A probabilistic view of Hershfield's method for estimating probable maximum precipitation

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Paper submitted to Water Resources Research

January 1998 / October 1998

Abstract. A simple alternative formulation of the Hershfield's statistical method for estimating probable maximum precipitation (PMP) is proposed. Specifically, it is shown that the published Hershfield's data do not support the hypothesis that there exists a PMP as a physical upper limit, and therefore a purely probabilistic treatment of the data is more consistent. In addition, using the same data set, it is shown that Hershfield's estimate of PMP may be obtained using the Generalized Extreme Value (GEV) distribution with shape parameter given as a specified linear function of the average value of annual maximum precipitation series, and for return period of about 60 000 years. This formulation substitutes completely the standard empirical nomograph that is used for the application of the method. The application of the method can be improved when long series of local rainfall data are available that support an accurate estimation of the shape parameter of the GEV distribution.

1. Introduction

The probable maximum precipitation (PMP) defined as "theoretically the greatest depth of precipitation for a given duration that is physically possible over a given size storm area at a particular geographical location at a certain time of year" [*World Meteorological Organization*, 1986, p. 1] has been widely used for the design of major flood protection works. Typically, PMP is used to estimate the largest flood that could occur in a given hydrological basin, the so-called probable maximum flood (PMF). In turn, PMF is used to determine the design characteristics of flood protection works. The PMP approach, which practically assumes a physical upper bound of precipitation amount, is contrary to the probabilistic approach, according to which any amount must be associated with a probability of exceedance or return period.

Despite its widespread acceptance, the concept of PMP has been criticized by many hydrologists. We quote, for example, *Dingman* [1994, p. 141]:

> «The concepts of PMP and PMF are controversial. Can we really specify an upper bound to the amount of rain that can fall in a given time? (…) we must recognize that the plotted values are only those that have been observed historically at the infinitesimal fraction of the earth covered by rain gages, and higher amounts must have fallen at ungaged locations at other times and places. And, conceptually, we can always imagine that a few more molecules of water could fall beyond any specified limit.»

Among the most neat criticisms of the concept of PMP is that of *Benson* [1973]:

> «The "probable maximum" concept began as "maximum possible" because it was considered that maximum limits exist for all the elements that act together to produce rainfall, and that these limits could be defined by a study of the natural processes. This was found to be impossible to accomplish – basically because nature is not constrained to limits (...). At this point, the concept should have been abandoned and admitted to be

a failure. Instead, it was salvaged by the device of renaming it "probable maximum" instead of "maximum possible". This was done, however, at a sacrifice of any meaning or logical consistency that may have existed originally (...). The only merit in the value arrived at is that it is a very large one. However, in some instances, maximum probable precipitation or flood values have been exceeded shortly after or before publication, whereas, in some instances, values have been considered by competent scientists to be absurdly high. (…) The method is, therefore, subject to serious criticism on both technical and ethical grounds – technical because of a preponderance of subjective factors in the computation process, and because of a lack of specific or consistent meaning in the result; ethical because of the implication that the design value is virtually free from risk.»

The defects of the PMP concept is vividly expressed by two of the *Wileeke*'s [1980] myths: the "Myth of infinitesimal probability", which reads "The probability of occurrence of probable maximum event is infinitesimal", and the "Myth of Impossibility", which reads "Hydrometeorological estimates of stream events are so large they cannot or will not occur". Here he points out a number of storms recorded in the USA that exceeded the PMP estimates [see also *Dooge*, 1986].

The justification of the wide use of the PMP approach is attributed to the "no-risk" aspect of the method. According to *Benson* [1973]:

«The method has been used and accepted for a long time, for one reason, not because of its merits, but because it provides a solution that removes responsibility for making important decisions as to degree of risk or protection.»

However, the removal of responsibility is an illusion because the adoption of the PMP approach by no means implies zero risk in reality. Therefore, not long ago, there has been an initiative for a movement away from the PMP-based methods to risk-based approaches for engineering design [e.g., *Stedinger and Grygier*, 1985; *Dawdy and Lettenmaier*, 1987; *National Research Council*, 1988, 1994].

Besides the general concept of PMP itself, other issues related to the methodology of determining the PMP amount have been criticized, mainly because there is no unique method for determining the upper bound of rainfall assuming that it really exists. A variety of procedures to determine PMP have been proposed [see *Wiesner*, 1970; *Schreiner and Reidel*, 1978; *World Meteorological Organization*, 1986; *Collier and Hardaker*, 1996; among others], and different procedures may result in different, higher or lower values.

Most procedures are based on a comprehensive meteorological analysis, while some are based on statistical analysis. Among the latter, the most widely used is *Hershfield's* [1961, 1965] procedure that has become one of the standard methods suggested by *World Meteorological Organization* [1986] for estimating PMP. It has the advantages of taking account of the actual historical data in the location of interest, expressing it in terms of statistical parameters, and being easy to use. The procedure is based on the general equation

$$h_\mathrm{m} = \bar{h}_n + k_\mathrm{m}\, s_n \tag{1}$$

where h_m is the maximum observed rainfall amount at the site of interest, $\bar{h}_n$ and s_n are the mean and standard deviation of a series of n annual rainfall maxima at that site, and k_m is a frequency factor. To evaluate this factor *Hershfield* [1961] initially analyzed a total of 95 000 station-years of annual maximum rainfall belonging to 2645 stations, of which about 90% were in the USA, and found that the maximum observed value of k_m was 15. Then, he concluded that an estimate of the PMP amount can be determined by setting k_m = 15 in (1) and substituting h_m for the PMP value. Subsequently, *Hershfield* [1965], proposed that k_m varies with the rainfall duration d and the mean $\bar{h}_n$. More specifically, he found that the value of k_m = 15 is too high for areas with heavy rainfall and too low for arid areas, whereas it is too high for rain durations shorter than 24 hours. Therefore, he constructed an empirical nomograph indicating that k_m varies between 5 and 20 depending on the rainfall duration d

and the mean $\overline{h}_n$. This nomograph along with equation (1) constitute the basis of the statistical method for estimating PMP, which was standardized by *World Meteorological Organization* [1986].

Undoubtedly, the data, analysis and results of Hershfield contain extremely useful information and additional validity has been appended to them by the widespread application of the method. However, after the discussion of the previous paragraphs, the question arises whether the huge amount of rainfall information used by Hershfield suggests the existence of a deterministic upper limit of precipitation or not. If the answer to this question is negative (and, in fact, is), we can maintain that there is no reason to consider the results of Hershfield's method as PMP. Then other questions arise, i.e., whether this standard method can be reformulated in a purely probabilistic manner, without postulating the existence of PMP, and what is the probability of exceedance of the method's results. The answers to these questions are the objectives of this paper, which, as we show in the following sections, are achievable and simple.

The reader may have a primary objection for the attempted probabilistic reformulation of Hershfield's method mainly because of the use of a specified type of probability distribution function for describing data from 2645 raingage stations which may experience a variety of climatic conditions. Indeed, a single type of distribution function may not be appropriate for all stations, but it would be very useful to have an idea of an "average" probabilistic behavior of maximum rainfall in such a tremendous number of stations. Besides, such an objection is justified only at the same grounds as that for the original Hershfield's method, because this method actually did the same sort of generalization. Moreover, we include in our analysis an investigation of the implications of such an assumption via simulation.

2. Statistical interpretation of Hershfield's data

Hershfield [1961] published a table with summary data of all 2645 records he used; this table serves as the basis of all analysis of the present study. More specifically, Hershfield divided the available records into 13 classes according to the length of record, which varied from 10-14 years to more than 70 years, as shown in Table 1. Furthermore, he calculated for each record the value of k_m (eqn. (1)) and discretized the range of k_m, which was extended from 1.0 to 15.0, using a step equal to 0.5 (thus having 28 intervals in total). In the aforesaid table, he published the number of occurrences of each interval of k_m for each class of record length. In Table 1 we reproduce the observed minimum and maximum interval of k_m for each class of record length.

Let us provisionally ignore the effect of the record length n on k_m (which apparently exists and we will return to it later) and unify all classes of record lengths adding the number of occurrences of all classes (as Hershfield already did, as well). Then, using the well-known Weibull formula, we can estimate the probability of non-exceedance of the random variable K_m whose realization is k_m, by

$$F^*(k_m) = \frac{r(k_m)}{r' + 1} \tag{2}$$

where $r(k_m)$ is the number of records with $K_m \leq k_m$, and r' is the total number of records (2645). In fact, this estimation is possible only for the upper bounds of the 28 intervals of k_m used by Hershfield.

A plot of k_m versus F^* (more precisely, versus the Gumbel reduced variate $-\ln(-\ln F^*)$), is given in Figure 1 on Gumbel probability paper. Contrary to the interpretation of *World Meteorological Organization* [1986, p. 96], which speaks about an "enveloping" value of k_m this figure suggests that there is no such an enveloping value. As in any finite sample, we have in Figure 1 a finite maximum value (k_m = 15 in our case), but there is no evidence to consider that value as an enveloping one. This would be justified only if there was a trend for k_m to stabilize (or saturate) at a certain value as the probability of non-exceedance $F^*(k_m)$ approaches unity. But this is not the case in Figure 1, where we observe an intensifying rate of increase of k_m versus the increase of the Gumbel reduced variate $-\ln(-\ln F^*)$. If this rate of increase were constant (i.e. the points in Figure 1 formed a straight line) this would indicate that k_m would have a Gumbel distribution. The observed curvature (i.e., intensifying rate of increase) suggests that a generalized extreme value (GEV) distribution with shape parameter $\neq 0$ is more appropriate (the curve shown in Figure 1, estimated by least squares, corresponds to $\kappa = 0.0857$). In conclusion, Figure 1 indicates that the maximum observed value $k_m = 15$ is not at all a physical upper limit and it would be greater in case that more records were available. To support this interpretation, we invoke the cautionary remarks of *World Meteorological Organization* [1986, p. 108] on the statistical method of PMP, which mention values of k_m equal to 25-30 for USA and Canada.

We remind that the Gumbel distribution of maxima is

$$F_X(x) = \exp(-\mathrm{e}^{-x/\lambda + \psi}) \tag{3}$$

and the GEV distribution is

$$F_X(x) = \exp\left\{-\left[1 + \kappa\left(\frac{x}{\lambda} - \psi\right)\right]^{-1/\kappa}\right\} \qquad \kappa\, x \geq \kappa\, \lambda\, (\psi - 1/\kappa) \tag{4}$$

In both (3) and (4) X and x denote the random variable and its value, respectively (in our case they represent K_m and k_m, respectively), $F_X(x)$ is the distribution function, and κ, λ, and ψ are shape, scale, and location parameters, respectively; κ and ψ are dimensionless whereas λ (> 0) has the same units as x. Note that (3) is the two-parameter special case of the three-parameter (4), resulting when $\kappa = 0$.

In the above analysis we have not considered the effect of the record length n on k_m. Apparently, there exists a serious such effect for two reasons. First, the more the available data values are the more likely is the occurrence of a higher value of k_m. Second, as k_m is an amount standardized by the sample mean and standard deviation, the larger the record length the more accurate the estimation of k_m. In Figure 2 we have plotted the empirical distributions of each class of record length separately, using again (2) but with r' being the total number of records of the specific class. Clearly, for low and moderate probabilities of non-exceedance (low and moderate values of the Gumbel reduced variate $–\ln(–\ln F^*)$) the empirical distribution of a class with small record length (e.g., 10-14 years) differs from that of a class with long record length (e.g., 65-69 years), the curves of the former being below those of the latter. This is absolutely justified by the first reason reported above. For very high probabilities of non-exceedance (very high values of the Gumbel reduced variate $–\ln(–\ln F^*)$) this situation is reversed in some cases (e.g., the empirical distributions of the classes of record length 10-14 and 20-24 years surpass those of classes with longer record length). Most probably, this is a consequence of the poor estimation of the mean and standard deviation for the classes with small record length.

The effect of the record length on the accuracy of estimation of the mean and standard deviation of a random variable is intrinsic and unavoidable. However, the effect of the record length on the value of maximum k_m can be easily averted if, instead of studying the distribution of the maximum observed k_m, we choose to study the distribution of all standardized annual maxima k_i within a record, defined by

$$h_i = \overline{h}_n + k_i\, s_n \tag{5}$$

where h_i, $i = 1, \dots n$, is the ith observed annual maximum rainfall within the record. Denoting by $F^*(\,)$ and $F(\,)$ the distribution function of K_m and K_i, respectively, and since K_i are independent, it is well known [e.g., *Gumbel*, 1958, p. 75] that $F^*(k) = [F(k)]^n$, so that we can find the empirical distribution $F(k)$, given $F^*(k)$ from (2), by

$$F(k) = [F^*(k)]^{1/n} \tag{6}$$

This we have done for all Hershfield's classes of record length by adopting a unique value of n for each class, equal to the arithmetic mean of the class's bounds (for the last class which does not have an upper bound we assumed $n = 85$, so that the total number of station-years be 94 523, i.e., very close to the value of 95 000 station-years mentioned by Hershfield). Again, the estimation of $F(k)$ is possible only for the upper bounds of the 28 intervals of k_m used by Hershfield. The estimated empirical distribution functions $F(k)$ are depicted in Figure 3.

Clearly, the departures in $F^*(k_m)$, among different classes observed in Figure 2, have disappeared in the $F(k)$ of Figure 3. Some departures of different classes appear in Figure 3 only for $-\ln(-\ln F) > 6$, or $F > 0.998$, or, equivalently, for return periods greater than 500 years. This is not so strange if we consider the high uncertainty for such high return periods, especially for the classes of small length. We note also that the Gumbel probability plot of Figure 3 enlarges greatly any difference in probability at the very right part of the graph. As a rough indication that these differences are not significant, we have plotted in Figure 3 a couple of confidence curves around a theoretical GEV distribution. The GEV distribution is representative for the unified record containing all classes, and its derivation is discussed in section 3. The 99% confidence limits at a certain value of F are $F \pm 2.576\, \sigma_F$ where 2.576 is the standard normal variate for confidence coefficient 99% and $\sigma_F = [F\,(1 - F)\,(1 / m')]^{0.5}$ is the sample standard deviation of F for a sample size m' [*Papoulis*, 1990, pp. 284, 299]; in our case m' equals the number of station-years. For the construction of the confidence curves of Figure 3, the value of m' was taken 2000 which is representative for class 3 with mean record length 22 years and 2024 station-years.

3. Proposed alternative formulation of Hershfield's method for 24 hour depths

The approximate analysis of the previous section indicates that the classes of different record lengths do not differ substantially in regard to the distribution of k. Apparently, the record length which was the criterion for separating classes is not the most appropriate one; other criteria such as climatic conditions would be more appropriate to test whether they affect the distribution of k. However, even if the existing Hershfield's 13 classes are considered as totally randomly selected, the absence of substantial difference among them provides a rough indication that the specific standardization of annual maximum rainfall (i.e., the use of k) is a useful analysis tool.

Therefore, with the reservations and explanations given in the last paragraph of section 1, we can proceed assuming that all records of standardized annual maximum rainfall k represent practically the same population. This assumption is also supported by the original work of *Hershfield* [1961] who tested the different records for randomness and independence.

It is rather simple to find the empirical distribution function $F(k)$ of the union of all records. Given from (6) the empirical distribution function $F_i(k)$ for class i which has m'_i station-years in total, we estimate the number $m_i(k)$ of station-years whose values are less than or equal to k by

$$m_i(k) = F_i(k)\,(m'_i + 1) \tag{7}$$

Adding $m_i(k)$ and for all i we find the total number $m(k)$ for the union of all records; similarly adding all m'_i we find the total number of station years ($m' = 94\ 523$), so that finally

$$F(k) = \frac{m(k)}{m' + 1} \tag{8}$$

Again, the estimation of $F(k)$ is possible only for the upper bounds of the 28 intervals of k_m used by Hershfield. Thus, the range of k where the estimation of $F(k)$ is possible extends from $k = 1.5$ to 15. The corresponding estimated range of $F(k)$ is from $F(k) = 0.7759$ to 0.9999904. We do not have any information for values of $F(k) < 0.7759$ (or, equivalently, for return periods less than 4.5 years), but this is not a significant gap because typically in engineering problems our interest is focused on high return periods.

The empirical distribution function of all records is given graphically in Figure 4 on Gumbel probability paper. The curvature of this empirical distribution function clearly shows that the Gumbel distribution is not appropriate. Therefore, we have fitted the more generalized GEV distribution which is also shown in Figure 4 (as well as in Figure 3), along with two 99% confidence curves derived with the method already described in section 2 for $m' = 95\ 000$. Due to the unusual situation about the available empirical information (only 28 values of the empirical distribution), typical fitting methods such as those of moments, L-moments, maximum likelihood, etc., are not applicable for fitting the theoretical distribution. Instead, we used a least square method aimed at the minimization of the mean square error among theoretical end empirical values of the magnitude $-\ln(-\ln F(k))$ for the specified values of k. The objective function (mean square error) is a function of the GEV distribution parameters, i.e., κ, λ and ψ of (4), and its minimization using nonlinear programming resulted in parameter values $\hat{\kappa} = 0.13$, $\hat{\lambda} = 0.6$, and $\hat{\psi} = 0.73$.

One may argue that, because k is a standardized variable, the parameters must be constrained so that the theoretical mean and standard deviation of k equal 0 and 1, respectively. We preferred not to introduce those constraints into the optimization process for two reasons. First, the union of many records with standard deviation 1, has no more standard deviation 1 as it can be easily verified. Second, *Hershfield* [1961] used adjusting factors for both the sample mean and standard deviation, so that the mean and standard deviation of k in each record do not actually equal 0 and 1. Therefore, the theoretical values of mean and standard deviation for the above parameter values are 0.87 and 0.94, respectively. If we force parameters to obey the requirements regarding the mean and standard deviation, by introducing the relevant constraints into the formulation of the optimization problem, the estimated parameters become $\hat{\kappa} = 0.13$, $\hat{\lambda} = 0.64$, and $\hat{\psi} = -0.69$, that is, κ remains constant, λ changes slightly and ψ changes significantly.

In fact, the only parameter that we practically need to know is the shape parameter κ because this is the only one that remains invariable when we apply the standardization transformation on the random variable representing the annual maximum rainfall depth. Moreover, κ is the most difficult to estimate accurately from a small record, whereas λ and ψ are more accurately estimated, e.g., from the sample mean and standard deviation.

Adopting the parameter set $\kappa = 0.13$, $\lambda = 0.6$, $\psi = 0.73$, and solving (4) for the value $k = 15$ which was specified by Hershfield as corresponding to PMP, we find that $F(k) = 0.999\,982$ which corresponds to a return period of 55 700 years. If we assume that k has zero mean and unit standard deviation then (3) and (4) after algebraic manipulations reduce to

$$F(k) = \exp\left[-\exp\left(-\frac{\pi\, k}{\sqrt{6}} - \gamma\right)\right] \qquad \kappa = 0 \tag{9}$$

$$F(k) = \exp\left\{-\left[\mathrm{sgn}(\kappa)\ \sqrt{\Gamma(1-2\kappa) - \Gamma^2(1-\kappa)}\ k + \Gamma(1-\kappa)\right]^{-1/\kappa}\right\} \qquad \kappa \neq 0 \qquad (10)$$

where sgn(κ) is the sign of κ, γ is the Euler's constant (= 0.57722) and $\Gamma(\,)$ is the gamma function. Apparently, $F(k)$ is a function of the value of the standardized variable k and the shape parameter κ only (note the distinction of the Latin k and the Greek κ). Setting $\kappa = 0.13$ and $k = 15$ to (10) we find $F(k) = 0.999\,983$ and $T = 58600$, i.e., very close to the previous results. By rounding these results we could say that Hershfield's value $k = 15$ corresponds to a return period of about 60 000 years. This is somehow different from the empirical estimation which would be 95 000 years (equal to the number of station-years). The difference is apparent in Figure 4 (last point to the right versus the solid line). However, it is not statistically significant: if we assume that the probability that k does not exceed 15 is $p = 1 - 1 / 60\,000$, then the probability that all 95 000 observations do not exceed 15 is $p^{95\,000} \approx 0.20$. Therefore, the hypothesis that the empirical and theoretical probabilities are the same, is not rejected at the typical levels of significance (e.g., 1%, 5%, or even more, up to 20%). This result is graphically verified by the confidence curves of Figure 4.

The above results may be summarized in the following three points, which provide the alternative interpretation to *Hershfield*'s [1961] statistical PMP method:

(1) The GEV distribution can be considered as appropriate, for annual maximum rainfall series.

(2) The value of the standardized annual maximum rainfall $k = 15$ (which was considered by Hershfield as corresponding to PMP) corresponds to a return period of about 60 000 years.

(3) The shape parameter κ of the GEV distribution is 0.13.

This formulation is more consistent than the original of *Hershfield* [1961] with the probabilistic nature of rainfall and, furthermore, it allows quantification of the risk when $k = 15$, as well as an assessment of the risk for different (greater or less) values of k.

We must emphasize that the above results are subject to at least two sources of possible bias, introduced first by the parameter estimation procedure used by Hershfield to estimate the mean and standard deviation for each record, and, second, by the unification of all records as if they were independently identically distributed. To quantify the bias from both sources we have performed simulations whose results are given in Table 2. In each simulation we generated an ensemble of 60 sets each containing 2645 records with randomly chosen lengths from the distribution that is derived from Table 1 of *Hershfield* [1961]. The about $60 \times 2645 \times 36 = 5.7 \times 10^6$ values of k (where 36 is the average record length) were generated from the distribution function (10) with population mean 0 and variance 1. These values were then re-standardized using the sample mean and standard deviation of each generated sample. These statistics were estimated both by the typical unbiased statistical estimators (method C in Table 2) and the Hershfield's adjusting factors (already mentioned above), as they are implied by his nomographs (method B). From each ensemble we estimated the value of k_m, as well as the empirical return period of the value $k = 15$, as the average values from the 60 generated sets. We examined five cases as shown in Table 2. In case 0 we assumed that all 2645 synthetic records have constant parameter κ equal to 0.13. In cases 1-4 we assumed a varying κ randomly chosen (for each synthetic record) from a gamma distribution with mean value 0.10 - 0.13 and plausible values of standard deviation and skewness as shown in Table 2.

From the results of Table 2 it is evident that the adoption of Hershfield's adjustment factors in all cases results in slight overestimation of the value of k_m (method B versus method A) and, consequently, in underestimation of the return period of the value $k = 15$. Notably, however, if these adjustments were not used the result would be a serious underestimation of k_m (method C versus method A). Furthermore, the adoption of varying κ results in an increase of k_m and decrease of the return period of the value $k = 15$ (case 1 versus case 0). Conversely, given that Hershfield's 2645 records certainly have not constant κ, and the parameters were estimated by method B, we can expect that the estimated value $\kappa = 0.13$ is too high as an average value if the value $k = 15$ must have a return period of about 60 000 years; a smaller average value of $\kappa \approx 0.10$ (case 4, method B) seems more consistent. Therefore, the adoption a constant value $\kappa = 0.13$, among with the use of Hershfield's adjusting factors, results in overestimation of k_m and safer design parameters on the average.

In his subsequent work, *Hershfield* [1965] replaced the unique value $k = 15$ with a nomograph giving k as a function of the mean value of annual maximum series $\bar{h}$, which is reproduced in Figure 5 (curve of 24-hour rainfall). The curve of this nomograph may be easily replaced with a mathematical relationship of the shape parameter κ with $\bar{h}$. To establish this relationship we can follow the following steps: (a) select numerous points $\bar{h}$ and estimate from the nomograph the corresponding values k; (b) for each k find from (10) the value of κ so that $F(k) = 1 - 1 / 60\,000$; (c) from the set of pairs $(\bar{h}, \kappa)$ establish a simple type of mathematical relationship and estimate its parameters. Using this procedure, it was found that this nomograph is practically equivalent to the following mathematically simple statement, which substitutes point 3 in the previously stated alternative formulation of the method:

(3a) The shape parameter κ of the GEV distribution is given as a function of the mean value of annual maximum series $\bar{h}$, by

$$\kappa = 0.183 - 0.00049\,\bar{h} \qquad (\bar{h} \text{ in mm}) \tag{11}$$

The curve $k = g(\bar{h})$, which is obtained by combining (11) and (10) with $T = 1 / 60\,000$, is shown in Figure 5 and agrees well with the empirical Hershfield's curve.

We observe that for very large values of $\bar{h}$, i.e., for $\bar{h} > 373.5$ mm, (11) results in $\kappa < 0$. This combined with (4) implies that, in that case, k will be upper bounded (and lower unbounded). However, to the author's opinion there is no sufficient physical or empirical reasoning to accept an upper bound for k. A direct solution would be to set $\kappa = 0$ and use the Gumbel distribution in such extremely unusual situations, even though this results in slight disagreement with Hershfield for $\bar{h} > 373.5$ mm. We note that, in his original study, *Hershfield* [1965] had only 5 out of about 2700 points located in that area ($\bar{h}$ > 373.5 mm) to draw his enveloping curve, and therefore the uncertainty is large, anyhow.

4. Verification of the proposed alternative formulation of the method

To verify the proposed method with historical data we used the longest available record of annual maximum daily rainfall in Greece [*Koutsoyiannis and Baloutsos*, 1998]. This comes from the National Observatory of Athens and extends through 1860-1995 (136 years). Its mean and standard deviation are 47.9 and 21.7 mm, respectively, and the maximum observed value is 150.8 mm (therefore, the observed k_m = (150.8 – 47.9) / 21.7 = 4.74). The direct application of Hershfield's method results in k_m = 17.2 and, consequently, in a PMP value 424.1 mm. From (11) we find $\hat{\kappa}$ = 0.160, which applies for both k and h. Then, adopting the GEV distribution for h and using the simplest method of moments for estimating the remaining two parameters (with mean and standard deviation of h 47.9 and 21.7 mm, respectively) we find $\hat{\lambda}$ = 12.95 mm and $\hat{\psi}$ = 2.93 (the corresponding parameters for k are $\hat{\lambda}$ = 0.60 and $\hat{\psi}$ = 0.77).

Alternatively, we fitted the GEV distribution to the given sample of h itself, without reference to the proposed method, using the standard methods of maximum likelihood and of L-moments. The estimated parameters are $\hat{\kappa}$ = 0.161, $\hat{\lambda}$ = 12.93 mm, and $\hat{\psi}$ = 2.94 for the method of maximum likelihood [*Papoulis*, 1990, p. 303] (the maximization was performed numerically), and $\hat{\kappa}$ = 0.185, $\hat{\lambda}$ = 12.64 mm, and $\hat{\psi}$ = 2.98 for the method of L-moments [*Stedinger et al.*, 1993, p. 18.18]. Interestingly, the parameters of the method of maximum likelihood almost coincide with those of the proposed method. This indicates that the estimate of the shape parameter κ by (11) is reliable, at least in the examined case.

The empirical distribution function of the 136-year record, in comparison with GEV distribution function with all three parameter sets are shown in Figure 6. As expected, the GEV distribution of the proposed method (as well as the indistinguishable method of maximum likelihood) verifies that the Hershfield's PMP value (424.1 mm) has a return period

60 000 years (Figure 6). For comparison, we have additionally plotted in Figure 6 the Gumbel distribution function, which, apparently, has very poor performance with regard to the empirical distribution and underestimates seriously the rainfall amount for large return periods.

In conclusion, in the examined case we may replace the statement “the PMP is 424.1 mm” with “the 60 000-year rainfall, as resulted from the GEV distribution is 424.1 mm”; the latter implies that greater amounts of rainfall are possible but with less probability.

5. Effect of rain duration

In addition to the curve for the 24-hour rainfall, *Hershfield* [1965] presented in his nomograph curves of k_{m} for shorter durations, i.e., 1 and 2 h, as a result of analysis of rainfall data from about 210 stations. These curves are also shown in Figure 5.

Generally, if we have available the intensity-duration-frequency (idf) relationships for the location of interest we can easily infer the relevant effect of the duration without the need of additional curves. The idf relationship may be expressed [*Koutsoyiannis et al.*, 1998] by the general form

$$i_d(T) = \frac{a(T)}{b(d)} \tag{12}$$

where $i_d(T)$ is the rainfall intensity corresponding to duration d and return period T, and $a(T)$ and $b(d)$ functions of T and d, respectively. Particularly, the function $b(d)$ is typically a power function of d and, as we will show, determines completely the effect of duration and can be used to infer curves such as those of Hershfield’s nomograph. Writing (12) for two durations, d and 24 h, eliminating $a(T)$, and also substituting $i_d(T) = h_d(T) / d$, where $h_d(T)$ is the rainfall depth corresponding to duration d and return period T, we find

$$h_d(T)\,\frac{b(d)}{d} = h_{24}(T)\,\frac{b(24)}{24} \tag{13}$$

which implies equality in probability for $[h_d\, b(d) / d]$ and $[h_{24}\, b(24) / 24]$. Therefore, if μ_d and σ_d denote the mean and standard deviation of the rainfall depth for duration d, then

$$\mu_d \frac{b(d)}{d} = \mu_{24} \frac{b(24)}{24}, \quad \sigma_d \frac{b(d)}{d} = \sigma_{24} \frac{b(24)}{24} \tag{14}$$

If we denote $k_d(T) = [h_d(T) - \mu_d] / \sigma_d$, the standardized variate of $h_d(T)$, we easily find from (13) and (14) that $k_d(T) = k_{24}(T) = k(T)$, that is, $k_d(T)$ does not depend on d, so that we can use (10) and (11) to determine it. Note that $\overline{h}$ that appears in (11) is the sample mean of h_{24}, i.e., $\overline{h} \equiv \overline{h}_{24}$ which corresponds to the true mean μ_{24}. The relationship $k = g(\overline{h}_{24})$ which resulted numerically from (10) and (11) for T = 60 000 (Figure 5, curve for 24 hours) may be reformulated so as to write as a function of $\overline{h}_d$, which is the case of Hershfield's curves. Assuming that the sample means $\overline{h}_d$ and $\overline{h}_{24}$ have the same relation as the true means μ_d and μ_{24} (eqn. (14)) we can write

$$k = g(\overline{h}_{24}) = g\left(\overline{h}_d \frac{24\, b(d)}{d\, b(24)}\right) = g_d(\overline{h}_d) \tag{15}$$

where

$$g_d(x) := g\left(x \frac{24\, b(d)}{d\, b(24)}\right) \tag{16}$$

Now, if we assume for simplicity that an "average" expression for $b(d)$ is $b(d) = d^{\,0.5}$, then (15) becomes

$$k = g_d(\overline{h}_d) = g\left(\overline{h}_d \frac{24^{\,0.5}}{d^{0.5}}\right) \tag{17}$$

Using (17) and starting with the known function $k = g(\overline{h}_{24})$ we calculated the function $g_d(\overline{h}_d)$ for d = 1 h and 2 h and we plotted the resulting curves in Figure 5 in comparison with the empirical curves of Hershfield. Interestingly, the two sets of curves almost coincide.

In conclusion, the analysis of this section shows that (a) there is no need to establish relations of the standardized annual maximum rainfall k with any of the rainfall characteristics for rain durations less than 24 h, because such relationships are directly derived from idf curves; (b) the particular Hershfield's curves of k_{m} for low durations are practically equivalent with the assumption that the rainfall depth is proportional to the square root of duration.

6. Conclusions

Hershfield's method of estimating probable maximum precipitation (PMP) is a very useful, widespread and reliable tool for hydrologic design because it is based on the analysis of a huge amount of rainfall information (2645 data records throughout the world containing 95 000 station-years). However, what the method estimates may not be PMP and there is no reason to consider it so. More specifically, the analysis performed with Hershfield's data provided no evidence that there exists an upper bound of precipitation amount and, besides, suggested that a simple alternative formulation of the method is possible. This formulation can be purely probabilistic and need not postulate the existence of PMP as an upper physical limit.

It is shown that Hershfield's estimate of PMP may be obtained by using the Generalized Extreme Value (GEV) distribution with shape parameter given as a specified linear function of the average of annual maximum precipitation, and for return period equal to 60 000 years. This formulation is supported by the published Hershfield's data and substitutes completely the standard empirical nomograph that is used for the application of the method. Moreover, the alternative formulation assigns a probability distribution function to annual maximum rainfall, thus allowing for the estimation of risk either for the Hershfield's "PMP" value or any other large rainfall amount.

The return period of about 60 000 years estimated here for Hershfield's PMP is rather small as compared to other estimates of the literature (although the other estimates may not be fully comparable as they refer to other PMP estimation methods). For example, according to *National Research Council* [1994, p. 14] the return period of PMP in the United States is estimated to 10^5-10^9 years, whereas *Foufoula-Georgiou* [1989] and *Fontaine and Potter* [1989] indicate that values of PMP of the literature have return periods of 10^5-10^6 years. Likewise, according to *Austin et al.* [1995, p. 74] the values of PMP estimated in Great Britain by a storm model are associated with return period of 200 000 years.

The verification of the proposed alternative formulation of the method was performed by applying it in Athens, Greece, where there exists a long (136-year) record of annual maximum daily rainfall. The available long record suggested that the GEV distribution is appropriate and made possible a relevantly accurate estimation (by standard statistical methods) of its shape parameter, which almost coincided with that obtained by the proposed method. This coincidence enhances our trust for the results of the typical statistical analysis in the examined case, which prove to be in agreement with the outcome of a comprehensive analysis of 95 000 station-years of rainfall information throughout the world. However, the examined case is not a typical one, because most often the available records have lengths of a few tens of years, thus not allowing a reliable estimate of the distribution's shape parameter. In those cases, the proposed alternative formulation of Hershfield's method provides at least a first approximation of the shape parameter based on the average of the annual maximum daily precipitation.

In cases where rain durations less than daily are of interest, Hershfield's nomograph provides additional curves for specified such durations. Our analysis showed that (a) there is no need to use separate curves for lower durations, because such curves can be directly derived from local intensity-duration-frequency curves; (b) the particular Hershfield's curves for low durations are practically equivalent with the assumption that the rainfall depth is proportional to the square root of duration.

Acknowledgements. The review comments of E. Foufoula-Georgiou, F. De Troch, and an anonymous reviewer are gratefully appreciated.

References

Austin, B. N., I. D. Cluckie, C. G. Collier, and P. J. Hardaker, *Radar-Based Estimation of Probable Maximum Precipitation and Flood*, report, Meteorological Office, Bracknell, UK, 1995.

Benson, M. A., Thoughts on the design of design floods, in *Floods and Droughts, Proc. 2nd Intern. Symp. in Hydrology*, pp. 27-33, Water Resources Publications, Fort Collins, Colorado, 1973.

Collier, C. G., and P. J. Hardaker, Estimating probable maximum precipitation using a storm model approach, *J. of Hydrol.*, 183, 227-306, 1996.

Dawdy, D. R., and D. P. Lettenmaier, An initiative for risk-based flood design, *J. Hydraul. Div. Am. Soc. Civ. Eng.*, 113(8), 1041-1051, 1987.

Dingman, S. L., *Physical Hydrology*, Prentice Hall, Englewood Cliffs, New Jersey, 1994.

Dooge, J. C. I., Looking for hydrologic laws, *Water Resour. Res.*, 22(9) pp. 46S-58S, 1986.

Fontaine, T. A., and K. W. Potter, Estimating probabilities of extreme rainfalls, *J. Hydraul. Eng., ASCE*, 115(11), 1562-1575, 1989.

Foufoula-Georgiou E., A probabilistic storm transposition approach for estimating exceedance probabilities of extreme precipitation events, *Water Resour. Res.*, 26(5), 799-815, 1989.

Gumbel, E. J., *Statistics of Extremes*, Columbia University Press, New York, 1958.

Hershfield, D. M., Estimating the probable maximum precipitation, *Proc. ASCE, J. Hydraul. Div.*, 87(HY5), 99-106, 1961

Hershfield, D. M., Method for estimating probable maximum precipitation, *J. American Waterworks Association*, 57, 965-972, 1965.

Koutsoyiannis, D., and G. Baloutsos, Analysis of a long record of annual maximum rainfall at Athens, Greece, unpublished, 1998.

Koutsoyiannis, D., D. Kozonis and A. Manetas, A mathematical framework for studying rainfall intensity-duration-frequency relationship, *J. of Hydrol.*, 206, 118-135, 1998.

National Research Council, *Estimating Probabilities of Extreme Floods: Methods and Recommended Research*, National Academy Press, Washington, D.C., 1988.

National Research Council, *Estimating Bounds on Extreme Precipitation Events*, National Academy Press, Washington, 1994.

Papoulis, A., *Probability and Statistics*, Prentice-Hall, 1990.

Schreiner, L. C., and J. T. Reidel, Probable maximum precipitation estimates, United States east of 105th meridian, *Hydrometeorological Report 51*, U.S. National Weather Service, Washington, DC, 1978.

Stedinger, J. R., and J. Grygier, Risk-cost analysis and spillway design, in *Computer Applications in Water Resourcers*, edited by H. C. Torno, pp. 1208-1217, ASCE, Buffalo, N.Y., 1985.

Stedinger, J. R., R. M. Vogel, and E. Foufoula-Georgiou, Frequency analysis of extreme events, Chapter 18 in *Handbook of Hydrology*, edited by D. R. Maidment, McGraw-Hill, 1993.

Willeke, G. E., Myths and uses of hydrometeorology in forecasting, in *Proceedings of March 1979 Engineering Foundation Conference on Improved Hydrological Forecasting – Why and How*, pp. 117-124, American Society of Civil Engineers, New York, 1980.

Wiesner, C. J., *Hydrometeorology*, Chapman & Hall, London, 1970.

World Meteorological Organization (WMO), *Manual for Estimation of Probable Maximum Precipitation*, Operational Hydrology Report 1, 2nd edition, Publication 332, World Meteorological Organization, Geneva, 1986.

List of Figures

Tables

Table 1 Summary table of the data published by *Hershfield* [1961].

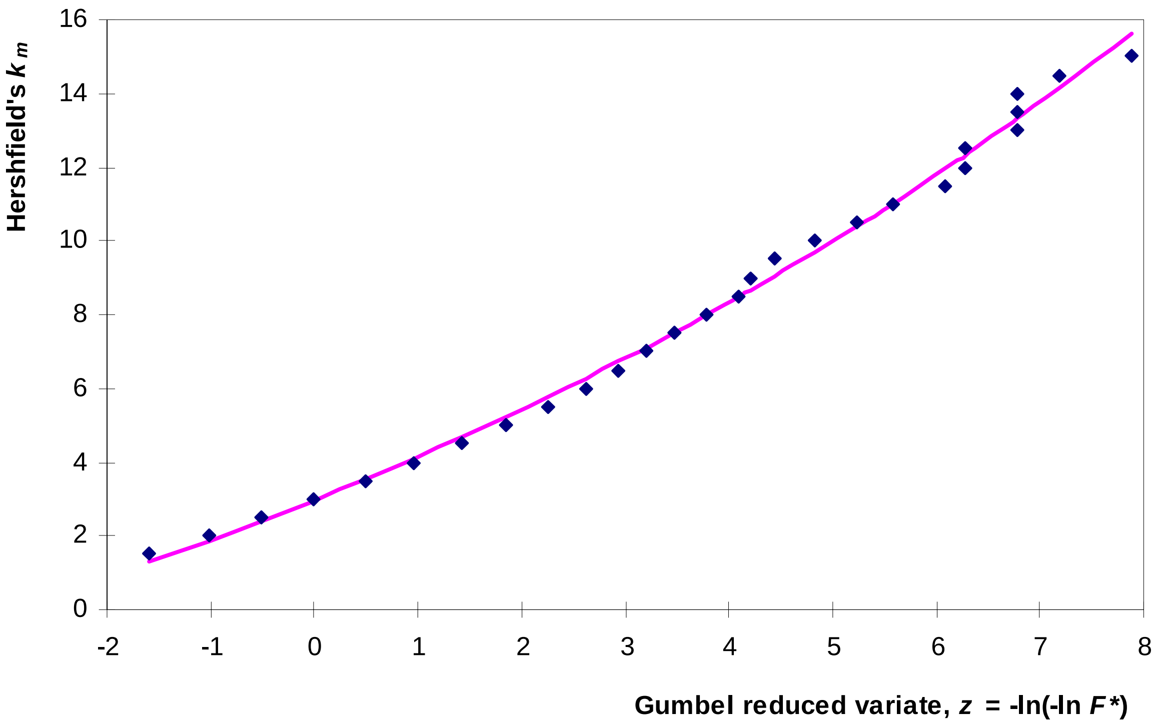
Class number	Length of record	Number of individual records	Minimum value of k_m (interval where it lies)	Maximum value of k_m (interval where it lies)
1	10 - 14	208	1.0 - 1.5	14.5 - 15.0
2	15 - 19	851	1.0 - 1.5	13.0 - 13.5
3	20 - 24	92	1.5 - 2.0	14.0 - 14.5
4	25 - 29	108	1.5 - 2.0	8.0 - 8.5
5	30 - 34	97	2.0 - 2.5	10.5 - 11.0
6	35 - 39	85	2.0 - 2.5	9.5 - 10.0
7	40 - 44	108	2.0 - 2.5	10.0 - 10.5
8	45 - 49	149	1.5 - 2.0	9.5 - 10.0
9	50 - 54	260	2.0 - 2.5	9.5 - 10.0
10	55 - 59	352	1.5 - 2.0	11.0 - 11.5
11	60 - 64	279	2.0 - 2.5	11.0 - 11.5
12	65 - 69	45	2.0 - 2.5	8.0 - 8.5
13	> 70	11	2.0 - 2.5	6.0 - 6.5
Total		2645	1.0 - 1.5	14.5 - 15.0

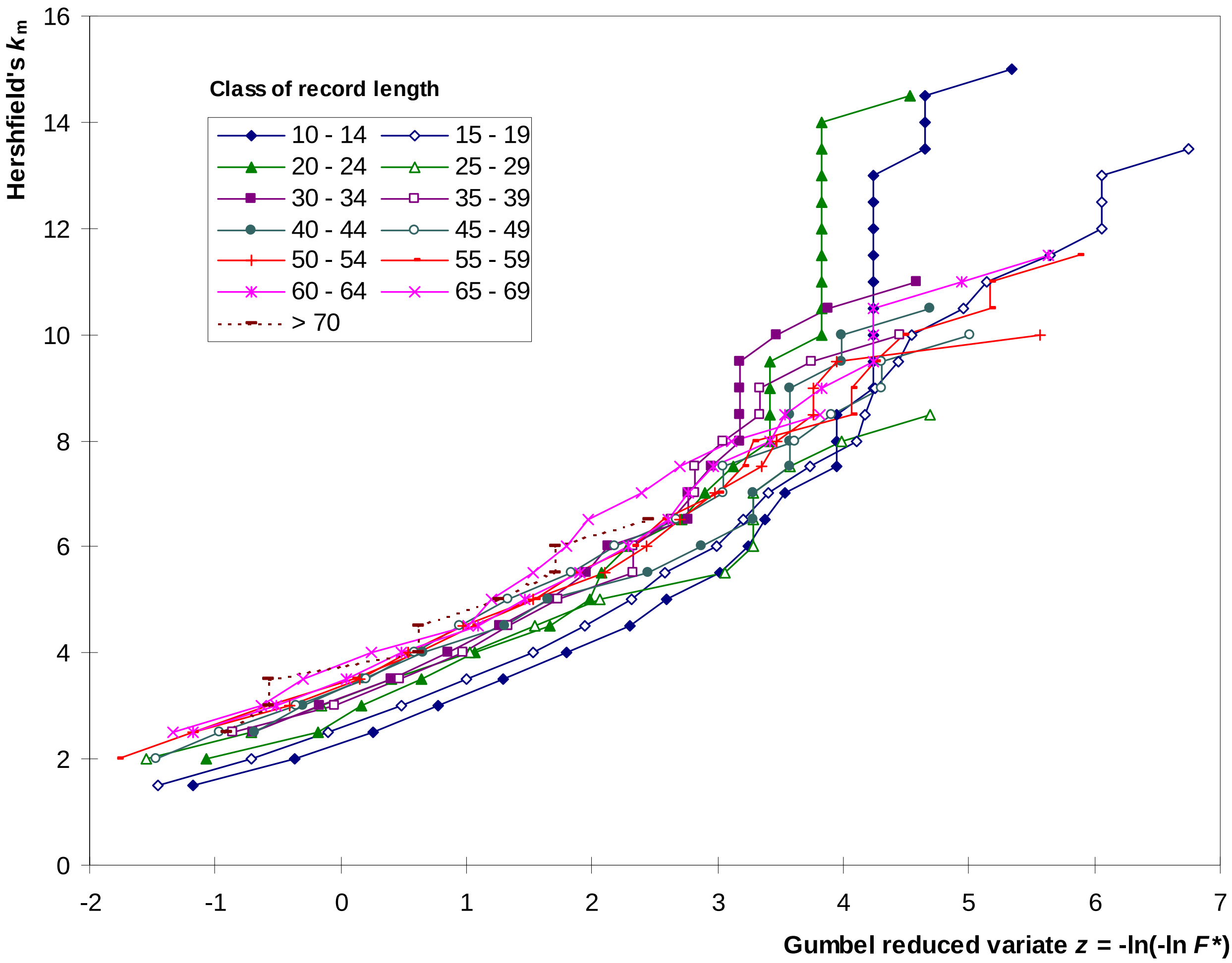
Table 2 Simulation results for the exploration of sources of bias in the proposed formulation of Hershfield's method.

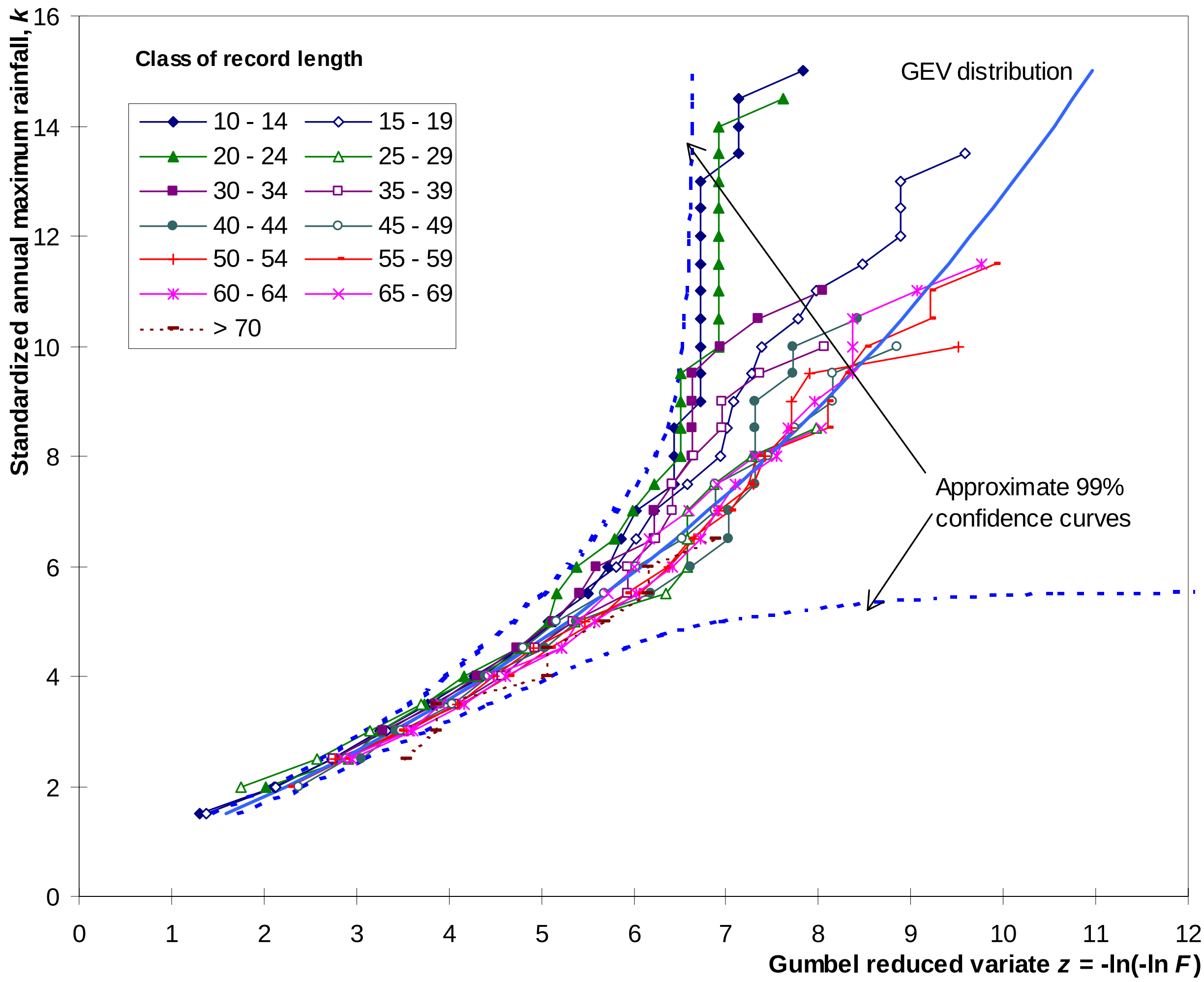
Case no.	Assumed statistical characteristics of κ			k_m using estimation method[†]			Return period of k = 15 using estimation method[†]		
	$E[\kappa]$	Std$[\kappa]$	$C_s[\kappa]$	A	B	C	A	B	C
0	0.13	0	0	17.6 (16.2)[‡]	18.8	7.1	66300 (58600)[‡]	42900	>> 95000
1	0.13	0.036	1.35	19.1	21.4	7.2	44900	28500	>> 95000
2	0.12	0.039	1.24	18.2	19.7	7.1	52800	38000	>> 95000
3	0.11	0.042	1.15	17.5	19.9	7.0	82600	47500	>> 95000
4	0.10	0.045	1.08	17.0	19.0	7.1	93400	53300	>> 95000

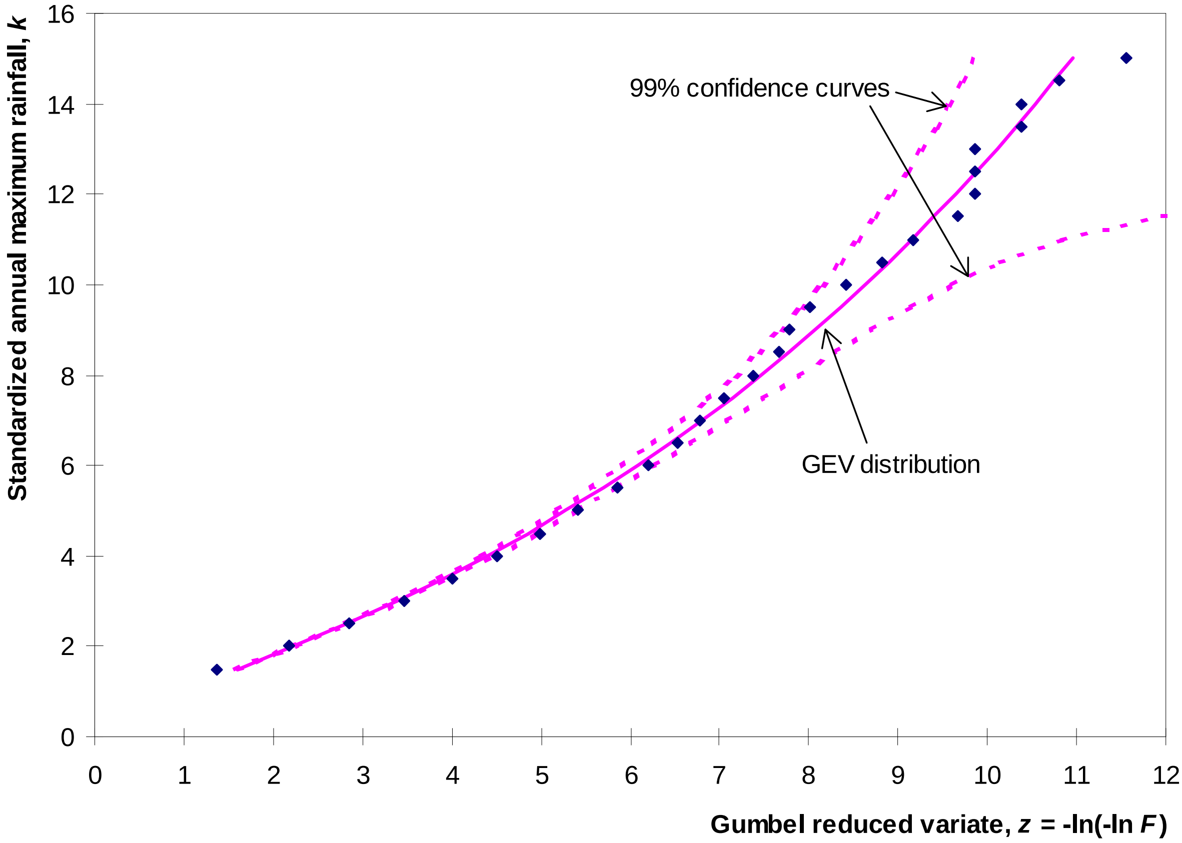
[†] Estimation methods: (A) using theoretical moments (mean 0 and standard deviation 1); (B) using adjusted moments by the Hershfield's procedure; (C) using typical statistical moments.

[‡] Theoretical values (where applicable).

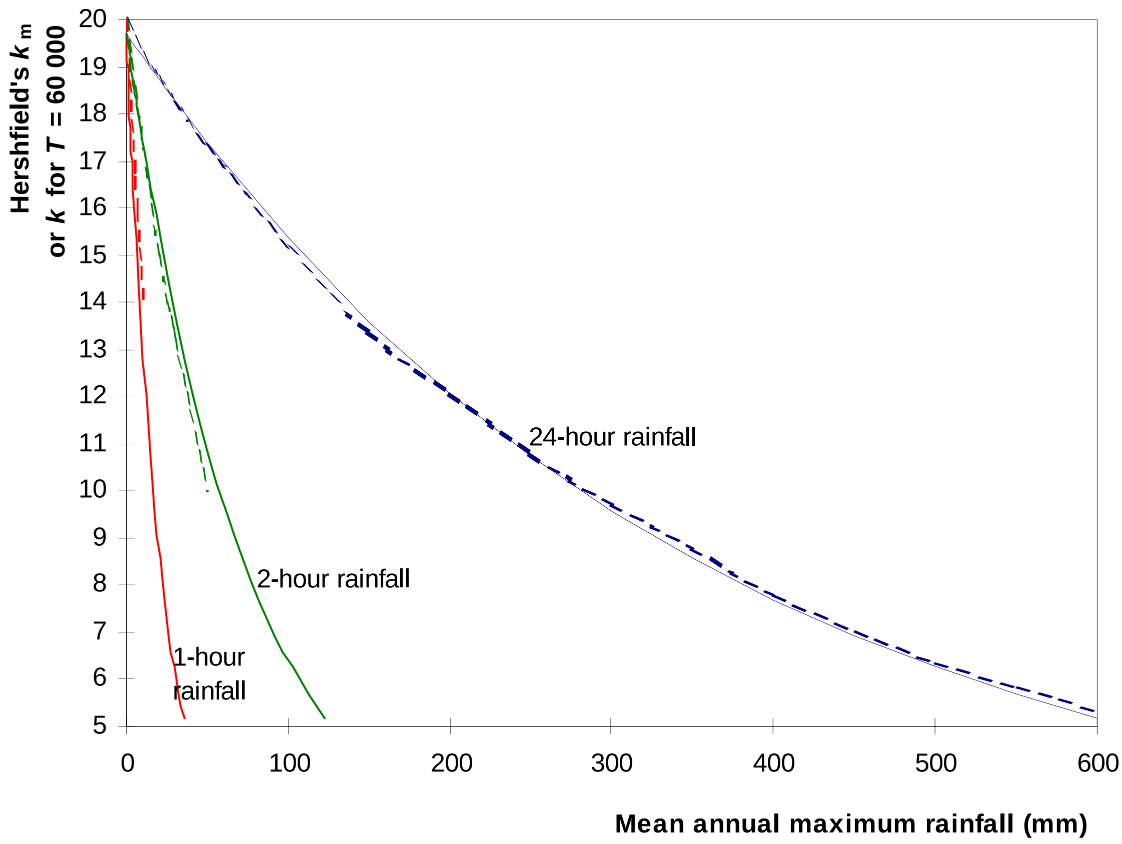

Hershfield's k_m
Gumbel reduced variate, z = -ln(-ln F*)
0
2
4
6
8
10
12
14
16
-2
-1
0
1
2
3
4
5
6
7
8

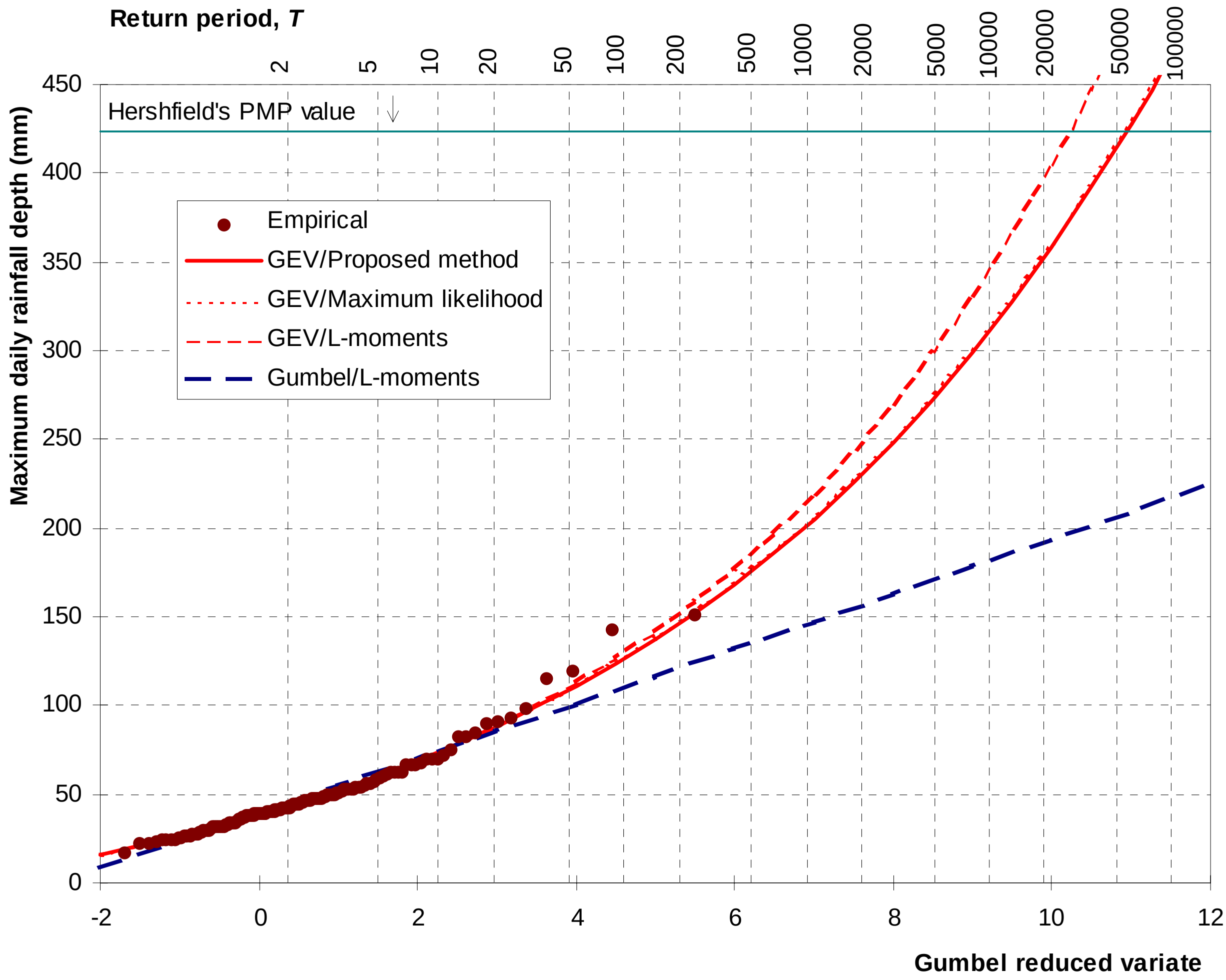

Hershfield's k_m
Class of record length
10 - 14
15 - 19
20 - 24
25 - 29
30 - 34
35 - 39
40 - 44
45 - 49
50 - 54
55 - 59
60 - 64
65 - 69
> 70
16
14
12
10
8
6
4
2
0
-2
-1
0
1
2
3
4
5
6
7
Gumbel reduced variate z = -ln(-ln F*)


Standardized annual maximum rainfall, k
Class of record length
10 - 14
15 - 19
20 - 24
25 - 29
30 - 34
35 - 39
40 - 44
45 - 49
50 - 54
55 - 59
60 - 64
65 - 69
> 70
GEV distribution
Approximate 99% confidence curves
Gumbel reduced variate z = -ln(-ln F)
0
2
4
6
8
10
12
14
16
0 1 2 3 4 5 6 7 8 9 10 11 12



Standardized annual maximum rainfall, k
Gumbel reduced variate, z = -ln(-ln F)
99% confidence curves
GEV distribution
0
2
4
6
8
10
12
14
16
0
1
2
3
4
5
6
7
8
9
10
11
12


Hershfield's k_m
or k for T = 60 000
20
19
18
17
16
15
14
13
12
11
10
9
8
7
6
5
0
100
200
300
400
500
600
24-hour rainfall
2-hour rainfall
1-hour rainfall
Mean annual maximum rainfall (mm)



Return period, T
2
5
10
20
50
100
200
500
1000
2000
5000
10000
20000
50000
100000
Hershfield's PMP value
Empirical
GEV/Proposed method
GEV/Maximum likelihood
GEV/L-moments
Gumbel/L-moments
Maximum daily rainfall depth (mm)
0
50
100
150
200
250
300
350
400
450
-2
0
2
4
6
8
10
12
Gumbel reduced variate