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A PROBLEM IN GEOMETRIC PROBABILITY

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Let N points be scattered at random on the surface of the unit sphere in n-space. The problem of the title is to evaluate $p_{n,N}$, the probability that all the points lie on some hemisphere. I shall show that

$$(1) \qquad p_{n,N} = 2^{-N+1}\sum_{k=0}^{n-1}\binom{N-1}{k} .$$

I first heard of the problem from L. J. Savage, who had been challenged by R. E. Machol to evaluate $p_{3,4}$. Savage showed that $p_{3,4}=\frac{7}{8}$, and more generally that

$$(2) \qquad p_{n,n+1} = 1-2^{-n} .$$

Then I was able to obtain the relation

$$(3) \qquad p_{n,n+2} = 1-(n+2)2^{-(n+1)} .$$

and D. A. Darling proved that $p_{2,N}=N\cdot 2^{-N+1}$, which on setting $N=n+2$ became

$$(4) \qquad p_{2,n+2} = (n+2)2^{-(n+1)} .$$

Equations (3) and (4) suggested the attractive "duality relation"

$$(5) \qquad p_{m,m+n}+p_{n,m+n} = 1 ,$$

which was found to hold generally. The results (2), (3) and (5) then led to the conjecture (1). Since (5) is a corollary to (1) it seems superflous to give a separate proof; instead I proceed now to the proof of (1), and in a slightly more general setting.

Let $x_1,x_2,\ldots,x_N$ be random vectors in E^n whose joint distribution is invariant under all reflections through the origin and is such that with probability one all subsets of size n are linearly independent; for example. the x_j may be uniformly and independently distributed over the surface of the unit sphere. The probability $p_{n,N}$ is now interpreted as the probability that all x_j lie in a half-space. i.e. that for some vector y the inner

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products (y, x_j) are all positive. I shall show that $p_{n,N}$ satisfies the recurrence relation

$$p_{n,N} = \tfrac{1}{2}(p_{n,N-1} + p_{n-1,N-1}) \, . \tag{6}$$

Since the right member of (1) also satisfies (6), together with the evident boundary conditions $p_{1,N} = 2^{-N+1}$, $p_{n,N} = 1$ if $N \leqq n$, this will complete the proof of (1).

PROOF OF (6). It is sufficient to evaluate the corresponding *conditional* probability when the x_j are non-zero and lie on fixed lines through the origin. Suppose that y is perpendicular to none of these lines. Then the sequence $s_y = \{\text{sgn}(y, x_j)\}$ is a random point in the set $S = \{s\}$ of all ordered N-tuples consisting of plus and minus signs. A specified s is said to *occur* if there is a y such that $s_y = s$. Let A_s be the event that s occurs, and let I_s be the indicator of A_s. By definition $p_{n,N} = \Pr\{A_{s_0}\}$, where $s_0 = (+, +, \ldots, +)$. Since any s can be changed into any other by reflecting appropriate x_j through the origin it follows that all A_s are equally likely. Hence

$$2^N p_{n,N} = \sum_s \Pr\{A_s\} = E\left(\sum_s I_s\right) = E(Q) \, ,$$

say, with $Q = Q_{n,N} = \sum_s I_s$ being the number of different s that occur.

Ostensibly Q is a random variable, but in fact a simple argument now shows that Q is a constant not depending on the directions of the fixed lines, providing of course that they are linearly independent in sets of n. Let X_j be the hyperplane perpendicular to x_j. Then Q is just the number of components (maximal connected subsets) complementary to all the X_j in E^n, because each component consists of all the vectors y for which s_y has a fixed value.

In order to count the components, consider the effect of deleting one hyperplane, say X_N. There remain $N-1$ hyperplanes, with complementary set composed of $Q_{n,N-1}$ components. These components are of two kinds: (i) those which meet X_N, and (ii) those not meeting X_N. In an obvious notation we have $Q_{n,N-1} = Q^{(i)} + Q^{(ii)}$. When X_N is restored it cuts each component of type (i) into two and does not disturb the others. Therefore $Q_{n,N} = 2Q^{(i)} + Q^{(ii)}$. It follows that

$$Q_{n,N} = Q_{n,N-1} + Q^{(i)} \, . \tag{7}$$

I claim now that $Q^{(i)} = Q_{n-1,N-1}$. In fact, the sets $X_j \cap X_N$ are hyperplanes in the $(n-1)$-dimensional space X_N, and their normals are linearly independent in sets of $n-1$. Therefore $X_N - \bigcup_{j=1}^{N-1}(X_j \cap X_N)$ has $Q_{n-1,N-1}$ components in X_N, and it is easy to see that these are just the intersections of the original type (i) components with X_N, establishing the claim. Substituting into (7) and recalling that $Q_{n,N} = 2^N p_{n,N}$ we obtain (6). This completes the proof.

The argument given above is essentially the same as that presented by Schläfli [1, pp. 209–212], but is included here for the sake of completeness. I am obliged to H. S. M. Coxeter for the reference. It may also be remarked that the form of the result (1) shows that $p_{n,N}$ equals the probability that in tossing an honest coin repeatedly the n'th "head" occurs on or after the N'th toss. But it does not seem possible to find an isomorphism between coin-tossing and the given problem that would make the result immediate.

REFERENCE

1. Ludwig Schläfli, *Gesammelte mathematische Abhandlungen* I, Basel, 1950.

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