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A Priori L_2 Error Estimates for Galerkin Approximations to Parabolic Partial Differential Equations — Source link

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A PRIORI L_2 ERROR ESTIMATES FOR GALERKIN APPROXIMATIONS TO PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

by

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Dedication

This thesis is dedicated to
my mother, Mary Milligan Fanett.

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§1 Introduction

In recent years much attention has been devoted to the formulation and analysis of Galerkin methods for approximating solutions of parabolic problems. In [9], Douglas and Dupont obtained H^1 error estimates for the continuous time and for several discrete time Galerkin procedures. Price and Varga in [18] obtained L_2 error estimates for the continuous time Galerkin procedure for linear parabolic problems. It appears however that their proofs are restricted to linear problems and the use of L-splines. Fix and Strang [14] have also obtained L_2 estimates for linear initial value problems.

This thesis is an extension of the work of Douglas and Dupont [9]. In this thesis L_2 error estimates for continuous time and several discrete time Galerkin approximations of some second order nonlinear parabolic boundary value problems are derived. It appears that this analysis carries over to higher order parabolic problems and to systems of parabolic equations.

This thesis is divided into three main chapters. In Chapter II the variational problem and the Galerkin procedure are described. The basic techniques of this thesis are developed in Chapter III. There L_2 error estimates for Galerkin approximations of linear elliptic problems are used to derive a priori L_2 error estimates for continuous time Galerkin approximations of nonlinear parabolic problems. These estimates are independent of the choice of basis functions used in the Galerkin procedure. However, they do depend on an L_∞ estimate of the derivative of a function which is the Galerkin solution to a certain linear elliptic problem. In Chapter III we also derive L_2 and L_∞ error estimates for Galerkin approximations and derivatives of Galerkin approximations where the region under consideration is a rectangular parallelepiped and the tensor products of piecewise Hermite polynomials of degree $2m-1$, $m \geq 1$, are used as a basis. In Chapter IV we use the techniques of Chapter III to obtain L_2 error estimates for several discrete time Galerkin procedures.

§2 Formulation of the Variational Problem and the Galerkin Procedure

2.1 Definitions and notation: In this thesis all functions are real-valued. Let Ω be a bounded domain in R^n.

Definition 2.1: $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with compact support in Ω.

Definition 2.2: $C^{s*}(\Omega)$, s a positive integer, is a subset of $C^s(\Omega)$ consisting of functions w with

$$\left(\sum_{|\alpha|\leq s} \|D^\alpha w\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}} < \infty$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a n-tuple with non-negative integer components and

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$

Definition 2.3: $H^s(\Omega)$, s a positive integer, denotes the completion of $C^{s*}(\Omega)$ with respect to the norm

$$\|\cdot\|_{H^s(\Omega)} = \Big(\sum_{|\alpha|\le s} \|D^\alpha \cdot\|^2_{L_2(\Omega)} \Big)^{\frac{1}{2}}$$

(see Agmon [1])

Definition 2.4: $H^1_0(\Omega)$ denotes the closure in $H^1(\Omega)$ of $C^\infty_0(\Omega)$.

If $w \in H^1_0(\Omega)$ then

$$\text{(2.0)} \qquad \|w\|^2_{L_2(\Omega)} \le C_\Omega \sum_{i=1}^{n} \int_\Omega w^2_{x_i}\, dx$$

(see [4]). We norm $H^1_0(\Omega)$ by

$$\|w\|^2_{H^1_0(\Omega)} = \sum_{i=1}^{n} \int_\Omega w^2_{x_i}\, dx$$

Definition 2.5: Let H be a normed linear space consisting of a set of functions defined on Ω. If w is a function defined on $[0,T] \times \Omega$ we say $w \in L_p([0,T],H)$, $1 \le p \le \infty$, if for $t \in [0,T]$ $w(\cdot,t) \in H$ and $\|w\|_H \in L_p([0,T])$. We define

$$\|w\|_{H\times L_p([0,T])} = \|F(t)\|_{L_p([0,T])}$$

where

$$F(t) = \|w\|_H(t).$$

In this thesis we use $\langle\ ,\ \rangle$ to denote

$$\langle w,v\rangle = \int_\Omega wv\ dx \qquad w,v \in L_2(\Omega)$$

and $H_0^1 = H_0^1(\Omega)$, $L_s = L_s(\Omega)$, and $H^s = H^s(\Omega)$, s a positive integer. Also $L_\infty = L_\infty(\Omega)$, $\|\cdot\|_\infty = \|\cdot\|_{L_\infty(\Omega)\times L_\infty[0,T]}$.

If $f_{x_i} \in L_\infty(\Omega \times [0,T])$, $1 \le i \le n$, we define

$$\|\nabla f\|_\infty = \max_{1\le i\le n} \|f_{x_i}\|_\infty.$$

2.2 <u>The variational problem and the Galerkin procedure</u>. Consider the parabolic partial differential equation

$$(2.1)\quad u_t = \nabla\cdot a(x,u)\nabla u \qquad x \in \Omega,\quad t > 0$$

with boundary condition

$$(2.2)\qquad u(x,t) = 0 \qquad x \in \partial\Omega,\quad t > 0$$

and initial condition

$$(2.3)\quad u(x,0) = \psi(x) \qquad x \in \Omega$$

where Ω is a bounded domain in R^n. $a(x,p)$, $(x,p) \in \Omega \times R$, is assumed to be positive and bounded. It is evident that if u is a solution to (2.1) - (2.3) then u satisfies

$$(2.4)\quad \langle u_t, v\rangle = -\int_\Omega a(x,u)\nabla u\cdot\nabla v\,dx, \qquad t > 0,\ v \in H_0^1(\Omega),$$

$$(2.5)\quad \langle u(x,0),\ v\rangle = \langle\psi, v\rangle \qquad v \in H_0^1(\Omega),$$

and

$$(2.6)\quad u(x,t) \in H_0^1(\Omega) \qquad t > 0$$

If u satisfies (2.4) - (2.6) we say that u is a weak solution to (2.1) - (2.3). In fact (2.1) - (2.3) and (2.4) - (2.6) are equivalent if $u(x,t)$ and $a(x,u)$ have sufficient regularity.

In the Galerkin method, we seek a differentiable function $U(\cdot,t) \in \mathcal{M}$, a finite dimensional subspace of $H_0^1(\Omega)$, such that

$$(2.7)\quad \begin{aligned} &\langle U_t, v\rangle = -\int_\Omega a(x,U)\nabla U\cdot\nabla v\,dx, && t > 0,\ v \in \mathcal{M} \\ &\langle U, v\rangle = \langle\psi, v\rangle && t = 0,\ v \in \mathcal{M} \end{aligned}$$

Let $\mathcal{M}$ denote the span $\{v_i\}_{i=1}^{M}$ where $v_1,\ldots,v_M$ are linearly independent and let

$$U(x,t) = \sum_{i=1}^{M} \xi_i(t)\, v_i(x)$$

Then (2.7) reduces to an initial value problem for the system of nonlinear ordinary differential equations

$$G\xi'(t) = -B(\xi)\xi$$

and

$$G\xi(0) = b.$$

Here b is a vector whose k^{th} component $b_k = \langle \psi, v_k \rangle$, $G = (G_{k\ell})$ with

$$G_{k\ell} = \langle v_k,\, v_\ell \rangle,$$

and $B(\xi) = (B_{k\ell}(\xi))$ with

$$B_{k\ell}(\xi) = \int_\Omega a(x, \sum_{i=1}^{M} \xi_i(t) v_i(x)) \nabla v_k \cdot \nabla v_\ell dx$$

The matrices G and $B(\xi)$ are positive definite since the v_i, $1 \le i \le M$, are linearly independent and $a(x,p)$ for $(x,p) \in \Omega \times R$ is positive and bounded. We will also assume that $a(x,\cdot)$ is uniformly Lipschitz continuous with respect to its (n+1)st variable. It then follows from the theory of ordinary differential equations that $\xi(t)$ exists and is unique for $t > 0$.

The solution to (2.7) is called the <u>continuous time</u>

Galerkin approximation to u. In Chapter IV, we discuss several procedures for approximating U in which the variable t will be discretized. The solution to the discrete problem is called the discrete time Galerkin approximation to u.

2.3 Basis functions for the Galerkin procedure. Throughout this thesis, S^h will denote the span of M linearly independent functions in V, where V is appropriately $H_0^1(\Omega)$ or $H^1(\Omega)$. For parabolic problems with homogeneous Dirichlet boundary conditions S^h will be an $S_{k,m}^{h,0}(\Omega)$ space and for Neumann problems S^h will be an $S_{k,m}^h(\Omega)$ space. The spaces $S_{k,m}^{h,0}(\Omega)$ and $S_{k,m}^h(\Omega)$ will now be defined. (see [5]).

Definition 2.6: Let h, $0 < h < 1$, be a parameter and G a bounded open set in R^n. For any two positive integers k and m with $k < m$, let $S_{k,m}^{h,0}(G)$ be any finite dimensional subspace of $H^k(G) \cap H_0^1(G)$ with norm $\| \ \|_{H^k(G)}$ which satisfies:

For any $v \in H^j(G) \cap H_0^1(G)$ there exists a constant Q independent of h and v such that

$$(2.8) \quad \inf_{X \in S_{k,m}^{h,0}(G)} \|v-X\|_{H^{\ell}(G)} \leq Q h^{j-\ell} \|v\|_{H^j(G)}$$

for all non-negative j and ℓ with $\ell \leq k$ and $\ell \leq j \leq m$.

Likewise let $S^h_{k,m}(G)$ denote any finite dimensional subspace of $H^k(G)$ such that if $u \in H^j(G)$ then (2.8) with $S^{h,0}_{k,m}(G)$ replaced by $S^h_{k,m}(G)$ holds for all non-negative integers j and ℓ such that $\ell \leq k$ and $\ell \leq j \leq m$.

We now describe an example of an $S^h_{m,2m}(B)$ space where B is a rectangular parallelepiped in R^n.

Let Δ denote a partition of $[a,b]$ with $a = x_0 < x_1 < \ldots < x_{N+1} = b$ and $x_{j+1} - x_j = h$. The set of piecewise Hermite polynomials of degree $2m-1$, $m \geq 1$, which are defined on Δ is $\{s_{ik}(x,m)\}_{i=0}^{N+1}{}_{k=0}^{m-1}$ where

$$D^\ell s_{ik}(x_j,m) = \delta_{ij}\,\delta_{\ell k} h^{-\ell},\ 0 \leq \ell \leq m-1,\ 0 \leq j \leq N+1$$

(see [7])

We note that $s_{ik}(x,m)$ has support in $[x_{i-1},x_{i+1}]$ and that $\|D^\ell s_{ik}\|_{L_\infty[a,b]} \leq C_m h^{-\ell}$, $0 \leq \ell \leq m$, where C_m is a constant independent of h. The $\{s_{ik}(x,m)\}_{i=0}^{N+1}{}_{k=0}^{m-1}$ is a basis for $H_m(\Delta)$ where $H_m(\Delta)$ is the collection of all real piecewise polynomial functions $w(x)$ on $[a,b]$ such that $w(x) \in C^{m-1}([a,b])$ and such that on each interval $[x_j,x_{j+1}]$, w is a polynomial of degree $2m-1$, $m \geq 1$.

Let $B = \mathop{X}\limits_{i=1}^{n} (a_i,b_i)$, Δ_i denote a partition of $[a_i,b_i]$, and h be the maximum interval length of Δ_i, $1 \leq i \leq n$.

In [21] Schultz shows that the tensor products of the piecewise Hermite polynomials of degree 2m-1, $m \geq 1$, which are defined on $\overset{n}{\underset{i=1}{X}} \Delta_i$ form a basis for an $S^h_{m,2m}(B)$ space.

§3. A Priori L_2 Error Estimates for the Continuous Time Galerkin Procedure

3.1 Estimates for the heat equation in one dimension with homogeneous Dirichlet boundary conditions. We will consider the heat equation in one dimension to illustrate the techniques used in this thesis without becoming involved in complex details. In this section we will also choose a particular basis for the Galerkin procedure in order to derive L_∞ as well as L_2 error estimates.

Let u be a unique solution to the one dimensional heat equation

$$(3.1) \qquad u_t = u_{xx} + f(x,u) \qquad (x,t) \in I \times (0,T]$$

with initial and boundary conditions

$$(3.2) \qquad u(x,0) = \Psi(x) \qquad x \in I$$

and

$$(3.3) \qquad u(0,t) = u(1,t) = 0 \qquad t \in (0,T]$$

where $I = (0,1)$. We will assume that f is uniformly Lipschitz continuous with respect to its second variable with Lipschitz constant K, and $f(\cdot,0) \in L_2(I)$. In addition, we assume $u, u_t \in L_2([0,T], H^r(I))$ for some positive r and $u \in C^2(I \times [0,T])$.

Let Δ denote a partition of [0,1] with uniform mesh h. We choose S^h to be the span $\{v_i(x)\}_{i=1}^M$, the piecewise Hermite polynomials of degree 2m-1, $m \geq 1$, which are defined on Δ and contained in $H_0^1(I)$. These basis functions are defined in Section 2.3. For simplicity we assume $\Psi \in S^h$.

The Galerkin approximation U to u is defined as follows:

$$(3.4) \qquad U(x,t) = \sum_{i=1}^{M} \xi_i(t)\, v_i(x) \qquad (x,t) \in \bar{I} \times [0,T],$$

$$(3.5) \qquad \langle U_t, v\rangle = \langle U_x, v_x\rangle + \langle f(x,U), v\rangle \qquad t > 0,\ v \in S^h$$

and

$$(3.6) \qquad U(x,0) = \Psi(x)$$

For $t \in [0,T]$ we denote the $H_0^1(I)$ projection of $u(x,t)$ onto S^h by $\tilde{u}(x,t)$, that is $\tilde{u}(x,t)$ is defined by

$$(3.7) \qquad \int_I (u(x,t)-\tilde{u}(x,t))_x v_x dx = 0 \qquad v \in S^h$$

We now determine an a priori estimate of $\|U-\tilde{u}\|_{L_2(I)}(\tau)$

for $\tau \in (0,T]$.

Theorem 3.1: Let u be the solution to (3.1)-(3.3). Assume that the above conditions imposed on u and f(x,u) hold. Let U and $\tilde{u}$ be defined by (3.4)-(3.6) and (3.7) respectively. Then for $\tau \in (0,T]$

$$\begin{aligned} &\|U-\tilde{u}\|^2_{L_2(I)}(\tau) + 2\|U-\tilde{u}\|^2_{H^1_0(I) \times L_2(0,\tau)} \\ (3.8) \qquad &\le e^{L\tau}(\|(u-\tilde{u})_t\|^2_{L_2(I)\times L_2(0,\tau)} + K\|u-\tilde{u}\|^2_{L_2(I)\times L_2(0,\tau)}) \end{aligned}$$

where $L = 3K + 1$.

Proof: Since u satisfies

$$\langle u_t, v\rangle = -\langle u_x, v_x\rangle + \langle f(x,u), v\rangle \qquad v \in S^h$$

we have by (3.7)

$$(3.9) \quad \langle \tilde{u}_t, v\rangle = -\langle \tilde{u}_x, v_x\rangle + \langle (\tilde{u}-u)_t, v\rangle + \langle f(x,u), v\rangle \quad v \in S^h$$

Subtracting (3.9) from (3.5) with $(U-\tilde{u})$ as a test function, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|U-\tilde{u}\|^2_{L_2}) = -\|U-\tilde{u}\|^2_{H^1_0} - \langle(\tilde{u}-u)_t, U-\tilde{u}\rangle +$$

$$\langle(f(x,U)-f(x,u)), U-\tilde{u}\rangle \le -\|U-\tilde{u}\|^2_{H^1_0} + \|(\tilde{u}-u)_t\|_{L_2}\|U-\tilde{u}\|_{L_2} + \tag{3.10}$$

$$K\|U-u\|_{L_2}\|U-\tilde{u}\|_{L_2}$$

Applying the triangle inequality and the inequality $ab \le (a^2 + b^2)/2$ to the right hand side of (3.10) we see that

$$\frac{d}{dt}(\|U-\tilde{u}\|^2_{L_2}) + 2\|U-\tilde{u}\|^2_{H^1_0} \le \|(u-\tilde{u})_t\|^2_{L_2} + K\|u-\tilde{u}\|^2_{L_2} + \tag{3.11}$$

$$L\|U-\tilde{u}\|^2_{L_2}$$

where $L = 3K + 1$. Integrating (3.11) with respect to t and using Gronwall's lemma [3], we obtain (3.8).

From Theorem 3.1 we see that if one determines estimates for $\|u-\tilde{u}\|_{L_2(I)}(t)$ and $\|(u-\tilde{u})_t\|_{L_2(I)}(t)$ for $t \in [0,T]$ then an a priori estimate of $\|U-u\|_{L_2(I)}(\tau)$ can be found for $\tau \in (0,T]$. We now obtain L_2 estimates of $(u-\tilde{u})$ and $(u-\tilde{u})_t$. Notice that $\tilde{u}_t(x,t)$ is the $H^1_0(I)$ projection of $u_t(x,t)$ onto S^h for $t \in [0,T]$ since

$$\int_I ((u-\tilde{u})_t)_x v_x dx = \frac{d}{dt}\int_I (u-\tilde{u})_x v_x dx = 0 \qquad v \in S^h$$

We remark that for $t \in (0,T]$ $\tilde{u}(x,t)$ is the Galerkin approximation to the linear elliptic boundary value problem

$$w_{xx} - u_t + f(x,u(x,t)) = 0 \qquad x \in (0,1)$$
$$w(0) = w(1) = 0 .$$

Thus for $\tau \in (0,T]$, L_2 error estimates of $(u-\tilde{u})(x,\tau)$ and $(u-\tilde{u})_t(x,\tau)$ can be obtained using known L_2 error estimates for the Galerkin approximation to a linear elliptic problem. This is done in Section 3.2 for an arbitrary S^h space when more general nonlinear parabolic problems are considered, (see Theorem 3.5). In this section however we obtain these error estimates by proving an interpolation result.

Lemma 3.1: Let $w \in H_0^1(I)$. If $\tilde{w}$ is the $H_0^1(I)$ projection of w onto S^h, then $\tilde{w}$ interpolates w and

(3.12) $$\|w-\tilde{w}\|_{L_2} \le 2h\|w-\tilde{w}\|_{H_0^1} .$$

Proof: This lemma holds for $m = 1$ since the H_0^1 projection of w onto S^h is the Hermite interpolate of w. (see [18]). We will assume that $m \geq 2$.

Let $w^* \in H_m(\Delta)$, $m \geq 2$, ($H_m(\Delta)$ is defined in Section 2.3) satisfy

$$\int_I (w-w^*)_x\, v_x dx = 0, \qquad v \in H_m(\Delta)\ .$$

w^* is not unique; however since $x \in H_m(\Delta)$ we have

$$\int_I (w-w^*)_x\, dx = 0$$

Since $(w-w^*) \in H^1(I)$ and is absolutely continuous on $[0,1]$, we conclude that $w^*(1) = w^*(0)$. Let $\bar{w}(x) = w^*(x) - w^*(1)$. Therefore $\bar{w} \in S^h$ and $\bar{w}$ satisfies

$$\int_I (w-\bar{w})_x v_x dx = 0, \qquad v \in H_m(\Delta)$$

Since $S^h \subset H_m(\Delta)$, $\bar{w} \in S^h$, and $\tilde{w}$ is unique, we have $\bar{w} = \tilde{w}$.

We now construct functions $F_j \in H_m(\Delta)$, $0 \leq j \leq N$, such that

$$\int_{jh}^{(j+1)h} (w-\tilde{w})(F_j)_{xx} dx = - \int_I (w-\tilde{w})_x (F_j)_x dx = 0$$

where $(F_j)_{xx}$ is of one sign in $(jh,(j+1)h)$. This would imply $\tilde{w}$ interpolates w.

For $0 \le j \le N$ define

$$F_j(x) = \begin{cases} 0 & 0 \le x \le jh \\ (x-jh)^m\{\gamma_{0,j}+\gamma_{1,j}(x-(j+1)h) & \\ \quad +\ldots+ \gamma_{m-3,j}\ (x-(j+1)h)^{m-3} & jh \le x \le (j+1)h \\ \quad +(x-(j+1)h)^{m-2}\} & \\ \alpha_j x + \beta_j & (j+1)h \le x \le 1 \end{cases}$$

The constants α_j, β_j and γ_{ij}, $0 \le i \le m-3$, are chosen so that $F_j \in C^{m-1}(I)$ and $F_j \in H_m(\Delta)$, $m \ge 2$. It is easy to verify that

$$(F_j)_{xx} = \begin{cases} c_m(x-jh)^{m-2}(x-(j+1)h)^{m-2} & jh \le x \le (j+1)h \\ 0 & x \in [0,1]-[jh,\ (j+1)h] \end{cases}$$

where

$$c_m = m(m-1) + 2m(m-2) + (m-2)(m-3)$$

Thus there exists $\{\xi_j\}_{j=0}^{N+2}$, with $\xi_0 = 0$, $\xi_{N+2} = 1$, and $\xi_j \in ((j-1)h,\ jh)$ $1 \le j \le N+1$ such that

$$(w-\tilde{w})\ (\xi_j) = 0.$$

Note that

$$|\xi_i - \xi_{i+1}| < 2h \qquad\qquad 0 \le i \le N+1$$

Now

$$\int_I (w-\tilde{w})^2 dx = \sum_{i=0}^{N+1} \int_{\xi_i}^{\xi_{i+1}} (w-\tilde{w})^2 dx$$

$$= \sum_{i=0}^{N+1} \int_{\xi_i}^{\xi_{i+1}} \left[\int_{\xi_i}^{x} (w-\tilde{w})_\eta(\eta)\,d\eta\right] (w-\tilde{w})(x)\,dx$$

$$\le \sum_{i=0}^{N+1} \int_{\xi_i}^{\xi_{i+1}} \|(w-\tilde{w})_\eta\|_{L_2(\xi_i,x)} \sqrt{x-\xi_i}\ |(w-\tilde{w})(x)|\,dx$$

$$\le 2h \sum_{i=0}^{N+1} \|(w-\tilde{w})_\eta\|_{L_2(\xi_i,\xi_{i+1})} \|w-\tilde{w}\|_{L_2(\xi_i,\xi_{i+1})}$$

$$\le 2h\ \|w-\tilde{w}\|_{H_0^1}\ \|w-\tilde{w}\|_{L_2}$$

The above inequalities follow from Hölder's inequality. The proof of the lemma is completed.

Applying Lemma 3.1 to u and u_t and recalling that S^h is a $S^{\hat{h},0}_{m,2m}(I)$ space we have

$$(3.13)\quad \|u-\tilde{u}\|_{L_2}(t) \le 2Qh^s \|u\|_{H^s}(t) \qquad t \in [0,T]$$

and

$$(3.14)\quad \|(u-\tilde{u})_t\|_{L_2}(t) \le 2Qh^s \|u_t\|_{H^s}(t) \qquad t \in [0,T]$$

where s = min (r,2m). The constant Q depends on m. (see [21]). Using the above estimates we can now obtain a priori L_2 estimates for $(U-u)(x,\tau)$, $\tau \in (0,T]$.

Theorem 3.2: Assume the hypotheses of Theorem 3.1. Then for $\tau \in (0,T]$

$$\|U-u\|^2_{L_2}(\tau) \le C_m h^{2s}[e^{L\tau}(K\|u\|^2_{H^s(I)\times L_2(0,\tau)} + \|u_t\|^2_{H^s(I)\times L_2(0,\tau)})$$
$$+ \|u\|_{H^s}(\tau)]$$

where C_m is a constant independent of u, u_t, and h, L = 3K + 1, and s = min (r,2m).

Proof: From Theorem 3.1 and (3.13) and (3.14) we have for $\tau \in (0,T]$

$$(3.15) \qquad \|U-\tilde{u}\|^2_{L_2}(\tau) \le 2Qe^{L\tau}h^{2s}[K\|u\|^2_{H^s(I)\times L_2(0,\tau)} + \|u_t\|^2_{H^s(I)\times L_2(0,\tau)}]$$

We note that

$$(3.16) \qquad \|U-u\|^2_{L_2}(\tau) \le 2[\|U-\tilde{u}\|^2_{L_2}(\tau) + \|\tilde{u}-u\|^2_{L_2}(\tau)]$$

The required result now follows by substituting (3.13) and (3.15) into the right hand side of (3.16).

Using Theorem 3.1 we can also obtain a priori L_∞ and L_2 estimates of $D^j(U-u)$, $0 \le j \le m$. We now prove

Theorem 3.3: Assume the same hypotheses as in Theorem 3.2 with $r \ge m$. Then for $\tau \in (0,T]$

$$(3.17) \qquad \|D^j(U-u)\|_{L_2}(\tau) \le C_1 h^{s-j}, \qquad 0 \le j \le m$$

If for $\tau \in (0,T]$, $(D^s u)(x,\tau) \in L_\infty(I)$, then

$$(3.18)\qquad \|D^j(U-u)\|_{L_\infty}(\tau) \le C_2 h^{s-j-\frac{1}{2}}, \qquad 0 \le j \le m$$

where $s = \min(r, 2m)$. Here C_1 and C_2 are positive constants which depend on m, K, Q, τ, $\|u\|_{H^s(I)}(\tau)$, $\|u_t\|_{H^s(I)\times L_2(0,\tau)}$. In addition, C_2 also depends on $\|D^s u\|_{L_\infty(I)}(\tau)$.

Proof: From [6] we have

$$(3.19)\qquad \|D^j(u-u_m)\|_{L_q(I)}(\tau) \le c_m^* h^{s-j} \|D^s u\|_{L_q(I)}(\tau),$$

$$0 \le j \le m,\ q = 2,\ \infty$$

where $u_m(x,\tau)$ is the piecewise Hermite interpolate of $u(x,\tau)$ on S^h and c_m^* is a constant independent of u and h. By the the triangle inequality we see that

(3.20) $$\|U-u_m\|_{L_2}(\tau) \le \|U-u\|_{L_2}(\tau) + \|u-u_m\|_{L_2}(\tau) \le \mathcal{K}_m h^s$$

where $\mathcal{K}_m$ is a constant depending on K, τ, $\|u\|_{H^s(I)}(\tau)$, $\|u_t\|_{H^s(I)\times L_2(0,\tau)}$, and $\|u\|_{H^s(I)\times L_2(0,\tau)}$.

Now $(U-u_m) \in S^h$ and has the expansion

$$(U-u_m)(x,\tau) = \sum_{i=1}^{M} \gamma_i(\tau)\ v_i(x) \qquad \tau \in (0,T]$$

We note that for $\tau \in (0,T]$

(3.21) $$\|D^j(U-u_m)\|^2_{L_2}(\tau) = (G_j\gamma(\tau),\gamma(\tau)), \quad 0 \le j \le m$$

where G_j is the Gramian matrix corresponding to the functions $\{D^j v_i\}_{i=1}^{M}$. It can be easily verified that G_j, $0 \le j \le m$, is a positive definite matrix with eigenvalues bounded above by $K_m h^{-2j+1}$ where K_m is a constant independent of h. In [15] Gardner proved that for $m = 2$ the eigenvalues of G_0 are bounded below by $c_2 h$ where c_2 is a positive constant independent of h. The techniques developed in [15] have an obvious extension to arbitrary

$m \geq 1$, and it can be shown that for $m \geq 1$ the eigenvalues of G_0 are bounded below by $c_m h$ where c_m is a positive constant independent of h. Thus from (3.20) and (3.21) we obtain

$$(3.22)\quad \|\gamma(\tau)\|_2^2 = \sum_{i=1}^{M} \gamma_i^2(\tau) \leq (\chi_m^2/c_m)h^{2s-1} \qquad \tau \in (0,T]$$

and for $0 \leq j \leq m$

$$\begin{aligned} \|D^j(U-u_m)\|_{L_2}^2(\tau) &= (G_j\gamma(\tau),\ \gamma(\tau)) \\ &\leq K_m\ h^{1-2j}\|\gamma(\tau)\|_2^2 \\ (3.23)\qquad &\leq (K_m\chi_m^2/c_m)h^{2(s-j)}, \qquad \tau \in (0,T] \end{aligned}$$

Inequality (3.17) now follows form (3.19) with $q = 2$ and (3.23).

Similarly for $0 \leq j \leq m$ and $\tau \in (0,T]$

$$\begin{aligned} \|D^j(U-u_m)\|_{L_\infty(I)}(\tau) &\leq \sup_{x\in I} \Big| \sum_{i=1}^{M} \gamma_i(\tau)\ D^j v_i(x)\Big| \\ &\leq \|\gamma(\tau)\|_2 \sup_{x\in I} \Big(\sum_{i=1}^{M} (D^j v_i(x))^2\Big)^{1/2} \end{aligned}$$

Using the definition of the v_i we note that $|D^j v_i(x)| \leq C_m h^{-j}$ $0 \leq j \leq m$, and that for $x \in I$ no more than 2m of $\{v_i(x)\}_{i=1}^{M}$ are not identically zero. Thus for $0 \leq j \leq m$

and $\tau \in (0,T]$

(3.24)
$$\|D^j(U-u_m)\|_{L_\infty(I)}(\tau) \le \sqrt{2m}\, C_m h^{-j} \|\gamma(\tau)\|_2$$
$$\le (C_m \chi_m \sqrt{2m/c_m})\, h^{s-j-\frac{1}{2}}$$

Inequality (3.18) now follows from (3.19) with $q = \infty$ and (3.24).

3.2 Some general estimates for nonlinear parabolic problems with homogeneous Dirichlet boundary conditions

In this section we shall derive L_2 error estimates for the Galerkin approximations to solutions of the boundary value problems:

(3.30)
$$u_t = \nabla\cdot a(x,u)\nabla u + \sum_{i=1}^{n} b_i(x,u)u_{x_i} + f(x,u)$$
$$(x,t) \in \Omega \times (0,T]$$

and

(3.31)
$$u_t = \sum_{i,j=1}^{n} (a_{ij}(x)p(x,u)u_{x_j})_{x_i} + \sum_{i=1}^{n} b_i(x,u)u_{x_i} + f(x,u)$$
$$(x,t) \in \Omega \times (0,T]$$

The boundary and initial conditions are respectively

(3.32) $u(x,t) = 0 \qquad (x,t) \in \partial\Omega \times (0,T]$

and

(3.33) $u(x,0) = \Psi(x) \qquad x \in \Omega$

We first consider the parabolic problem defined by (3.30), (3.32), and (3.33). We make several assumptions which will be referred to as Condition A.

<u>Condition A</u>

(1) For $(x,p) \in \Omega \times R$

$0 < \eta \le a(x,p) \le C_0$

$|b_i(x,p)| \le C_0, \; 1 \le i \le n$

(3.34) (2) a, f, and b_i, $1 \le i \le n$, are uniformly Lipschitz continuous with respect to their (n+1)st variable with Lipschitz constant K, and $f(\cdot,0) \in L_2(\Omega)$. Further assume $a_u(x,u(x,t))$ exists for $(x,t) \in \Omega \times [0,T]$.

(3) $\Psi \in H^r(\Omega) \cap H_0^1(\Omega)$ for some positive integer r.

(4) $u \in C^2(\Omega \times [0,T])$ is a unique solution to (3.30), (3.32), and (3.33), and u, $u_t \in L_2([0,T], H^r(\Omega) \cap H_0^1(\Omega))$ (same r as in (3)).

(5) For $1 \leq i \leq n$, $(b_i(x,u))_{x_i}$ exists and is bounded by M^* for $t \in [0,T]$.

For convenience we will define

$$a(p; w,v) = \int_\Omega a(x,p)\nabla w \cdot \nabla v dx$$

Let S^h be a $S_{k,m}^{h,0}(\Omega)$ space. The Galerkin approximation $U(\cdot,t) \in S^h$ to the solution u of (3.30), (3.32), and (3.33) is defined by

$$(3.35) \quad \langle U_t,v\rangle = -a(U; U,v) + \langle f(x,U), v\rangle + \sum_{i=1}^{n} \langle b_i(x,U)U_{x_i},v\rangle, \qquad t > 0,\ v \in S^h$$

and

$$(3.36) \qquad U(x,0) = \Psi_0(x)$$

where $\Psi_0 \in S^h$ and $\|\Psi-\Psi_0\|_{L_2} \leq C^*h^s\|\Psi\|_{H^s}$, $s = \min(r,m)$.

For example, one could define Ψ_0 as the L_2 projection of Ψ onto S^h.

For $t \in [0,T]$, we define $\tilde{u}(x,t) \in S^h$

by

$$(3.37) \qquad a(u(x,t);\ (u-\tilde{u})(x,t),\ v) = 0 \qquad v \in S^h$$

Let S^h be the span $\{v_i\}_{i=1}^{M}$ where $v_1, v_2, \ldots, v_M$ are linearly independent, and let

$$\tilde{u}(x,t) = \sum_{i=1}^{M} \gamma_i(t) v_i(x) \qquad (x,t) \in \bar{\Omega} \times [0,T]$$

Then (3.37) reduces to a system of linear algebraic equations

$$B(t)\ \gamma(t) = q(t) \qquad t \in [0,T]$$

where

$$B(t) = \langle b_{k\ell}(t) \rangle = \langle \int_\Omega a(x,u(x,t)) \nabla v_k \cdot \nabla v_\ell dx \rangle \qquad t \in [0,T]$$

and

$$q_\ell(t) = \langle \int_\Omega a(x,u(x,t))\ \nabla u \cdot \nabla v_\ell dx \rangle \qquad t \in [0,T]$$

The matrix $B(t)$, $t \in [0,T]$, is positive definite since $\{v_i\}_{i=1}^{M}$ is linearly independent and $a(x,u(x,t)) \geq \eta > 0$.

Notice that $\tilde{u}$ has as many t derivatives as $a(x,u(x,t))$ and u_{x_j}, $1 \leq j \leq n$, have since

$$\gamma(t) = B^{-1}(t)\ q(t)$$

and

$$B^{-1}(t) = (\det B(t))^{-1}\ (\operatorname{adj} B)$$

where $\operatorname{adj} B = (B_{\ell k}(t))$ and $B_{k\ell}(t)$ is the cofactor of $b_{k\ell}(t)$. Also, we note that for $\tau \in (0,T]$ $\tilde{u}(x,\tau)$ is the Galerkin approximation in S^h to the weak solution of the linear elliptic boundary value problem

$$\nabla \cdot a(x,u(x,\tau))\nabla\varphi = g(x,\tau) \qquad x \in \Omega$$

$$\varphi = 0 \qquad x \in \partial\Omega$$

where

$$g(x,\tau) = u_t(x,\tau) - f(x,u(x,\tau)) - \sum_{i=1}^{n} b_i(x,u(x,\tau))u_{x_i}(x,\tau)$$

We now obtain an L_2 estimate of $(U-\tilde{u})(x,\tau)$ for $\tau \in (0,T]$.

Theorem 3.4: Let u be the solution to (3.30), (3.32), and (3.33). Assume Condition A, (3.34). Let U and $\tilde{u}$ be defined by (3.35)-(3.36) and (3.37) respectively.

Then for $\tau \in (0,T]$

$$\|U-\tilde{u}\|^2_{L_2(\Omega)}(\tau) + \eta\|U-\tilde{u}\|^2_{H_0^1(\Omega)\times L_2(0,\tau)}$$

$$\leq C_1^*\|u-\tilde{u}\|^2_{L_2(\Omega)\times L_2(0,\tau)} + C_2^*\|U-\tilde{u}\|^2_{L_2(\Omega)}(0)$$

$$+ C_3^*\|(u-\tilde{u})_t\|^2_{L_2(\Omega)\times L_2(0,\tau)}$$

where C_1^*, C_2^*, and C_3^* are positive constants which depend on τ, n, K, η, M^*, C_0, and $\|\nabla\tilde{u}\|_{L_\infty(\Omega)\times L_\infty[0,T]}$.

(In Section 3.4 we will give examples in which $\|\nabla\tilde{u}\|_{L_\infty(\Omega)\times L_\infty[0,T]}$ is bounded independently of h.)

Proof: From (3.30) and (3.37) we have for $\tau \in (0,T]$

$$\langle\tilde{u}_t,v\rangle = -a(u;\tilde{u},v) + \langle(\tilde{u}-u)_t,v\rangle \tag{3.38}$$

$$+ \sum_{i=1}^{n}\langle b_i(x,u)u_{x_i},v\rangle + \langle f(x,u),v\rangle, \quad v \in S^h$$

Subtracting (3.38) from (3.35) with $U-\tilde{u}$ as a test function, we obtain

$$\langle(U-\tilde{u})_t,U-\tilde{u}\rangle = -a(U;U,U-\tilde{u}) + \sum_{i=1}^{n}\langle b_i(x,U)U_{x_i},U-\tilde{u}\rangle$$

$$+ \langle f(x,U),U-\tilde{u}\rangle + a(u;\tilde{u},U-\tilde{u}) + \tag{3.39}$$

$$+ \langle (u-\tilde{u})_t, U-\tilde{u}\rangle - \sum_{i=1}^{n} \langle b_i(x,u)u_{x_i}, U-\tilde{u}\rangle$$

$$- \langle f(x,u), U-\tilde{u}\rangle$$

On rearranging terms, the right hand side of (3.39) becomes

$$-a(U; U-\tilde{u}, U-\tilde{u}) + \langle (a(x,u)-a(x,U)), \nabla\tilde{u}\cdot\nabla(U-\tilde{u})\rangle$$

$$+ \sum_{i=1}^{n} [\langle b_i(x,U)(U-\tilde{u})_{x_i}, U-\tilde{u}\rangle + \langle (b_i(x,U)-b_i(x,u))\tilde{u}_{x_i}, U-\tilde{u}\rangle$$

(3.40)

$$+ \langle b_i(x,u)(\tilde{u}-u)_{x_i}, U-\tilde{u}\rangle] + \langle (u-\tilde{u})_t, U-\tilde{u}\rangle$$

$$+ \langle (f(x,U)-f(x,u)), U-\tilde{u}\rangle$$

Now

(3.41) $$\sum_{i=1}^{n} \langle b_i(x,u)(\tilde{u}-u)_{x_i}, U-\tilde{u}\rangle = \sum_{i=1}^{n} \langle (b_i(x,u)(U-\tilde{u}))_{x_i}, u-\tilde{u}\rangle$$

Thus from (3.39), (3.40), and (3.41) and assumptions (1), (2), and (5) made in Condition A, (3.34),

$$(\tfrac{1}{2}) \frac{d}{dt}(\|U-\tilde{u}\|^2_{L_2}) \le -\eta\|U-\tilde{u}\|^2_{H^1_0} + \int_\Omega \sum_{i=1}^{n} K|u-U\|\tilde{u}_{x_i}\|(U-\tilde{u})_{x_i}|dx$$

(3.42) $$+C_0\langle \sum_{i=1}^{n}|(U-\tilde{u})_{x_i}|, |U-\tilde{u}|\rangle + K \sum_{i=1}^{n}\langle|\tilde{u}_{x_i}||u-U|, |U-\tilde{u}|\rangle +$$

$$+nM^{*}\ \langle |U-\tilde{u}|,\ |u-\tilde{u}|\rangle + C_0\langle \sum_{i=1}^{n} |(U-\tilde{u})_{x_i}|,\ |u-\tilde{u}|\rangle$$

$$+ K\ \langle |u-U|,\ |U-\tilde{u}|\rangle + \langle |(u-\tilde{u})_t|,\ |U-\tilde{u}|\rangle$$

Applying the Hölder inequality to the integrals on the right hand side of (3.42), we obtain

$$\begin{aligned}(\tfrac{1}{2})\ \frac{d}{dt}(\|U-\tilde{u}\|^2_{L_2}) \le\ & -\eta\|U-\tilde{u}\|^2_{H_0^1} + K\sqrt{n}\ \|\nabla\tilde{u}\|_\infty \|U-u\|_{L_2} \|U-\tilde{u}\|_{H_0^1} \\ & + C_0\sqrt{n}\ \|U-\tilde{u}\|_{H_0^1} \|U-\tilde{u}\|_{L_2} + nK\|\nabla\tilde{u}\|_\infty \|u-U\|_{L_2} \|U-\tilde{u}\|_{L_2} \\ & + nM^{*}\ \|U-\tilde{u}\|_{L_2} \|u-\tilde{u}\|_{L_2} + C_0\sqrt{n}\ \|U-\tilde{u}\|_{H_0^1} \|u-\tilde{u}\|_{L_2} \\ & + K\ \|u-U\|_{L_2} \|U-\tilde{u}\|_{L_2} + \|(u-\tilde{u})_t\|_{L_2} \|U-\tilde{u}\|_{L_2}\end{aligned} \tag{3.43}$$

If we now use the triangle inequality and the inequality $ab \le (\tfrac{1}{2})[\epsilon a^2 + (1/\epsilon)b^2]$($\epsilon = \eta/4$ when $\|U-\tilde{u}\|_{H_0^1}$ is a factor; otherwise $\epsilon = 1$), we see that

$$\begin{aligned}(\tfrac{1}{2})\ \frac{d}{dt}(\|U-\tilde{u}\|^2_{L_2}) \le\ & -\eta/2\,\|U-\tilde{u}\|^2_{H_0^1} \\ & + (2K^2\|\nabla\tilde{u}\|^2_\infty n\eta^{-1})[\,\|u-\tilde{u}\|^2_{L_2} + \|U-\tilde{u}\|^2_{L_2}) \\ & + 2C_0^2 n\eta^{-1}\|U-\tilde{u}\|^2_{L_2} + nK\|\nabla\tilde{u}\|_\infty\|U-\tilde{u}\|^2_{L_2}\ +\end{aligned} \tag{3.44}$$

$$+(nK\|\nabla\tilde{u}\|_\infty/2)\ [\|u-\tilde{u}\|^2_{L_2} + \|U-\tilde{u}\|^2_{L_2}]$$

$$+ (nM^*/2)[\|U-\tilde{u}\|^2_{L_2} + \|u-\tilde{u}\|^2_{L_2}]$$

$$+ 2C_0^2 n\eta^{-1}\|u-\tilde{u}\|^2_{L_2} + 3K/2\|U-\tilde{u}\|^2_{L_2}$$

$$+ K/2\|u-\tilde{u}\|^2_{L_2} + (\tfrac{1}{2})[\|U-\tilde{u}\|^2_{L_2} + \|(u-\tilde{u})_t\|^2_{L_2}]$$

Thus

$$\begin{aligned} &\frac{d}{dt}(\|U-\tilde{u}\|^2_{L_2}) + \eta\|U-\tilde{u}\|^2_{H_0^1} \\ (3.45)\qquad & \\ &\le C_1\|U-\tilde{u}\|^2_{L_2} + C_2\|u-\tilde{u}\|^2_{L_2} + \|(u-\tilde{u})_t\|^2_{L_2} \end{aligned}$$

Here

$$\begin{aligned} C_1 &= nK\|\nabla\tilde{u}\|_\infty (3 + 4K\eta^{-1}\|\nabla\tilde{u}\|_\infty) \\ &\quad + nM^* + 3K + 1 + 4C_0^2 n\eta^{-1} \end{aligned}$$

and

$$\begin{aligned} C_2 &= nK\|\nabla\tilde{u}\|_\infty (4K\eta^{-1}\|\nabla\tilde{u}\|_\infty + 1) \\ &\quad + nM^* + K + 4C_0^2 n\eta^{-1}. \end{aligned}$$

Integrating (3.45) with respect to t and using Gronwall's lemma [3] we obtain the desired estimate.

Recall that for $t \in (0,T]$, $\tilde{u}$ is the Galerkin approximation to a solution of a linear elliptic partial differential equation. We now discuss the error of the approximate solutions of boundary value problems for linear elliptic partial differential equations by Galerkin's method.

Let G be a sufficiently smooth bounded open subset of R^n with boundary ∂G. Let $V = H_0^1(G)$ or $V = H^1(G)$. We introduce the bilinear form

$$(3.46) \qquad a(w,v) = \sum_{|p|,|q|\leq 1} \int_G c_{pq}(x)\, D^p w(x) D^q v(x) \qquad w,\ v \in V$$

where the coefficients c_{pq} are real and belong to $L_\infty(G)$. We will assume that

$$|a(w,v)| \leq C_0 \|w\|_{\bar{V}} \|v\|_{\bar{V}} \qquad w,v \in V$$

and

$$a(w,w) \geq \eta \|w\|_V^2 \qquad w \in V$$

We define $a^*(\ ,\ )$ by

$$a^*(w,v) = a(v,w) \qquad w,\ v \in V$$

and shall assume that $a^*(\ ,\ )$ is 0-regular on $\bar{V}$ where k-regularity on V is defined as follows.

Definition 3.1: The form $a(\ ,\ )$ is k-regular on V if given $f \in H^s(G)$, $0 \le s \le k$, any solution $w \in V$ of $a(w,v) = \langle f,v \rangle$ for every $v \in V$ satisfies $w \in H^{s+2}(G)$.

We now prove Theorem 3.5. The proof given here is due Nitsche [17]. In [8], Dendy gives an alternate proof based upon the work of Aubin [2].

Theorem 3.5: Let S^h denote a $S^{h,0}_{k,m}(G)$ space if $V = H^1_0(G)$ or an $S^h_{k,m}(G)$ space if $V = H^1(G)$. Assume $a^*(\ ,\ )$ is 0-regular on V. Then if $w \in H^p(G) \cap V$, p a positive integer, and $w^h \in S^h$ such that

$$(3.47) \qquad a(w-w^h,\ v) = 0 \qquad\qquad v \in S^h$$

we have

$$\begin{aligned} \|w-w^h\|_V &\le C_0\eta^{-1} \inf_{\hat{w}\in V} \|w-\hat{w}\|_V \\ &\le C_0\eta^{-1}Qh^{s-1}\|w\|_{H^s(G)} \end{aligned}$$

and

$$\|w-w^h\|_{L_2(G)} \le C'C_0^2Q^2\eta^{-1}h^s\|w\|_{H^s(G)}$$

where C' is some positve constant independent of w and h and s = min (p,m).

Proof: Let $\epsilon = w-w^h$. We deduce from (3.47) that

$$a(\epsilon,\epsilon) = a(\epsilon, w-\hat{w}) \qquad \hat{w} \in S^h,$$

This implies that

$$\eta\|\epsilon\|_V \le C_0 \inf_{\hat{w}\in S^h} \|w-\hat{w}\|_V$$

Since S^h is a $S^{h,0}_{k,m}(G)$ or a $S^h_{k,m}(G)$ space

$$\|\epsilon\|_V \le \eta^{-1} C_0 Q\, h^{s-1} \|w\|_{H^s(G)}$$

where s = min(p,m).

Let $\xi\epsilon V$ such that

$$a^*(\xi,v) = \langle\epsilon,v\rangle \qquad v \in V$$

Since a* is 0-regular on V, $\xi \in H^2(G)$ and

(3.48) $$\|\xi\|_{H^2(G)} \le C'\|\epsilon\|_{L_2(G)}$$

where C' is some positive constant. (See [8] for example).

We note that

$$\langle \epsilon, \epsilon \rangle = a^*(\xi, \epsilon) = a(\epsilon, \xi)$$

Thus from (3.47) we have

$$\langle \epsilon, \epsilon \rangle = a(\epsilon, \ \xi - \hat{w}) \qquad \hat{w} \in S^h$$

Therefore

$$(3.49) \qquad \|\epsilon\|^2_{L_2(G)} \leq C_0 \|\epsilon\|_V \inf_{\hat{w} \in S^h} \|\xi - \hat{w}\|_V$$

Since $\xi \in H^2(G)$,

$$(3.50) \qquad \inf_{\hat{w} \in S^h} \|\xi - \hat{w}\|_V \leq Qh \|\xi\|_{H^2(G)}$$

From (3.48), (3.49), and (3.50), we conclude that

$$\|w - w^h\|_{L_2(G)} \leq C' C_0 Qh \|w - w^h\|_V$$

Proof of theorem is now completed.

In Section 3.4, we will show that A(,) and P(,) where

$$P(w,v) = \int_B \nabla w \cdot \nabla v dx \qquad w,v \in H_0^1(\Omega)$$

and

$$A(w,v) = \int_B (\nabla w \cdot \nabla v + wv) dx \qquad w,v \in H^1(\Omega)$$

are 0-regular on $H_0^1(B)$ and $H^1(B)$ respectively where B is a rectangular parallelepiped in R^n.

We now wish to apply Theorem 3.5 to

(3.51) $$a_\tau(w,v) = \int_\Omega a(x,u(x,\tau)) \nabla w \cdot \nabla v dx \quad w, v \in H_0^1(\Omega)$$

to obtain estimates of $\|u-\tilde{u}\|_{L_2}(\tau)$ and $\|(u-\tilde{u})_t\|_{L_2}(\tau)$, $\tau \in [0,T]$. For $\tau \in [0,T]$ we recall that $\tilde{u}(x,\tau) \in S^h$ and $a_\tau(u,v) = a_\tau(\tilde{u},v)$, $v \in S^h$. We assume that $P(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ where

(3.52) $$P(w,v) = \int_\Omega \nabla w \cdot \nabla v dx \qquad w,v \in H_0^1(\Omega)$$

and that for $t \in [0,T]$, $a(\cdot, u(\cdot,t)) \in C^1(\Omega)$ and $(a(\cdot,u(\cdot,\cdot)))_{x_i} \in L_\infty([0,T], L_\infty(\Omega))$, $1 \le i \le n$.

Let $t \in [0,T]$ and $g \in L_2(\Omega)$. From the theory of linear elliptic equations (see Agmon [1]) there exists a unique $w \in H_0^1(\Omega)$ such that

$$(3.52') \qquad a_t^*(w,v) = a_t(w,v) = \langle g,v\rangle \qquad v \in H_0^1(\Omega).$$

Substituting $v/a(x,u(x,t))$, $v \in H_0^1(\Omega)$, for v in (3.52'), we see that

$$P(w,v) = \langle g^*,v\rangle \qquad v \in H_0^1(\Omega)$$

where

$$g^*(x) = g(x)/a(x,u(x,t))-a(x,u(x,t))\nabla w(x)\cdot\nabla(1/a(x,u(x,t)))$$

Since $P(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ and $g^*\in L_2(\Omega)$, $w \in H^2(\Omega)$ and

$$\|w\|_{H^2} \le C_p\|g^*\|_{L_2}$$

where C_p is some constant which depends on $P(\ ,\ )$ and is independent of t. We conclude that $a_t(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ for $t \in [0,T]$. It is easily verified that

$$\|g^*\|_{L_2} \le \eta^{-1}\|g\|_{L_2} + \eta^{-1}\sqrt{n}\ \|\nabla a(x,u)\|_\infty\|w\|_{H_0^1}$$

$$\le \eta^{-1}[1 + \sqrt{C_\Omega}\ \eta^{-1}\sqrt{n}\ \|\nabla a(x,u)\|_\infty]\ \|g\|_{L_2}$$

where C_Ω is defined by (2.0). Thus

$$\|w\|_{H^2} \le C'\|g\|_{L_2}$$

where C' is a constant independent of t. We can now apply Theorem 3.5 to $a_t(\ ,\ )$, $t \in [0,T]$, to obtain

Lemma 3.2: Assume Condition A, (3.34), and assume $a(\cdot,\ u(\cdot,t)) \in C^1(\Omega)$ for $t \in [0,T]$ and $(a(\cdot,u(\cdot,\cdot))_{x_i} \in L_\infty([0,T],\ L_\infty(\Omega))$, $1 \le i \le n$. Further assume that $P(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ where P is defined by (3.52). Let $\tilde{u}$ be defined by (3.37). Then for $t \in [0,T]$,

$$\|u-\tilde{u}\|_{H_0^1}(t) \le C_0\eta^{-1}Qh^{s-1}\|u\|_{H^s}(t)$$

and

$$\|u-\tilde{u}\|_{L_2}(t) \le C_0^2\ C'\eta^{-1}Q^2h^s\|u\|_{H^s}(t)$$

where s = min (r,m).

We now wish to obtain an L_2 estimate for $(u-\tilde{u})_t$ using the above analysis. We first obtain an H_0^1 estimate for $(u-\tilde{u})_t$.

Lemma 3.3: Assume that hypotheses of Lemma 3.2 and assume $u_t \in L_\infty([0,T], L_\infty(\Omega))$. Then for $\tau \in [0,T]$,

$$\|(u-\tilde{u})_t\|_{H_0^1}(\tau) \le C_0\eta^{-1}Qh^{s-1}[\eta^{-1}K\|u_t\|_\infty\|u\|_{H^s} + \|u_t\|_{H^s}]$$

where $s = \min(r,m)$.

Proof: Differentiating (3.37) with respect to t we see that

$$(3.53) \qquad a_t((u-\tilde{u})_t,v) + \int_\Omega a(x,u)_t\nabla(u-\tilde{u})\cdot\nabla v dx = 0 \qquad v \in S^h$$

Define $u^* \in S^h$ by

$$(3.54) \qquad a_t(u_t-u^*,v) = 0 \qquad v \in S^h$$

Since $a_t(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ from Theorem 3.5 we have for $\tau \in [0,T]$

$$(3.55) \qquad \|u_t-u^*\|_{H_0^1}(\tau) \le C_0\eta^{-1}Qh^{s-1}\|u_t\|_{H^s}(\tau)$$

where $s = \min(r,m)$. From (3.53) and (3.54) with $v = u^*-\tilde{u}_t$ we deduce that

$$(3.56) \quad \|u^*-\tilde{u}_t\|_{H_0^1}(\tau) \le K\eta^{-1}\|u_t\|_\infty \|u-\tilde{u}\|_{H_0^1}$$
$$\le K(\eta^{-1})^2 \|u_t\|_\infty \, C_0 Q h^{s-1} \|u\|_{H^s}(\tau)$$

where $s = \min(r,m)$. The last inequality follows from Lemma 3.2. Proof of the lemma now follows from estimates (3.55) and (3.56) and the triangle inequality.

Using the above H_0^1 estimate on $(u-\tilde{u})_t$ we obtain

Lemma 3.4: Assume the hypotheses of Lemma 3.3. Further assume that $(a(x,u)_t)_{x_i} \in L_\infty([0,T], L_\infty\Omega))$, $1 \le i \le n$. Then for $\tau \in [0,T]$,

$$\|(u-\tilde{u})_t\|_{L_2(\Omega)}(\tau) \le h^s[K_1\|u\|_{H^s}(\tau) + K_2\|u_t\|_{H^s}(\tau)]$$

where $s = \min(r,m)$,

$$K_2 = C_0^2\eta^{-1}C'Q^2$$

and K_1 is a positive constant which depends on K, η, C_0, C', Q, n, $\|u_t\|_\infty$ and $\|\nabla(\frac{a(x,u)_t}{a(x,u)})\|_\infty$.

Proof: From equation (3.53) we easily obtain

$$a_t(q,v) = g^*(t,v) \qquad v \in S^h$$

where

(3.57) $$g^*(t,v) = \int_\Omega a(x,u(x,t))(u-\tilde{u})(x,t)\nabla\left(\frac{a(x,u)_t}{a(x,u)}\right)\cdot\nabla v dx$$

and

$$q(x,t) = (u-\tilde{u})_t(x,t) + \left(\frac{a(x,u)_t}{a(x,u)}\right)(x,t)(u-\tilde{u})(x,t)$$

Let $p(x,t) \in S^h$ which satisfies

(3.58) $$a_t(p,v) = g^*(t,v) \quad v \in S^h,\ t \in [0,T]$$

By Theorem 3.5, we have

(3.59) $$\|q-p\|_{L_2}(t) \le C_0 C' Qh \|q-p\|_{H_0^1}(t) \qquad t \in [0,T]$$

From (2.0), (3.57), and (3.58) with $v = p$ we see that for $t \in [0,T]$,

$$\eta C_\Omega^{-1}\|p\|_{L_2}^2 \le \eta\|p\|_{H_0^1}^2$$

$$(3.60)\qquad \begin{aligned} &\leq \int_\Omega a(x,u(x,t))(u-\tilde{u})\nabla\left(\frac{a(x,u)_t}{a(x,u)}\right)\cdot\nabla p\,dx \\ &\leq \sqrt{n}\ \left\|a(x,u)\nabla\left(\frac{a(x,u)_t}{a(x,u)}\right)\right\|_\infty \|u-\tilde{u}\|_{L_2}(t)\,\|p\|_{H_0^1}(t) \end{aligned}$$

By the triangle inequality we have

$$(3.61)\qquad \begin{aligned} \|(u-\tilde{u})_t\|_{L_2} &\leq \left\|\frac{a(x,u)_t}{a(x,u)}(u-\tilde{u})\right\|_{L_2}(t) + \|q\|_{L_2}(t) \\ &\leq K\eta^{-1}\|u_t\|_\infty\|u-\tilde{u}\|_{L_2}(t) + \|p\|_{L_2}(t) \\ &\quad + \|q-p\|_{L_2}(t) \end{aligned}$$

From (3.59) and (3.60) we see that

$$(3.62)\qquad \begin{aligned} \|p\|_{L_2}(t) + \|q-p\|_{L_2}(t) &\leq (\sqrt{C_\Omega} + C_0C'Qh)\|p\|_{H_0^1}(t) \\ &\quad + C_0C'Qh\,\|q\|_{H_0^1}(t) \\ &\leq (\sqrt{C_\Omega} + C_0C'Qh)C_0\sqrt{n}\ \eta^{-1}\left\|\nabla\left(\frac{a(x,u)_t}{a(x,u)}\right)\right\|_\infty\|u-\tilde{u}\|_{L_2}(t) \\ &\quad + C_0C'Qh\|q\|_{H_0^1}(t) \end{aligned}$$

Now

$$\|q\|_{H_0^1}(t) \le \|(u-\tilde{u})_t\|_{H_0^1}(t) + \left\|\frac{a(x,u)_t}{a(x,u)}(u-\tilde{u})\right\|_{H_0^1}(t)$$

$$\le \|(u-\tilde{u})_t\|_{H_0^1}(t) + \sqrt{2n}\,\left\|\nabla\left(\frac{a(x,u)_t}{a(x,u)}\right)\right\|_\infty \|u-\tilde{u}\|_{L_2}(t)$$

$$+ \sqrt{2}\,K\eta^{-1}\|u_t\|_\infty \|u-\tilde{u}\|_{H_0^1}(t)$$

Therefore from (3.61), (3.62), and (3.63) we see that

$$\|(u-\tilde{u})_t\|_{L_2}(t) \le C_1\|u-\tilde{u}\|_{L_2}(t) + C_0C'Qh[\|(u-\tilde{u})_t\|_{H_0^1}(t)$$

(3.64)

$$+ 2K\eta^{-1}\|u_t\|_\infty\|u-\tilde{u}\|_{H_0^1}(t)]$$

where C_1 is a constant which depends on K, η, $\|u_t\|_\infty$, C_0, C', Q, n, $\|\nabla(\frac{a(x,u)_t}{a(x,u)})\|_\infty$.

The proof of this lemma now follows from Lemmas 3.2 and 3.3 and (3.64).

Since for $t \in [0,T]$, we have bounds for $\|u-\tilde{u}\|_{L_2}(t)$ and $\|(u-\tilde{u})_t\|_{L_2}(t)$ we can now determine for $\tau \in (0,T]$ a priori $L_2(\Omega)$ estimates for $(U-\tilde{u})(x,\tau)$.

Theorem 3.6: Let u be the solution to (3.30), (3.32), and (3.33). Assume Condition A, (3.34), and assume $a(x,u(x,t)) \in C^1(\Omega)$ for $t \in [0,T]$ and $u_t, a(x,u)_{x_i}$, $(a(x,u)_t)_{x_i} \in L_\infty([0,T], L_\infty(\Omega))$ for $1 \le i \le n$. In addition, assume $P(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ where $P(\ ,\ )$ is defined by (3.52). If U, the Galerkin approximation to u, satisfies (3.35) and (3.36) then for $\tau \in (0,T]$,

$$\|U-u\|^2_{L_2(\Omega)}(\tau) \le h^{2s}[K_1^*\|u\|^2_{H^s(\Omega)\times L_2(0,\tau)} + K_3^*\|\Psi\|^2_{H^s}$$
$$+ K_2^*\|u_t\|^2_{H^s(\Omega)\times L_2(0,\tau)} + K_4^*\|u\|^2_{H^s}(\tau)]$$

where $s = \min(r,m)$, $K_4^* = 2C_0^2C'\eta^{-1}Q^2$, and K_1^*, K_2^*, and K_3^* are positive constants which depend on τ, n, η, K, M^*, C_0, C^*, C', Q, and $\|\nabla\tilde{u}\|_\infty$. K_1^* also depends on $\|u_t\|_\infty$ and $\|\nabla(\frac{a(x,u)_t}{a(x,u)})\|_\infty$.

We now consider the boundary value problem (3.31)-(3.33). We make several assumptions which will be referred to as Condition B.

<u>Condition</u> <u>B</u>

(1) $0 < \sqrt{\eta}\|\xi\|_2^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \le \sqrt{C_0}\|\xi\|_2^2 \qquad \xi \ne 0 \in R^n$

where $A = (a_{ij}(x))$ is a symmetric matrix.

Also $|b_i(x,w)| \le C_0$, $1 \le i \le n$, $(x,w) \in \Omega \times R$,

and $0 < \sqrt{\eta} \le |p(x,w)| \le \sqrt{C_0}$

$(x,w) \in \Omega \times R$

(3.64')

(2) f and b_i, $1 \le i \le n$ satisfy (2) in Condition A, (3.34). p is Lipschitz continuous with respect to its (n+1)st variable with Lipschitz constant K, and $p_u(x,u(x,t))$ exists for $(x,t) \in \Omega \times [0,T]$.

(3) Ψ satisfies (3) in Condition A, (3.34).

(4) $u \in C^2(\Omega \times [0,T])$ is a unique solution to (3.31)-(3.33) and $u, u_t \in L_2([0,T], H^r(\Omega) \cap H_0^1(\Omega))$.

(5) $b_i(x,u)$, $1 \le i \le n$ satisfies (5) in Condition A, (3.34).

For convenience, we define

$$(3.65) \quad a(w;\, q,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)p(x,w)q_{x_j} v_{x_i}\, dx$$

$$q,\ v \in H_0^1(\Omega)$$

The Galerkin approximation $U(\cdot,t) \in S^h$ satisfies for $t \in [0,T]$,

$$(3.66)\qquad \langle U_t,v\rangle = -a(U;U,v) + \langle f(x,U),v\rangle + \langle \sum_{i=1}^{n} b_i(x,U)\, U_{x_i},v\rangle,\quad t > 0,\ v \in S^h$$

and

$$(3.67)\qquad U(x,0) = \Psi_0(x)$$

where $\Psi_0 \in S^h$ and $\|\Psi-\Psi_0\|_{L_2} \le C^* h^s \|\Psi\|_{H^s}$, $s = \min(r,m)$.

The next theorem provides an a priori estimate for $\|U-u\|_{L_2(\Omega)}(\tau)$ for $\tau \in (0,T]$.

Theorem 3.7: Let u be the solution to (3.31)-(3.33). Assume Condition B, (3.64'), and assume $p(x,u(x,t)) \in C^1(\Omega)$ for $t \in [0,T]$ and u_t, $p(x,u)_{x_i}$, $(p(x,u)_t)_{x_i} \in L_\infty([0,T]), L_\infty(\Omega))$ for $1 \le i \le n$. Further assume that $P(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ where

$$P(w,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\, w_{x_j} v_{x_i}\, dx \qquad w,v \in H_0^1(\Omega)$$

If U is the Galerkin approximation to u defined by (3.66) and (3.67), then for $\tau \in (0,T]$

$$\|U-u\|^2_{L_2(\Omega)}(\tau) \le h^{2s}\{K_1^*\|u\|^2_{H^s(\Omega)\times L_2(0,\tau)} + K_2^*[\|\Psi\|^2_{H^s(\Omega)} + \|u_t\|^2_{H^s(\Omega)\times L_2(0,\tau)} + \|u\|^2_{H^s(\Omega)}(\tau)]\}$$

where $s = \min(r,m)$ and K_1^* and K_2^* are positive constants which depend on τ, n, K, η, M^*, C_0, C^*, Q, $\|\nabla p(x,u)\|_\infty$, and $\|\nabla\tilde{u}\|_\infty$. K_1^* also depends on $\|u_t\|_\infty$ and $\|\nabla(\frac{p(x,u)_t}{p(x,u)})\|_\infty$.

Proof: Let $\tilde{u}(x,t) \in S^h$ for $t \in [0,T]$ which satisfies

$$a(u(x,t);\ (u-\tilde{u})(x,t),v) = 0 \qquad v \in S^h. \tag{3.68}$$

We easily see that

$$\begin{aligned}\langle\tilde{u}_t, v\rangle &= \langle(\tilde{u}-u)_t, v\rangle + \langle f(x,u),v\rangle \\ &+ \langle \sum_{i=1}^{n} b_i(x,u)u_{x_i}, v\rangle \\ &- a(u;\ \tilde{u},v) \qquad\qquad v \in S^h,\ t > 0\end{aligned} \tag{3.69}$$

Subtracting (3.69) from (3.66) with $(U-\tilde{u})$ as a test function, we obtain

$$\begin{aligned}\tfrac{1}{2}\frac{d}{dt}(\|U-\tilde{u}\|_{L_2}^2) &= -a(U;\ U-\tilde{u},\ U-\tilde{u}) \\ &+ \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)(p(x,u)-p(x,U))\tilde{u}_{x_j}(U-\tilde{u})_{x_i}dx \\ &+ \langle f(x,U)-f(x,u),\ U-\tilde{u}\rangle \\ &+ \sum_{i=1}^{n} \langle b_i(x,U)U_{x_i}-b_i(x,u)u_{x_i},\ U-\tilde{u}\rangle \\ &+ \langle(u-\tilde{u})_t,\ U-\tilde{u}\rangle\end{aligned} \tag{3.70}$$

We note that the term

$$-a(U;\ U-\tilde{u},\ U-\tilde{u}) \geq -\eta\|U-\tilde{u}\|^2_{H_0^1}$$

and that the term

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)(p(x,u)-p(x,U))\tilde{u}_{x_j}(U-\tilde{u})_{x_i}dx$$

$$\leq \sqrt{C_0}\ \|\nabla\tilde{u}\|_\infty\ n\ K \int_\Omega \sum_{i=1}^{n} |U-u|\,|(U-\tilde{u})_{x_i}|dx$$

$$\leq \sqrt{C_0}\ \|\nabla\tilde{u}\|_\infty\ n^{3/2}K\|U-u\|_{L_2}\|U-\tilde{u}\|_{H_0^1}$$

The remaining terms in (3.70) are bounded as in Theorem 3.4. Following the same analysis as in Theorem 3.4, we obtain for $\tau \in (0,T]$,

$$\|U-\tilde{u}\|^2_{L_2(\Omega)}(\tau) + \eta\|U-\tilde{u}\|^2_{H_0^1(\Omega)\times L_2(0,\tau)}$$

(3.71)
$$\leq C_1^*\|u-\tilde{u}\|^2_{L_2(\Omega)\times L_2(0,\tau)} + C_2^*\|U-\tilde{u}\|^2_{L_2(\Omega)}(0)$$

$$+ C_3^*\ \|(u-\tilde{u})_t\|^2_{L_2(\Omega)\times L_2(0,\tau)}$$

Here C_1^*, C_2^*, and C_3^* are positive constants which depend on τ, n, η, K, M^*, C_0, and $\|\nabla\tilde{u}\|_{L_\infty(\Omega)\times L_\infty[0,T]}$.

Since $P(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$, we deduce that

$$a_t(w,v) = a(u;\ w,v) \qquad w,\ v \in H_0^1(\Omega)$$

is 0-regular on $H_0^1(\Omega)$ for $t \in [0,T]$. By Theorem 3.5

(3.72) $$\|u-\tilde{u}\|_{L_2}(t) \le C_0^2 \eta^{-1} Q^2 C' h^s \|u\|_{H^s}(t)$$

where C' is some constant independent of t for $t \in [0,T]$ (C' does depend on n, η, C_0, and $\|\nabla p(x,u)\|_\infty$.) Upon differentiating (3.68) with respect to t, we see that

$$a(u;\ ((u-\tilde{u})_t + \frac{p(x,u)_t}{p(x,u)}\ (u-\tilde{u})),\ v)$$

$$= \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\ p(x,u)\left(\frac{p(x,u)_t}{p(x,u)}\right)_{x_j} (u-\tilde{u}) v_{x_i} dx$$

$$v \in S^h$$

Following the proofs of Lemmas 3.3 and 3.4, we obtain

(3.73) $$\|(u-\tilde{u})_t\|_{L_2}(t) \le h^s [K_1 \|u\|_{H^s}(t) + K_2 \|u_t\|_{H^s}(t)]$$

where $s = \min(r,m)$,

$$K_2 = C_0^2 \eta^{-1} C' Q^2,$$

and K_1 is a positive constant which depends on K, η, C_0, C', Q, n, $\|u_t\|_\infty$, and $\|\nabla(\frac{p(x,u)_t}{p(x,u)})\|_\infty$.

Proof of the theorem now follows from (3.71), (3.72), and (3.73).

3.3 Some general estimates for nonlinear parabolic problems with Neumann boundary conditions. In this section we consider the boundary value problem

$$(3.80)\qquad u_t = \sum_{i,j=1}^{n} (a_{ij}(x)p(x,u)u_{x_j})_{x_i} + f(x,u)$$

$$(x,t) \in \Omega \times (0,T]$$

with initial and boundary conditions

$$(3.81)\qquad u(x,0) = \Psi(x)$$

and

$$(3.82)\qquad \sum_{i,j=1}^{n} a_{ij}(x)p(x,u)\gamma_j u_{x_i} = 0 \qquad (x,t) \in \partial\Omega\times(0,T]$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is the unit exterior normal to $\partial\Omega$. We make several assumptions which will be referred to as Condition C.

Condition C

(1) a_{ij} and p satisfy assumption (1) in Condition B, (3.64').

(2) f and p satisfy assumption (2) in Condition B, (3.64').

(3.82')

(3) $\Psi \in H^r(\Omega)$ for some positive integer r.

(4) $u \in C^2(\Omega \times [0,T])$ is a unique solution to (3.80)-(3.82), and $u, u_t \in L_2([0,T], H^r(\Omega))$.

Let S^h denote a $S^h_{k,m}(\Omega)$ space. The Galerkin approximation $U(\cdot,t) \in S^h$ to u is defined by

$$(3.83) \quad \langle U_t, v\rangle = -a(U;U,v) + \langle f(x, U), v) \quad t > 0, \ v \in S^h$$

$$(3.84) \quad U(x,0) = \Psi_0(x)$$

where $\Psi_0 \in S^h$ and $\|\Psi-\Psi_0\|_{L_2} \le C^* h^s \|\Psi\|_{H^s}$, $s = \min(r,m)$ and $a(\ ; \ , \ )$ is defined by (3.65).

To obtain the error bounds for this problem we simply modify the techniques developed in previous sections. For $t \in [0,T]$ define $\tilde{u}(\cdot,t) \in S^h$ by

$$(3.85) \quad a(u;u-\tilde{u},v) + \sqrt{\eta} < p(x,u)(u-\tilde{u}),v\rangle = 0 \quad v \in S^h$$

We now obtain L_2 error estimates for $(U-u)(x,\tau)$ for $\tau \in (0,T]$.

Theorem 3.8: Let u be the solution to (3.80)-(3.82). Assume Condition C, (3.82'), and assume $p(\cdot,u(\cdot,t)) \in C^1(\Omega)$ for $t \in [0,T]$ and u_t, $(p(x,u))x_i$, $(p(x,u)_t)_{x_i} \in L_\infty([0,T], L_\infty(\Omega))$. Further assume that $P(\ ,\ )$ is 0-regular on $H^1(\Omega)$ where

$$P(w,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) w_{x_j} v_{x_i} dx + \sqrt{\eta} \int_\Omega wv dx$$

If U is the Galerkin approximation to u defined by (3.83) and and (3.84) then for $\tau \in (0,T]$

$$\|U-u\|^2_{L_2(\Omega)}(\tau) \le h^{2s}\{K_1^* \|u\|^2_{H^s(\Omega)\times L_2(0,\tau)}$$

$$+ K_2^*[\|\Psi\|^2_{H^s(\Omega)} + \|u_t\|^2_{H^s(\Omega)\times L_2(0,\tau)} + \|u\|^2_{H^s(\Omega)}(\tau)]\}$$

where $s = \min(r,m)$ and K_1^* and K_2^* are positive constants which depend on τ, n, K, η, C_0, C^*, Q, $\|\nabla p(x,u)\|_\infty$, and $\|\nabla\tilde{u}\|_\infty$. K_1^* also depends on $\|u_t\|_\infty$ and $\|\nabla(\frac{p(x,u)_t}{p(x,u)})\|_\infty$.

Proof: For $t \in (0,T]$ let $\tilde{u}$ be defined by (3.85). From (3.85) and (3.80), we have

$$\langle\tilde{u}_t,v\rangle = \langle(\tilde{u}-u)_t,v\rangle - a(u;\tilde{u},v\rangle + \sqrt{\eta}\langle p(x,u)(u-\tilde{u}),v\rangle +$$

$$+ \langle f(x,u),v\rangle \quad t \in (0,T],\ v \in S^h$$

Subtracting (3.86) from (3.83) and setting $v = U-\tilde{u}$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|U-\tilde{u}\|^2_{L_2}) = -a(U;U-\tilde{u}, U-\tilde{u})$$

$$+ \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)(p(x,u)-p(x,U))\tilde{u}_{x_j}(U-\tilde{u})_{x_i}$$

$$+ \langle f(x,U) - f(x,u), U-\tilde{u}\rangle$$

$$+ \langle (u-\tilde{u})_t, U-\tilde{u}\rangle$$

$$+ \sqrt{\eta}\ \langle p(x,u)(\tilde{u}-u), U-\tilde{u}\rangle$$

Using the same techniques as in Theorems 3.4 and 3.7, we have for $\tau \in (0,T]$

$$\|U-\tilde{u}\|^2_{L_2(\Omega)}(\tau) + \eta\|U-\tilde{u}\|^2_{H^1(\Omega)\times L_2(0,\tau)}$$

$$\leq C_1^*\|u-\tilde{u}\|^2_{L_2(\Omega)\times L_2(0,\tau)} + C_2^*\|U-\tilde{u}\|^2_{L_2(\Omega)}(0) \tag{3.87}$$

$$+ C_3^*\|(u-\tilde{u})_t\|^2_{L_2(\Omega)\times L_2(0,\tau)}$$

where C_1^*, C_2^*, and C_3^* are positive constants which depend on τ, n, K, C_0, and $\|\nabla\tilde{u}\|_\infty$.

We now determine L_2 estimates of $(u-\tilde{u})(x,t)$ and $(u-\tilde{u})_t(x,t)$ for $t \in [0,T]$. For $t \in [0,T]$ define

$$a_t(w,v) = a(u(x,t);\, w,v) + \sqrt{\eta}\, \langle p(x,u)w,v\rangle,$$
$$w,\ v \in H^1(\Omega)$$

and let $\xi \in H^1(\Omega)$ which satisfies

$$(3.88) \qquad a_t(\xi,v) = \langle g,v\rangle \qquad v \in H^1(\Omega)$$

where $g \in L_2(\Omega)$. Substituting $v/p(x,u(x,t))$ for v, $v \in H^1(\Omega)$, in (3.88) we obtain

$$P(\xi,v) = \langle g^*,v\rangle \qquad v \in H^1(\Omega)$$

where

$$g^*(x) = g(x)/p(x,u(x,t))$$
$$- \sum_{i,j=1}^{n} a_{ij}(x)p(x,u(x,t))\,\xi_{x_j}(1/p(x,u(x,t)))_{x_i}$$

Since $g^* \in L_2(\Omega)$ and $P(\ ,\ )$ is 0-regular on $H^1(\Omega)$, $\xi \in H^2(\Omega)$, $a_t(\ ,\ )$ is 0-regular on $H^1(\Omega)$ for $t \in [0,T]$, and

$$\|\xi\|_{H^2} \le C'\|g\|_{L_2}$$

where C' is a constant independent of t. From Theorem 3.5, we obtain the estimate (3.72). The L_2 estimate (3.73) for $\|(u-\tilde{u})_t\|_{L_2}$ is obtained in the same manner as described in Theorem 3.7 and Lemmas (3.3) and (3.4).

Proof of this theorem then follows from (3.72), (3.73), and (3.87).

3.4. Estimates for some nonlinear parabolic problems on rectangular parallelepipeds. We now consider the nonlinear boundary value problems defined by (3.30), (3.32) and (3.33) and (3.80) with $(a_{ij}(x)) = I$, the identity matrix, (3.81), and (3.82) where the region Ω is a rectangular parallelepiped in R^n. Theorems 3.6 and 3.8 provide estimates for $\|U-u\|_{L_2(\Omega)}(\tau)$, $\tau \in (0,T]$, where u is the solution to (3.30), (3.32), and (3.33) or to (3.80)-(3.82) and U is the corresponding Galerkin approximation to u. Both theorems require a given bilinear form to be 0-regular on V where V is either $H_0^1(\Omega)$ or $H^1(\Omega)$. In addition, the estimates in Theorems (3.6) and (3.8) involve a constant $\|\nabla\tilde{u}\|_\infty$. In this section we prove the above 0-regularity assumptions are valid on rectangular parallelepipeds. If S^h is the span of the tensor products of piecewise Hermite polynomials of degree 2m-1, $m \geq 1$, we show that $\|\nabla\tilde{u}\|_\infty$ is bounded under certain restrictions. L_∞ and L_2 error estimates for Galerkin approximations and derivatives of Galerkin approximations are also derived for this choice of basis.

Let $B = \overset{n}{\underset{i=1}{X}} (a_i, d_i)$, $a_i < d_i$. We first prove that $P(\ ,\ )$ and $A(\ ,\ )$ where

$$P(w,v) = \int_B \nabla w \cdot \nabla v dx \qquad w,\ v \in H_0^1(B)$$

and

$$A(w,v) = \int_B (\nabla w \cdot \nabla v + wv) dx \qquad w,\ v \in H^1(B)$$

are 0-regular on $H_0^1(B)$ and $H^1(B)$ respectively.[1]

Theorem 3.9: Let $B' = \overset{n}{\underset{i=1}{X}} (0,\pi)$. If $w \in H_0^1(B)$ satisfying

$$\int_{B'} \nabla w \cdot \nabla v dx = \int_{B'} gv dx$$

where $g \in L_2(B')$, then $w \in H^2(B')$.

Proof: Let $B'' = \overset{n}{\underset{i=1}{X}} (-\pi,\pi)$. Extend g to be an odd function over B'' and call this extension g. Since $g \in L_2(B'')$ g has a Fourier series expansion

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1 The approach used in proving the next two theorems was suggested to the author by Rod Dunn and B.F. Jones, Jr.

$$g(x) = \left(\frac{1}{2\pi}\right)^{n/2} \sum_{\substack{m=(m_1,m_2,\ldots,m_n) \\ m\neq 0}} G_m e^{im\cdot x}$$

Define

$$\varphi(x) = \left(\frac{1}{2\pi}\right)^{n/2} \sum_{\substack{m \\ m \neq 0}} G_m/\|m\|^2 \; e^{im\cdot x}$$

where $\|m\|^2 = m_1^2 + m_2^2 + \ldots + m_n^2$. By the Riesz-Fischer theorem, $\varphi \in L_2(B'')$. Consider φ and g as distributions over B''. Since the Fourier series of a distribution can be differentiated termwise (see [13]) the mth coefficient of the Fourier series corresponding to the distribution $-\Delta\varphi$ equals G_m. Moreover, since two distributions having the same Fourier series are equal (see [13]), we see that $-\Delta\varphi = g$. Thus $\nabla\varphi \in L_2(B'')$, $\Delta\varphi \in L_2(B'')$, and $\varphi \in H^2(B'')$. If $v \in H^1(B'')$, then v has a Fourier series expansion

$$v(x) = \left(\frac{1}{2\pi}\right)^{n/2} \sum V_m e^{im\cdot x}$$

By Parseval's formula

$$\int_{B''} \Delta\varphi \cdot \nabla v dx = \sum_{\substack{m \\ m\neq 0}} G_m V_{-m}$$

$$= \int_{B''} gvdx$$

Since φ is the limit of an absolutely convergent series of odd functions, we deduce that φ is an odd function. Let $v^* \in C_0^\infty(B')$ and extend v^* to be an odd function over B''. Notice that $\nabla\varphi \cdot \nabla v^*$ and gv^* are even functions over B''. Thus

$$\int_{B'} \nabla\varphi \cdot \nabla v^* dx = \int_{B'} gv^* dx$$

Since $C_0^\infty(B')$ is dense in $H_0^1(B')$, the above equation holds for every $v^* \in H_0^1(B')$. We conclude that $\varphi = w$ and $w \in H^2(B')$.

Theorem 3.10: Let $B' = \mathop{X}_{i=1}^{n} (0,\pi)$. If $w \in H'(B')$ satisfying

$$\int_{B'} (\nabla w \cdot \nabla v + wv) dx = \int_{B'} gv dx \qquad v \in H^1(B')$$

where $g \in L_2(B')$, then $w \in H^2(B')$.

Proof: Let $B'' = \mathop{X}_{i=1}^{n} (-\pi,\pi)$. Extend g to be an even function over B'', and call this extension g. g has a Fourier series

$$g(x) = \left(\frac{1}{2\pi}\right)^{n/2} \sum_m G_m e^{im\cdot x} \quad .$$

Define Φ_m by

$$\Phi_m = G_m/(\|m\|^2 + 1)$$

where $\|m\| = m_1^2 + m_2^2 + \dots + m_n^2$, and let φ be the inverse Fourier transform of Φ. Considering φ and g as distributions, it is easy to verify that $-\Delta\varphi + \varphi = g$. Thus $\varphi \in H^2(B'')$.
By Parseval's formula

$$\int_{B''} (\nabla\varphi \cdot \nabla v + \varphi v)dx = \int_{B''} gvdx \qquad v \in H^1(B'')$$

Let $v^* \in C^1(\bar{B}')$ and extend v^* to be an even function over B''. We claim that $v^* \in H^1(B'')$. Define for $1 \le j \le n$

$$A_j = (\overset{j-1}{\underset{i=1}{X}} (-\pi,\pi)) \times [0,\pi] \times (\overset{n}{\underset{i=j+1}{X}} (-\pi,\pi))$$

and

$$B_j = (\overset{j-1}{\underset{i=1}{X}} (-\pi,\pi)) \times [-\pi,0] \times (\overset{n}{\underset{i=j+1}{X}} (-\pi,\pi))$$

For $\Psi \in C_0^\infty(B'')$ we note that

$$-\int_{B''} v^* \Psi_{x_j} dx = \int_{A_j} v^*_{x_j} \Psi(x)dx + \int_{B_j} v^*_{x_j} \Psi(x)dx$$

$$= \int_{B''} v^*_{x_j} \Psi(x)dx$$

Thus $v^*_{x_j} \in L_2(B'')$, $1 \le j \le n$, and $v^* \in H^1(B'')$. Since $\nabla\varphi\cdot\nabla v^* + \varphi v^*$ and gv^* are even functions we obtain

$$\int_{B'} (\nabla\varphi\cdot\nabla v^* + \varphi v^*)dx = \int_{B'} gv^* dx$$

This equation holds for $v^* \in H^1(B')$ since $C^1(\bar{B}')$ is dense in $H^1(B')$. Thus $\varphi = w$ and $w \in H^2(B')$.

Obviously these theorems hold for $B = \overset{n}{\underset{i=1}{X}} (a_i,d_i)$. In this section $L_p = L_p(B)$ and $H^p = H^p(B)$.

Let Δ_i denote a partition of $[a_i,d_i]$ with uniform mesh h_i. Define $h = \max\limits_{1\le i\le n} \{h_i\}$ and $\underline{h} = \min\limits_{1\le i\le n} \{h_i\}$. Let S^h be the span of the tensor products of piecewise Hermite polynomials of degree $2m-1$, $m \ge 1$, which are defined on $\overset{n}{\underset{i=1}{X}} \Delta_i$ and are in $\mathring{V}$. Let $\tilde{u}$ be defined by (3.37) or by (3.85) with $\Omega = B$. We now show that with restrictions on u, $a(x,u)$, $p(x,u)$, and n, $\|\nabla\tilde{u}\|_{L_\infty(B)\times L_\infty[0,T]}$ is bounded independently of h.

We first give several definitions;

Definition 3.2: A collection C of partitions $\rho = \overset{n}{\underset{i=1}{X}} \rho_i$ of $\bar{B}$, where ρ_i is a partition of $[a_i,d_i]$, is said to be <u>quasi-uniform</u> if and only if there exists a constant ξ such that $\bar{\rho}_i/\underline{\rho}_j \le \xi$. $1\le i, j\le n$ where $\bar{\rho}_k$ and $\underline{\rho}_k$ denote the maximum and minimum interval lengths in ρ_k respectively, (see [20]).

Definition 3.3: For k a positive integer, let $K^{k,\infty}(\bar{B})$ be the set of all functions $g \in C^{k-1}(\bar{B})$ such that $D_i^{k-1} g$ is absolutely continuous for $1 \le i \le n$ and $D_i^k g \in L_\infty(\bar{B})$, $1 \le i \le n$. (see [20]).

Definition 3.4: Let w be a function defined on $\bar{B} \times [0,T]$. We say $w \in L_\infty([0,T], K^{k,\infty}(\bar{B}))$ if for $t \in [0,T]$ $w(x,t) \in K^{k,\infty}(\bar{B})$ and if

$$F_{i\ell}(t) = \|D_\ell^i\, w(x,t)\|_{L_\infty(\bar{B})}$$

then $F_{i\ell} \in L_\infty([0,T])$ for $1 \le \ell \le n$ and $0 \le i \le k$.

The following theorem is proved by Schultz in [20].

Theorem 3.11: Let $\rho = \mathop{X}\limits_{i=1}^{n} \rho_i$ be a quasi-uniform partition of $\bar{B}$ and let M be the span of the tensor products of piecewise Hermite polynomials of degree 2m-1, $m \ge 1$, which are defined on ρ and are in V ($V = H_0^1(B)$ or $V = H^1(B)$). Then if $w \in K^{k,\infty}(\bar{B}) \cap V$, there exists $w_m \in M$ such that

$$\|w-w_m\|_{L_\infty} \le \tilde{C}_m \bar{\rho}^s \sup_{1\le i\le n} \|D_i^s w\|_{L_\infty}$$

for $1 \le s \le \min(k,2m)$ and

$$\|D_\ell^j(w-w_m)\|_{L_\infty} \le \tilde{C}_m \bar{\rho}^{s-j} \sup_{1\le i\le n} \|D_i^s w\|_{L_\infty}$$

for $2 \le s \le (k, 2m)$, $1 \le j \le s-1$, and $1 \le \ell \le n$. Here $L_\infty = L_\infty(\bar{B})$ and $\bar{\rho}$ denotes the maximum interval length of ρ_i, $1 \le i \le n$. If $D_\ell^j w_m \notin C(\bar{B})$, $\|D_\ell^j(w-w_m)\|_{L_\infty}$ is to be interpreted as the uniform norm of the interior of the cells defined by the partition.

We now prove

Lemma 3.5: Let $\Omega = B$ and assume the hypotheses of Theorem 3.6 or Theorem 3.8 with $(a_{ij}(x)) = \bar{I}$, the identity matrix. In addition, assume $u \in L_\infty([0,T], K^{q,\infty}(\bar{B}))$, $2 \le q \le 2m$, and let $\tilde{u}$ be defined by (3.37) or (3.85). Then if $\mathop{X}\limits_{i=1}^{n} \Delta_i$ is quasi-uniform and

$$q \ge (n+2)/2,$$

$\|\nabla\tilde{u}\|_{L_\infty(B)\times L_\infty[0,T]}$ is bounded independently of the h_i, $1 \le i \le n$.

Proof: From Lemma 3.2 (or proof of Theorem 3.8 following (3.87)) we have

$$\|u-\tilde{u}\|_{L_2}(t) \leq C_0^2 C' \eta^{-1} Q^2 h^s \|u\|_{H^s}(t)$$

for $t \in [0,T]$ and $0 \leq s \leq \min(2m,r)$. By Theorem 3.11 there exists for $t \in [0,T]$ $u_m(x,t) \in S^h$ such that

$$(3.90) \quad \|D_\ell^j(u-u_m)\|_{L_\infty(\bar{B})}(t) \leq \tilde{C}_m h^{q-j} \sup_{1\leq i\leq n} \|D_i^q u\|_{L_\infty(\bar{B})}(t)$$

for $0 \leq j \leq 1$, $1 \leq \ell \leq n$. Thus

$$(3.90') \quad \|u-u_m\|_{L_2}(t) \leq \tilde{C}_m h^q \left(\prod_{i=1}^{n}(d_i-a_i)\right)^{\frac{1}{2}} \sup_{1\leq i\leq n} \|D_i^q u\|_{L_\infty(\bar{B})}(t)$$

and

$$\|\tilde{u}-u_m\|_{L_2}(t) \leq C(t)h^s$$

where $s = \min(r,2m,q)$ and

$$C(t) = \eta^{-1} C_0^2 C' Q^2 \|u\|_{H^s}(t) + \tilde{C}_m\left(\prod_{i=1}^{n}(d_i-a_i)\right)^{\frac{1}{2}} \sup_{1\leq i\leq n} \|D_i^s u\|_{L_\infty(\bar{B})}(t).$$

Since for $t \in [0,T]$, $\tilde{u}-u_m \in S^h$ we can write

$$(\tilde{u}-u_m)(x,t) = \sum_{i=1}^{M} \gamma_i(t) v_i(x) \qquad (x,t)\in\bar{B}\times[0,T]$$

where $\{v_i(x)\}_{i=1}^{M}$ forms a basis for S^h and is defined in

Section 2.3. Let $G_0 = (g_{k\ell})$ where

$$g_{k\ell} = \int_B v_k(x) v_\ell(x)\,dx$$

G_0 is the tensor product of matrices G_0^j, $1 \le j \le n$, where G_0^j is the Gramian matrix corresponding to the piecewise Hermite polynomials of degree 2m-1, $m \ge 1$, which are defined on Δ_j and are in V. Here V is defined appropriately as $H_0^1(B)$ or $H^1(B)$. The G_0^j are positive definite matrices and the minimum eigenvalue of G_0^j is bounded below by $c_m h_j$ where c_m is a constant independent of h_j (see for example [15]). Thus

$$\|\tilde{u}-u_m\|_{L_2}^2(t) = (G_0\gamma(t),\ \gamma(t))$$

which implies

$$(3.91) \qquad \|\gamma(t)\|_2^2 \le C^2(t)(1/c_m)^n \left(\frac{h}{\underline{h}}\right)^n h^{2s-n} \quad t \in [0,T]$$

where $s = \min(r,q,2m)$. Now

$$(\tilde{u}-u_m)_{x_j} = \sum_{i=1}^{M} \gamma_i (v_i)_{x_j}(x) \quad (x,t) \in \bar{B} \times [0,T]$$

Notice if $x \in \bar{B}$ at most $(2m)^n$ of the $\{v_i(x)\}_{i=1}^{M}$ are not identically zero. In addition $|(v_i(x))_{x_j}| \le C_m/h_j$ where

C_m is a constant independent of the h_i, $1 \le i \le n$.
Therefore

$$(3.92) \qquad |(\tilde{u}-u_m)_{x_j}(x,t)| \le C_m h_j^{-1}(2m)^n \|\gamma(t)\|_2,$$

$$(x,t) \in \bar{B} \times [0,T]$$

For $t \in [0,T]$

$$\|\tilde{u}_{x_j}\|_{L_\infty(B)}(t) \le \|(\tilde{u}-u_m)_{x_j}\|_{L_\infty(B)}(t) + \|(u_m-u)_{x_j}\|_{L_\infty(t)} +$$

$$+ \|u_{x_j}\|_{L_\infty(B)}(t) \le Ch^{s-(n+2)/2} + \|u_{x_j}\|_{L_\infty(B)}(t)$$

where $s = \min(2m,r,q) = q$. The above inequality follows from (3.90), (3.91), and (3.92). The proof of the lemma now follows.

From Lemma 3.5 and Theorems 3.6, 3.8, 3.9, and 3.10, we obtain the following result.

Theorem 3.12: Let $\Omega = B = \mathop{X}\limits_{i=1}^{n}(a_i,d_i)$ with $a_i < d_i$.
Assume

(1°) Condition A, (3.34) (Condition C,(3.82'), with $(a_{ij}(x)) = I$, the identity matrix)

(2°) $u_t \in L_\infty([0,T], L_\infty(B))$

(3°) $a(\cdot,u(\cdot,t)) \in C^1(\Omega)$ for $t \in [0,T]$ and $a(x,u)_{x_i}, (a(x,u)_t)_{x_i} \in L_\infty([0,T], L_\infty(B))$ (or $p(x,u)$ satisfies the same assumptions).

(4°) Δ_i is a partition of $[a_i,d_i]$ with uniform mesh h_i, $1 \le i \le n$, and S^h, $h = \max\limits_{1\le i\le n} h_i$, is the span of the tensor products of piecewise Hermite polynomials of degree $2m-1$, $m \ge 1$, which are defined on $\mathop{X}\limits_{i=1}^{n} \Delta_i$ and are in $H_0^1(B)$ $(H^1(B))$.

If the Galerkin approximation U to u is defined by (3.35) and (3.36) (or (3.83) and (3.84)), $\mathop{X}\limits_{i=1}^{n} \Delta_i$ is quasi-uniform, and $q \ge (n+2)/2$, then for $\tau \in (0,T]$

$$\begin{aligned}\|U-u\|^2_{L_2(B)}(\tau) \le K_1^* h^{2s}[&\|u\|^2_{H^s(B)\times L_2(0,\tau)} \\ &+ \|\Psi\|^2_{H^s(B)} + \|u_t\|^2_{H^s(B)\times L_2(0,\tau)} \\ &+ \|u\|^2_{H^s(B)}(\tau)]\end{aligned}$$

where $s = \min(r,2m)$ and K_1^* is a constant which depends on τ, n, $\bar{K}$, M^*, C_0, C^*, Q, m, $\tilde{C}_m$, $\mu(B)$, $\|\nabla u\|_\infty$, $\|u\|_{H^q(B)\times L_\infty[0,T]}$, $\|u_t\|_\infty$, $\|D_i^q u\|_\infty$, $1 \le i \le n$, and

$\|\nabla(a(x,u))\|_\infty$ and $\|\nabla(a(x,u)_t)\|_\infty$ (or $\|\nabla(p(x,u))\|_\infty$ and $\|\nabla(p(x,u)_t)\|_\infty$.) Here $\|\ \|_\infty = \|\ \|_{L_\infty(B)\times L_\infty[0,T]}$.

Corollary: Let u be the solution to (3.30), (3.32), and (3.33) with $B = \overset{n}{\underset{i=1}{X}}(0,1)$ with $1 \le n \le 6$. Assume $u, u_t \in C^4(\bar{B} \times [0,T])$, $a(x,u) \in C^2(\bar{B} \times [0,T])$, $b_i(x,u) \in C^1(\bar{B} \times [0,T])$, and $f(x,u) \in C^1(\bar{B} \times [0,T])$. Also assume (1) and (2) of Condition A hold. Let S^h be the span of the tensor products of the piecewise Hermite cubics defined on $\overset{n}{\underset{i=1}{X}} \Delta$ where Δ is a partition of [0,1] with uniform mesh h. Let U be defined by (3.35) and (3.36). Then for $\tau \in (0,T]$

$$\|U-u\|_{L_2(B)}(\tau) \le 0(h^4).$$

Using Theorems 3.11 and 3.12 we can also obtain a priori estimates of $\|D^j_\ell(U-u)\|_{L_2(B)}(t)$ and $\|D^j_\ell(U-u)\|_{L_\infty(B)}(t)$ for $t \in [0,T]$.

Theorem 3.13: Assume the hypotheses of Theorem 3.12 with $2 \le q = r \le 2m$. Then if $\overset{n}{\underset{i=1}{X}} \Delta_i$ is quasi-uniform, U is defined by (3.35) and (3.36) (or (3.83) and (3.84)), and $r \ge (n+2)/2$

$$\|D^j_\ell(U-u)\|_{L_2(B)}(\tau) \le K_1(\tau)h^{s-j}$$

and

$$\|D_\ell^j(U-u)\|_{L_\infty(\bar{B})}(\tau) \le K_2(\tau)h^{s-(n+2j)/2}$$

for $\tau \in [0,T]$, $1 \le \ell \le n$, and $0 \le j \le \min(m,s-1)$ where $s = \min(r,2m)$. $K_1(\tau)$ and $K_2(\tau)$ are constants which depend on the same constants as given for K_1^* in Theorem 3.12 and $\|u\|_{H^r(B)\times L_2(0,\tau)}$, $\|u_t\|_{H^r(B)\times L_2(0,\tau)}$, $\|\Psi\|_{H^r(B)}$, and $\|u\|_{H^r(B)}$.

Here if $D_\ell^j U \notin C(\bar{B})$, $\|D_\ell^j(U-u)\|_{L_\infty(\bar{B})}$ is to be interpreted as the uniform norm of the interior of the cells defined by the partition $\overset{n}{\underset{i=1}{X}} \Delta_i$.

Proof: By assumption $u(x,t) \in K^{r,\infty}(\bar{B})$, $t \in [0,T]$. From Theorem 3.11, there exists $u_m(x,t) \in S^h$, $t \in [0,T]$, such that (3.90) holds for $q = r$, $0 \le j \le r-1$, and $1 \le \ell \le n$. From Theorem 3.12 and (3.90'), we obtain

$$\|U-u_m\|_{L_2(B)}(t) \le K_2^*(t)h^r, \qquad t \in [0,T]$$

where $K_2^*(t)$ depends on the above constants listed for $K_1(t)$ and $K_2(t)$. Since $U-u_m \in S^h$, we can write

$$(U-u_m)(x,t) = \sum_{i=1}^{M} \alpha_i(t)v_i(x) \qquad (x,t)\in\bar{B}\times[0,T]$$

where $\{v_i(x)\}_{i=1}^{M}$ is a basis for S^h and is defined in Section

2.3. Thus

$$(3.92') \qquad \|U-u_m\|^2_{L_2(B)}(t) = (G_0\alpha(t),\ \alpha(t))$$

where G_0 is the Gramian matrix corresponding to the basis $\{v_i(x)\}_{i=1}^M$. As in Lemma 3.5, (3.92') implies that

$$(3.93) \qquad \|\alpha(t)\|_2^2 \le K_3^*(t)h^{2r-n}$$

where $K_3^*(t)$ depends on the above listed constants. Now for $1 \le j \le m$

$$\|D_\ell^j(U-u_m)\|^2_{L_2(B)}(t) = (B_{\ell,j}\alpha(t),\ \alpha(t))$$

where the matrix $B_{\ell,j}$ is the tensor product of matrices $G_0^1,\ G_0^2,\ldots,G_0^{\ell-1},\ G_j^\ell,\ G_0^{\ell+1},\ldots$ and G_0^n where G_s^k is the Gramian matrix corresponding to the sth derivative $0 \le s \le m$, of the piecewise Hermite polynomials of degree $2m-1$, $m \ge 1$, which are defined on Δ_k and are in V (V defined appropriately as $H_0^1(B)$ or $H^1(B)$.) It is easy to verify that G_s^k is a non-negative definite matrix and the eigenvalues of G_s^k are bounded above by $\chi_m h_k^{1-2s}$, $0 \le s \le m$, $1 \le k \le n$.

Thus

$$(3.94) \qquad \|D_\ell^j(U-u_m)\|^2_{L_2(B)}(t) \le \chi_m^n \prod_{k=1}^n h_k\, h_\ell^{-2j} \|\alpha(t)\|_2^2, \quad t \in [0,T]$$

for $1 \le j \le m$ and $1 \le \ell \le n$. Since $\overset{n}{\underset{i=1}{X}} \Delta_i$ is quasi-uniform, we obtain from (3.93) and (3.94)

$$(3.95) \qquad \|D_\ell^j(U-u_m)\|^2_{L_2(B)}(t) \le (\chi_m \xi)^n K_3^*(t)h^{2(r-j)}, \quad t \in [0,T]$$

$1 \le j \le m$, $1 \le \ell \le n$. Estimates of $\|D_\ell^j(U-u)\|_{L_2(B)}(t)$, now follow from (3.95) and (3.90) with $q = r$.

Now for $0 \le j \le m$

$$|D_\ell^j(U-u_m)(x,t)| \le \sum_{i=1}^{M} |\alpha_i(t)||D_\ell^j v_i(x)| \quad (x,t)\in\bar{B}\times[0,T]$$

Since at most only $(2m)^n$ of $\{v_i(x)\}_{i=1}^{M}$ are not identically zero and $|D_\ell^j v_i(x)| \le C_m h_\ell^{-j}$, we see by Hölder's inequality that

$$(3.96) \qquad |D_\ell^j(U-u_m)(x,t)| \le (2m)^n \|\alpha(t)\|_2 \, C_m h_\ell^{-j}$$

$$(x,t) \in \bar{B} \times [0,T]$$

Thus by (3.96) and (3.93)

$$(3.97) \qquad |D_\ell^j(U-u_m)(x,t)| \le (2m)^n (K_3^*(t))^{\frac{1}{2}} \xi^j h^{r-n/2-j}$$

$$(x,t) \in \bar{B} \times [0,T]$$

$1 \le \ell \le n$, $0 \le j \le m$. Estimates for $\|D_\ell^j(U-u)\|_{L_\infty(\bar{B})}(t)$ now follow from (3.97) and (3.90).

3.5 Estimates for more general parabolic problems. In this section we extend the results of Section 3.3 to include problems where the term u_t is replaced by the term $\varphi(x,u)_t$, which may be nonlinear. L_2 estimates are also determined for problems with inhomogeneous Dirichlet and Neumann boundary conditions.

Consider the problem

$$\varphi(x,u)_t = \sum_{i,j=1}^{n} (a_{ij}(x)p(x,u)u_{x_j})_{x_i} + \sum_{i=1}^{n} b_i(x,u)u_{x_i} \tag{3.100}$$
$$+ f(x,u) \qquad (x,t) \in \Omega\times(0,T]$$

with initial and boundary conditions given by

$$(3.101) \qquad u(x,0) = \psi(x) \qquad x \in \Omega$$

and

$$(3.102) \qquad u(x,t) = 0 \qquad (x,t) \in \partial\Omega \times (0,T]$$

We make the following assumptions:

(3.102')

(1°) (1)-(3) of Condition B, (3.64').

(2°) φ_u, the partial derivative of φ with respect to its (n+1)st variable, exists, is continuous, and

$$\eta \le \varphi_u(x,s) \le C_0 \qquad (x,s) \in \Omega \times R$$

φ_{uu} also exists and

$$|\varphi_{uu}(x,s)| \le K \qquad (x,s) \in \Omega \times R$$

(3°) $u \in C^2(\Omega \times [0,T])$ is a unique solution to (3.100)-(3.102) and $u, u_t \in L_2([0,T], H^r(\Omega) \cap H_0^1(\Omega))$

(4°) (5) of Condition B, (3.64')

(5°) u_t, $p(x,u)_{x_i}, (p(x,u)_t)_{x_i} \in L_\infty(\Omega \times [0,T])$, $1 \le i \le n$.

(6°) $P(\ ,\ )$ is 0-regular on $H_0^1(\Omega)$ where

$$P(w,v) = \int_\Omega \sum_{i,j=1}^{n} a_{i_j}(x) w_{x_j} v_{x_i} dx \qquad w, v \in H_0^1(\Omega)$$

Let S^h be a $S_{k,m}^{h,0}(\Omega)$ space. Let $U \in S^h$ be the Galerkin approximation to u defined by

$$\langle \varphi(x,U)_t, v\rangle = -a(U;U,v) + \langle f(x,U),v\rangle$$

$$+ \sum_{i=1}^{n} \langle b_i(x,U)U_{x_i}, v\rangle, \qquad t > 0 \tag{3.103}$$

and

(3.104) $\langle U,v\rangle = \langle\psi,v\rangle \qquad t = 0$

for $v \in S^h$ where $a(\ ;,\ )$ is defined by (3.65)

We now modify Theorem 3.7

Theorem 3.14. Assume (1°)-(6°) in (3.102') hold. Let U be defined by (3.103) and (3.104). Then for $\tau \in (0,T]$,

$$\|U-u\|^2_{L_2} \le h^{2s}\{K_1^*\|u\|^2_{H^s(\Omega)\times L_2(0,\tau)} + K_2^*[\|\psi\|^2_{H^s(\Omega)} + \|u_t\|^2_{H^s(\Omega)\times L_2(0,\tau)} + \|u\|^2_{H^s(\Omega)}(\tau)]\}$$

where $s = \min(r,m)$ and K_1^* and K_2^* are positive constants which depend on $\tau, n, K, \eta, M^*, C_0, Q$, $\|\nabla p(x,u)\|_\infty$, $\|\nabla\tilde{u}\|_\infty$, $\|\tilde{u}_t\|_\infty$, and $\|u_t\|_\infty$. K_1^* also depends on $\|\nabla(p(x,u)_t)\|_\infty$.

Proof: For $t \in [0,T]$ define $\tilde{u}(x,t)\in S^h$ by (3.68). From (3.100) and (3.68) we have $t > 0$

$$\langle\varphi_u(x,\tilde{u})\tilde{u}_t,v\rangle = \langle\varphi_u(x,\tilde{u})\tilde{u}_t - \varphi_u(x,u)u_t,\ v\rangle - a(u;\ \tilde{u},v)$$

(3.105) $$+ \langle f(x,u),v\rangle + \sum_{i=1}^{n}\langle b_i(x,u)u_{x_i},v\rangle$$

Subtracting (3.105) from (3.103) with $v=(U-\tilde{u})$ we obtain

$$\langle\varphi_u(x,U)U_t - \varphi_u(x,\tilde{u})\tilde{u}_t,\ U-\tilde{u}\rangle + \eta\|U-u\|^2_{H_0^1} \le -a(U;\ u,U-\tilde{u}) +$$

$$a(u;\tilde{u},U-\tilde{u}) + \langle(\varphi_u(x,u)u_t - \varphi_u(x,\tilde{u})\tilde{u}_t),\ U,\tilde{u}\rangle + G_1 + G_2$$

where

$$G_1 = \langle f(x,U) - f(x,u), v\rangle$$

and

$$G_2 = \sum_{i=1}^{n} \langle b_i(x,U)U_{x_i} - b_i(x,u)u_{x_i},\ U-\tilde{u}\ \rangle$$

We note that

$$\langle \varphi_u(x,u)u_t - \varphi_u(x,\tilde{u})\tilde{u}_t,\ U-\tilde{u}\rangle$$

$$= \langle(\varphi_u(x,u)-\varphi_u(x,\tilde{u}))u_t,\ U-\tilde{u}\rangle$$

$$+ \langle\varphi_u(x,\tilde{u})(u-\tilde{u})_t,\ U-\tilde{u}\rangle$$

$$\leq (K+1)\|u_t\|_\infty[\|u-\tilde{u}\|_{L_2}^2 + \|U-\tilde{u}\|_{L_2}^2]$$

$$+ C_0[\|(u-\tilde{u})_t\|_{L_2}^2 + \|U-\tilde{u}\|_{L_2}^2]$$

Bounding the terms G_1, G_2 and $-a(U;\tilde{u},U-\tilde{u}) + a(u;\tilde{u},U-\tilde{u})$ as before we have

$$\langle\varphi_u(x,U)U_t-\varphi_u(x,\tilde{u})\tilde{u}_t,\ U-\tilde{u}\rangle + \eta\|U-\tilde{u}\|_{H_0^1}^2$$

$$\leq C_1[\|U-\tilde{u}\|_{L_2}^2 + \|u-\tilde{u}\|_{L_2}^2] + C_2\|(u-\tilde{u})_t\|_{L_2}^2$$

where C_1 and C_2 are positive constants which depend on

η, C_0, K, $\|\nabla\tilde{u}\|_\infty$, M^*, n. C_1 also depends on $\|u_t\|_\infty$. Using the same technique as in [11], we note that

$$\langle\varphi_u(x,U)(U-\tilde{u})_t,\ U-\tilde{u}\rangle = \frac{d}{dt}\left(\int_\Omega R(x,\tilde{u},\tilde{u}-U)dx\right) - \int_\Omega\int_0^{\tilde{u}-U}\varphi_{uu}(x,\tilde{u}-\mu)\tilde{u}_t\mu d\mu dx$$

where

$$R(x,s,\xi) = \int_0^\xi \varphi_u(x,s-\mu)\mu\ d\mu$$

Since

$$\left|\int_\Omega\int_0^{\tilde{u}-U}\varphi_{uu}(x,\tilde{u}-\mu)\tilde{u}_t\mu d\mu\right| \leq K\|\tilde{u}_t\|_\infty\ \|U-\tilde{u}\|_{L_2}^2$$

and

$$|\langle(\varphi_u(x,U)-\varphi_u(x,\tilde{u}))\tilde{u}_t\ ,\ U-\tilde{u}\rangle| \leq (K+1)\|\tilde{u}_t\|_\infty\ \|U-\tilde{u}\|_{L_2}^2$$

we deduce that

$$\begin{aligned}\frac{d}{dt}\left(\int_\Omega R(x,\tilde{u},\tilde{u}-U)dx\right) = &\ \langle\varphi_u(x,U)U_t - \varphi_u(x,\tilde{u})\tilde{u}_t,\ U-\tilde{u}\rangle \\ &+ \int_\Omega\int_0^{\tilde{u}-U}\varphi_{uu}(x,\tilde{u}-\mu)\tilde{u}_t\ \mu d\mu dx \\ &- \langle(\varphi_u(x,U)-\varphi_u(x,\tilde{u}))\tilde{u}_t, U-\tilde{u}\rangle \\ \leq &\ 2(K+1)\|\tilde{u}_t\|_\infty\|U-\tilde{u}\|_{L_2}^2 + \langle\varphi_u(x,U)U_t - \varphi_u(x,\tilde{u})\tilde{u}_t,\ U-\tilde{u}\rangle\end{aligned}$$

$$\le (2(K+1)\|\tilde{u}_t\|_\infty + C_1)\ \|U-\tilde{u}\|^2_{L_2} + C_1\|u-\tilde{u}\|^2_{L_2} + C_2\|(u-\tilde{u})_t\|^2_{L_2}$$

Integrating (3.107) with respect to t, we obtain for $\tau \in (0,T]$

$$\int_\Omega R(x,u(x,\tau),(\tilde{u}-U)(x,\tau))dx - \int_\Omega R(x,u(x,0),(\tilde{u}-U)(x,0)dx$$

$$\le (2(K+1)\ \|\tilde{u}_t\|_\infty + C_1)\ \|U-\tilde{u}\|^2_{L_2(\Omega)\times L_2(0,\tau)}$$

$$+ C_1\|u-\tilde{u}\|^2_{L_2(\Omega)\times L_2(0,\tau)} + C_2\|(u-\tilde{u})_t\|^2_{L_2(\Omega)\times L_2(0,\tau)}$$

Notice that

$$\left|\int_\Omega R(x,u(x,0),(\tilde{u}-U)(x,0)dx\right| \le C_0\ \|U-\tilde{u}\|^2_{L_2}(0)$$

and

$$\int_\Omega R(x,u(x,\tau),(\tilde{u}-U)(x,\tau)dx \ge \eta\|U-\tilde{u}\|^2_{L_2}(\tau)$$

Therefore

$$\eta\ \|U-\tilde{u}\|^2_{L_2}(\tau) \le (2(K+1)\|\tilde{u}_t\|_\infty + C_1)\|U-\tilde{u}\|^2_{L_2(\Omega)\times L_2(0,\tau)}$$

$$+ C_1\|u-\tilde{u}\|^2_{L_2(\Omega)\times L_2(0,\tau)} + C_0\|U-\tilde{u}\|^2_{L_2}(0)$$

$$+ C_2\|(u-\tilde{u})_t\|^2_{L_2(\Omega)\times L_2(0,\tau)}$$

By Gronwall's lemma, we obtain inequality (3.71). The constants C_1^*, C_2^*, and C_3^* also depend on $\|u_t\|_\infty$ and $\|\tilde{u}_t\|_\infty$. Proof of theorem follows by using the same arguments following (3.71) in Theorem 3.7.

The estimate of $\|U-u\|_{L_2}$ derived in Theorem 3.14 involves a constant $\|\tilde{u}_t\|_\infty$. We now give an example where $\|\tilde{u}_t\|_\infty$ is bounded independently of h. Assume $\Omega = B = \overset{n}{\underset{i=1}{X}} (a_i, d_i)$ with $a_i < d_i$, $1 \le i \le n$. Let Δ_i be a partition of $[a_i, d_i]$ with uniform mesh h_i, $1 \le i \le n$, and let S^h, $h = \max_{1 \le i \le n} h_i$, be the span of the tensor products of piecewise Hermite polynomials of degree $2m-1$, $m \ge 1$, which are defined on $\overset{n}{\underset{i=1}{X}} \Delta_i$. If $u_t \in L_\infty([0,T], K^{q,\infty}(\bar{B}))$ where $\min(1, n/2) \le q \le 2m$, then $\|\tilde{u}_t\|_\infty$ is bounded independently of h_i, $1 \le i \le n$. For proof we use an argument similar to the one given in Lemma 3.5. Replace u and $\tilde{u}$ by u_t and $\tilde{u}_t$ respectively in proof of Lemma 3.5 and note that

$$\left| \sum_{i=1}^{M} \gamma_i(t) v_i(x) \right| \le C_m (2m)^n \|\gamma\|_2$$

where

$$|v_i(x)| \le C_m \qquad x \in \bar{B},\ 1 \le i \le M.$$

We now briefly discuss the inhomogeneous Dirichlet problem. Consider the boundary value problem defined by (3.100), (3.101), and

$$(3.110) \qquad u(x,t) = g(x,t) \qquad (x,t) \in \partial\Omega \times (0,T]$$

We assume (1°)-(6°), (3.102'), with $H^1(\Omega)$ replacing $H_0^1(\Omega)$ in (1°) and (3°) and u being defined as the unique solution of (3.100), (3.101), and (3.110) in (3°). Let S^h be a $S^h_{k,m}(\Omega)$ space. We further assume that for $t \in [0,T]$ $g(x,t) \in S^h$ and that if $w \in H^r(\Omega)$ there exists a $w^*(x,t) \in S^h$ such that $(w-w^*)(x,t) \in H_0^1(\Omega)$ and

$$(3.111) \qquad \|w-w^*\|_{H_0^1} \leq Q^* h^{s-1} \|w\|_{H^s}$$

where $s = \min(r,m)$ and Q^* is a constant independent of w. In the case where $\Omega = B$, a rectangle in R^2, and S^h is the span of the tensor products of piecewise Hermite polynomials of degree 2m-1, $m \geq 1$, defined on a partition $\rho = \underset{i=1}{\overset{n}{X}} \Delta_i$ of B, Birkhoff, et al [6] show that if u is sufficiently continuous

$$\|u-u_m\|_{H_0^1}(t) \leq Q^* \, h^{2m-1} \|u\|_{H^{2m}}(t),$$

where $u_m(x,t)$ denotes the Hermite interpolate of $u(x,t)$ on S^h and h is the maximum interval length of Δ_i, 1,2.

We define the Galerkin approximation to u, U, by

$$U(x,t) = g(x,t) + \sum_{i=1}^{M} c_i(t)v_i(x) \quad (x,t)\in\bar{\Omega}\times[0,T]$$

where U satisfies (3.103) and (3.104) for $v \in S^h \cap H_0^1(\Omega)$. Here $\{v_i(x)\}_{i=1}^{M}$ is a linearly independent set which spans $S^h \cap H_0^1(\Omega)$.

For $t \in [0,T]$ define $\tilde{u}(x,t)\in S^h$ by

$$\tilde{u}(x,t) = g(x,t) + \sum_{i=1}^{M} d_i(t)v_i(x) \qquad (x,t) \in \bar{\Omega} \times [0,T]$$

where

$$a(u;(u-\tilde{u})(x,t),v) = 0 \qquad v \in S^h \cap H_0^1(\Omega)$$

From Theorem 3.5 and (3.111) we have

$$\|u-\tilde{u}\|_{L_2}(t) \le C' C_0^2 Q\eta^{-1} h \inf_{\hat{v}\in S^h \cap H_0^1(\Omega)} \|u-g-\hat{v}\|_{H_0^1}$$

$$\le Q^* C' C_0^2 Q\eta^{-1} h^s \|u\|_{H^s}(t)$$

where $s = \min(m,r)$. Following the proofs of Lemmas 3.3 and 3.4 we obtain the L_2 estimate of $(u-\tilde{u})_t$ given by (3.73).

We remark that $(U-\tilde{u})(x,t) \in S^h \cap H_0^1(\Omega)$ for $t \in (0,T]$.

Using the same argument as in the treatment of the homogeneous Dirichlet problem we now obtain the same estimates of $\|U-u\|^2_{L_2}(\tau)$ as given in Theorem 3.14 and in Theorem 3.7 if $\varphi(x,u) = u$.

The analysis given in Section 3.3, where we assumed no flux, also applies to problems with non-zero flux. Consider the parabolic problem defined by

$$(3.112)\quad \varphi(x,u)_t = \sum_{i,j=1}^{n} (a_{ij}(x)p(x,u)u_{x_j})_{x_i} + f(x,u)$$

$$(x,t) \in \Omega \times (0,T]$$

with initial and boundary conditions

$$(3.113)\quad u(x,0) = \psi(x)$$

and

$$(3.114)\quad \sum_{i,j=1}^{n} a_{ij}(x)p(x,u)\gamma_j u_{x_i} = g(x,t)$$

where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is the unit exterior normal to $\partial\Omega$.

We assume Condition C, (3.82') (u is now a solution to (3.112)-(3.114)), and (1°) and (5°) in (3.102'). We also assume $P(\ ,\ )$ is 0-regular on $H^1(\Omega)$ where

$$P(w,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) w_{x_j} v_{x_i} dx + \sqrt{\eta} \int_\Omega wv dx$$

Let S^h be a $S^h_{k,m}(\Omega)$ space. Let $U(\ ,t) \in S^h$ be the Galerkin approximation to u defined by

$$\langle \varphi(x,U)_t, v\rangle = -a(U; U,v) + \langle f(x,U), v\rangle$$

$$+ \int_{\partial\Omega} g(x,t) v\, d\sigma \qquad t > 0,\ v \in S^h$$

$$\langle U,v\rangle = \langle \psi, v\rangle \qquad t = 0,\ v \in S^h$$

Multiplying (3.112) by v, $v \in S^h$, and subtracting from (3.116), we note that $\int_{\partial\Omega} g(x,t)vdx$ terms cancel. We then conclude that the analysis given for problems with no flux can be applied to problems with non-zero flux. Thus, estimates of $\|U-u\|^2_{L_2}(\tau)$, $\tau \in (0,T]$ are given in Theorem 3.8. The constants K_1^* and K_2^* also depend on $\|\tilde{u}_t\|_\infty$ if $\varphi(x,u) \neq u$. (If $\varphi(x,u) \neq u$ Theorem 3.8 is modified in the same way as described for the homogeneous Dirichlet problem in Theorem (3.14).)

§4. A Priori L_2 Error Estimates for Several Discrete Time Galerkin Procedures

4.1: Several discrete time Galerkin procedures.

In this chapter we will consider several procedures formulated in [9] for approximating the solution of (3.31)-(3.33) in which the variable t will be discretized.

Let $\Delta t = T/N$ where N is a positive integer and let $t_j = j\Delta t$. We use the following notation:

$$g_j = g(x,t_j) \qquad 0 \le j \le N$$

$$g_{j,\theta} = \tfrac{1}{2}(1+\theta)\, g_{j+1} + \tfrac{1}{2}(1-\theta)g_j, \quad 0 \le j \le N-1$$

where $\theta \in [0,1]$. $a(\cdot;\cdot,\cdot)$ is defined by (3.65) and S^h will denote a $S^{h,0}_{k,m}(\Omega)$ space.

Consider the following discrete Galerkin procedure: Let $\{U_j\}_{j=0}^{N} \in S^h$ satisfying

$$\left\langle \frac{U_{j+1}-U_j}{\Delta t}, v \right\rangle + a(U_{j,\theta};\, U_{j,\theta}, v)$$

$$= \left\langle \sum_{i=1}^{n} b_i(x,U_{j,\theta})(U_{j,\theta})_{x_i}, v \right\rangle \tag{4.2}$$

$$+ \langle f(x,U_{j,\theta}), v\rangle \qquad v \in S^h,\ j \ge 0$$

and

$$(4.3) \qquad \langle U_0, v\rangle = \langle \psi, v\rangle \qquad v \in S^h$$

where $\theta \in [0,1]$. If $\theta = 0$, (4.2) yields the so-called Crank Nicolson Galerkin approximation; for $\theta = 1$ (4.2) is a backward difference Galèrkin approximation. We remark that (4.2) and (4.3) have solutions (possibly nonunique) if Δt is sufficiently small (see Lenma 7.1 in [9]).

The solution to (4.2) and (4.3) requires the solution of a system of nonlinear algebraic equations at each time level t_j. A scheme which requires the solution of two linear algebraic systems at each time step is the predictor-corrector-Galerkin Method

$$(4.4) \qquad \left\langle \frac{W_{j+1}-U_j}{\Delta t}, v\right\rangle + a(U_j;\, W_j^{\theta}, v\rangle = \langle f(x,U_j), v\rangle + \left\langle \sum_{i=1}^{n} b_i(x,U_j)(W_j^{\theta})_{x_i}, v\right\rangle, \qquad j \geq 0,\ v \in S^h$$

$$(4.5) \qquad \left\langle \frac{U_{j+1}-U_j}{\Delta t}, v\right\rangle + a(W_j^{\theta};\, U_{j,\theta}, v) = \langle f(x,W_j^{\theta}), v\rangle + \left\langle \sum_{i=1}^{n} b_i(x,W_j^{\theta})(U_{j,\theta})_{x_i}, v\right\rangle, \qquad j \geq 0,\ v \in S^h$$

and

$$(4.6) \qquad \langle U_0, v\rangle = \langle \psi, v\rangle \qquad v \in S^h$$

where $\theta \in [0,1]$ and

$$W_j^\theta = \frac{1}{2}(1+\theta)W_{j+1} + \frac{1}{2}(1-\theta)U_j$$

with $W_{j+1} \in S^h$.

In Sections 4.2 and 4.3 we obtain estimates of $\|(U-u)_N\|_{L_2(\Omega)}$ for both (4.2)-(4.3) and (4.4)-(4.6). We show that if Condition B is assumed and additional restrictions are made on u and p(x,u) then

$$\|(\bar{U}-u)_N\|_{L_2(\Omega)} \le C[h^s + (\Delta t)^k]$$

where $s = \min(r,m)$. For $\theta \in (0,1]$, $k = 1$ and for $\theta = 0$, $k = 2$.

The analysis given in this chapter can easily be extended to problems with inhomogeneous boundary conditions in the span of the chosen basis functions or with Neumann boundary conditions (The proof given of Theorem 4.4 holds only for Dirichlet boundary conditions). The modifications in the arguments presented here which are necessary to treat these extensions so closely resemble those used in extending the homogeneous Dirichlet boundary conditions for the continuous time Galerkin procedure that they will not be repeated.

In this chapter we will use C as a generic constant. Also $H \times L_S = H(\Omega) \times L_S[0,T]$, $S = 2,\infty$ where $H = H_0^1$, H^s, or L_s, s a positive integer.

4.2. <u>Estimates for a family of generalized discrete Galerkin procedures.</u> In this section we determine L_2 error estimates for the Galerkin procedures defined by (4.2) and (4.3). We assume Condition B, $p(\cdot,u(\cdot,\cdot)) \in C^2(\Omega\times[0,T])$ and $(u_{x_i})_{tt} \in C(\Omega \times [0,T])$ for $1 \le i \le n$.

Let S^h, a $S^{h,0}_{\underset{\sim}{k},\underset{\sim}{m}}(\Omega)$ space, be the span $\{v_{\underset{\sim}{i}}(x)\}_{\underset{\sim}{i}=\underset{\sim}{1}}^{M}$, a linearly independent set where $D_\ell v_i(x) \in L_\infty(\bar{\Omega})$, $1 \le i \le M$, $1 \le \ell \le n$. For $t \in [0,T]$, define $\tilde{u}(x,t) \in S^h$ by (3.68).

Now

$$\tilde{u}(x,t) = \sum_{\underset{\sim}{i}=\underset{\sim}{1}}^{M} \gamma_i(t)v_i(x).$$

From Condition B and the above assumptions made on p and u, we deduce that $\gamma_i \in C^2[0,T]$, $1 \le i \le M$. Thus $\tilde{u}_{tt}$ and $\tilde{u}_{tx_i}$, $1 \le i \le n$, are bounded on $\bar{\Omega} \times [0,T]$ (not necessarily independently of h). We note that for $t = t_{j,\theta}$, $j \geq 0$,

$$\langle \frac{\tilde{u}_{j+1}-\tilde{u}_j}{\Delta t}, v\rangle = \langle \tilde{u}_t(x,t_{j,\theta}) + \rho_{j,\theta}, v\rangle \qquad v \in S^h$$

where $\|\rho_{j,\theta}\|_{L_2} \leq \Delta t \|\tilde{u}_{tt}\|_{L_2 \times L_\infty}$. From (3.68) and (3.31) we see that for $t = t_{j,\theta}$, $0 \leq j \leq N-1$, and $v \in S^h$

$$\begin{aligned}\langle \frac{\tilde{u}_{j+1}-\tilde{u}_j}{\Delta t}, v\rangle = {} & \langle \tilde{u}_t(x,t_{j,\theta}) + \rho_{j,\theta}, v\rangle \\ & - a(u_{j,\theta}+\xi_{j,\theta}; \tilde{u}_{j,\theta} + \alpha_{j,\theta}, v) \\ & - \langle u_t(x,t_{j,\theta}), v\rangle \\ & + \langle f(x, u_{j,\theta} + \xi_{j,\theta}), v\rangle \\ & + \langle \sum_{i=1}^{n} b_i(x, u_{j,\theta}+\xi_{j,\theta})(u_{j,\theta}+\xi_{j,\theta})_{x_i}, v\rangle\end{aligned}$$

Here $\|\xi_{j,\theta}\|_{L_2} \leq \Delta t \|u_t\|_{L_2 \times L_\infty}$ and $\|\alpha_{j,\theta}\|_{H_0^1} \leq \Delta t \|u_t\|_{H_0^1 \times L_\infty}$, $0 \leq j \leq N-1$.

Subtracting (4.8) from (4.2) and taking $v = (U-\tilde{u})_{j,\theta}$, we obtain

$$\frac{1}{2\Delta t}[\|(U-\tilde{u})_{j+1}\|^2_{L_2} - \|(U-\tilde{u})_j\|^2_{L_2}]$$

$$\leq \langle \frac{(U-\tilde{u})_{j+1}-(U-\tilde{u})_j}{\Delta t}, (U-\tilde{u})_{j,\theta}\rangle$$

$$= -a(U_{j,\theta};U_{j,\theta},(U-\tilde{u})_{j,\theta}) + \langle f(x,U_{j,\theta}),(U-\tilde{u})_{j,\theta}\rangle$$

$$+ a(u_{j,\theta}+\xi_{j,\theta};\tilde{u}_{j,\theta}+\alpha_{j,\theta},(U-\tilde{u})_{j,\theta})$$

$$+ \langle u_t(x,t_{j,\theta}) - \tilde{u}_t(x,t_{j,\theta}) - \rho_{j,\theta},(U-\tilde{u})_{j,\theta}\rangle$$

$$- \langle f(x,u_{j,\theta} + \xi_{j,\theta}), (U-\tilde{u})_{j,\theta}\rangle + G_1$$

where

$$G_1 = \sum_{i=1}^{N} \langle b_i(x,U_{j,\theta})(U_{j,\theta})_{x_i} - b_i(x,u_{j,\theta}+\xi_{j,\theta})(u_{j,\theta}+\xi_{j,\theta})_{x_i}, (U-\tilde{u})_{j,\theta}\rangle$$

Estimating terms as before, we have

$$\frac{1}{2\Delta t}[\|(U-\tilde{u})_{j+1}\|^2_{L_2} - \|(U-\tilde{u})_j\|^2_{L_2}] + \eta\|(U-\tilde{u})_{j,\theta}\|^2_{H_0^1}$$

$$\leq -a(U_{j,\theta};\tilde{u}_{j,\theta},(U-\tilde{u})_{j,\theta}) + a(u_{j,\theta}+\xi_{j,\theta};\tilde{u}_{j,\theta}+\alpha_{j,\theta},(U-\tilde{u})_{j,\theta})$$

$$+ K[\|(U-u)_{j,\theta}-\xi_{j,\theta}\|_{L_2}\|(U-\tilde{u})_{j,\theta}\|_{L_2}]$$

$$+ \|(u-\tilde{u})_t\|_{L_2}(t_{j,\theta}) \|(U-\tilde{u})_{j,\theta}\|_{L_2} + |G_1|$$

$$+ \|\rho_{j,\theta}\|_{L_2} \|(U-\tilde{u})_{j,\theta}\|_{L_2}$$

We note that

$$-a(U_{j,\theta};\tilde{u}_{j,\theta},(U-\tilde{u})_{j,\theta}) + a(u_{j,\theta} + \xi_{j,\theta};\tilde{u}_{j,\theta} + \alpha_{j,\theta},(U-\tilde{u})_{j,\theta})$$

$$= a(u_{j,\theta} + \xi_{j,\theta};\ \alpha_{j,\theta},\ (U-\tilde{u})_{j,\theta})$$

$$- \int_\Omega \sum_{i,k=1}^{n} a_{ik}(x)[p(x,U_{j,\theta})-p(x,u_{j,\theta}+\xi_{j,\theta})](\tilde{u}_{j,\theta})_{x_k}((U-\tilde{u})_{j,\theta})_{x_i}dx$$

$$\le C_0\ \|\alpha_{j,\theta}\|_{H_0^1}\ \|(U-\tilde{u})_{j,\theta}\|_{H_0^1}$$

$$+ \sqrt{C}_0\ \|\nabla\tilde{u}\|_\infty n^{3/2}\ K\|(U-u)_{j,\theta} - \xi_{j,\theta}\|_{L_2}\ \|(U-\tilde{u})_{j,\theta}\|_{H_0^1}$$

$$\le C[\|\alpha_{j,\theta}\|^2_{H_0^1} + \|(U-u)_{j,\theta} - \xi_{j,\theta}\|^2_{L_2}] \quad +$$

$$\eta/4\|(U-\tilde{u})_{j,\theta}\|^2_{H_0^1}$$

where C is a constant which depends on C_0, $\|\nabla\tilde{u}\|_\infty$, n, K and η. This last inequaltiy follows from the inequality $ab \le \frac{1}{2}[\epsilon a^2 + (1/\epsilon)b^2]$ with $\epsilon = \eta/4$

$$G_1 = \sum_{i=1}^{n}[\langle b_i(x,U_{j,\theta})((U-\tilde{u})_{j,\theta})_{x_i},\quad (U-\tilde{u})_{j,\theta}\rangle$$

$$+ \langle(b_i(x,U_{j,\theta})-b_i(x,u_{j,\theta}+\xi_{j,\theta}))(\tilde{u}_{j,\theta})_{x_i},\ (U-\tilde{u})_{j,\theta}\rangle$$

$$- \langle b_i(x,u_{j,\theta}+\xi_{j,\theta})((u-\tilde{u})_{j,\theta}+\xi_{j,\theta})_{x_i},(U-\tilde{u})_{j,\theta}\rangle]$$

Using assumptions made on the b_i (Condition B) and integrating the third term by parts, we obtain

$$|G_1| \le C_0\sqrt{n}\ \|(U-\tilde{u})_{j,\theta}\|_{H_0^1}\ \|(U-\tilde{u})_{j,\theta}\|_{L_2}$$

$$+\ Kn\|\nabla\tilde{u}\|_\infty\|(U-\tilde{u})_{j,\theta}\|_{L_2}\|(U-u)_{j,\theta}-\xi_{j,\theta}\|_{L_2}$$

$$+\ \|(u-\tilde{u})_{j,\theta}+\xi_{j,\theta}\|_{L_2}\ [C_0\sqrt{n}\ \|(U-\tilde{u})_{j,\theta}\|_{H_0^1}$$

$$+\ M^*n\ \|(U-\tilde{u})_{j,\theta}\|_{L_2}\ ]$$

Applying the inequality $ab \le \frac{1}{2}(\epsilon a^2 + (1/\epsilon)b^2)$ with $\epsilon = \eta/4$ if $\|(U-\tilde{u})_{j,\theta}\|_{H_0^1}$ is a factor and $\epsilon = 1$ otherwise, we see that

$$|G_1| \le \eta/4\|(U-\tilde{u})_{j,\theta}\|^2_{H_0^1} + C[\|(U-u)_{j,\theta}-\xi_{j,\theta}\|^2_{L_2}$$

$$+\ \|(u-\tilde{u})_{j,\theta}\|^2_{L_2} + \|\xi_{j,\theta}\|^2_{L_2} + \|(U-\tilde{u})_{j,\theta}\|^2_{L_2}\ ]$$

where C is a constant which depends on η, K, M^*, n, C_0, and $\|\nabla\tilde{u}\|_\infty$. A little manipulation yields

$$\frac{1}{2\Delta t}[\|(U-\tilde{u})_{j+1}\|^2_{L_2} - \|(U-\tilde{u})_j\|^2_{L_2}] + \eta/2\|(U-\tilde{u})_{j,\theta}\|^2_{H_0^1}$$

$$\leq C[\|(U-\tilde{u})_j\|^2_{L_2} + \|(U-\tilde{u})_{j+1}\|^2_{L_2} + \|(u-\tilde{u})_j\|^2_{L_2} + \|(u-\tilde{u})_{j+1}\|^2_{L_2}$$

$$(4.11) \qquad + \|(u-\tilde{u})_t\|^2_{L_2}(t_{j,\theta}) + \|\rho_{j,\theta}\|^2_{L_2}$$

$$+ \|\xi_{j,\theta}\|^2_{L_2} + \|\alpha_{j,\theta}\|^2_{H_0^1}]$$

where C is a constant which depends on n, η, K, M^*, C_0, and $\|\nabla\tilde{u}\|_\infty$. Multiplying (4.11) by $2\Delta t$ and then summing for $j = 0,1,..N-1$ we obtain

$$\|(U-\tilde{u})_N\|^2_{L_2} + \eta\Delta t \sum_{j=0}^{N-1} \|(U-\tilde{u})_{j,\theta}\|^2_{H_0^1}$$

$$\leq \Delta t C \{\sum_{j=0}^{N} [\|(U-\tilde{u})_j\|^2_{L_2} + \|(u-\tilde{u})_j\|^2_{L_2}]$$

$$+ \sum_{j=0}^{N-1}[\|(u-\tilde{u})_t\|^2(t_{j,\theta}) + \|\alpha_{j,\theta}\|^2_{H_0^1} + \|\xi_{j,\theta}\|^2_{L_2}$$

$$+ \|\rho_{j,\theta}\|^2_{L_2}]\} + \|(U-\tilde{u})_0\|^2_{L_2}$$

If Δt is sufficiently small we obtain by Gronwall's lemma [16]

$$\|(U-\tilde{u})_N\|^2_{L_2} + \eta\Delta t \sum_{j=0}^{N-1} \|(U-\tilde{u})_{j,\theta}\|^2_{H_0^1}$$

$$\le C\{\|(U-\tilde{u})_0\|^2_{L_2} + \Delta t\ [\sum_{j=0}^{N} \|(u-\tilde{u})_j\|^2_{L_2}$$

$$+ \sum_{j=0}^{N-1} (\|(u-\tilde{u})_t\|^2_{L_2}(t_{j,\theta}) + \|\alpha_{j,\theta}\|^2_{H_0^1}$$

$$+ \|\rho_{j,\theta}\|^2_{L_2} + \|\xi_{j,\theta}\|^2_{L_2})]\}$$

where C is a constant which depends on T, n, K, η, C_0, M^*, and $\|\nabla u\|_\infty$.

We now require bounds for $\|u-\tilde{u}\|_{L_2}(t)$ and $\|(u-\tilde{u})_t\|_{L_2}(t)$, $t \in [0,T]$, and for $\|\alpha_{j,\theta}\|_{H_0^1}$, $\|\xi_{j,\theta}\|_{L_2}$, and $\|\rho_{j,\theta}\|_{L_2}$, $0 \le j \le N-1$.

We now further assume that u_t, $(p(x,u))_{x_i}$, $(p(x,u)_t)_{x_i} \in L_\infty(\Omega\times[0,T])$ and that P(,) is 0-regular on $H_0^1(\Omega)$ where

(4.13') $$P(w,v) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x) w_{x_i} v_{x_j}\, dx \quad w, v \in H_0^1(\Omega)$$

Estimates for $\|u-\tilde{u}\|_{L_2}(t)$ and $\|(u-\tilde{u})_t\|_{L_2}(t)$, $t \in [0,T]$, are derived in Theorem 3.7 and are given by (3.72) and (3.73) respectively.

Since $u_t \in L_\infty(\Omega \times [0,T])$ we find that

$$(4.13) \qquad \|\xi_{j,\theta}\|_{L_2} \leq \mu(\Omega)\Delta t \; \|u_t\|_\infty$$

where $\mu(\Omega)$ denotes the measure of Ω. To estimate $\|\alpha_{j,\theta}\|_{H_0^1}$ we require the following lemma.

Lemma 4.1: If $u, u_t \in L_\infty([0,T], H^1(\Omega))$ then

$$(4.14) \qquad \|\alpha_{j,\theta}\|_{H_0^1} \leq \Delta t\{(1+C_0\eta^{-1}Q)\|u_t\|_{H^1\times L_\infty} + C_0^{3/2}(\eta^{-1})^2 QK\|u\|_{H^1\times L_\infty}\}.$$

Proof: Using an argument similar to the one given in Lemma 3.3, we have

$$(4.15) \quad \|(u-\tilde{u})_t\|_{H_0^1\times L_\infty} \leq C_0\eta^{-1}Q[\|u_t\|_{H^1\times L_\infty} + \eta^{-1}\sqrt{C_0}\; K\|u_t\|_\infty \|u\|_{H^1\times L_\infty}]$$

Thus

$$\|\tilde{u}_t\|_{H_0^1\times L_\infty} \leq (1 + C_0\eta^{-1}Q)\|u_t\|_{H^1\times L_\infty} + QC_0^{3/2}(\eta^{-1})^2 K\|u_t\|_\infty \|u\|_{H^1\times L_\infty}$$

Since for $0 \le j \le N-1$ and $\theta \in [0,1]$

$$\|\alpha_{j,\theta}\|_{H_0^1} \le \Delta t \ \|\tilde{u}_t\|_{H_0^1 \times L_\infty}$$

we obtain (4.14)

A bound for $\|\rho_{j,\theta}\|_{L_2}$ is derived in

Lemma 4.2: If $u_{x_i tt} = u_{ttx_i}$, $1 \le i \le n$, u, u_t, $u_{tt} \in L_\infty([0,T], H^1(\Omega))$ and $p(x,u)_{tt} \in L_\infty([0,T], L_\infty(\Omega))$, then for $0 \le j \le N-1$ and $\theta \in [0,1]$

$$(4.16) \qquad \|\rho_{j,\theta}\|_{L_2} \le C\Delta t[\|u\|_{H^1 \times L_\infty} + \|u_t\|_{H^1 \times L_\infty} + \|u_{tt}\|_{H^1 \times L_\infty}]$$

where C is a constant which depends on n, K, η, Q, $\|u_t\|_\infty$, $\|p(x,u)_{tt}\|_\infty$, and C_0.

Proof: Differentiating (3.68) twice with respect to t, we see that for $v \in S^h$

$$(4.17) \qquad \begin{aligned} &a(u;(u-\tilde{u})_{tt},v)+\sum_{i,j=1}^{n} 2\int_\Omega a_{ij}(x)p(x,u)_t((u-\tilde{u})_t)_{x_j} v_{x_i}\, dx \\ &+ \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x)p(x,u)_{tt}(u-\tilde{u})_{x_j} v_{x_i} dx = 0 \end{aligned}$$

For $t \in [0,T]$ define $u^*(x,t) \in S^h$ by

(4.18) $$a(u;\, u_{tt}-u^*,\, v) = 0 \qquad v \in S^h$$

By Theorem 3.5,

(4.19) $$\|u_{tt}-u^*\|_{H_0^1 \times L_\infty} \le C_0 \eta^{-1} Q\, \|u_{tt}\|_{H^1 \times L_\infty}$$

From (4.17) and (4.18) we deduce that

(4.20) $$\begin{aligned} \|u^*-\tilde{u}_{tt}\|_{H_0^1\times L_\infty} \le \eta^{-1} n[2\sqrt{C_0}\, K\|u_t\|_\infty\, \|(u-\tilde{u})_t\|_{H_0^1\times L_\infty} \\ + \|p(x,u)_{tt}\|_\infty \sqrt{C_0}\|u-\tilde{u}\|_{H_0^1\times L_\infty}] \end{aligned}$$

Since for $v \in H_0^1(\Omega)$, $\|v\|_{L_2} \le C_\Omega^{\frac{1}{2}}\|v\|_{H_0^1}$ we have from (4.19) and (4.20)

$$\begin{aligned} \|(u-\tilde{u})_{tt}\|_{L_2\times L_\infty} &\le C_\Omega^{\frac{1}{2}}\|(u-\tilde{u})_{tt}\|_{H_0^1\times L_\infty} \\ &\le C[\|(u-\tilde{u})_t\|_{H_0^1\times L_\infty} + \|u-\tilde{u}\|_{H_0^1\times L_\infty} + \|u_{tt}\|_{H^1\times L_\infty}] \end{aligned}$$

where C is a constant which depends on n, C_Ω, η, K, C_0,

$\|u_t\|_\infty$, Q, and $\|p(x,u)_{tt}\|_\infty$. Thus

(4.21)

$$\|(u-\tilde{u})_{tt}\|_{L_2 \times L_\infty} \leq C[\|u_t\|_{H^1 \times L_\infty} + \|u\|_{H^1 \times L_\infty} + \|u_{tt}\|_{H^1 \times L_\infty}]$$

where C depends on the above constants. Since for $0 \leq j \leq N-1$ and $\theta \in [0,1]$

$$\|\rho_{j,\theta}\|_{L_2} \leq \Delta t \ \|\tilde{u}_{tt}\|_{L_2 \times L_\infty}$$

we now obtain (4.16) from (4.21). We now summarize the above results

Theorem 4.1: Assume

(1°) Condition B, (3.64')

(2°) $u_{x_i tt} \in C(\Omega\times[0,T])$ and

$u_{x_i tt} = u_{ttx_i}$, $\quad 1 \leq i \leq n$

(3°) u, u_t, $u_{tt} \in L_\infty([0,T], H^1(\Omega))$

(4°) $p(x,u) \in C^2(\Omega\times[0,T])$ and

$p(x,u)_{x_i}$, $p(x,u)_{tx_i}$, $p(x,u)_{tt} \in L_\infty(\Omega\times[0,T])$

(5°) $P(\ ,\ )$ defined by (4.13') is 0-regular on $H_0^1(\Omega)$

Let U_j, $0 \leq j \leq N$, be defined by (4.2) and (4.3) for $\theta \in [0,1]$. Then if Δt is sufficiently small, we have

$$\|(U-u)_N\|_{L_2} \le C_0^2 C' \eta^{-1} Q h^s \|u_N\|_{H^s}$$

$$(4.22) \qquad + C h^s \{ [\sum_{j=0}^{N} \Delta t \|u_j\|_{H^s}^2]^{1/2}$$

$$+ [\sum_{j=0}^{N-1} \Delta t (\|u\|_{H^s}^2 (t_{j,\theta}) + \|u_t\|_{H^s}^2 (t_{j,\theta}))]^{1/2} \}$$

$$+ \bar{C} \Delta t$$

where $s = \min(m,r)$. Here C and $\bar{C}$ are positive constants which depend on T, n, η, K, C_0, $\|\nabla\tilde{u}\|_\infty$, Q, M^*, $\|u_t\|_\infty$, C', $\|\nabla p(x,u)\|_\infty$ and $\|\nabla(p(x,u)_t)\|_\infty$. $\bar{C}$ also depends on $\mu(\Omega)$ (measure of Ω), $\|u\|_{H^1 \times L_\infty}$, $\|u_t\|_{H^1 \times L_\infty}$, $\|u_{tt}\|_{H^1 \times L_\infty}$, and $\|p(x,u)_{tt}\|_\infty$.

We now consider the Crank-Nicolson-Galerkin procedure, that is, (4.2) with $\theta = 0$. Under additional assumptions on the continuity of u and p(x,u) we obtain a $(\Delta t)^2$ term in place of Δt in (4.22).

Theorem 4.2: Assume the hypotheses (1°)-(5°) in Theorem 4.1. Further assume

(6°) $u_{x_{itt}} \in C(\Omega \times [0,T])$ and

$$u_{x_{itt}} = u_{tttx_i}, \quad 1 \le i \le n$$

(7°) $u_{ttt} \in L_\infty([0,T], H^1(\Omega))$

(8°) $p(x,u)_{ttt} \in L_\infty(\Omega \times [0,T])$

Let U_j, $0 \le j \le N$, be defined by (4.2) and (4.3) with $\theta = 0$. Then if Δt is sufficiently small

$$\|(U-\tilde{u})_N\|_{L_2} \le C_0^2 C'\eta^{-1}Q^2h^s\|u_N\|_{H^s} + \bar{C}(\Delta t)^2$$

$$+ Ch^s\{[\sum_{j=0}^{N} \Delta t(\|u\|^2_{H^s}(t_{j,\theta}) + \|u_t\|^2_{H^s}(t_{j,\theta}))]^{\frac{1}{2}} \tag{4.23}$$

$$+ [\sum_{j=0}^{N} \Delta t(\|u_j\|^2_{H^s}]^{\frac{1}{2}}\}$$

where $s = \min(m,r)$. C and $\bar{C}$ are positive constants which depend on the same constants given in Theorem 4.1. $\bar{C}$ also depends on $\|u_t\|_{H^1\times L_\infty}$ and $\|p(x,u)_{ttt}\|_\infty$.

Proof: We note that for $0 \le j \le N-1$

$$\|\alpha_{j,0}\|_{H_0^1} \le (\Delta t)^2\|\tilde{u}_{tt}\|_{H_0^1\times L_\infty},$$

$$\|\xi_{j,0}\|_{L_2} \le (\Delta t)^2\|u_{tt}\|_{L_2\times L_\infty}, \tag{4.24}$$

$$\|\rho_{j,0}\|_{L_2} \le (\Delta t)^2\|\tilde{u}_{ttt}\|_{L_2\times L_\infty}$$

From (4.21) we have for $0 \le j \le N-1$

$$(4.25) \quad \|\alpha_{j,0}\|_{H_0^1} \le C(\Delta t)^2 [\|u\|_{H^1 \times L_\infty} + \|u_t\|_{H^1 \times L_\infty} + \|u_{tt}\|_{H^1 \times L_\infty}]$$

where C is a constant which depends on η, K, n, C_0, $\|u_t\|_\infty$, Q, and $\|p(x,u)_{tt}\|_\infty$. Differentiating (3.68) three times with respect to t and using an argument similar to the one given in Lemma 4.2, we see that

$$(4.26) \qquad \|\rho_{j,0}\|_{L_2} \le C(\Delta t)^2 [\sum_{i=0}^{3} \| \frac{\partial^i u}{\partial t^i} \|_{H^1 \times L_\infty}], \quad 0 \le j \le N-1$$

where C depends on the above listed constants and $\|p(x,u)\|_\infty$. The proof of the theorem now follows.

4.3 Estimates for a family of predictor-corrector-Galerkin procedures. In this section we obtain L_2 estimates for the predictor-corrector-Galerkin procedures defined by (4.4)-(4.6). We prove

Theorem 4.3. Assume that hypotheses (1°)-(5°) of Theorem 4.1 hold. Let U_j, $0 \le j \le N$, be defined by (4.4)-(4.6) with $\theta \in [0,1]$. Then if Δt is sufficiently small we have inequality (4.22).

Proof: The proof of this theorem proceeds in two parts. First for $t = t_{j,\theta}$, $0 \le j \le N-1$, and $v \in S^h$

$$\langle \frac{\tilde{u}_{j+1}-\tilde{u}_j}{\Delta t}, v\rangle = \langle \tilde{u}_t(x,t_{j,\theta}) + \rho_{j,\theta}, v\rangle$$

$$-a(u_j + \beta_{j,\theta}; \tilde{u}_{j,\theta} + \alpha_{j,\theta}, v)$$

$$-\langle u_t(x,t_{j,\theta}), v\rangle + \langle f(x,u_j+\beta_{j,\theta}), v\rangle$$

$$+\langle \sum_{i=1}^{n} b_i(x,u_j + \beta_{j,\theta})(u_{j,\theta} + \xi_{j,\theta})_{x_i}, v\rangle$$

where

$$\|\rho_{j,\theta}\|_{L_2} \le \Delta t\|\tilde{u}_{tt}\|_{L_2\times L_\infty}, \quad \|\alpha_{j,\theta}\|_{H_0^1} \le \Delta t\|\tilde{u}_t\|_{H_0^1\times L_\infty},$$

$$\|\beta_{j,\theta}\|_{L_2} \le \Delta t\|u_t\|_{L_2\times L_\infty}, \text{ and } \|\xi_{j,\theta}\|_{L_2} \le \Delta t\|u_t\|_{L_2\times L_\infty}, \quad 0\le j\le N-1.$$

Note that (4.30) differs from (4.8) in that we are expanding $u(x,t_{j,\theta})$ in a truncated Taylor series about t_j, that is $u(x,t_{j,\theta}) = u_j + \beta_{j,\theta}$, in the terms $p(x,u(x,t))$, $b_i(x,u(x,t))$, and $f(x,u(x,t))$ where $t = t_{j,\theta}$.

Subtracting (4.30) from (4.4) and letting $v = V_{j,\theta} = W_j^\theta - \tilde{u}_{j,\theta}$, $0 \le j \le N-1$, we obtain

$$\frac{1}{2\Delta t}[\|(W-\tilde{u})_{j+1}\|^2_{L_2} - \|(U-\tilde{u})_j\|^2_{L_2}] \le \langle \frac{(W-\tilde{u})_{j+1}-(U-\tilde{u})_{j+1}}{\Delta t}, V_{j,\theta}\rangle$$

$$= -a(U_j; W_j^\theta, V_{j,\theta}) + a(u_j+\beta_{j,\theta}; \tilde{u}_{j,\theta} + \alpha_{j,\theta}, V_{j,\theta})$$

$$+ \langle (u-\tilde{u})_t(x,t_{j,\theta}) - \rho_{j,\theta}, V_{j,\theta}\rangle + G_1 + G_2$$

where

$$G_1 = \langle f(x,U_j) - f(x,u_j + \rho_{j,\theta}), V_{j,\theta}\rangle$$

and

$$G_2 = \sum_{i=1}^{n} \langle b_i(x,U_j)(W_j^\theta)_{x_i} - b_i(x,u_j+\beta_{j,\theta})(u_{j,\theta}+\xi_{j,\theta})_{x_i}, V_{j,\theta}\rangle$$

A little manipulation yields

$$\frac{1}{2\Delta t}[\|(W-\tilde{u})_{j+1}\|_{L_2}^2 - \|(U-\tilde{u})_j\|_{L_2}^2 + \eta\|V_{j,\theta}\|_{H_0^1}^2$$

(4.31) $$\leq \tfrac{1}{2}[\|(u-\tilde{\tilde{u}})_t\|_{L_2}^2(t_{j,\theta}) + \|\rho_{j,\theta}\|_{L_2}^2] + \|V_{j,\theta}\|_{L_2}^2$$

$$+ G_1 + G_2 + G_3$$

where

$$G_3 = -a(U_j;\tilde{\tilde{u}}_{j,\theta}, V_{j,\theta}) + a(u_j + \beta_{j,\theta}; \tilde{\tilde{u}}_{j,\theta}+\alpha_{j,\theta}, V_{j,\theta})$$

Using the triangle inequality and the inequality $ab \leq \tfrac{1}{2}(\epsilon a^2 + (1/\epsilon)b^2)$ with $\epsilon = 1/8\Delta t$ if $\|\rho_{j,\theta}\|_{L_2}$ is a factor and $\epsilon = 1$ otherwise, we obtain

$$|G_1| \leq K\|(U-u)_j - \rho_{j,\theta}\|_{L_2} \|V_{j,\theta}\|_{L_2}$$

$$\leq K[\|V_{j,\theta}\|_{L_2}^2 + \|(U-\tilde{u})_j\|_{L_2}^2 + \|(u-\tilde{u})_j\|_{L_2}^2]$$

$$+ 4K^2\Delta t \ \|\beta_{j,\theta}\|^2_{L_2} + \frac{1}{16\Delta t}\ \|V_{j,\theta}\|^2_{L_2}$$

Now

$$G_2 = \sum_{i=1}^{n}[\langle b_i(x,U_j)(V_{j,\theta})_{x_i}, V_{j,\theta}\rangle$$

$$+ \langle(b_i(x,U_j) - b_i(x,u_j + \beta_{j,\theta}))(\tilde{u}_{j,\theta})_{x_i}, V_{j,\theta}\rangle$$

$$- \langle b_i(x,u(x,t_{j,\theta})((u-\tilde{u})_{j,\theta} + \xi_{j,\theta})_{x_i}, V_{j,\theta}\rangle]$$

Integration by parts yields

$$\langle b_i(x,u(x,t_{j,\theta}))((u-\tilde{u})_{j,\theta} + \xi_{j,\theta})_{x_i}, V_{j,\theta}\rangle$$

$$= -\langle(b_i(x,u(x,t_{j,\theta})V_{j,\theta})_{x_i}, (u-\tilde{u})_{j,\theta} + \xi_{j,\theta}\rangle$$

Hence

$$|G_2| \le C_0\sqrt{n}\ \|V_{j,\theta}\|_{H_0^1}\ \|V_{j,\theta}\|_{L_2} + nK\|\nabla\tilde{u}\|_\infty\|(U-u)_j - \beta_{j,\theta}\|_{L_2}\|V_{j,\theta}\|_{L_2}$$

$$+ \|(u-\tilde{u})_{j,\theta} + \xi_{j,\theta}\|_{L_2}[M^*n\|V_{j,\theta}\|_{L_2} + C_0\sqrt{n}\ \|V_{j,\theta}\|_{H_0^1}]$$

$$\le C_0\sqrt{n}\ \|V_{j,\theta}\|_{H_0^1}\ \|V_{j,\theta}\|_{L_2} + nK\|\nabla\tilde{u}\|_\infty\|V_{j,\theta}\|_{L_2}[\|\beta_{j,\theta}\|_{L_2}$$

$$+ \|(U-\tilde{u})_j\|_{L_2} + \|(u-\tilde{u})_j\|_{L_2}]$$

$$+ [M^*n\|V_{j,\theta}\|_{L_2} + C_0\sqrt{n}\ \|V_{j,\theta}\|_{H_0^1}]\|(u-\tilde{u})_{j,\theta} + \xi_{j,\theta}\|_{L_2}$$

Using the inequality $ab \leq \frac{1}{2}[\epsilon a^2 + (1/\epsilon)b^2]$ with $\epsilon = 1/8\Delta t$ if $\|\beta_{j,\theta}\|_{L_2}$ is a factor, $\epsilon = \eta/4$ if $\|V_{j,\theta}\|_{H_0^1}$ is a factor, and $\epsilon = 1$ otherwise, we obtain

$$
\begin{aligned}
|G_2| \leq\ & \eta/4\|V_{j,\theta}\|^2_{H_0^1} + \left(C + \frac{1}{16\Delta t}\right)\|V_{j,\theta}\|^2_{L_2} \\
&+ 4n^2K^2\|\nabla\tilde{u}\|^2_\infty\ \Delta t\|\beta_{j,\theta}\|^2_{L_2} \\
&+ \bar{C}[\|(U-\tilde{u})_j\|^2_{L_2} + \|(u-\tilde{u})_j\|^2_{L_2} + \|\xi_{j,\theta}\|^2_{L_2} \\
&+ \|(u-\tilde{u})_{j,\theta}\|^2_{L_2}]
\end{aligned}
$$

where C and $\bar{C}$ are positive constants which depend on C_0, n, η, K, M^*, and $\|\nabla\tilde{u}\|_\infty$. We next bound G_3. Now

$$
\begin{aligned}
G_3 =\ & -\int_\Omega \sum_{i,k=1}^n a_{ik}(x)(p(x,U_j)-p(x,u_j+\beta_{j,\theta}))(\tilde{u}_{j,\theta})_{x_k}(V_{j,\theta})_{x_i}dx \\
&+ a(u_j + \beta_{j,\theta};\ \alpha_{j,\theta},\ V_{j,\theta})
\end{aligned}
\tag{4.32}
$$

Thus

$$
\begin{aligned}
|G_3| \leq\ & K\sqrt{C_0}\ n^{3/2}\ \|\nabla\tilde{u}\|_\infty\ \|(U-u)_j - \beta_{j,\theta}\|_{L_2}\|V_{j,\theta}\|_{H_0^1} \\
&+ C_0\|\alpha_{j,\theta}\|_{H_0^1}\ \|V_{j,\theta}\|_{H_0^1}
\end{aligned}
$$

$$\leq C[\|(U-\tilde{u})_j\|_{L_2}^2 + \|\beta_{j,\theta}\|_{L_2}^2 + \|(u-\tilde{u})_j\|_{L_2}^2$$

$$+ \|\alpha_{j,\theta}\|_{H_0^1}^2] + \eta/4\ \|V_{j,\theta}\|_{H_0^1}^2$$

where C is a constant defined as before. Substituting the bounds for G_1, G_2, and G_3 into (4.31) we obtain

$$\frac{1}{2\Delta t}[\|(W-\tilde{u})_{j+1}\|_{L_2}^2 - \|(U-\tilde{u})_j\|_{L_2}^2] + \eta/2\|V_{j,\theta}\|_{H_0^1}^2$$

$$\leq \tfrac{1}{2}[\|(u-\tilde{u})_t\|_{L_2}^2(t_{j,\theta}) + \|\rho_{j,\theta}\|_{L_2}^2] + (C+1/8\Delta t)\|V_{j,\theta}\|_{L_2}^2$$

(4.33')

$$+ C[(1+\Delta t)\|\beta_{j,\theta}\|_{L_2}^2 + \|\xi_{j,\theta}\|_{L_2}^2 + \|(u-\tilde{u})_j\|_{L_2}^2 + \|(u-\tilde{u})_{j+1}\|_{L_2}^2$$

$$+ \|\alpha_{j,\theta}\|_{H_0^1}^2]$$

where C is defined as before. Noting that

$$\|V_{j,\theta}\|_{L_2}^2 \leq 2[\|(W-\tilde{u})_{j+1}\|_{L_2}^2 + \|(U-\tilde{u})_j\|_{L_2}^2]$$

and replacing $\|V_{j,\theta}\|_{L_2}^2$ by this bound, we see that

$$\|(W-\tilde{u})_{j+1}\|^2_{L_2} \le C\|(U-\tilde{u})_j\|^2_{L_2} + C\Delta t[\|(u-\tilde{u})_t\|^2_{L_2}(t_{j,\theta})$$

$$+ \|\rho_{j,\theta}\|^2_{L_2} + (1+\Delta t)\|\beta_{j,\theta}\|^2_{L_2} + \|\xi_{j,\theta}\|^2_{L_2}$$

$$+ \|\alpha_{j,\theta}\|^2_{H^1_0} + \|(u-\tilde{u})_j\|^2_{L_2} + \|(u-\tilde{u})_{j+1}\|^2_{L_2}]$$

where C is a constant defined as before.

We now use (4.8) and (4.5) and proceed exactly as in Theorem 4.1. Notice that (4.5) differs from (4.2) in that the functions $p(x,U_{j,\theta})$, $f(x,U_{j,\theta})$, and $b_i(x,U_{j,\theta})$, $1 \le i \le n$, are replaced by $p(x,W_j^\theta)$, $f(x,W_j^\theta)$, and $b_i(x,W_j^\theta)$, $1 \le i \le n$. Also note that

$$\|W_j^\theta - u_{j,\theta} - \xi_{j,\theta}\|_{L_2} \le \|(W-\tilde{u})_{j+1}\|_{L_2} + \|(u-\tilde{u})_{j+1}\|_{L_2}$$

$$+ \|(U-\tilde{u})_j\|_{L_2} + \|(u-\tilde{u})_j\|_{L_2}$$

$$+ \|\xi_{j,\theta}\|_{L_2}$$

and that a bound for $\|(W-\tilde{u})_{j+1}\|_{L_2}$ is given by (4.33). Thus the only modifications in the proof of Theorem 4.1 are in replacing $\|(U-u)_{j,\theta}-\xi_{j,\theta}\|_{L_2}$ by $\|W_j^\theta-\tilde{u}_{j,\theta}-\xi_{j,\theta}\|_{L_2}$, adding $\Delta t\|\rho_{j,\theta}\|^2_{L_2}$ to the right hand side of (4.11') and $C\Delta t^2 \sum_{j=0}^{N-1}\|\rho_{j,\theta}\|^2_{L_2}$ to the right hand sides of (4.11') and (4.12)

Proof of Theorem 4.3 is completed.

We now obtain estimates for the predictor-corrector version of the Crank-Nicolson Galerkin procedure, that is (4.4) and (4.5) with $\theta = 0$.

Theorem 4.4. Assume the hypotheses of Theorem 4.2. Further assume that $|(a_{ij}(x))_{x_k}| \le M^*$, $1 \le i, j \le n$, $1 \le k \le n$. Let U_j, $0 \le j \le N$, be defined by (4.4)-(4.6) with $\theta = 0$. Then if Δt is sufficiently small, we have inequality (4.23) The constants C and $\bar{C}$ also depend on $\mu(\Omega)$ and $\|\tilde{u}_{x_i x_k}\|_\infty$, $1 \le i, k \le n$.

Proof: From (4.24)-(4.26) we obtain $O((\Delta t)^2)$ estimates for $\|\alpha_{j,0}\|_{H_0^1}$, $\|\xi_{j,0}\|_{L_2}$ and $\|\rho_{j,0}\|_{L_2}$, $0 \le j \le N-1$. We find that

$$\|\beta_{j,0}\|_{L_2} \le \Delta t \mu(\Omega)^{\frac{1}{2}} \|u_t\|_\infty$$

Theorem 4.3 now yields the estimate

$$\|(U-u)_N\|_{L_2} \le O(h^s) + O(\Delta t^{3/2});$$

the estimate $O(h^s) + O((\Delta t)^2)$ is not obtained since $\|\rho_{j,\theta}\|_{L_2}$ is only $O(\Delta t)$. We modify Theorem 4.3.

From (4.32) we have

$$G_3 = -\int_\Omega \sum_{i,k=1}^{n} a_{ik}(x)[p(x,U_j)-p(x,u_j)](\tilde{u}_{j+\frac{1}{2}})_{x_k}(V_{j,0})_{x_i}dx$$

$$+ a(u_j + \beta_{j,0};\ \alpha_{j,0}, V_{j,0})$$

Now $u_j + \rho_{j,0} = u(x,(j+\frac{1}{2})\Delta t)$. Performing integration by parts, we obtain

$$\int_\Omega \sum_{i,k=1}^{n} a_{ik}(x)[p(x,u_j+\rho_{j,0})-p(x,u_j)](\tilde{u}_{j+\frac{1}{2}})_{x_k}(V_{j,0})_{x_i}\,dx$$

$$= -\int_\Omega \sum_{i,k=1}^{n} [a_{ik}(x)(p(x,u(x,(j+\tfrac{1}{2})\Delta t)) - p(x,u_j))(\tilde{u}_{j+\frac{1}{2}})_{x_k}]_{x_i} V_{j,0}\,dx$$

$$\le C\Delta t\ \|V_{j,0}\|^2_{L_2} \le 4C^2\ (\Delta t)^3 + \frac{1}{16\Delta t}\|V_{j,0}\|^2_{L_2}$$

where C is a constant which depends on n, M^*, $\mu(\Omega)$, $\|u_t\|_\infty$, $\|\nabla(p(x,u)_t)\|_\infty$, and $\|\tilde{u}_{x_ix_j}\|_\infty$, $1 \le i,\ j \le n$. Bounding the first and third terms of G_3 as before, we have

$$|G_3| \le K\sqrt{C}_0\ n^{3/2}\|\nabla\tilde{u}\|_\infty\ \|(U-u)_j\|_{L_2}\|V_{j,0}\|_{H_0^1}$$

$$+ 4C^2(\Delta t)^3 + \frac{1}{16\Delta t}\ \ \|V_{j,0}\|^2_{L_2}$$

$$+ C_0\|\alpha_{j,0}\|_{H_0^1}\ \ \|V_{j,0}\|_{H_0^1}$$

$$\le C[(\Delta t)^3 + \|(U-\tilde{u})_j\|^2_{L_2} + \|(u-\tilde{u})_j\|^2_{L_2} + \|\alpha_{j,0}\|^2_{H_0^1}]$$

$$+ \frac{1}{16\Delta t} \|V_{j,0}\|^2_{L_2} + \eta/4 \|V_{j,0}\|^2_{H^1_0}$$

where C is a constant which depends on n, M^*, $\mu(\Omega)$, $\|u_t\|_\infty$, $\|\nabla\tilde{u}\|_\infty$, K, C_0, η, $\|u_t\|_\infty$, $\|\nabla(p(x,u)_t)\|_\infty$, and $\|\tilde{u}_{x_i x_j}\|_\infty$, $1 \le i,\ j \le n$. Substituting the bounds for G_1 and G_2 given in Theorem 4.3 and the above bound for G_3 into (4.31) we obtain

$$\frac{1}{2\Delta t}[\|(W-\tilde{u})_{j+1}\|^2_{L_2} - \|(U-\tilde{u})_j\|^2_{L_2}] + \eta/2\|V_{j,0}\|^2_{H^1_0}$$
$$\le \tfrac{1}{2}[(u-\tilde{u})_t\|^2_{L_2}(t_{j,0}) + \|\rho_{j,0}\|^2_{L_2}] + (C+\frac{3}{16\Delta t})\|V_{j,0}\|^2_{L_2}$$
$$+ C[\Delta t\|\rho_{j,0}\|^2_{L_2} + \|\xi_{j,0}\|^2_{L_2} + (\Delta t)^3 + \|\alpha_{j,0}\|^2_{H^1_0}$$
$$+ \|(u-\tilde{u})_j\|^2_{L_2} + \|(u-\tilde{u})_{j+1}\|^2_{L_2}]$$

where C is defined as before. We now use the same argument given in Theorem 4.3 following (4.33').

The results derived in Theorem 4.4 involve constants $\|\tilde{u}_{x_i x_k}\|_\infty$, $1 \le i,\ k \le n$. Using an argument similar to the one given in Lemma 3.5, one can show that $\|\tilde{u}_{x_i x_k}\|_\infty$, $1 \le i$, $k \le n$ is bounded independently of h if $\Omega = B = \mathop{X}_{i=1}^{n} (a_i,d_i)$, S^h

is the span of the tensor products of piecewise Hermite polynomials of degree $2m-1$, $m \geq 2$, defined on a partition ρ of B, and $u \in L_{\infty}([0,T], K^{q}(\bar{B}))$ with $n/2 + 2 \leq q \leq 2m$.

References

[1] S. Agmon, Lectures on Elliptic Boundary Value Problems, Van Nostrand, Princeton, 1965.

[2] J. P. Aubin, "Behavior of the error of the approximate solutions of boundary value problems for linear elliptic operators by Galerkin's and finite difference methods", Annali della Scuola Normale Superiore di Pisa 2, 599-637, (1967).

[3] R. Bellman, Stability Theory of Differential Equations, McGraw Hill, New York 1952.

[4] L. Bers, F. John, and M. Schechter, Partial Differential Equations, Interscience, New York, 1966.

[5] J. H. Bramble and A. H. Schatz, "Raleigh-Ritz-Galerkin methods for Dirichlet's problem using subspaces without boundary conditions" to appear.

[6] G. Birkhoff, M. H. Schultz, and R. S. Varga, "Piecewise Hermite interpolation in one or two variables with applications to partial differential equations", Num. Math. 11, 232-256, (1968).

[7] P. G. Ciarlet, M. H. Schultz, and R. S. Varga, "Numerical Methods of high-order accuracy for non-linear boundary value problems", Num. Math. 9, 394-430, (1967).

[8] J. Dendy, Ph.D. Thesis, Rice University, 1971.

[9] J. Douglas, Jr. and T. Dupont, "Galerkin methods for parabolic equations", SIAM J. Numer. Anal. 7, 575-626, (1970).

[10] J. Douglas, Jr. and T. Dupont, "The numerical solution of waterflooding problems in petroleum engineering", Studies in Numerical Analysis, vol. 2, SIAM, Philadelphia, 53-63, (1970).

[11] J. Douglas and T. Dupont, "Galerkin Methods for Parabolic Problems II" to appear.

[12] T. Dupont (private communication).

[13] R. E. Edwards, Fourier Series, Holdt, Rinehart, and Winston, Dallas, 1967.

[14] G. Fex and G. Strang, "A Fourier analysis of the finite element variational method", to appear

[15] C. Gardner, "Rounding errors in the solution of the one dimensional heat equation using a Galerkin technique", M.A. thesis, Rice University, 1970.

[16] M. Lees, "A priori estimates for the solution of a difference approximation to parabolic differential equations", Duke Math J., Vol. 27, no. 3, 297-312, (1960).

[17] J. Nitsche, "Ein Kriteruim fur die quasi-optimilital des Ritzschen verfahrend", Numer. Math. 11, 346-348 (1968).

[18] H. S. Price and R. S. Varga, "Error bounds for semi-discrete Galerkin approximations of parabolic problems with applications to petrolum reservoir mechanics", Numerical Solution of Field Problems in Continuum Physics, Amer. Society, Providence, 74-95, (1970).

[19] M. H. Schultz and R. S. Varga, "L-Splines", <u>Numer. Math</u>. 10 345-369, (1967),

[20] M. H. Schultz, "L^∞ Multivariate Approximation theory", <u>SIAM J</u>. <u>Numer.</u> <u>Anal.</u> 6, 161-183, (1969).

[21] M. H. Schultz, "Approximation theory of multivariate spline functions in Sobolev spaces", <u>SIAM J</u>. <u>Numer. Anal</u>. 6, 570-582, 1969.