A Probabilistic Analysis for the Range Assignment Problem in Ad Hoc Networks[†]

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ABSTRACT

In this paper we consider the following problem for ad hoc networks: assume that n nodes are distributed in a d-dimensional region, with $1 \leq d \leq 3$, and assume that all the nodes have the same transmitting range r; how large must r be to ensure that the resulting network is strongly connected? We study this problem by means of a probabilistic approach, and we establish lower and upper bounds on the probability of connectedness. For the one-dimensional case, these bounds allow us to determine a suitable magnitude of r for a given number of nodes and displacement region size. In an alternate formulation, the bounds allow us to calculate how many nodes must be distributed should the transmitting range be fixed. Finally, we investigate the required magnitude of r in the two and three-dimensional cases through simulation. Based on the bounds provided and on the simulation analysis, we conclude that, as compared to the deterministic case, a probabilistic solution to this range assignment problem achieves substantial energy savings. A number of other potential uses for our analyses are discussed as well.

1. INTRODUCTION

Recent emergence of affordable, portable, wireless communication and computation devices, and concomitant advances in the communication infrastructure, have resulted in the rapid growth of mobile wireless networks. Among these, *ad hoc networks*, i.e. networks of mobile, untethered units communicating with each other via radio transceivers, are receiving increasing attention in the scientific community. Ad hoc networks, also called *multi-hop packet radio networks*, can be used wherever a wired backbone is not viable, e.g. in mobile computing applications in areas where other infrastructure is unavailable, or to provide communications during emergencies.

When designing protocols for ad hoc networks, the following characteristics peculiar to these networks have to be taken into account:

- *shared communications*: since the stations in the network communicate via radio transceivers, the most natural communication paradigm is one-to-many: when a unit transmits, all the units within its transmitting range receive the message. On the contrary, wired networks use selective transmission (one-to-one) as the natural communication paradigm.
- *energy constraints*: since the stations are equipped with limited energy supplies, one of the primary goals is to reduce the overall energy consumption of the network, thus increasing its lifetime.

Routing, broadcast and clustering protocols explicitly designed for ad hoc networks have been recently proposed in the literature [1,2,5,6,9,11,12,13,15]. Some of these protocols are designed for energy-efficient operation in an existing network topology, while others attempt to deal with the effects of mobility, and still others consider both of these aspects.

It should be observed that further energy can be saved if the network topology itself is energy-efficient, i.e. if the transmitting ranges of the units are set in such a way that a target property (e.g. strong connectivity[1]) of the resulting network topology is guaranteed, while the global energy consumption is minimal. For this reason, *topology control* protocols have been recently introduced [10,14,16] in the literature. Informally speaking, a topology control protocol is an algorithm in which units adjust their transmitting ranges in order to achieve a desired topological property, while optimizing energy consumption. The problem of ensuring strong connectivity while minimizing some measure of energy consumption has also been considered in a more theoretical framework, where it is referred to as the *range assignment problem*. In particular, it has been shown that determining an optimal range assignment is solvable in polynomial time in the one-dimensional case, while it is NP-hard

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1 A directed graph $G=(V,E)$ is said to be *strongly connected* if, for every $u,v \in V$, there exist directed paths from u to v and from v to u.

in the two and three-dimensional cases [3,8] [2].

In this paper, we consider the range assignment problem under a probabilistic model in which we assume the assignment to be homogeneous, i.e. each node sets its transmitting range to the same value r. Note that, in this formulation, r is a critical parameter to optimize because the energy consumed by a node for communication is directly dependent on its transmitting range. In the probabilistic model that we adopt, n nodes are distributed uniformly and independently in a d-dimensional cube having sides of length l, with $1 \le d \le 3$. One question to ask in this setting is: "How large must r be to ensure strong connectivity of the resulting network?". It can be easily seen that the deterministic solution to this problem is $r \approx l\sqrt{d}$, which accounts for the fact that nodes could be concentrated at opposite corners of the displacement region. However, this scenario appears to be very unlikely in most realistic situations.

We investigate the tradeoffs between the values of r, n and l that are necessary or sufficient in order to produce a strongly connected graph with high probability. In particular, we establish lower bounds on the magnitude of $r^d n$ (with respect to l) that is necessary to produce a high probability of connectedness. Furthermore, we derive an upper bound on rn that is sufficient to ensure connectedness with high probability for the case d=1. The derived bounds are validated by means of simulation, which also gives experimental tradeoffs between r, n and l for the cases d=2 and d=3. Based on the bounds provided and on the simulation analysis, we conclude that, as compared to the deterministic case, a probabilistic solution to the range assignment problem achieves substantial energy savings. In light of existing work on related topics, our analyses can also be used in a number of other ways, as described in the next section.

2. RELATED WORK

In this section, we review several topology control protocols that have been presented recently in the literature.

In [16], Rodoplu and Meng presented a distributed topology control protocol aimed at minimizing the energy required to communicate with a given master node. They assume that nodes are equipped with low-power GPS receivers, which provide position information to each node. The protocol is divided in two phases. Initially, every node iteratively broadcasts its position to different *search regions*. This process ends when the node, based on the position information gathered from neighbors, is able to calculate a set of nodes referred to as its *enclosure*. The authors prove that the *enclosure graph*, i.e. the graph in which every node is connected to all the nodes in its enclosure, is strongly connected. In the second phase of the protocol, the enclosure graph is pruned in a distributed fashion so that the energy needed to communicate with a given master node is minimum. The protocol has been simulated in both static and dynamic networks, and the simulation results show that it achieves significant energy savings.

In [14], Ramanathan and Rosales-Hain considered the problem of minimizing the maximum of node transmitting ranges while achieving connectedness. They also considered the stronger requirement of bi-connectivity. They present centralized topology control algorithms that provide the optimal solution for both versions of the problem. The range assignment returned by the algorithm has the additional property of being per-node minimal, i.e. no transmitting range can be reduced further without impairing connectedness (or bi-connectivity). The authors also present two heuristics to deal with mobility. The first heuristic, called LINT, aims at maintaining a specified number of neighbors. Every node is configured with three parameters: the desired number of neighbors, and high and low thresholds. When the actual number of nodes is below (above) the threshold, the transmitting range is increased (decreased) until the number of neighbors is in the proper range. A second heuristic, called LILT, is considered. LILT exploits the global topology information that is available with some routing protocols: if the topology change indicated by the routing update results in a disconnected graph, LILT overrides the high threshold on the number of neighbors in order to restore connectedness. The performance of the algorithms and of the heuristics proposed are evaluated through simulations, which show that they reduce the energy consumption or improve the network throughput.

In [10], Li, et al., analyze the properties of a distributed topology control protocol based on directional information. The basic idea is that a node u transmits with the minimum power $p_{u,\alpha}$ such that there is at least one neighbor in every cone of angle α centered at u. In order to reduce the power, α should be as large as possible. The authors show that setting $\alpha=5\pi/6$ is a necessary and sufficient condition to guarantee connectedness, while the stronger requirement $\alpha=2\pi/3$ ensures the network is strongly connected. A set of optimizations aimed at further reducing the power consumption without impairing (strong) connectedness is also presented. Furthermore, the authors discuss how the algorithm could be modified to deal with mobility. The effectiveness of the proposed protocol for static networks is demonstrated by means of simulation.

Although the protocols described above have been proven to be effective, all of them rely on rather strong assumptions. In fact, the protocol of [16] assumes that every node is equipped with a GPS receiver, while in [14] it is assumed that the relative distances of all the nodes are given as input to a centralized topology control algorithm. Finally, the protocol of [10] relies on directional information available at each node, which is made possible by using more than one directional antenna. Such assumptions are not achievable in many settings, such as sensor networks, where nodes must be extremely simple, and where no centralized communication facility is available.

It should also be observed that, even in those settings where the proposed topology control mechanisms are feasible, some aspects of their implementation have not been considered. For instance, in [16] the number of iterations needed to determine the enclosure is based on the definition of the initial search region. Hence, the search region is a critical aspect, which affects the energy consumption of the protocol. However, no clues are given on how this region should be defined. In [14], the authors give no insight

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[2] It should be noted that the topology control protocols proposed in [16] and [14] minimize a given objective function, which is not the overall energy consumption. In particular, in [16] the authors minimize the energy needed to communicate with a single master node, while in [14] the authors minimize the maximum transmitting range of nodes. In [10], no explicit cost function is minimized.

into how the appropriate number of neighbors should be selected for the LINT heuristic. Finally, in [10] every node starts by broadcasting a "hello" message with some minimal power p_0. If the nodes that respond to the message cover all the cones with the given angle, then the protocol stops, otherwise the transmitting power is increased and a new "hello" message is broadcast. This process is iterated until the desired coverage is guaranteed. However, what the initial power and its step increase should be is not discussed.

These observations provide the following additional motivations to study the probabilistic range assignment problem, beyond simply reducing energy consumption as compared to the deterministic solution:

- to determine an initial range assignment for nodes from which a simple (and therefore energy-efficient) search for an optimized assignment can be initiated; such an assignment could be the initial "best guess" for distributed protocols such as those in [16] and [10];
- to study simple topological properties (e.g. minimum degree) that are associated with strong connectivity; these properties can be useful, for instance, in determining appropriate parameters for the LINT protocol [14].

A final motivation stems from the observation that inexpensive radio transceivers might not allow the transmitting range to be adjusted [14]. This type of transceiver is likely to be used in sensor networks, where the individual unit should cost as little as possible. Under this scenario, a fundamental question to the system designer would be: for a given transmitter technology, how many nodes must be distributed over a given region to ensure connectedness with high probability? Such a question can be addressed using the probabilistic model presented herein.

3. PRELIMINARIES

A multi-hop wireless d-dimensional network is represented by a pair M_d=(N,D), where N is a set of nodes, and $D: N \times T \to [0,l]^d$, for some l>0, is the *displacement function*. $T \subset \mathbf{R}$ is a set of moments of time. The parameter d, with $1 \le d \le 3$, is called the *dimension* of the network. The cardinality of set N, denoted n, is called the *order* of the network. In the following it is assumed that n>1. The displacement function assigns to every element of N and to any time t a set of coordinates in the d-dimensional cube of width l, representing the node's physical position at time t. The choice of limiting the admissible physical locations of units to a bounded region of $\mathbf{R}^d$ is realistic and will ease the probabilistic analysis of Section 5. For the sake of simplicity, we assume that the bounded region is of the form $[0,l]^d$, for some l>0. Parameter l is called the *size* of the system. If the physical location of nodes does not vary with time, the network is said to be *static*, and function D can be redefined simply as $D: N \to [0,l]^d$.

A *range assignment* for a d-dimensional network M_d=(N,D) is a function $RA: N \to (0,R]$ that assigns to every element of N a value in (0,R], representing its transmitting range. Parameter R is called the *maximum transmitting range* of the nodes in the network and depends on the features of the radio transceivers equipping the mobile stations. We assume that all the stations are equipped with transceivers having the same features; hence, we have a single value of R for all the nodes in the network.

Given a d-dimensional network M_d=(N,D) and a range assignment RA for M_d, the *communication graph* of M_d induced by RA at time t, denoted $G_M(RA,t)$, is defined as $G_M(RA,t)=(N,E(t))$, where the directed edge (u,v) is in E(t) if and only if v is in the transmitting range of u at time t, i.e. if and only if $\delta_d(D(u,t),D(v,t)) \le RA(u)$, where δ_d denotes the Euclidean distance in the d-dimensional space. In this case node v is said to be a *neighbor* of u. Note that $G_M(RA,t)$ as defined here corresponds to the *point graph* as defined in [17].

The communication graph represents the set of all possible communication links in the network and is used to describe the desired properties of the wireless network. One common requirement is that the communication graph be strongly connected. Given a d-dimensional network M_d, a range assignment RA is said to be *connecting at time t* if $G_M(RA,t)$ is strongly connected. Conversely, RA is said to be *disconnecting at time t* if $G_M(RA,t)$ is not strongly connected.

A range assignment RA is said to be *r-homogeneous* if $RA(u)=r$ for every $u \in N$, i.e. if all the units have the same transmitting range r, for some $r \le R$.

If a topology control mechanism is available, the range assignment may vary with time in order to ensure target properties (e.g. strong connectivity, or a given diameter $h<n$) of the communication graph. Hence, a sequence of range assignments $RA_{t_1}, RA_{t_2},\ldots$ can be defined, where RA_{t_i} is the range assignment at time t_i and the transition between range assignments is determined by the topology control mechanism. If topology control is not available, the range assignment is said to be *static* and cannot vary with time. Observe that if the range assignment is static but the network is not static, it is very likely that the target properties of the communication graph will be impaired at least temporarily.

4. A RANGE ASSIGNMENT PROBLEM FOR AD HOC NETWORKS

In the remainder of this paper we consider the following problem for static ad hoc networks with homogeneous transmitting range:

MINIMUM HOMOGENEOUS RANGE ASSIGNMENT (MHRA):

> Suppose n nodes are placed in $[0,l]^d$; let R be the maximum transmitting range of the nodes, and let the R-homogeneous range assignment be connecting. What is the minimum r such that the r-homogeneous range assignment is connecting?

Solving this problem is of primary importance in all those settings in which a topology control mechanism is not feasible and the physical location of units is not known in advance, as could be the case in a sensor network whose elements are spread from a moving vehicle (airplane, ship or spacecraft). In this scenario, setting the same transmitting range for all the units is a reasonable choice, and the transmitting range should be as small as possible in order to reduce both the power consumption and the interference between node transmissions while preserving connectedness.

It can be easily seen that the deterministic solution to MHRA is $r \in \Theta(l)$ for any n, since the worst-case scenario occurs when all the units are concentrated at opposite corners of $[0,l]^d$. However, this scenario appears to be very unlikely in most realistic situations. For this reason we study MHRA by means of a probabilistic model, which establishes tradeoffs between the magnitudes of r, n and l that yield a high probability of connectedness. Observe that, when dealing with the magnitude of l, the choice of unit is important. In the following, we assume that r and l are measured using the same arbitrary unit, which is therefore canceled out when discussing the relative sizes of r and l.

5. A PROBABILISTIC MODEL FOR MHRA

Consider the probability space $(\Omega_l, \mathcal{F}_l, P_l)$, where $\Omega_l=[0,l]^d$, $\mathcal{F}_l$ is the family of all closed subsets of Ω_l and P_l is a probability distribution on Ω_l. In this paper, we assume that P_l is the uniform distribution on Ω_l, i.e. the distribution of a vector-valued random variable Z with values in Ω_l such that for any $V \in \mathcal{F}_l$,

$$P_l(Z \in V) = \frac{vol(V)}{l^d},$$

where $vol(V)$ is the d-dimensional volume of region V. Under this setting, nodes in N can be modeled as independent random variables with values in Ω_l, which will be denoted $Z_1,\ldots,Z_n$.

We say that an event E_k, describing a property of a random structure depending on a parameter k, holds *asymptotically almost surely* (a.a.s. for short), if $P(E_k) \to 1$ as $k \to \infty$.

We recall the standard notation regarding the asymptotic behavior of functions. Let f and g be functions of the same parameter x. We have:

- $f(x) \in O(g(x))$ if there exist constants C and x_0 such that $f(x) \le C \cdot g(x)$ for any $x \ge x_0$;
- $f(x) \in \Omega(g(x))$ if $g(x) \in O(f(x))$;
- $f(x) \in \Theta(g(x))$ if $f(x) \in O(g(x))$ and $f(x) \in \Omega(g(x))$;
- $f(x) \in o(g(x))$ if $f(x)/g(x) \to 0$ as $x \to \infty$.
- $f(x) << g(x)$ or $g(x) >> f(x)$ if $f(x) \in o(g(x))$.

In the following we consider the asymptotic behavior of the event CONNECTED_l on the random structures $(\Omega_l, \mathcal{F}_l, P_l)$ as $l \to \infty$. Informally speaking, event CONNECTED_l corresponds to all the values of the random variables $Z_1,..,Z_n$ for which the r-homogeneous range assignment is connecting.

5.1 An upper bound on the probability of a connecting range assignment

The formal definition of CONNECTED_l in terms of $Z_1,..,Z_n$ and r is a difficult task. For this reason, we first study the asymptotic behavior of the simpler event $\text{ISOLATEDNode}_l(i)$, which corresponds to all the values of the random variables $Z_1,..,Z_n$ such that node i is isolated in the communication graph. It is immediate that $\text{ISOLATEDNode}_l(i) \subseteq \text{DISCONNECTED}_l$, where $\text{DISCONNECTED}_l = \Omega_l - \text{CONNECTED}_l$. As a consequence, if $P_l(\text{ISOLATEDNode}_l(i)) \to \varepsilon > 0$ as $l \to \infty$, then CONNECTED_l does not hold a.a.s.. Hence, lower bounds on the order of magnitude of r and n that enable an a.a.s. connected communication graph can be derived by studying the asymptotic behavior of $\text{ISOLATEDNode}_l(i)$.

Upper and lower bounds to the probability of the event $\text{ISOLATEDNode}_l(i)$ are stated in the following Lemma.

Lemma 1

$\forall l>0, \forall i \in \{1,..,n\}$,

$$\left(1-c_1\left(\frac{r}{l}\right)^d\right)^{n-1} \le P_l(\text{ISOLATEDNode}_l(i)) \le \left(1-c_2\left(\frac{r}{l}\right)^d\right)^{n-1}$$

for networks of dimension d, where c_1 and c_2 are constants depending on d.

Proof.

We report the proof for the case d=1. The proofs for the cases d=2 and d=3 are similar. Consider an arbitrary node i, whose displacement is given by the value of the random variable Z_i. Node i is isolated if none of the remaining n-1 nodes is within its transmitting range. The upper bound accounts for the fact that i is at coordinate 0 or l. In this case, the probability that any other node is out of i's transmitting range is $\left(1-\frac{r}{l}\right)$, and the upper bound follows observing that the random variables are independent (in this case, c_2=1). The lower bound accounts for the fact that when node i is displaced in $[r,l-r]$ the probability that any other node is out of i's transmitting range is $\left(1-2\frac{r}{l}\right)$. ■

As a consequence of Lemma 1, the asymptotic behavior of $\text{ISOLATEDNode}_l(i)$ is given by $\lim_{l\to\infty}\left(1-\left(\frac{r}{l}\right)^d\right)^n$. The actual value of this limit depends on the magnitudes of r and n expressed as a function of l. First, we observe that if $r \in \Theta(l)$ and n is any increasing function of l, then the limit is at most 0, thus not contradicting the magnitude of r for the deterministic case. The following theorem establishes the value of the limit for $r \in o(l)$.

Theorem 1

Let $r \in o(l)$. Then:

$$\lim_{l\to\infty}\left(1-\left(\frac{r}{l}\right)^d\right)^n = \begin{cases} 1 & \text{if } r^d n << l^d \\ e^{-C} & \text{for some constant } C>0 \text{ if } r^d n \in \Theta(l^d) \\ 0 & \text{if } r^d n >> l^d \end{cases}$$

Proof.

The proof of this theorem is reported in the Appendix. ■

Corollary 1

Suppose n nodes are displaced in $[0,l]^d$ according to the uniform distribution. Then the r-homogeneous range assignment is not a.a.s. connecting if $r^d n \in O(l^d)$, and it is a.a.s. disconnecting if $r^d n << l^d$.

Corollary 1 states that the magnitude of $r^d n$ must be *strictly greater* than the volume of the displacement region, to have a possibility that the r-homogenous range assignment is a.a.s. connecting.

5.2 A lower bound on the probability of a connecting range assignment for d=1

The characterization of the event CONNECTED_l is not straightforward. In this sub-section we give a formal definition of its complementary event DISCONNECTED_l in terms of $Z_1,..,Z_n$ and r for the case d=1. The definition of CONNECTED_l in terms of DISCONNECTED_l follows trivially.

Let $x_1,..,x_n$ be the values assigned (according to the uniform distribution in $[0,l]$) to the random variables $Z_1,..,Z_n$. Let us order values $x_1,..,x_n$ starting from the smallest. We obtain a permutation ρ of the indices such that the sequence $x_{\rho(1)},\ldots x_{\rho(n)}$ is ordered. It is straightforward that the r-homogeneous range assignment on the network whose nodes are located at $x_1,..,x_n$ is disconnecting if and only if there exist $1<i\leq n$ such that $x_{\rho(i)}-x_{\rho(i-1)}>r$. Thus, the range assignment is disconnecting if and only if there exists a segment $S=[s,s+\lambda]$, with $\lambda\geq r$, such that no unit is displaced in S, at least one unit is in $[0,s)$ and at least one unit is in $(s+\lambda,l]$ (see Figure 1). The segment S is said to be a λ-*hole*.

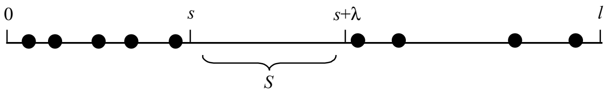

Figure 1. **Node displacement generating a disconnecting range assignment ($\lambda \geq r$).**

In order to upper bound the probability of the event DISCONNECTED_l, we consider the probability of the sub-event $\text{DISCONNECTED}_l^{s,\lambda}$, in which the left point of the λ-hole is at coordinate s. Observe that $\text{DISCONNECTED}_l^{s,\lambda} \subset$ DISCON-$\text{NECTED}_l^{s,\lambda'}$ for every $\lambda' < \lambda$, hence we can restrict our attention to the largest event, i.e. $\text{DISCONNECTED}_l^{s,r}$.

Let us define random variables LH, H, and HR representing the number of nodes to the left of the hole, in the hole, and to the right of the hole, respectively. Since random variables $Z_1,..,Z_n$ are independent, the probability $P(LH=k_1,H=k_2,HR=k_3)$ with $k_1+k_2+k_3=n$ is given by:

$$\frac{n!}{k_1!k_2!k_3!}p_1^{\,k_1}p_2^{\,k_2}p_3^{\,k_3},$$

i.e. it is the multinomial distribution [4] with parameters $p_1=s/l$, $p_2=r/l$ and $p_3=(l-s-r)/l$. We have:

$$P(\text{DISCONNECTED}_l^{s,r})=\sum_{i=1}^{n-1}P(LH=i,H=0,HR=n-i)=$$

$$\sum_{i=1}^{n-1}\frac{n!}{(n-i)!i!}p_1^{\,i}p_3^{\,n-i}=\sum_{i=1}^{n-1}\binom{n}{i}p_1^{\,i}p_3^{\,n-i}$$

From the equality $(a+b)^n=\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i}$ we obtain:

$$P(\text{DISCONNECTED}_l^{s,r})=(p_1+p_3)^n-p_1^n-p_3^n\leq$$

$$\leq(p_1+p_3)^n=\left(1-\frac{r}{l}\right)^n$$

An upper bound to $P(\text{DISCONNECTED}_l)$ can be derived by summing the probabilities $P(\text{DISCONNECTED}_l^{s,r})$ for all possible values of s. We thus have:

$$P(\text{DISCONNECTED}_l)\leq\int_0^{l-r}P\left(\text{DISCONNECTED}_l^{s,r}\right)ds\leq$$

$$\leq\int_0^{l-r}\left(1-\frac{r}{l}\right)^n ds=(l-r)\left(1-\frac{r}{l}\right)^n \qquad (1)$$

Based on the preceding discussion, we can state the following theorem:

Theorem 2
Assume n nodes are displaced at random in $[0,l]$. Then, the probability that the r-homogeneous range assignment is connecting is at least $1-(l-r)\left(1-\frac{r}{l}\right)^n$.

The result stated in Theorem 2 can be useful in the design of sensor networks. If sensors are equipped with inexpensive transceivers that do not allow the transmitting range to be adjusted, and a certain region must be covered, a natural question would be: "how many sensors have to be deployed in order to obtain a strongly connected network?". Theorem 2 gives a probabilistic answer to this question in the case of one-dimensional networks. As an example, Table 1 reports the minimum value of n ensuring a probability of connectedness of at least 0.95 for values of l ranging from 1024 to 4194304, with range assignments $r=l/100$, $r=\sqrt{l}$ and $r=l/\log^2 l$.[3] It should be observed that if the transmitting range is proportional to l, then the number of nodes required to achieve a high probability of connectedness grows very slowly with l. Conversely, if $r\in o(l)$, then the number of nodes required grows rapidly with l. Since the transmitting range is expected to be fairly small with the inexpensive transmitters likely to be used in sensor networks, this implies possibly large numbers of nodes will be required in those settings.

Since we are assuming $r\in o(l)$, the asymptotic behavior of the expression (1) is the same as that of the simpler expression $l\left(1-\frac{r}{l}\right)^n$. The following theorem establishes the condition under which $\lim_{l\to\infty}l\left(1-\frac{r}{l}\right)^n=0$.

Theorem 3
Let r and n be positive increasing functions of l, with $l>0$. Let $r\in o(l)$ and assume that $\exists l_0$ and $\varepsilon>0$, such that $rn\geq l(1+\varepsilon)\ln l$ for all $l\geq l_0$. Then $\lim_{l\to\infty}l\left(1-\frac{r}{l}\right)^n=0$.

Proof. The proof is reported in the Appendix. ■

[3] Unless otherwise specified, all the logarithms in this paper have a base of 2.

l	$r=l/100$	$r=\sqrt{l}$	$r=l/\log^2 l$
1024	987	312	987
4096	1125	718	1623
16384	1263	1619	2482
65536	1401	3598	3598
262144	1539	7914	5005
1048576	1677	17254	6735
4194304	1815	37356	8821

Table 1. **Minimum value of n ensuring a probability of connectedness of at least 0.95. We considered values of l ranging from 1024 to 4194304, and three different range assignments: $r=l/100$, $r=\sqrt{l}$ and $r=l/\log^2 l$.**

We can conclude this subsection with the following theorem:

Theorem 4

Suppose n nodes are placed in $[0,l]$ according to the uniform distribution. If $rn\in\Omega(l \log l)$, then the r-homogeneous range assignment is a.a.s. connecting.

Proof.

We have $\lim_{l\to\infty} P(\text{CONNECTED}_l) \geq 1-\lim_{l\to\infty} l\left(1-\frac{r}{l}\right)^n = 1$, and the proof follows trivially by observing that $P(\text{CONNECTED}_l)\leq 1$ for every l. ■

6. SIMULATION RESULTS

The probabilistic model introduced in the previous section has been experimentally evaluated by means of extensive simulations. The objective of the simulations was twofold: on one hand, to validate the conditions, obtained by the theoretical analysis, under which range assignments are a.a.s. disconnecting and a.a.s. connecting (d=1 only); on the other hand, to provide simulation-based analysis of the probability of connectedness for two and three-dimensional networks.

The simulator distributes n nodes in $[0,l]^d$ according to the uniform distribution, then generates the communication graph assuming that all the nodes have the same transmitting range r. Parameters n, l, d and r are given as input to the simulator, along with the number of iterations to run. The simulator returns the percentage of connecting range assignments and the average number of neighbors of a node (i.e., the average degree of the communication graph). The average is evaluated over all the iterations, including those that yielded a disconnecting range assignment.

A first set of simulations was aimed at confirming Corollary 1, which gives conditions under which range assignments are a.a.s disconnecting. We considered increasing values of l ranging from 256 to 1048576. For every value of l, we chose r and n in such a way that $r^d n=l^d$ and we ran 250 simulations. Two cases were considered: $n=\sqrt{l}$ and $n=l/\log^2 l$, thus obtaining values of n ranging from 16 to 1024 and from 4 to 2621, respectively[4]. Simulating beyond n=2621 would have required excessive simulation time.

[4] In the latter case, the simulations for n=4 were not considered, due to their scarce significance.

The results of these simulations confirmed the Corollary 1 result: the percentages of connecting range assignments are always quite low, and tend to decrease as l increases. These results are not shown because the percentage of connecting assignments was quite close to 0 for all simulation runs. We also considered the impact of a multiplicative constant of r or n on the percentages of connecting assignments. The results showed a more significant effect due to a multiplicative constant of r than one of n. Although showing higher percentages of connecting range assignments with respect to the previous simulations, the asymptotic behavior was confirmed: as l increases, the percentage of connecting assignments decreases (see Figure 2).

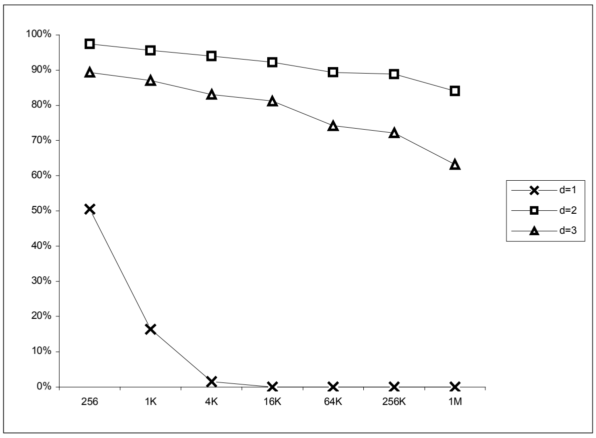

Figure 2. **Percentage of connecting range assignments for increasing values of l. Parameter n was set to $\sqrt{l}$. Parameter r was set to $3\sqrt{l}$ for d=1, to $2l^{3/4}$ for d=2, and to $1.5l^{5/6}$ for d=3.**

A second set of simulations was aimed at verifying Theorem 4 for one-dimensional networks. Recall that Theorem 4 states conditions under which range assignments are a.a.s. connecting for d=1. We ran simulations with the same values of l and n, this time setting r in such a way that $rn=l \log l$. The results were that 100% of the range assignments were connecting for all the simulations. We also considered the effect of multiplicative constants of r on the percentage of connecting assignments. We ran simulations with $r'=kr$, for values of k ranging from 0.5 to 0.9 in steps of 0.1. As shown in Figure 3, the results once more confirmed the expected asymptotic behavior.

Finally, we investigated the relation between $r^d n$ and the percentage of connecting assignments for two and three-dimensional networks. Quite surprisingly, the simulations indicated that the bound for one-dimensional networks might also hold in two and three dimensions. In fact, we ran simulations for values of l ranging from 256 to 4194304, with values of n ranging from 16 to 2048 and from 4 to 8666. The larger value of l (and, consequently, of n) was needed in order to better investigate the asymptotic behavior. For every simulation, we set r in such a way that $r^d n=l^d \log l$. 100% of the assignments were connecting for all the simulations. We also ran the simulations with $r' = kr$, for values of k ranging from 0.5 to 0.9 in steps of 0.1. As shown in Figures 4 and 5, the results showed that a $l^d \log l$ bound is sufficient to ensure increasing percentages of connecting range assignments. Note for d=3 (Figure 5), that when the multiplicative constant on r gets small, percentages of connecting assignments can be quite low but that the asymptotic trend is still increasing in this case.

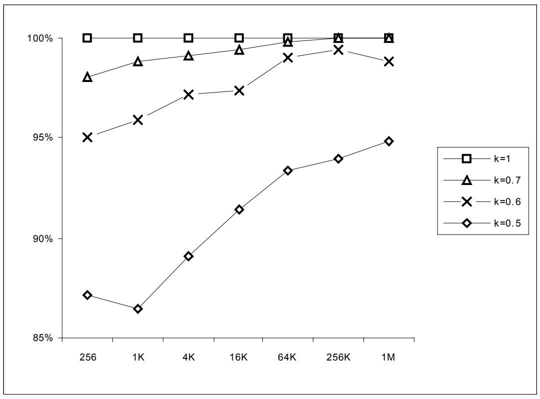

Figure 3. **Percentage of connecting range assignments for increasing values of l in one-dimensional networks. Parameters n and r were set to $\sqrt{l}$ and $k\sqrt{l}\log l$, respectively.**

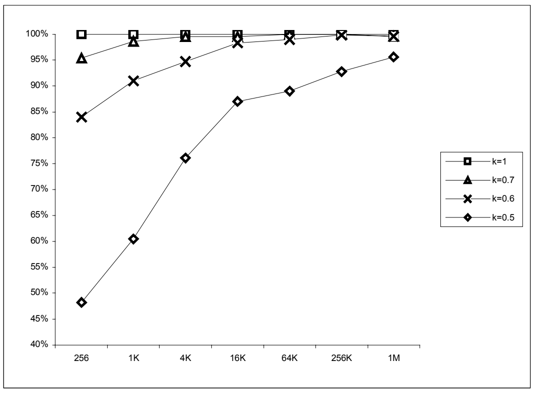

Figure 4. **Percentage of connecting range assignments for increasing values of l in two-dimensional networks. Parameters n and r were set to $\sqrt{l}$ and $kl^{3/4}\sqrt{\log l}$, respectively.**

The simulations evidenced a close relation between the average number of neighbors, denoted by *#Neigh,* and log l, i.e. the expected number of neighbors in the case d=1 (and, presumably, also in the cases d=2 and d=3). When the ratio *#Neigh*/log l was far below 1, the percentage of connecting range assignments was negligible; as *#Neigh*/log l approached 1, the percentage of connecting assignments approached 100%. Furthermore, *#Neigh*/log l had the same asymptotic behavior as the probability of obtaining a connecting assignment: if $r^d n \in O(l^d)$, it decreased as l increased; conversely, if $r^d n \in \Omega(l^d \log l)$, it increased (see Figure 6). This relationship could be an important guideline in the design of topology control protocols aimed at maintaining connectivity in highly dynamic networks. For example, the desired number of neighbors in the LINT heuristic of [14] could be set to log l.

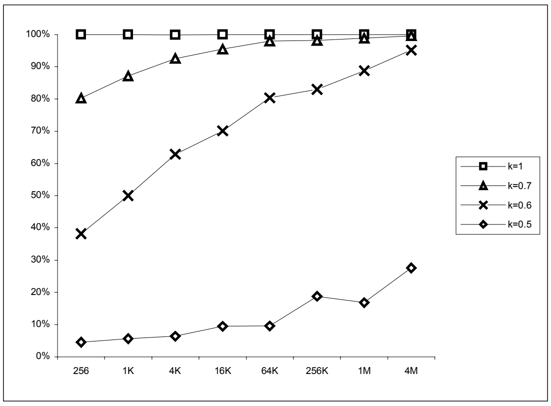

Figure 5. **Percentage of connecting range assignments for increasing values of l in three-dimensional networks. Parameters n and r were set to $\sqrt{l}$ and $kl^{5/6}\sqrt[3]{\log l}$, respectively.**

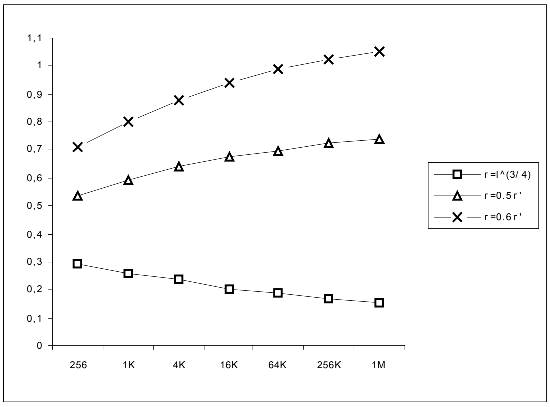

Figure 6. **Value of #*Neigh*/log l for increasing values of l in two-dimensional networks. Parameter n was set to $\sqrt{l}$. The plots for $r=l^{3/4}$, $r=0.5r'$ and $r=0.6r'$, with $r' = l^{3/4}\sqrt{\log l}$, are displayed.**

We also evaluated the transmitting range reduction achieved by a probabilistic solution to MHRA, by comparing the transmitting range $l\sqrt{d}$ of the deterministic solution with that obtained by setting r in such a way that $r^d n = l^d \log l$. Since the transmitting power is proportional to r^α, for some $2 \leq \alpha \leq 6$, a reduction in the transmitting range results in an even higher reduction in the power needed to transmit data. Figure 7 shows the ratio of the probabilistic to the deterministic transmitting ranges for d=1, d=2 and d=3. As can be seen, the ratio is always below 0.5, and can be as small as 0.02. It should also be noted that the relative reduction achieved by the probabilistic solution tends to increase as the system size increases.

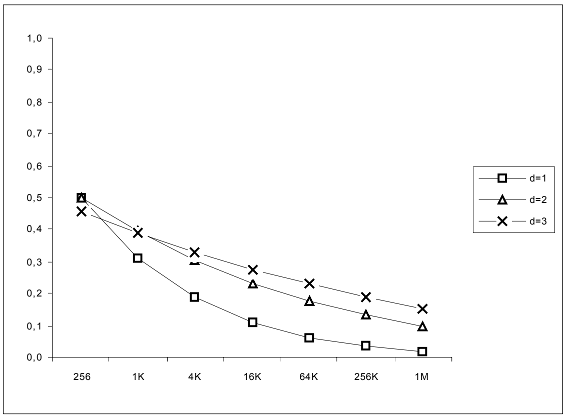

***Figure 7.* Ratio of the probabilistic to the deterministic transmitting ranges for increasing values of *l*. Parameter *n* was set to $\sqrt{l}$.**

7. CONCLUSIONS

The results obtained in this paper give positive answers to a number of the problems that motivated our study.

First, we showed that if a high probability of connectedness is sufficient, then the transmitting range (hence, the energy consumption) of nodes can be reduced substantially from the deterministic requirements. For the parameter values studied through simulation, the probabilistic transmitting range varies from 50% down to 2% of the deterministic value as *l* increases.

Our analysis can also be used to determine the appropriate number of nodes to distribute over a given interval in order to achieve a connecting assignment with high probability. To be specific, we have shown that, if *n* nodes are displaced at random in $[0,l]$, then the *r*-homogeneous range assignment is connecting with probability at least $1-(l-r)\left(1-\frac{r}{l}\right)^n$.

We also showed that $r \approx l\sqrt[d]{\frac{\log l}{n}}$ is a good choice for the initial range assignment for nodes from which the search for an optimized range assignment can be initiated. This result is useful to a number of existing topology control algorithms that require each node to search for its proper transmission range. Such a search could itself consume a great deal of energy without a good initial starting value.

The results presented in this paper also indicate that a degree of log *l* is a topological parameter closely related to the connectedness of the communication graph. The extent to which this relation can be exploited in the development of simple local topology control mechanisms is a matter of ongoing research. This relation also suggests an interesting analogy between our probabilistic analysis and the random graph model [7]. In the theory of random graphs, it has been shown that if the expected number of neighbors of a node is slightly above log *n* then the graph is connected with high probability, while in our model a high probability of connectedness appears to require log *l* neighbors. Hence, the connectedness of a random graph in our model depends on a geometric parameter, rather than on the number of nodes in the graph. This observation confirms our feeling (which is supported also by the discussion in [2]) that the classical theory of random graphs is not adequate to study ad hoc networks, and validates the model proposed in this paper as a promising alternative to classical random graphs theory and to the Random Network Model proposed in [2].

A final comment concerns the accuracy of the model presented in this paper. Although it is known that distance alone does not faithfully model the propagation of the radio signal in the air, we believe that the basic nature of the results would not change if more detailed propagation models were considered. Our feeling is supported by the simulation results presented in [18], where the performance of an energy-efficient routing protocol called GAF is evaluated using two different propagation models: the tworayground model, which corresponds to the model used in this paper, and the shadowing model, which accounts also for multi-path effects (fading). The simulation results show that the performance of GAF is not significantly impacted by the choice of the propagation model.

8. ACKNOWLEDGEMENT

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APPENDIX

Proof of Theorem 1.

We report the proof for the case d=1. The proofs for the cases d=2 and d=3 are similar.

The limit is a 1^∞ indeterminate form, which can be reduced to the indeterminate form 0/0 by taking the logarithm. We have:

$$\lim_{l\to\infty}\ln\left(\left(1-\frac{r}{l}\right)^n\right)=\lim_{l\to\infty} n\ln\left(1-\frac{r}{l}\right)=\lim_{l\to\infty}\frac{\ln\left(1-\frac{r}{l}\right)}{\frac{1}{n}},$$

which, since $r\in o(l)$, is a 0/0 indeterminate form. We apply the first l'Hopital's rule, obtaining:

$$\lim_{l\to\infty}\frac{\ln\left(1-\frac{r}{l}\right)}{\frac{1}{n}}=\lim_{l\to\infty}\frac{\frac{\partial\left(\ln\left(1-\frac{r}{l}\right)\right)}{\partial l}}{\frac{\partial\left(\frac{1}{n}\right)}{\partial l}}=\lim_{l\to\infty}\frac{\frac{-\frac{\partial\left(\frac{r}{l}\right)}{\partial l}}{1-\frac{r}{l}}}{\frac{\partial\left(\frac{1}{n}\right)}{\partial l}},$$

which, given that $r\in o(l)$, can be rewritten as:

$$\lim_{l\to\infty}\frac{-\frac{\partial\left(\frac{r}{l}\right)}{\partial l}}{\frac{\partial\left(\frac{1}{n}\right)}{\partial l}} \qquad (2)$$

The value of (2) depends of the magnitudes of r and n with respect to l. Assume $rn\in\Theta(l)$; then $r/l\in\Theta(1/n)$ and (2) equals $-C$ for some positive constant C. Hence, $\lim_{l\to\infty}\left(1-\left(\frac{r}{l}\right)^d\right)^n = e^{-C}$. Assume now $rn<<l$; then $r/l<<1/n$ and (2) equals 0, thus implying $\lim_{l\to\infty}\left(1-\left(\frac{r}{l}\right)^d\right)^n=1$. Finally, if $rn>>l$ then $r/l>>1/n$ and (2) equals $-\infty$, thus implying $\lim_{l\to\infty}\left(1-\left(\frac{r}{l}\right)^d\right)^n=0$. ■

Proof of Theorem 3.

Using an equality $\ln(1-x)=-x-\varphi(x)$, where $\varphi(x)=o(x)$ and $\varphi(x)\geq 0$ for $x\geq 0$, we obtain:

$$l\left(1-\frac{r}{l}\right)^n = e^{\ln l}e^{n\ln(1-\frac{r}{l})} = e^{\ln l-n(\frac{r}{l}+\varphi(\frac{r}{l}))} = e^{\ln l-\frac{nr}{l}-n\varphi(\frac{r}{l})}.$$

Denote by $a=\ln l-\frac{rn}{l}-n\varphi\left(\frac{r}{l}\right)$. It is clear that $\lim_{l\to\infty} l\left(1-\frac{r}{l}\right)^n=0$ if and only if $\lim_{l\to\infty} a=-\infty$. On the other hand, for $l\geq l_0$,

$$a=\ln l-\frac{rn}{l}-n\varphi\left(\frac{r}{l}\right)\leq \ln l-\frac{l(1+\varepsilon)\ln l}{l}-n\varphi\left(\frac{r}{l}\right)=$$

$$=-\varepsilon\ln l-n\varphi\left(\frac{r}{l}\right)\leq -\varepsilon\ln l.$$

Therefore, $\lim_{l\to\infty} a\leq\lim_{l\to\infty}(-\varepsilon\ln l)=-\infty$ and the theorem has been proved. ■