A PROBLEM OF FINITE BENDING OF CIRCULAR RING PLATES*

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1. Introduction. We consider axi-symmetrical transverse bending of a circular ring plate under the action of transverse edge forces. Let a be the inner radius of the plate and $a + b$ the outer radius. If b is sufficiently small compared with a and if the conditions of support permit it it is generally considered allowable to obtain deflections and stresses in the ring plate by beam theory rather than by plate theory.

The purpose of the present note is a more detailed analysis of this problem. We shall show that the question of whether beam theory and plate theory give essentially the same results depends not only on the ratio b/a but also on the magnitude of the deflections of the plate.

Briefly, let ϕ_0 be a quantity of the order of magnitude of the deflection of the plate according to beam theory divided by the width b of the plate, and let μ be a dimensionless parameter of the form $[12(1 - \nu^2)]^{1/2} b^2/ah$, where ν is Poisson's ratio and h the thickness of the plate.

We find that beam theory is applicable as long as $\phi_0\mu \ll 1$. When $\phi_0\mu = O(1)$ then plate theory must be used and the appropriate equations are those of finite bending. We find further that when $\phi_0\mu \gg 1$ a boundary layer effect is encountered.

In order to have a continuous transition between finite bending of beams and finite bending of plates we must use a system of equations for bending of plates which is more general than the equations of Kirchhoff and von Kármán for small finite plate bending. A more general system of this nature, applicable to axi-symmetrical bending of circular plates has recently been given.**

The results which are obtained in what follows may be of some direct practical significance in connection with the analysis of expansion joints and of corrugated cylindrical shells.

2. The equations of finite axi-symmetrical transverse bending of plates of constant thickness. The differential equations which must be solved are of the form**

$$\phi'' + \frac{1}{r}\phi' - \frac{1}{r^2}\cos\phi\sin\phi = \frac{1}{Dr}[\Psi\sin\phi - (rV)\cos\phi], \tag{1}$$

$$\begin{aligned}\Psi'' + \frac{1}{r}\Psi' - \left[\frac{1}{r^2}\cos^2\phi - \frac{\nu}{r}\phi'\sin\phi\right]\Psi &= \frac{C}{r}[\cos\phi - 1] - \frac{(r^2p_H)'}{r} - \nu p_H\cos\phi \\ &\quad + \left[\frac{1}{r^2}\cos\phi\sin\phi + \frac{\nu}{r}\phi'\cos\phi\right](rV) + \frac{\nu}{r}\sin\phi(rV)'.\end{aligned} \tag{2}$$

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**E. Reissner, *On finite deflections of circular plates*, Proc. Symp. Appl. Math. vol. 1 (1949) pp. 213-219 and *On axisymmetrical deformations of thin shells of revolution*, Proc. Symp. Appl. Math. vol. 3 (1950) pp. 27-52.

In equations (1) and (2) r is the radial distance of a point of the middle surface of the plate before deformation, ϕ is the sloping angle of the deflected middle surface, Ψ is a stress function, p_H the horizontal load intensity, V the vertical stress resultant, primes indicate differentiation with respect to r, and D and C are defined in the usual way by

$$C = Eh, \qquad D = Eh^3/12(1 - \nu^2). \tag{3}$$

Stress resultants and couples and displacements of the points of the middle surface of the shell are given as follows

$$\begin{aligned}
rV &= -\int rp_V\,dr, & rN_r &= \Psi\cos\phi + (rV)\sin\phi,\\
rQ &= -\Psi\sin\phi + (rV)\cos\phi, & N_\theta &= \Psi' + rp_H\\
M_r &= -D\left(\phi' + \frac{\nu}{r}\sin\phi\right), & M_\theta &= -D\left(\frac{1}{r}\sin\phi + \nu\phi'\right)\\
u &= \frac{r}{C}(N_\theta - \nu N_r), & w &= \int \sin\phi\,dr.
\end{aligned} \tag{4}$$

The well known equations for small finite deflections follow from (1), (2) and (4) by writing $\sin\phi \sim \phi$, $\cos\phi \sim 1 - \frac{1}{2}\phi^2$, by omitting third and higher power terms in the dependent variables ϕ and Ψ and by omitting terms containing V in equation (2).

3. Bending by transverse edge forces. In what follows we assume that

$$p_H = 0, \qquad p_V = 0. \tag{5}$$

We write further

$$rV = Pa, \tag{6}$$

so that P is the transverse edge load at the boundary $r = a$ of the plate, per unit of circumferential length.

We introduce dimensionless variables in (1) and (2) by writing

$$r = a + bx, \qquad \phi = \phi_0 f(x), \qquad \Psi = \Psi_0 g(x). \tag{7}$$

This gives

$$\begin{aligned}
&\frac{\phi_0}{b^2}\left[f'' + \frac{b}{a+bx}f' - \left(\frac{b}{a+bx}\right)^2\left(1 - \frac{2}{3}\phi_0^2 f^2 + \cdots\right)f\right]\\
&\qquad = \frac{\phi_0\Psi_0}{D(a+bx)}\left(1 - \frac{1}{6}\phi_0^2 f^2 + \cdots\right)fg - \frac{Pa}{D(a+bx)}\left(1 - \frac{1}{2}\phi_0^2 f^2 + \cdots\right),
\end{aligned} \tag{8}$$

$$\begin{aligned}
&\frac{\Psi_0}{b^2}\left[g'' + \frac{b}{a+bx}g' - \frac{b^2(1 - \phi_0^2 f^2 + \cdots)}{(a+bx)^2} - \frac{\nu b\phi_0^2(1 - \frac{1}{6}\phi_0^2 f^2 + \cdots)ff'}{(a+bx)}g\right]\\
&\qquad = -\frac{C}{a+bx}\frac{\phi_0^2 f^2}{2}\left[1 - \frac{1}{12}\phi_0^2 f^2 + \cdots\right]\\
&\qquad\qquad + \left[\frac{\phi_0 f(1 - \frac{2}{3}\phi_0^2 f^2 + \cdots)}{(a+bx)^2} + \frac{\nu\phi_0 f'(1 - \frac{1}{2}\phi_0^2 f^2 + \cdots)}{b(a+bx)}\right.\\
&\qquad\qquad\qquad \left. + \frac{\nu\phi_0 f(1 - \frac{1}{6}\phi_0^2 f^2 + \cdots)}{(a+bx)^2}\right]Pa,
\end{aligned} \tag{9}$$

where primes now indicate differentiation with respect to x.

We now assume that the ratio b/a of width to inner radius of the plate is negligibly small compared to unity.* With this assumption equations (8) and (9) may be written in the form

$$f'' = \frac{\Psi_0 b^2}{Da}\left(1 - \frac{1}{6}\phi_0^2 f^2 + \cdots\right)fg - \frac{Pb^2}{\phi_0 D}\left(1 - \frac{1}{2}\phi_0^2 f^2 + \cdots\right), \tag{10}$$

$$g'' = -\frac{1}{2}\frac{Cb^2\phi_0^2}{\Psi_0 a}\left(1 - \frac{1}{12}\phi_0^2 f^2 + \cdots\right)f^2 + \frac{Pb\phi_0}{\Psi_0}\nu\left(1 - \frac{1}{2}\phi_0^2 f^2 + \cdots\right)f'. \tag{11}$$

Inspection of (10) and (11) indicates that the assumption $b/a \ll 1$ implies that the equations of beam theory can be used for the solution of the ring-plate problem in the *linear*, small-deflection range.

In order to see to what *extent* the linear small-deflection theory applies we may proceed as follows. We set

$$\frac{Pb^2}{\phi_0 D} = 1, \tag{12}$$

and we take account of the fact that in non-linear plate theory non-linear terms both in (10) and (11) are important by setting

$$\frac{\Psi_0 b^2}{Da} = \frac{Cb^2\phi_0^2}{\Psi_0 a}. \tag{13}$$

From (12) and (13) follows

$$\phi_0 = \frac{Pb^2}{D}, \qquad \Psi_0 = \frac{Eh^2}{[12(1 - \nu^2)]^{1/2}}\,\phi_0\,, \tag{14}$$

and

$$\frac{\Psi_0 b^2}{Da} = [12(1 - \nu^2)]^{1/2}\,\frac{b^2}{ah}\,\phi_0\,, \tag{15}$$

$$\frac{Pb\phi_0}{\Psi} = \frac{1}{[12(1 - \nu^2)]^{1/2}}\,\frac{h}{b}\,\phi_0\,. \tag{16}$$

We set as an abbreviation

$$\mu = [12(1 - \nu^2)]^{1/2}\,\frac{b^2}{ah} \tag{17}$$

and take account of the fact that always $h/b \ll 1$ and $\phi_0 = O(1)$. Equations (10) and (11) then reduce to the following system

$$f'' = -\left(1 - \frac{1}{2}\phi_0^2 f^2 + \cdots\right) + \mu\phi_0\left(1 - \frac{1}{6}\phi_0^2 f^2 + \cdots\right)fg, \tag{18}$$

$$g'' = -\frac{1}{2}\mu\phi_0\left(1 - \frac{1}{12}\phi_0^2 f^2 + \cdots\right)f^2. \tag{19}$$

*This is equivalent to saying that the leading terms only are retained in a development of the solution in powers of b/a.

The following conclusions may be drawn from equations (18) and (19).

(1) The condition for beam theory to be applicable is of the form

$$\mu\phi_0 \ll 1. \tag{i}$$

(1a) When in addition

$$\phi_0^2 \ll 1, \tag{ii}$$

then *linear* beam theory is applicable.

(2) When

$$\mu\phi_0 = O(1), \tag{iii}$$

the governing equations are those of non-linear *plate* theory no matter how small ϕ_0 by itself may be

(2a) Applicability of the Kirchhoff-von Kármán theory of small finite deflections of plates requires satisfaction of (*ii*).

Let w_0 be a measure of the transverse deflection of the plate. We may set $w_0 = \phi_0 b$. Condition (i) becomes

$$w_0 \ll \frac{b}{\mu} = \frac{1}{[12(1-\nu^2)]^{1/2}}\frac{a}{b}h, \tag{i'}$$

indicating that for the present plate problem linearity does not require that the deflection be small compared with the thickness of the plate.

4. Perturbation method for moderate values of $\mu\phi_0$. In order to see for specific examples to what extent the non-linearity for small $\mu\phi_0$ manifests itself the following series developments may be used

$$\begin{aligned} f &= f_0 + (\mu\phi_0)^2 f_2 + (\mu\phi_0)^4 f_4 + \cdots, \\ g &= \mu\phi_0 g_1 + (\mu\phi_0)^3 g_3 + \cdots. \end{aligned} \tag{20}$$

In what follows we shall for simplicity's sake assume that ϕ_0^2 by itself is negligibly small compared with unity so that the differential equations (18) and (19) may be taken in the form

$$f'' = -1 + \mu\phi_0 fg, \qquad g'' = -\tfrac{1}{2}\mu\phi_0 f^2. \tag{21}$$

We take as example a plate of width b with horizontal slope at $x = \pm\frac{1}{2}$ and with no horizontal edge forces. These boundary conditions correspond to what would be encountered if the ring plate were part of a corrugated cylindrical shell.

We have then as boundary conditions for the solutions (20) of (21)

$$f_n(\pm\tfrac{1}{2}) = 0, \qquad g_n(\pm\tfrac{1}{2}) = 0. \tag{22}$$

We find

$$\begin{aligned} f_0 &= -\frac{1}{2}\left(x^2 - \frac{1}{4}\right), \\ g_1 &= -\frac{1}{240}\left(x^6 - \frac{5}{4}x^4 + \frac{15}{16}x^2 - \frac{11}{64}\right), \\ f_2 &= \frac{1}{4800.9}\left(x^{10} - \frac{135}{56}x^8 + \frac{15}{4}x^6 - \frac{195}{64}x^4 + \frac{495}{256}x^2 - \frac{351}{1024} - \frac{5}{14.1024}\right). \end{aligned} \tag{23}$$

The relative edge deflection δ is given by

$$\delta = \int_{a-b/2}^{a+b/2} \phi\, dr = 2\phi_0 b \int_0^{1/2} f\, dx$$

$$= 2\phi_0 b\left[\int_0^{1/2} f_0\, dx + (\mu\phi_0)^2 \int_0^{1/2} f_2\, dx + \cdots\right]. \tag{24}$$

From (24) follows

$$\delta = \frac{\phi_0 b}{12}\,[1 - 0.000059(\mu\phi_0)^2 + \cdots]. \tag{25}$$

We see that while order of magnitude considerations require that $\mu\phi_0 \ll 1$ in order that there be no noticeable non-linear effect on load displacement relation the actual computations show that this is numerically conservative and that actually $\mu\phi_0$ can be appreciably larger than unity before an appreciable non-linear effect occurs.

Having f_0, f_2 and g_1 we may if we wish also determine the influence of small non-linearity on bending and direct stresses in the plate. We obtain in particular

$$\sigma_{r,B}(\pm\tfrac{1}{2}) = 6M_r(\pm\tfrac{1}{2})/h^2 = \pm(3bP/h^2)[1 + 0.00005(\mu\phi_0)^2 + \cdots],$$

$$\sigma_{\theta,D}(\pm\tfrac{1}{2}) = N_\theta(\pm\tfrac{1}{2})/h = \pm(3bP/h^2)[0.0012(1 - \nu^2)^{1/2}(\mu\phi_0) + \cdots].$$

5. Boundary layer equations for large values of $\mu\phi_0$. In order to investigate the character of the solutions of equations (18) and (19) when $\mu\phi_0 \gg 1$ we introduce new dependent and independent variables by setting

$$x = \lambda y, \qquad f = f_0\, F(y), \qquad g = g_0\, G(y). \tag{26}$$

We have then as differential equations

$$F'' = -\left[1 - \frac{1}{2}(\phi_0 f_0)^2 F^2 + \cdots\right]\frac{\lambda^2}{f_0} + \lambda^2\mu\phi_0 g_0\left[1 - \frac{1}{6}(\phi_0 f_0)^2 F^2 + \cdots\right]FG, \tag{27}$$

$$G'' = -\frac{1}{2}\lambda^2\mu\phi_0\,\frac{f_0^2}{g_0}\left[1 - \frac{1}{12}(\phi_0 f_0)^2 F^2 + \cdots\right]F^2. \tag{28}$$

We set

$$\frac{\lambda^2}{f_0} = 1, \qquad \lambda^2\mu\phi_0 g_0 = 1, \qquad \lambda^2\mu\phi_0\,\frac{f_0^2}{g_0} = 1 \tag{29}$$

or

$$\lambda = (\mu\phi_0)^{-1/4}, \qquad f_0 = (\mu\phi_0)^{-1/2}, \qquad g_0 = (\mu\phi_0)^{-1/2}. \tag{30}$$

Differential equations (27) and (28) now read

$$F'' = -\left[1 - \frac{1}{2}\frac{\phi_0}{\mu}F^2 + \cdots\right] + \left[1 - \frac{1}{6}\frac{\phi_0}{\mu}F^2 + \cdots\right]FG, \tag{31}$$

$$G'' = -\frac{1}{2}\left[1 - \frac{1}{12}\frac{\phi_0}{\mu}F^2 + \cdots\right]F^2. \tag{32}$$

The order of magnitude of the change of slope ϕ is now $\phi_0 f_0 = \sqrt{\phi_0/\mu}$ and consequently we know that

$$\phi_0/\mu = O(1). \tag{33}$$

Accordingly all coefficients of (31) and (32) are $O(1)$ and differentiation with respect to y does not change the order of magnitude of the quantities which are differentiated. Consequently significant changes of F and G occur over y-distances which are $O(1)$. But this is the same as saying that significant changes of f and g occur over x-distances which are $O(\lambda) = O([\mu\phi_0]^{-1/4})$. Since $\mu\phi_0 \gg 1$ we have then the existence of a boundary layer.* The actual width of the boundary layer with respect to the r-coordinate is $b(\mu\phi_0)^{-1/4}$.

The question may be asked in which way the relative deflection δ of the edges and bending and direct stresses depend on ϕ_0 and μ for the limiting case $\mu\phi_0 \to \infty$.

We may, again for simplicity's sake, assume that ϕ_0/μ is negligibly small compared to unity and restrict attention to the system

$$F'' = -1 + FG, \qquad G'' = -\tfrac{1}{2}F^2. \tag{34}$$

If we take the same boundary value problem as in the preceding section but now let the edges of the plate be at $x = 0$ and $x = 1$, then we have at the inner edge

$$x = y = 0: \qquad F = 0, \qquad G = 0. \tag{35}$$

At $x = \frac{1}{2}$ we have the symmetry conditions of vanishing F' and G' or with y as independent variable

$$y = \frac{1}{2\lambda} = \frac{1}{2}(\mu\phi_0)^{-1/4}: \qquad F' = 0, \qquad G' = 0. \tag{36}$$

As λ tends to zero the outer boundary $y = \frac{1}{2}\lambda^{-1}$ tends to infinity. We expect that for sufficiently large values of λ^{-1} we should be able to replace condition (36) by the limiting condition

$$F'(\infty) = G'(\infty) = 0, \tag{37}$$

and the problem then is to solve the system (34) subject to the boundary conditions (35) and (37).

We expect that $F'(y)$, $G'(y)$ are $O(1)$ and consequently bending and direct stresses have orders of magnitude determined as follows.

$$\sigma_{rB} = \frac{h}{2}E\phi'(r) = \frac{h}{2}E\frac{\phi_0 f_0}{\lambda b}F'(y) = \frac{E}{2}\frac{h}{b}\frac{\phi_0^{3/4}}{\mu^{1/4}}F'(y)$$

or

$$\begin{aligned}\frac{\sigma_{rB}}{E} &= \frac{1}{2}\left(\frac{P}{Eh}\right)^{3/4}[12(1-\nu^2)]^{5/8}\left(\frac{a}{h}\right)^{1/4}F'(y)\\ \sigma_{\theta D} &= \frac{\Psi'(r)}{h} = \frac{\Psi_0 g_0}{hb\lambda}G'(y) = \frac{Eh}{b[12(1-\nu^2)]^{1/2}}\frac{\phi_0^{3/4}}{\mu^{1/4}}G'(y)\end{aligned} \tag{38}$$

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*Boundary layer solutions for quite different problems of finite bending of circular plates have previously been discovered by Friedrichs and Stoker. See Am. J. Math. **63**, 839-888 (1941).

or

$$\frac{\sigma_{\theta D}}{E} = \left(\frac{P}{Eh}\right)^{3/4} [12(1 - \nu^2)]^{1/8} \left(\frac{a}{h}\right)^{1/4} G'(y) \tag{39}$$

It is remarkable that now $\sigma_{\theta D}$ and σ_{rB} are of the same order of magnitude. Furthermore we see that in this range the stresses vary with the three-fourth power of the applied load and are independent of the actual width of the plate. They are inversely proportional to the thickness of the plate and proportional to the fourth root of the inner (or equally well average) radius of the plate.

In order to determine the relative deflection of inner and outer edge we must calculate

$$\delta = \int_a^{a+b} \phi \, dr = 2 \int_0^{1/2\lambda} \phi_0 f_0 F(y) \lambda b \, dy = 2b \frac{\phi_0^{1/4}}{\mu^{3/4}} \int_0^{1/2\lambda} F(y) \, dy. \tag{40}$$

Now, before letting λ tend to zero we must know the behaviour of $F(y)$ for large y. It seems that one might reason as follows. Assume that for large y

$$F(y) = c_1 y^n, \qquad G(y) = c_2 y^m, \tag{41}$$

where both n and m are smaller than unity in order to satisfy (37). Equations (34) then become

$$c_1 n(n - 1)y^{n-2} = -1 + c_1 c_2 y^{n+m}, \qquad c_2 m(m - 1)y^{m-2} = -\tfrac{1}{2}c_1^2 y^{2n}. \tag{42}$$

The first of these equations gives

$$c_1 c_2 = 1, \qquad n + m = 0, \tag{43a}$$

and the second gives

$$c_2 m(m - 1) = -\tfrac{1}{2}c_1^2, \qquad m - 2 = 2n. \tag{43b}$$

Equations (43) are satisfied by

$$m = \frac{2}{3}, \qquad n = -\frac{2}{3}, \qquad c_1 = \left(\frac{4}{9}\right)^{1/3}, \qquad c_2 = \left(\frac{9}{4}\right)^{1/3}. \tag{44}$$

We have then for large y

$$F(y) \sim \left(\frac{4}{9}\right)^{1/3} y^{-2/3}, \qquad G(y) \sim \left(\frac{9}{4}\right)^{1/3} y^{2/3} \tag{45}$$

If the first of these two formulas is introduced into equation (40) for δ we conclude that since (45) should be valid over a predominant portion of the interval of integration we should have

$$\delta \sim 2b \frac{\phi_0^{1/4}}{\mu^{3/4}} \int_0^{1/2\lambda} \left(\frac{4}{9}\right)^{1/3} y^{-2/3} \, dy = 2b \frac{\phi_0^{1/4}}{\mu^{3/4}} 3\left(\frac{4}{9}\right)^{1/3} \left(\frac{1}{2\lambda}\right)^{1/3} = 6\left(\frac{2}{9}\right)^{1/3} b \frac{\phi_0^{1/3}}{\mu^{2/3}} \tag{46a}$$

or

$$\frac{\delta}{b} \sim 6\left(\frac{2}{9}\right)^{1/3} \left(\frac{P}{Eh}\right)^{1/3} \left(\frac{a}{b}\right)^{2/3} \tag{46b}$$

Clearly, the argument beginning with equations (41) is not rigorous. The results however are not implausible, and one may hope that future more rigorous considerations will confirm them.