sigma \\ + \frac{1}{p_i+1} \int_{|x| = \frac{\rho_1}{2}} K f_i^{-\tau_i} |x| u_i^{p_i+1} d\sigma.$$

q.e.d.

From Claim 1, we get

$$(4.16) \quad \left|P\left(\frac{\rho_1}{2}, u_i\right)\right| \leq c M_i^{-2\lambda_i}.$$

Define

$$(4.17) \quad \hat{A}_i(y) = (g^{kl} - \delta^{kl}) \left(M_i^{-\frac{p_i-1}{2}} y\right) \partial_{kl} v_i \\ + M_i^{-\frac{p_i-1}{2}} (\partial_k g^{kl} + |g|^{-\frac{1}{2}} \partial_k (|g|^{\frac{1}{2}}) g^{kl}) \left(M_i^{-\frac{p_i-1}{2}} y\right) \partial_l v_i \\ - c(n) M_i^{-(p_i-1)} R_g \left(M_i^{-\frac{p_i-1}{2}} y\right) v_i.$$

The change of variables $y = M_i^{-\frac{p_i-1}{2}} x$, the inequalities (4.14) and the fact that the metric is euclidean up to first order in normal coordinates

yield

$$\begin{aligned}
 (4.18) \quad & \left| \int_{|x| \leq \frac{\rho_1}{2}} \left(x^m \partial_m u_i + \frac{n-2}{2} u_i \right) A_i(x) dx \right| \\
 &= M_i^{p \frac{2-n}{2}} M_i^{\frac{n+2}{2}} \left| \int_{|y| \leq \frac{\rho_1}{2} M^{\frac{p_i-1}{2}}} \left(y^m \partial_m v_i + \frac{n-2}{2} v_i \right) \hat{A}_i(y) dy \right| \\
 &\leq C M_i^{p \frac{2-n}{2}} M_i^{\frac{n+2}{2}} M_i^{-(p_i-1)} M_i^{\delta(p_i-1)} \int_{|y| \leq \frac{\rho_1}{2} M^{\frac{p_i-1}{2}}} (1+|y|)^{4-2n} dy \\
 &\leq \begin{cases} C M_i^{(-1+\delta)\frac{4}{n-2}+o(1)} & \text{if } n > 4, \\ C M_i^{(-1+\delta)2+o(1)} \log M_i & \text{if } n = 4. \end{cases}
 \end{aligned}$$

Therefore, from the inequalities (4.16) and (4.18) and the identity (4.15),

$$\begin{aligned}
 (4.19) \quad & \frac{(n-2)\tau_i}{2(p_i+1)} \int_{|x| \leq \frac{\rho_1}{2}} K f_i^{-\tau_i} u_i(x)^{p_i+1} dx \\
 & \quad - \frac{\tau_i}{p_i+1} \int_{|x| \leq \frac{\rho_1}{2}} K f_i^{-\tau_i-1} (x^m \partial_m f_i) u_i^{p_i+1} dx \\
 & \leq \begin{cases} C M_i^{(-1+\delta)\frac{4}{n-2}+o(1)} & \text{if } n > 4, \\ C M_i^{(-1+\delta)2+o(1)} \log M_i & \text{if } n = 4. \end{cases}
 \end{aligned}$$

Since, from the Proposition 4.3,

$$\int_{|x| \leq R_i M_i^{-\frac{p_i-1}{2}}} u_i(x)^{p_i+1} dx \geq c > 0,$$

we conclude that if we choose ρ_1 sufficiently small, then

$$\begin{aligned}
 (4.20) \quad & \frac{n-2}{2} \int_{|x| \leq \frac{\rho_1}{2}} K f_i^{-\tau_i} u_i(x)^{p_i+1} dx \\
 & \quad - \int_{|x| \leq \frac{\rho_1}{2}} K f_i^{-\tau_i-1} (x^m \partial_m f_i) u_i^{p_i+1} dx \geq c > 0.
 \end{aligned}$$

The result follows immediately from the inequalities (4.19). q.e.d.

Claim 3. Given a small $\sigma > 0$, there exists a constant $C > 0$ such that

$$\int_{B_\sigma(x_i)} u_i^{p_i}(x) dx \leq C M_i^{-1}.$$

Proof of Claim 3. Set $s_i = R_i M_i^{-\frac{p_i-1}{2}}$.

First note that, changing variables, and then using $v_i(y) \leq cU_0(y)$ for $|y| \leq R_i$, we get

$$(4.21) \quad \int_{|x| \leq s_i} u_i^{p_i}(x) dx = M_i^{-\frac{(p_i-1)n}{2}} M_i^{p_i} \int_{|y| \leq R_i} v_i^{p_i}(y) dy \leq CM_i^{-1}.$$

On the other hand, by Claim 1,

$$(4.22) \quad \begin{aligned} \int_{s_i \leq |x| \leq \sigma} u_i^{p_i}(x) dx &\leq CM_i^{-\lambda_i p_i} \int_{s_i \leq |x| \leq \sigma} |x|^{(2-n+\delta)p_i} dx \\ &\leq CM_i^{-\lambda_i p_i} s_i^{(2-n+\delta)p_i+n} \leq o(1)M_i^{-1}. \end{aligned}$$

Claim 3 now follows from inequalities (4.21) and (4.22). q.e.d.

Claim 4. There exists $\sigma_1 > 0$ such that for all $0 < \sigma < \sigma_1$, there exists a constant $C = C(\sigma)$ with, for every i ,

$$u_i(x_i)u_i(x) \leq C(\sigma)$$

if $d(x, x_i) = \sigma$.

Proof of Claim 4. From the discussion in Section 3, if we choose $\sigma_1 > 0$ small, we can suppose that $R_{g_i} \geq 0$.

Choose $0 < \sigma < \sigma_1$ small and define

$$w_i(x) = u_i(x_\sigma)^{-1}u_i(x),$$

where x_σ is chosen so that $d(x_\sigma, x_i) = \sigma$. Note that

$$L_{g_i}w_i + Ku_i(x_\sigma)^{p_i-1}f_i^{-\tau_i}w_i^{p_i} = 0.$$

The Harnack inequality implies that, for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$C_\epsilon^{-1} \leq w_i(x) \leq C_\epsilon$$

if $d(x, \bar{x}) > \epsilon$. From Claim 1, we know that $u_i(x_\sigma)^{p_i-1} \rightarrow 0$ as $i \rightarrow \infty$, and then standard elliptic theory shows that, after maybe passing to a subsequence,

$$w_i \rightarrow w \text{ in } C_{loc}^2(B_\sigma(\bar{x})),$$

and w satisfies

$$L_g w = 0, w > 0.$$

Since the blowup is isolated simple, the function $\hat{u}_i(r)$ is decreasing in the interval $(R_i M_i^{-\frac{p_i-1}{2}}, \rho)$. Taking the limit, we conclude that $\hat{w}(r)$ is decreasing in the whole interval $(0, \rho)$.

As a consequence, w is singular at the origin.

It follows from the results contained in the appendix in [10] that

$$(4.23) \quad -\int_{B_\eta(x_i)} \Delta_{g_i} w_i = -\int_{\partial B_\eta(x_i)} \frac{\partial w_i}{\partial \nu} = -\int_{\partial B_\eta(\bar{x})} \frac{\partial w}{\partial \nu} + o(1) > c > 0$$

for each i , where $\eta > 0$ is sufficiently small.

On the other hand,

$$(4.24) \quad - \int_{B_\eta(x_i)} \Delta_{g_i} w_i = \int_{B_\eta(x_i)} [K u_i(x_\sigma)^{-1} f_i^{-\tau_i} u_i^{p_i} - c(n) R_{g_i} w_i] \\ \leq K \int_{B_\eta(x_i)} u_i(x_\sigma)^{-1} f_i^{-\tau_i} u_i^{p_i} \leq c u_i(x_\sigma)^{-1} M_i^{-1}.$$

Here, we have used Claim 3. Claim 4 now follows from inequalities (4.23) and (4.24).

Now, we are ready to prove Proposition 4.5.

Suppose the inequality (4.9) is not true. Then there exists a sequence \tilde{x}_i , with $d(\tilde{x}_i, x_i) \leq \frac{\rho_1}{2}$, ρ_1 small, such that

$$(4.25) \quad u_i(\tilde{x}_i) u_i(x_i) d(\tilde{x}_i, x_i)^{n-2} \rightarrow \infty$$

as $i \rightarrow \infty$.

Then, from Proposition 4.3, $R_i u_i(x_i)^{-\frac{p_i-1}{2}} \leq \tilde{r}_i \leq \frac{\rho_1}{2}$, where $\tilde{r}_i = d(\tilde{x}_i, x_i)$. Define

$$\tilde{v}_i(y) = \tilde{r}_i^{\frac{2}{p_i-1}} u_i(\tilde{r}_i y), \quad |y| < 2.$$

Now, it is not difficult to see that the origin is an isolated simple blowup point for \tilde{v}_i , and Claim 4 implies, together with the Harnack inequality,

$$\max_{|y|=1} \tilde{v}_i(0) \tilde{v}_i(y) \leq C.$$

This contradicts the limit in (4.25) and we finish the proof of inequality (4.9).

For the proof of inequality (4.10), recall that the Green function always exists when ρ_1 is sufficiently small. Now, observe that the inequality holds where $d_{g_i}(x, x_i) = R_i M_i^{-\frac{p_i-1}{2}}$ because $G_i(x_i, x) = O(r^{2-n})$ and where $d_{g_i}(x, x_i) = \rho_1$ because the Green function vanishes in this case and we are dealing with positive functions. Since

$$L_{g_i}(u_i(x_i) u_i) \leq 0 = L_{g_i} G_i,$$

we can apply the maximum principle on the region $\{x : R_i M_i^{-\frac{p_i-1}{2}} \leq d_{g_i}(x, x_i) \leq \rho_1\}$ to get the desired inequality. q.e.d.

Corollary 4.6. *Under the hypotheses of Proposition 4.5, after maybe passing to a subsequence,*

$$M_i u_i(x) \rightarrow h \text{ in } C_{\text{loc}}^2(B_{\rho_1}(\bar{x}) \setminus \{\bar{x}\}),$$

where h is a positive solution to the linear equation $L_g(h) = 0$, with a non-removable singularity at \bar{x} . (Here, g stands for the limit metric.)

Proof. Observe that the function $M_i u_i(x)$ satisfies

$$L_{g_i}(M_i u_i(x)) + K M_i^{1-p_i} f_i^{-\tau_i}(x) (M_i u_i(x))^{p_i} = 0.$$

The previous Proposition implies $M_i u_i(x)$ is uniformly bounded in compact sets contained in $B_{\rho_1}(\bar{x}) \setminus \{\bar{x}\}$, and then standard elliptic estimates show that, after extracting a subsequence, $M_i u_i(x) \rightarrow h$ in $C_{loc}^2(B_{\rho_1}(\bar{x}) \setminus \{\bar{x}\})$. Since $M_i \rightarrow \infty$, $L_g h = 0$.

Because of inequality (4.10), taking the limit, one sees that h is singular. This finishes the proof. q.e.d.

5. Symmetry estimates

In this section, we will estimate the difference between solutions to our equation and standard symmetric functions, which will be solutions to the corresponding critical equation in the Euclidean setting.

The next lemma gives us an estimate on $|v_i - U_0|$, depending on M_i and $\tau_i = \frac{n+2}{n-2} - p_i$. This is the first step towards the symmetry estimates, not depending on τ_i , we will prove later.

Lemma 5.1. *Let u_i be a sequence of positive functions satisfying equation (4.1) and $x_i \rightarrow \bar{x}$ be an isolated simple blowup point. Then there exists $\delta > 0$ such that*

$$|v_i(y) - U_0(y)| \leq C \begin{cases} \max\{M_i^{-2}, \tau_i\} & \text{if } n = 4, 5, \\ \max\{(\log M_i) M_i^{-2}, \tau_i\} & \text{if } n = 6, \\ \max\left\{M_i^{-2} M_i^{\frac{2(n-6)}{n-2}}, \tau_i\right\} & \text{if } n \geq 7, \end{cases}$$

for $|y| \leq \delta M_i^{\frac{p_i-1}{2}}$, where $\tau_i = \frac{n+2}{n-2} - p_i$.

Proof. Set $l_i = \delta M_i^{\frac{p_i-1}{2}}$, and $\Lambda_i = \max_{|y| \leq l_i} |v_i - U_0| = v_i(y_i) - U_0(y_i)$, for a certain $|y_i| \leq l_i$.

We observe that if there exists a constant $c > 0$ such that $|y_i| \geq cl_i$ for every i , then inequality $v_i \leq cU_0$ automatically implies the stronger inequality

$$|(v_i - U_0)(y)| \leq CM_i^{-2},$$

since

$$\Lambda_i = |v_i - U_0|(y_i) \leq C|y_i|^{2-n} \leq Cl_i^{2-n} \leq CM_i^{-2}.$$

So for large i , we will have $|y_i| \leq \frac{l_i}{2}$. We are using that $M_i^{\tau_i} \rightarrow 1$ as $i \rightarrow \infty$.

Define

$$w_i(y) = \Lambda_i^{-1}(v_i(y) - U_0(y)).$$

Then w_i satisfies

$$L_{h_i} w_i + b_i w_i = Q_i(y),$$

where

$$b_i(y) = K \tilde{f}_i^{-\tau_i} \frac{v_i^{p_i} - U_0^{p_i}}{v_i - U_0}(y),$$

and

$$(5.1) \quad Q_i(y) = \Lambda_i^{-1} \left\{ c(n)M_i^{-(p_i-1)}R_{g_i} \left(M_i^{-\frac{p_i-1}{2}}y \right) U_0(y) + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}} + K \left(U_0^{\frac{n+2}{n-2}} - \tilde{f}_i^{-\tau_i}U_0^{p_i} \right) \right\},$$

where $\tilde{f}_i(y) = f_i(M_i^{-\frac{p_i-1}{2}}y)$, $(h_i)_{kl}(y) = (g_i)_{kl}(M_i^{-\frac{p_i-1}{2}}y)$ and $O(|y|^N)$ comes from the expansion of the volume element in conformal normal coordinates and N is as big as we want.

Since the blowup is isolated simple, from inequality $v_i \leq cU_0$, it is easy to check, for example,

$$(5.2) \quad b_i(y) \leq c(1+|y|)^{-3}$$

for $|y| \leq l_i$.

We will choose δ small enough to guarantee the existence of the Green's function for the conformal Laplacian on a ball of radius δ , with respect to a Dirichlet boundary condition.

The Green's representation formula gives

$$(5.3) \quad w_i(y) = \int_{B_i} G_{i,L}(y, \eta)(b_i(\eta)w_i(\eta) - Q_i(\eta))d\eta - \int_{\partial B_i} \frac{\partial G_{i,L}}{\partial \nu}(y, \eta)w_i(\eta)ds$$

where B_i stands for $B_{l_i}(0)$ and $G_{i,L}$ is the Green function of L_{h_i} in B_i .

We will need the following lemma proved in [3]:

Lemma 5.2. *Suppose w is a solution to the equation*

$$(5.4) \quad \Delta w + n(n+2)U_0^{\frac{4}{n-2}}w = 0 \text{ in } \mathbb{R}^n.$$

If $\lim_{|y| \rightarrow \infty} w(y) = 0$, then there exist constants c_0, c_1, \dots, c_n such that

$$w(y) = c_0 \left(\frac{n-2}{2}U_0 + y \cdot \nabla U_0 \right) + \sum_{j=1}^n c_j \frac{\partial U_0}{\partial y_j}.$$

Remark. The functions $\frac{n-2}{2}U_0 + y \cdot \nabla U_0$ and $\frac{\partial U_0}{\partial y_j}$, $j = 1, \dots, n$, are solutions to the equation (5.4).

The proof of the Lemma 5.1 is by contradiction. Set

$$(5.5) \quad t_i = \begin{cases} M_i^{-2} & \text{if } n = 4, 5, \\ (\log M_i)M_i^{-2} & \text{if } n = 6, \\ M_i^{-2}M_i^{\frac{2(n-6)}{n-2}} & \text{if } n \geq 7. \end{cases}$$

If the proposition is false, we necessarily have

$$\Lambda_i^{-1} \max\{t_i, \tau_i\} \rightarrow 0$$

as $i \rightarrow \infty$, which implies that

$$\Lambda_i^{-1}t_i \rightarrow 0, \quad \Lambda_i^{-1}\tau_i \rightarrow 0.$$

Since $R = O(r^2)$ in conformal normal coordinates, we can get the following estimate:

$$(5.6) \quad |Q_i(y)| \leq c\Lambda_i^{-1} \left\{ M_i^{-\frac{8}{n-2}}|y|^2(1+|y|)^{2-n} \right. \\ \left. + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1+|y|^2)^{-\frac{n}{2}} \right. \\ \left. + \tau_i(|\log U_0| + |\log \tilde{f}_i|)(1+|y|)^{-n-2} \right\}.$$

Using the estimates (5.2) and (5.6), we get from the Green's representation formula (5.3) that w_i is bounded in C_{loc}^2 , and

$$(5.7) \quad |w_i(y)| \leq c[(1+|y|)^{-1} + c\Lambda_i^{-1}t_i]$$

for $|y| \leq \frac{\delta}{2}M_i^{\frac{p_i-1}{2}}$. We are using that $|w_i(y)| \leq C\Lambda_i^{-1}M_i^{-2}$ when $|y| = \frac{\delta}{2}M_i^{\frac{p_i-1}{2}}$, and also that $|G_{i,L}(y, \eta)| \leq C|y - \eta|^{2-n}$ for $|y| \leq \frac{l_i}{2}$.

Then, by standard elliptic estimates, there exists a subsequence, also denoted w_i , converging to w satisfying

$$\begin{cases} \Delta w + n(n+2)U_0^{\frac{4}{n-2}}(y)w = 0 & \text{in } \mathbb{R}^n, \\ |w(y)| \leq c(1+|y|)^{-1}. \end{cases}$$

So, Lemma 5.2 implies that

$$w(y) = c_0 \left(\frac{n-2}{2}U_0 + y \cdot \nabla U_0 \right) + \sum_{j=1}^n c_j \frac{\partial U_0}{\partial y_j}.$$

The conditions $w(0) = \frac{\partial w}{\partial y_j}(0) = 0$ show that $c_j = 0$ for every j , in other words, $w(y) \equiv 0$. From here, we conclude that $|y_i| \rightarrow \infty$ as $i \rightarrow \infty$.

This contradicts the estimate (5.7) since $w_i(y_i) = 1$ and $\Lambda_i^{-1}t_i \rightarrow 0$, and this finishes the proof. q.e.d.

In the next lemma, we estimate τ_i . This result and Lemma 5.1 give us an estimate on $|v_i - U_0|$ independent of τ_i .

Lemma 5.3. *Under the same hypotheses in Lemma 5.1,*

$$\tau_i \leq C \begin{cases} M_i^{-2} & \text{if } n = 4, 5, \\ (\log M_i)M_i^{-2} & \text{if } n = 6, \\ M_i^{-2}M_i^{\frac{2(n-6)}{n-2}} & \text{if } n \geq 7. \end{cases}$$

Proof. The proof will be again by contradiction and recall the definition (5.5). If the lemma is not true, then Lemma 5.1 implies that

$$|v_i(y) - U_0(y)| \leq C\tau_i.$$

Define

$$w_i(y) = \tau_i^{-1}(v_i - U_0)(y),$$

so w_i is uniformly bounded. The equation satisfied by w_i is

$$L_{h_i} w_i + b_i w_i = \tilde{Q}_i(y),$$

where

$$b_i(y) = K \tilde{f}_i^{-\tau_i} \frac{v_i^{p_i} - U_0^{p_i}}{v_i - U_0}(y)$$

and

$$(5.8) \quad \tilde{Q}_i(y) = \tau_i^{-1} \left\{ c(n) M_i^{-(p_i-1)} R_{g_i} \left(M_i^{-\frac{p-1}{2}} y \right) U_0(y) \right. \\ \left. + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N) |y| (1 + |y|^2)^{-\frac{n}{2}} \right. \\ \left. + K \left(U_0^{\frac{n+2}{n-2}} - \tilde{f}_i^{-\tau_i} U_0^{p_i} \right) \right\}.$$

If the lemma is not true, then $\tau_i^{-1} t_i \rightarrow 0$ as $i \rightarrow \infty$.

We have

$$(5.9) \quad |\tilde{Q}_i(y)| \leq c \tau_i^{-1} \left\{ M_i^{-\frac{8}{n-2}} |y|^2 (1 + |y|)^{2-n} \right. \\ \left. + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N) |y| (1 + |y|^2)^{-\frac{n}{2}} \right. \\ \left. + \tau_i (|\log U_0| + |\log \tilde{f}_i|) (1 + |y|)^{-n-2} \right\}.$$

By elliptic linear theory, we can suppose $w_i \rightarrow w$ in compact subsets.

If $\psi(y) = \frac{n-2}{2} U_0(y) + y \cdot \nabla U_0(y)$, then,

$$(5.10) \quad \int_{|y| \leq \frac{t_i}{2}} \psi(y) \tau_i^{-1} \left(M_i^{-\frac{8}{n-2}} |y|^2 (1 + |y|)^{2-n} \right. \\ \left. + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N) |y| (1 + |y|^2)^{-\frac{n}{2}} \right) \rightarrow 0.$$

Note that when $i \rightarrow \infty$, we have:

$$\tau_i^{-1} K \left(U_0^{\frac{n+2}{n-2}} - \tilde{f}_i^{-\tau_i} U_0^{p_i} \right) \rightarrow K (\log U_0(y) + \log f(\bar{x})) U_0^{\frac{n+2}{n-2}}$$

pointwise. It is not difficult to check that

$$\int_{\mathbb{R}^n} \psi(y) U_0^{\frac{n+2}{n-2}}(y) dy = 0.$$

Therefore, we can conclude

$$\lim_{i \rightarrow \infty} \int_{|y| \leq \frac{t_i}{2}} \psi(y) \tilde{Q}_i(y) dy = n(n-2) \int_{\mathbb{R}^n} \psi(y) (\log U_0(y)) U_0^{\frac{n+2}{n-2}}(y) dy.$$

On the other hand, integration by parts shows that

$$\begin{aligned}
 (5.11) \quad & \int_{|y| \leq \frac{l_i}{2}} \psi(y) \tilde{Q}_i(y) dy \\
 &= \int_{|y| \leq \frac{l_i}{2}} \psi(y) (L_{h_i} w_i + b_i w_i) dy \\
 &= \int_{|y| \leq \frac{l_i}{2}} (L_{h_i} \psi(y) + b_i \psi) w_i dy + \int_{|y| = \frac{l_i}{2}} \left(\psi \frac{\partial w_i}{\partial r} - w_i \frac{\partial \psi}{\partial r} \right) d\sigma.
 \end{aligned}$$

The integral on the boundary goes to zero when $i \rightarrow \infty$ because

$$\begin{cases} |\psi| = O(r^{2-n}), & |\nabla \psi| = O(r^{1-n}) \\ |w_i(\frac{l_i}{2})| \leq c\tau_i^{-1} M_i^{-2}, & |\nabla w_i(\frac{l_i}{2})| \leq c\tau_i^{-1} M_i^{-2} l_i^{-1}. \end{cases}$$

Taking the limit when $i \rightarrow \infty$, we would have

$$\lim_{i \rightarrow \infty} \int_{|y| \leq \frac{l_i}{2}} \psi(y) \tilde{Q}_i(y) dy = \int_{\mathbb{R}^n} (\Delta \psi(y) + n(n+2)U_0^{\frac{4}{n-2}} \psi) w dy = 0$$

because $\Delta \psi(y) + n(n+2)U_0^{\frac{4}{n-2}} \psi = 0$.

This is a contradiction because

$$(5.12) \quad n(n-2) \int_{\mathbb{R}^n} \psi(y) (\log U_0(y)) U_0^{\frac{n+2}{n-2}}(y) dy > 0.$$

To see this, first note that

$$\psi(y) = \frac{n-2}{2} \frac{1-r^2}{(1+r^2)^{\frac{n}{2}}}.$$

Observe that

$$\begin{aligned}
 (5.13) \quad & \int_{\mathbb{R}^n} \psi(y) (\log U_0(y)) U_0^{\frac{n+2}{n-2}}(y) dy \\
 &= -\frac{(n-2)^2}{4} \sigma_{n-1} \int_0^\infty \frac{1-r^2}{(1+r^2)^{n+1}} r^{n-1} \log(1+r^2) dr,
 \end{aligned}$$

and after changing variables $r = s^{-1}$, we get

$$\int_0^\infty \frac{1-r^2}{(1+r^2)^{n+1}} r^{n-1} \log(1+r^2) dr = 2 \int_1^\infty \frac{1-r^2}{(1+r^2)^{n+1}} r^{n-1} \log r dr.$$

Now, inequality (5.12) follows immediately, and that finishes the proof of the lemma. q.e.d.

The Lemmas 5.1 and 5.3 together imply our symmetry estimate:

Proposition 5.4. *Let u_i be a sequence of positive functions satisfying equation (4.1) and $x_i \rightarrow \bar{x}$ be an isolated simple blowup point. Then*

there exists $\delta > 0$ such that

$$(5.14) \quad |v_i(y) - U_0(y)| \leq C \begin{cases} M_i^{-2} & \text{if } n = 4 \text{ or } 5, \\ (\log M_i)M_i^{-2} & \text{if } n = 6, \\ M_i^{-2}M_i^{\frac{2(n-6)}{n-2}} & \text{if } n \geq 7 \end{cases}$$

for $|y| \leq \delta M_i^{\frac{p_i-1}{2}}$.

When $n \geq 6$, by applying the same technique, we can also get:

Proposition 5.5. *Let u_i be a sequence of positive functions satisfying equation (4.1) and $x_i \rightarrow \bar{x}$ be an isolated simple blowup point. Then*

$$(5.15) \quad |v_i(y) - U_0(y)| \leq C \begin{cases} M_i^{-2}M_i^{\frac{2}{n-2}}(1 + |y|)^{-1} & \text{if } n = 6 \\ M_i^{-2}M_i^{\frac{2(n-6)}{n-2}}(1 + |y|)^{6-n} & \text{if } n \geq 7. \end{cases}$$

Proof. Set

$$(5.16) \quad A_i = \begin{cases} M_i^{-2}M_i^{\frac{2}{n-2}} & \text{if } n = 6 \\ M_i^{-2}M_i^{\frac{2(n-6)}{n-2}} & \text{if } n \geq 7, \end{cases}$$

and define

$$w_i(y) = A_i^{-1}(v_i - U_0)(y)$$

for $|y| \leq \delta M_i^{\frac{p_i-1}{2}}$. Then our previous proposition implies w_i is uniformly bounded. The equation satisfied is

$$L_{h_i}w_i + b_iw_i = \tilde{Q}_i(y),$$

where

$$b_i(y) = K \frac{\tilde{f}_i^{-\tau_i} v_i^{p_i} - U_0^{p_i}}{v_i - U_0}(y)$$

and

$$(5.17) \quad \tilde{Q}_i(y) = A_i^{-1} \left\{ c(n)M_i^{-(p-1)} R_g \left(M_i^{-\frac{p-1}{2}} y \right) U_0(y) + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1 + |y|^2)^{-\frac{n}{2}} + K \left(U_0^{\frac{n+2}{n-2}} - \tilde{f}_i^{-\tau_i} U_0^{p_i} \right) \right\}.$$

Then

$$(5.18) \quad |\tilde{Q}_i(y)| \leq cA_i^{-1} \left\{ M_i^{-\frac{8}{n-2}} |y|^2(1 + |y|)^{2-n} + M_i^{-(1+N)\frac{p_i-1}{2}} O(|y|^N)|y|(1 + |y|^2)^{-\frac{n}{2}} + \tau_i(|\log U_0| + |\log \tilde{f}_i|)(1 + |y|)^{-n-2} \right\}.$$

The Green’s representation formula says that

$$(5.19) \quad w_i(y) = \int_{B_i} G_{i,L}(y, \eta)(b_i(\eta)w_i(\eta) - \tilde{Q}_i(\eta))d\eta - \int_{\partial B_i} \frac{\partial G_{i,L}}{\partial \nu}(y, \eta)w_i(\eta)ds,$$

where B_i stands for $B_{l_i}(0)$ and $G_{i,L}$ is the Green function of L_{h_i} in B_i .

Since $|G_{i,L}(y, \eta)| \leq C|y - \eta|^{2-n}$, for $|y| \leq \frac{l_i}{2}$, we get

$$(5.20) \quad |w_i(y)| \leq c\{(1 + |y|)^{-1} + cM_i^{-2}A_i^{-1}\} \leq c(1 + |y|)^{-1}.$$

If $n = 6$ or 7 the result follows multiplying the inequality (5.20) by A_i . If $n \geq 8$, we plug the estimate (5.20) in the representation formula (5.19) until we reach

$$|w_i(y)| \leq c(1 + |y|)^{6-n}.$$

Multiplying by A_i , we get the result.

q.e.d.

Remark 1. Once we have estimates (5.14), (5.15) on $v_i - U_0$, we can also get:

$$(5.21) \quad |\nabla(v_i - U_0)(y)| \leq C \begin{cases} M_i^{-2}(1 + |y|)^{-1} & \text{if } n = 4 \text{ or } 5, \\ M_i^{-2}M_i^{\frac{2}{n-2}}(1 + |y|)^{-2} & \text{if } n = 6 \\ M_i^{-2}M_i^{\frac{2(n-6)}{n-2}}(1 + |y|)^{5-n} & \text{if } n \geq 7 \end{cases}$$

and

$$(5.22) \quad |\nabla^2(v_i - U_0)(y)| \leq C \begin{cases} M_i^{-2}(1 + |y|)^{-2} & \text{if } n = 4 \text{ or } 5, \\ M_i^{-2}M_i^{\frac{2}{n-2}}(1 + |y|)^{-3} & \text{if } n = 6 \\ M_i^{-2}M_i^{\frac{2(n-6)}{n-2}}(1 + |y|)^{4-n} & \text{if } n \geq 7. \end{cases}$$

Remark 2. If h is as in the Corollary 4.6, the estimates (5.14) and (5.15) imply that:

$$(5.23) \quad |h(x) - |x|^{2-n}| \leq C \begin{cases} 1 & \text{if } n = 4, 5, \\ |x|^{-1} & \text{if } n = 6, \\ |x|^{6-n} & \text{if } n \geq 7. \end{cases}$$

Since that gives the asymptotic behavior of the Green function of the conformal Laplacian in conformal normal coordinates (see [7]), in some sense, our symmetry estimates cannot be improved.

6. Local blowup analysis

Now, let us turn our attention to the applications of these symmetry estimates.

In the first application, we will show that the Weyl tensor of the metric has to vanish at an isolated simple blowup point, when $n \geq 6$.

This result had been proposed by Schoen ([15]). This will allow us to use the Positive Mass Theorem in the proof of compactness of solutions to the Yamabe problem, when the dimension is 6 or 7.

Theorem 6.1. *Let u_i be a sequence of positive functions satisfying equation (4.1) and $x_i \rightarrow \bar{x}$ be an isolated simple blowup point. If $n \geq 6$, then*

$$W_g(\bar{x}) = 0.$$

Proof. We will use the Pohozaev identity (2.8) to u_i in a ball of radius r :

(6.1)

$$\begin{aligned} &P(r, u_i) \\ &= - \int_{|x| \leq r} \left(x^m \partial_m u_i + \frac{n-2}{2} u_i \right) ((g^{kl} - \delta^{kl}) \partial_{kl} u_i + \partial_k g^{kl} \partial_l u_i) dx \\ &\quad - c(n) \int_{|x| \leq r} \left(\frac{1}{2} x^k \partial_k R + R \right) u_i^2 dx + c(n) \frac{r}{2} \int_{|x|=r} R u_i^2 d\sigma_r \\ &\quad + \left(\frac{n}{p_i + 1} - \frac{n-2}{2} \right) \int_{|x| \leq r} K f_i^{-\tau_i} u_i^{p_i+1} dx \\ &\quad - \frac{\tau_i}{p_i + 1} \int_{|x| \leq r} K f_i^{-\tau_i-1} (x^m \partial_m f_i) u_i^{p_i+1} dx. \end{aligned}$$

Using conformal normal coordinates, we can get rid of the terms involving $|g|$.

Recall

(6.2)

$$\begin{aligned} &P(r, u_i) \\ &= \int_{|x|=r} \left(\frac{n-2}{2} u_i \frac{\partial u_i}{\partial r} - \frac{r}{2} |\nabla u_i|^2 + r \left| \frac{\partial u_i}{\partial r} \right|^2 + \frac{1}{p_i + 1} K f_i^{-\tau_i} r u_i^{p_i+1} \right) d\sigma_r. \end{aligned}$$

Since we have that $M_i u_i \rightarrow h$ in the C^2 topology on compact subsets of $B_\rho(\bar{x}) \setminus \{\bar{x}\}$, we conclude

$$(6.3) \quad M_i^2 |P(r, u_i)| \leq c < \infty.$$

The same holds for

$$(6.4) \quad M_i^2 \left| \int_{|x|=r} R u_i^2 d\sigma(r) \right| \leq c < \infty.$$

The change of variables $y = M_i^{\frac{p_i-1}{2}} x$ yields

$$\begin{aligned} &A_i(r) \\ &:= M_i^2 \left\{ - \int_{|x| \leq r} \left(x^m \partial_m u_i + \frac{n-2}{2} u_i \right) ((g^{kl} - \delta^{kl}) \partial_{kl} u_i + \partial_k g^{kl} \partial_l u_i) dx \right. \\ &\quad \left. - c(n) \int_{|x| \leq r} \left(\frac{1}{2} x^k \partial_k R + R \right) u_i^2 dx \right\} \\ &= -M_i^2 M_i^{2+(2-n)\frac{p_i-1}{2}} \int_{|y| \leq r M_i^{\frac{p_i-1}{2}}} \left\{ \left(y^m \partial_m v_i + \frac{n-2}{2} v_i \right) \right. \\ &\quad \cdot \left((g^{kl} - \delta^{kl}) \left(M_i^{-\frac{p_i-1}{2}} y \right) \partial_{kl} v_i + M_i^{-\frac{p_i-1}{2}} \partial_k g^{kl} \left(M_i^{-\frac{p_i-1}{2}} y \right) \partial_l v_i \right) \\ &\quad \left. + c(n) M_i^{-(p_i-1)} \left(\frac{1}{2} y^k \partial_k R \left(M_i^{-\frac{p_i-1}{2}} y \right) + R \left(M_i^{-\frac{p_i-1}{2}} y \right) \right) v_i^2 \right\} dy. \end{aligned}$$

Note that $M_i^{2+(2-n)\frac{p_i-1}{2}} \rightarrow 1$. Define

(6.5)

$$\begin{aligned} &\hat{A}_i(r) \\ &= -M_i^2 M_i^{2+(2-n)\frac{p_i-1}{2}} \int_{|y| \leq r M_i^{\frac{p_i-1}{2}}} \left\{ \left(y^m \partial_m U_0 + \frac{n-2}{2} U_0 \right) \right. \\ &\quad \cdot \left((g^{kl} - \delta^{kl}) \left(M_i^{-\frac{p_i-1}{2}} y \right) \partial_{kl} U_0 + M_i^{-\frac{p_i-1}{2}} \partial_k g^{kl} \left(M_i^{-\frac{p_i-1}{2}} y \right) \partial_l U_0 \right) \\ &\quad \left. + c(n) M_i^{-(p_i-1)} \left(\frac{1}{2} y^k \partial_k R \left(M_i^{-\frac{p_i-1}{2}} y \right) + R \left(M_i^{-\frac{p_i-1}{2}} y \right) \right) U_0^2 \right\} dy. \end{aligned}$$

Then one can check

$$\begin{aligned} (6.6) \quad &|A_i(r) - \hat{A}_i(r)| \\ &\leq c M_i^2 M_i^{-\frac{4}{n-2}} \int_{|y| \leq r M_i^{\frac{p_i-1}{2}}} \left\{ |v_i - U_0|(y) (1 + |y|)^{2-n} \right. \\ &\quad \left. + |\nabla(v_i - U_0)|(1 + |y|)^{3-n} + |\nabla^2(v_i - U_0)|(1 + |y|)^{4-n} \right\} dy. \end{aligned}$$

If $n = 6$, then

$$|A_i(r) - \hat{A}_i(r)| \leq c M_i^{-\frac{2}{n-2}} \int_{|y| \leq r M_i^{\frac{p_i-1}{2}}} (1 + |y|)^{1-n} dy.$$

When $n \geq 7$,

$$|A_i(r) - \hat{A}_i(r)| \leq c M_i^{-\frac{4}{n-2}} M_i^{(n-6)\frac{2}{n-2}} \int_{|y| \leq r M_i^{\frac{p_i-1}{2}}} (1 + |y|)^{8-2n} dy.$$

Then

$$(6.7) \quad |A_i(r) - \hat{A}_i(r)| \leq \begin{cases} C & \text{if } n = 6, 7, \\ C(\log M_i) & \text{if } n = 8, \\ CM_i^{(n-8)\frac{2}{n-2}} & \text{if } n \geq 9. \end{cases}$$

Also, observe that if we choose r sufficiently small,

$$(6.8) \quad \left(\frac{n}{p_i + 1} - \frac{n-2}{2}\right) \int_{|x| \leq r} K u_i^{p_i+1} dx - \frac{\tau_i}{p_i + 1} \int_{|x| \leq r} K f_i^{-\tau_i-1} (x^m \partial_m f_i) u_i^{p_i+1} dx \geq 0,$$

so we obtain, from identity (6.1), estimates (6.3), (6.4), (6.7) and inequality (6.8) that

$$\hat{A}_i(r) \leq \begin{cases} C & \text{if } n = 6, 7, \\ C(\log M_i) & \text{if } n = 8, \\ CM_i^{(n-8)\frac{2}{n-2}} & \text{if } n \geq 9. \end{cases}$$

We will need the Taylor series, for each i ,

$$R(x) = p_2(x) + p_3(x) + e(x) \\ \left(R + \frac{1}{2}x^k \partial_k R\right)(x) = 2p_2(x) + \frac{5}{2}p_3(x) + e'(x)$$

where p_i is a homogeneous polynomial of degree i and $|e(x)|, |e'(x)| \leq c|x|^4$.

Let us denote by

$$\tilde{u}_i(x) = M_i^{\frac{n-2}{4}\tau_i} \left(\frac{M_i^{-\frac{p_i-1}{2}}}{M_i^{-(p_i-1)} + |x|^2} \right)^{\frac{n-2}{2}}$$

and it is not difficult to see that, changing variables, estimates on $|v_i - U_0|$ yield estimates on $|u_i - \tilde{u}_i|$.

Note that

$$M_i^2 \int_{|x| \leq r} \left(\frac{1}{2}x^k \partial_k e + e\right) \tilde{u}_i^2 dx \leq \begin{cases} C & \text{if } n = 6, 7, \\ C(\log M_i) & \text{if } n = 8, \\ CM_i^{(n-8)\frac{2}{n-2}} & \text{if } n \geq 9. \end{cases}$$

Then, using symmetry of \tilde{u}_i

$$-c(n) \int_{|x| \leq r} 2p_2(x)(M_i \tilde{u}_i)^2 dx \leq c \begin{cases} C & \text{if } n = 6, 7, \\ C(\log M_i) & \text{if } n = 8, \\ CM_i^{(n-8)\frac{2}{n-2}} & \text{if } n \geq 9. \end{cases}$$

But, on the other hand, since $\Delta R(0) = -\frac{1}{6}|W(0)|^2$ in conformal normal coordinates, we obtain

$$\begin{aligned}
 & -c(n)M_i^2 \int_{|x|\leq r} 2p_2(x)\tilde{u}_i^2 dx \\
 & \geq \begin{cases} C|W(x_i)|^2 M_i^2 M_i^{-4\frac{2}{n-2}} (\log M_i) & \text{if } n = 6, \\ C|W(x_i)|^2 M_i^2 M_i^{-4\frac{2}{n-2}} & \text{if } n \geq 7. \end{cases}
 \end{aligned}$$

So

$$|W_{g_i}(x_i)|^2 \leq \begin{cases} c(\log M_i)^{-1} & \text{if } n = 6, \\ cM_i^{-\frac{2}{n-2}} & \text{if } n = 7, \\ cM_i^{-\frac{4}{n-2}} (\log M_i) & \text{if } n = 8, \\ cM_i^{-\frac{4}{n-2}} & \text{if } n \geq 9. \end{cases}$$

And taking the limit, we are done.

q.e.d.

The next result concerns the local asymptotic analysis at a blowup point. It will be used together with the Positive Mass Theorem to exclude the possibility of blowup phenomenon on manifolds not conformally diffeomorphic to the sphere.

Define

$$(6.9) \quad P'(r, v) = \int_{|x|=r} \left(\frac{n-2}{2} v \frac{\partial v}{\partial \nu} - \frac{r}{2} |\nabla v|^2 + r \left| \frac{\partial v}{\partial \nu} \right|^2 \right) d\sigma(r).$$

Theorem 6.2. *Let u_i be a sequence of positive functions satisfying equation (4.1) and $x_i \rightarrow \bar{x}$ be an isolated simple blowup point. and suppose $4 \leq n \leq 7$. If $u_i(x_i)u_i \rightarrow h$ away from the origin, then*

$$\liminf_{r \rightarrow 0} P'(r, h) \geq 0.$$

Proof. That is another application of the Pohozaev identity (2.7) and the symmetry estimates:

$$\begin{aligned}
 (6.10) \quad P(r, u_i) = & - \int_{|x|\leq r} \left(x^m \partial_m u_i + \frac{n-2}{2} u_i \right) \\
 & \cdot ((g^{kl} - \delta^{kl}) \partial_{kl} u_i + \partial_k g^{kl} \partial_l u_i - c(n) R u_i) dx \\
 & + \left(\frac{n}{p_i + 1} - \frac{n-2}{2} \right) \int_{|x|\leq r} K f_i^{-\tau_i} u_i^{p_i+1} dx \\
 & - \frac{\tau_i}{p_i + 1} \int_{|x|\leq r} K f_i^{-\tau_i-1} (x^m \partial_m f_i) u_i^{p_i+1} dx.
 \end{aligned}$$

Firstly, observe that

$$M_i^2 P(r, u_i) \rightarrow P'(r, h)$$

as $i \rightarrow \infty$.

Secondly, as in the previous result, if r is sufficiently small,

$$(6.11) \quad \left(\frac{n}{p_i + 1} - \frac{n-2}{2} \right) \int_{|x| \leq r} K f_i^{-\tau_i} u_i^{p_i+1} dx \\ - \frac{\tau_i}{p_i + 1} \int_{|x| \leq r} K f_i^{-\tau_i-1} (x^m \partial_m f_i) u_i^{p_i+1} dx \geq 0.$$

If

$$(6.12) \quad A_i(r) := M_i^2 \left\{ - \int_{|x| \leq r} \left(x^m \partial_m u_i + \frac{n-2}{2} u_i \right) ((g^{kl} - \delta^{kl}) \partial_{kl} u_i \right. \\ \left. + \partial_k g^{kl} \partial_l u_i - c(n) R u_i) dx \right\}$$

and

$$(6.13) \quad \hat{A}_i(r) := M_i^2 \left\{ - \int_{|x| \leq r} \left(x^m \partial_m \tilde{u}_i + \frac{n-2}{2} \tilde{u}_i \right) ((g^{kl} - \delta^{kl}) \partial_{kl} \tilde{u}_i \right. \\ \left. + \partial_k g^{kl} \partial_l \tilde{u}_i - c(n) R \tilde{u}_i) dx \right\},$$

then one can check

$$|A_i(r) - \hat{A}_i(r)| \leq \begin{cases} cr^2 & \text{if } n = 4, 5, \\ cr & \text{if } n = 6, 7. \end{cases}$$

So

$$(6.14) \quad \liminf_{r \rightarrow 0} P'(r, h) \\ \geq c(n) M_i^2 \liminf_{r \rightarrow 0} \int_{|x| \leq r} \left(x^m \partial_m \tilde{u}_i + \frac{n-2}{2} \tilde{u}_i \right) R \tilde{u}_i dx.$$

If $n = 4$ or 5 , since $R = O(r^2)$, this automatically gives

$$\liminf_{r \rightarrow 0} P'(r, h) \geq 0.$$

If $n = 6$ or 7 , the fact that $R = O(r^2)$ tells us that we only need to consider the second order term in the expansion of R . We are using the symmetry of \tilde{u}_i to get rid of the third order term.

The change of variables $y = M_i^{\frac{p_i-1}{2}} x$ implies that

$$(6.15) \quad M_i^2 \int_{|x| \leq r} \left(x^m \partial_m \tilde{u}_i + \frac{n-2}{2} \tilde{u}_i \right) R_{,ab} x^a x^b \tilde{u}_i dx \\ = (1 + o(1)) M_i^{\frac{2(n-6)}{n-2}} \int_{|y| \leq r M_i^{\frac{p_i-1}{2}}} \left(y^m \partial_m U_0 + \frac{n-2}{2} U_0 \right) R_{,ab} y^a y^b U_0 dy.$$

But

$$\begin{aligned}
 (6.16) \quad & \int_{|y| \leq rM_i^{\frac{p_i-1}{2}}} \left(y^m \partial_m U_0 + \frac{n-2}{2} U_0 \right) R_{,ab} y^a y^b U_0 dy \\
 &= \int_0^{rM_i^{\frac{p_i-1}{2}}} \left(r \partial_r U_0 + \frac{n-2}{2} U_0 \right) U_0 r^{n+1} \frac{\Delta R}{n} \sigma_{n-1} dr \\
 &= -\frac{1}{6n} \sigma_{n-1} |W(0)|^2 \int_0^{rM_i^{\frac{p_i-1}{2}}} \left(r \partial_r U_0 + \frac{n-2}{2} U_0 \right) U_0 r^{n+1} dr \\
 &= -\frac{1}{6n} \sigma_{n-1} |W(0)|^2 \left(-\frac{n-2}{2} \int_0^{rM_i^{\frac{p_i-1}{2}}} \frac{r^{n+3}}{(1+r^2)^{n-1}} dr \right. \\
 &\quad \left. + \frac{n-2}{2} \int_0^{rM_i^{\frac{p_i-1}{2}}} \frac{r^{n+1}}{(1+r^2)^{n-1}} dr \right).
 \end{aligned}$$

When $n = 6$ and i is large, this last expression is non-negative because the first integral diverges while the second one is finite. If $n = 7$ and i is large, again the expression is non-negative since

$$(6.17) \quad \int_0^\infty \frac{r^{n+3}}{(1+r^2)^{n-1}} dr = \frac{n+2}{n-6} \int_0^\infty \frac{r^{n+1}}{(1+r^2)^{n-1}} dr.$$

To see this, first observe that, if $m + 1 < 2k$, integration by parts gives

$$\int_0^\infty \frac{t^m}{(1+t^2)^k} dt = \frac{m-1}{2(k-1)} \int_0^\infty \frac{t^{m-2}}{(1+t^2)^{k-1}} dt.$$

Since

$$\int_0^\infty \frac{t^{m-2}}{(1+t^2)^{k-1}} dt = \int_0^\infty \frac{t^{m-2}}{(1+t^2)^k} dt + \int_0^\infty \frac{t^m}{(1+t^2)^k} dt,$$

one gets

$$\int_0^\infty \frac{t^m}{(1+t^2)^k} dt = \frac{m-1}{2k-m-1} \int_0^\infty \frac{t^{m-2}}{(1+t^2)^k} dt.$$

Just choose $m = n + 4$ and $k = n + 1$ to obtain (6.17).

The proof is finished by using the inequality (6.14). q.e.d.

7. Compactness theorem

In this section, we will prove the *a priori* estimates for the Yamabe problem in the non-locally conformally flat case, for $4 \leq n \leq 7$.

The next Proposition is fundamental since it allows us to use the symmetry estimates we proved before.

Proposition 7.1. *Suppose the blowup $x_i \rightarrow \bar{x}$ is isolated, $4 \leq n \leq 7$. Then it is also isolated simple.*

The proof of this statement is just as in [10], based on Proposition 6.2.

Once we have established Proposition 7.1, again following [10], we have:

Theorem 7.2. *Suppose $u_i \in \mathbb{M}_{p_i}$ is a sequence satisfying $\max_M u_i \rightarrow \infty$, as $i \rightarrow \infty$. Then $p_i \rightarrow \frac{n+2}{n-2}$. Moreover, after passing to a subsequence,*

- 1) *the set $S = \{\text{blowup points of } u_i\}$ is finite;*
- 2) *every blowup point of u_i is an isolated simple blowup point.*

Now, let us turn to the statement and proof of the compactness theorem.

Theorem 7.3. *Let (M^n, g) be a smooth closed Riemannian manifold with positive Yamabe quotient, not conformally equivalent to (\mathbb{S}^n, g_0) . Assume $4 \leq n \leq 7$. Then, for every $\epsilon > 0$, there exists a positive constant $C = C(\epsilon, g)$ so that*

$$\begin{cases} 1/C \leq u \leq C & \text{and} \\ \|u\|_{C^{2,\alpha}(M)} \leq C \end{cases}$$

for every $u \in \cup_{1+\epsilon \leq p \leq \frac{n+2}{n-2}} \mathbb{M}_p$.

Proof. Standard elliptic estimates and the Harnack inequality imply that it suffices to estimate $\|u\|_{C^0(M)}$. Suppose, by contradiction, that $\cup_{1+\epsilon \leq p \leq \frac{n+2}{n-2}} \mathbb{M}_p$ is not bounded in $C^0(M)$. This means that there exist $1 + \epsilon \leq p_i \leq \frac{n+2}{n-2}$ and $u_i \in \mathbb{M}_{p_i}$ with

$$\max_M u_i \rightarrow \infty \text{ as } i \rightarrow \infty.$$

From Theorem 7.2, we know that this is only possible if $p_i \rightarrow \frac{n+2}{n-2}$.

Now, Theorem 7.2 implies that, after possibly passing to a subsequence, u_i has N isolated simple blow-up points $x_i^{(1)} \rightarrow x^{(1)}, \dots, x_i^{(N)} \rightarrow x^{(N)}$, for some integer N .

Define $w_i(x) = u_i(x_i^{(1)})u_i(x)$. We can suppose, for example, that

$$u_i(x_i^{(1)}) = \min\{u_i(x_i^{(1)}), \dots, u_i(x_i^{(N)})\} \text{ for all } i.$$

Proposition 4.5 then implies that there exists $\rho, c > 0$ such that

$$(7.1) \quad w_i(x) \leq cd(x, x_i^{(j)})^{2-n} \text{ when } d(x, x_i^{(j)}) \leq \rho, 1 \leq j \leq N.$$

On the other hand, we know that the sequence u_i is uniformly bounded in $M \setminus \cup_{j=1}^N B_{\frac{\rho}{4}}(x^{(j)})$, since there is no blowup point in that region. Then the Harnack inequality implies w_i is uniformly bounded

in $M \setminus \cup_{j=1}^{j=N} B_{\frac{r}{2}}(x^{(j)})$. This and inequality (7.1) imply, after passing to a subsequence,

$$u_i(x_i^{(1)})u_i(x) \rightarrow h(x) = \sum_{j=1}^N a_j G_{x^{(j)}}(x) + b(x)$$

in $C_{loc}^2(M \setminus \{x^{(1)}, \dots, x^{(N)}\})$, where a_1, \dots, a_N are non-negative constants, $G_{x^{(j)}}$ is the Green function of the conformal Laplacian with pole at $x^{(j)}$ and $b(x)$ is a regular C^2 function satisfying $L_g(b) = 0$ in M . Note that, since the first eigenvalue of the conformal Laplacian L_g is positive, $b \equiv 0$.

Since the Green functions considered are positive and $a_1 > 0$ because of the Corollary 4.6, we obtain the following expansion ([7]):

$$h(x) = ad(x, x^{(1)})^{2-n} + A + O(r), \text{ for } 4 \leq n \leq 6$$

or

$$h(x) = ad(x, x^{(1)})^{2-n} - cR_{;ij}x_i x_j r^{-3} + A + O(r), \text{ for } n = 7.$$

The Positive Mass Theorem asserts that $A > 0$.

We want to compute the limit

$$\liminf_{r \rightarrow 0} P'(r, h).$$

Since r^{2-n} is harmonic, definition (6.9) gives

$$(7.2) \quad P'(r, h) = \int_{|x|=r} \left\{ -\frac{(n-2)^2}{2} a A r^{1-n} + O(r^{2-n}) \right\} d\sigma_r.$$

When $n = 7$, we use symmetry and the fact that $\Delta R(0) = -\frac{1}{6}|W(0)|^2 = 0$.

Then

$$\liminf_{r \rightarrow 0} P'(r, h) = -\frac{(n-2)^2}{2} a A \sigma_{n-1} < 0,$$

by the Positive Mass Theorem.

This contradicts Theorem 6.2 and it finishes the proof. q.e.d.

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INSTITUTO DE MATEMÁTICA PURA E APLICADA (IMPA)
 ESTRADA DONA CASTORINA 110
 22460-320, RIO DE JANEIRO – RJ
 BRAZIL
E-mail address: coda@impa.br