J. Austral. Math. Soc. 21 (Series A) (1976), 467–472.

A PROBLEM OF PAUL ERDÖS ON GROUPS

Dedicated to George Szekeres for his 65th birthday

B. H. NEUMANN

(Received 24 January 1975)

Communicated by Jennifer Seberry Wallis

1. Introduction

Let G be a group, and associate with G a graph[1] $\Gamma = \Gamma(G)$ as follows: the vertices of Γ are the elements of G, and two vertices g, h of Γ are joined by an undirected edge if, and only if, g and h do not commute as elements of G, that is $[g, h] \neq 1$ [where $[g, h]$ is the commutator $g^{-1}h^{-1}gh$, and 1 is the unit element of the groups that occur as well as the integer, according to context]. We are interested in complete subgraphs of Γ, or equivalently in sets of elements of G no two of which commute.

Paul Erdös recently[2] posed the following problem:

Let G be such that Γ contains no infinite complete subgraph; is there then a finite bound on the cardinality of complete subgraphs of Γ?

The purpose of this note is to answer this question affirmatively. It turns out, in fact, that the class of groups whose graph contains no infinite complete subgraph — let us call them Paul Erdös groups, or PE-groups for short — coincides with the class of groups whose centre has finite index; these latter groups are called FIZ-groups for short, and have been studied in other contexts; see, for example, Neumann (1955) and the literature there quoted, or Chapter 4 of Robinson's book (1972), where FIZ-groups are called *central-by-finite*.

[1] There are other, classical — and group-theoretically more significant — ways of associating a graph with a group. We here restrict attention to the graphs Γ, and "the graph of G" will mean $\Gamma(G)$.

[2] At the 15th Summer Research Institute of the Australian Mathematical Society, at the University of New South Wales, 13 January–14 February 1975.

2. Preliminaries

We make the convention that [PE] stands for the class of PE-groups, [FIZ] for the class of FIZ-groups, and similarly for other group-theoretical properties. In particular we denote by [FC] the class of groups in which all classes of conjugate elements are finite. It is a well-known fact, and indeed almost trivial, that all FIZ-groups are FC-groups. Thus the following lemma is a step in the right direction.

LEMMA 1. *All PE-groups are FC-groups.*

PROOF. Let G be a group not in [FC]; we show that G is not in [PE] either. Let $g \in G$ be an element with an infinite class of conjugates, and let T be an infinite set of elements of G such that distinct elements of T produce distinct conjugates of g: thus if $s, t \in T$ and $s \neq t$, then $s^{-1}gs \neq t^{-1}gt$. Consider the restriction $\Gamma(T)$ of $\Gamma(G)$ to T. By Ramsey's Theorem $\Gamma(T)$ either contains an infinite complete subgraph — in which case $G \notin [PE]$ and we have finished — or an infinite independent subset U, that is an infinite set of vertices without any edges. The elements of U, as elements of G, commute with each other. Now consider the set

$$gU = \{gu \mid u \in U\}.$$

If u, v are distinct elements of U, then the commutator

$$\begin{aligned}[gu, gv] &= (gu)^{-1}(gv)^{-1}gugv \\ &= u^{-1}g^{-1}v^{-1}ugv \\ &= u^{-1}g^{-1}uv^{-1}gv \neq 1,\end{aligned}$$

as $u^{-1}gu \neq v^{-1}gv$, by the choice of T. Thus no two distinct elements of gU commute, $\Gamma(gU)$ is an infinite complete graph, and $G \notin [PE]$; and the lemma follows.

We denote, as usual, the centralizer of an element $g \in G$ by $C_G(g)$, and the centralizer of a set S of elements of G by $C_G(S)$; thus

$$C_G(S) = \bigcap_{s \in S} C_G(s).$$

The index $|G: C_G(g)|$ of the centralizer of $g \in G$ equals the cardinal of the class of conjugates of g. Thus $G \in [FC]$ if, and only if, $|G: C_G(g)|$ is finite for all $g \in G$.

The reader is reminded that the intersection of a finite set of subgroups of a group G has finite index in G if each of the subgroups in the set has finite index in G. Thus $G \in [FC]$ if, and only if, $|G: C_G(S)|$ is finite for every finite set S of elements of G.

We also remind the reader that if the set S of elements of G generates G, then $C_G(S) = Z(G)$, the centre of G. It follows at once that — as is well known — every finitely generated FC-group is a FIZ-group.

LEMMA 2. *Let $G \in [FC]$ have an abelian subgroup A of finite index. Then $G \in [FIZ]$.*

PROOF. We can generate G by a set Q, say, of generators of A and a finite set R, say, of further generators, for example one element out of each (right) coset of A in G. Put $S = Q \cup R$. The centre of G then is

$$Z(G) = C_G(S) = C_G(Q) \cap C_G(R).$$

Now Q lies in the abelian subgroup A, hence $A \subseteq C_G(Q)$, and $C_G(Q)$ has finite index in G. But so has $C_G(R)$, as R is finite and $G \in [\mathrm{FC}]$. Thus also $Z(G)$ has finite index in G, and the lemma follows.

The converse of the lemma, namely that if $G \in [\mathrm{FIZ}]$, then G has an abelian subgroup of finite index, is trivial. More important for us is a negative reformulation of the lemma.

COROLLARY 3. *Let $G \in [FC] - [FIZ]$ have a subgroup A of finite index. Then A is not abelian.*

3. The main result

As we already know that both PE-groups and FIZ-groups are FC-groups, we now have to show that a group $G \in [\mathrm{FC}] - [\mathrm{FIZ}]$ cannot be a PE-group. The rest of the characterization of [PE] will be easy.

LEMMA 4. *Let $G \in [FC] - [FIZ]$, and assume that G contains two finite sequences of n elements*

$$(a_1, a_2, \cdots, a_n), \qquad (b_1, b_2, \cdots, b_n)$$

with the following properties:

(i) *if $i \neq j$, then $[a_i, a_j] \neq 1$;*
(ii) *if $i \neq j$, then $[a_i, b_j] = 1$;*
(iii) *for all i, $[a_i, b_i] \neq 1$;*
(iv) *for all i, j, $[b_i, b_j] = 1$.*

Then G contains two further elements a_{n+1}, b_{n+1} such that (i), (ii), (iii), (iv) *remain valid for the sequences*

$$(a_1, a_2, \cdots a_{n+1}), \qquad (b_1, b_2, \cdots, b_{n+1})$$

of length $n + 1$.

Before we embark on the proof, we pause to look at the restriction of the graph $\Gamma(G)$ to the set of elements $\{a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}$: we have a complete subgraph with vertices $a_1, a_2, \cdots, a_n$, and each a_i is joined to the vertex b_i, and there are no other edges.

PROOF OF LEMMA 4. Put

$$A = C_G(a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n\}).$$

Then A has finite index in G, and as $G \in [\mathrm{FC}]$–$[\mathrm{FIZ}]$ by assumption, A is, by Corollary 3, not abelian. Thus we may pick two elements from A, say a and b, such that $[a, b] \neq 1$. Now put

$$a_{n+1} = a \cdot b_1 b_2 \cdot \cdots \cdot b_n,$$

$$b_{n+1} = b.$$

Then, for $1 \leqq i \leqq n$,

(i′) $$[a_i, a_{n+1}] = [a_i, b_i] \neq 1,$$

as a and all b_j other than b_i commute with a_i;

(ii′) $$[a_{n+1}, b_i] = [a_i, b_{n+1}] = 1,$$

as a and all b_j commute with b_i and a_i commutes with b;

(iii′) $$[a_{n+1}, b_{n+1}] \neq 1,$$

as a and $b = b_{n+1}$ do not commute, but b and all b_i do;

(iv′) $$[b_i, b_{n+1}] = 1,$$

obviously. The lemma thus follows.

COROLLARY 5. *If $G \in [FC] - [FIZ]$, then $G \notin [PE]$.*

We only have to use Lemma 4 to build up, inductively, an infinite sequence $(a_1, a_2, a_3, \cdots)$ of pairwise non-commuting elements, starting from a pair a_1, b_1 of non-commuting elements, as G is obviously not abelian. This is now, but for the almost trivial converse, our main result.

THEOREM 6. *The group G is a PE-group if, and only if, it is a FIZ-group.*

PROOF. We have already seen that every PE-group must be a FIZ-group: this follows from Lemma 1 and Corollary 5. However, it is obvious that every FIZ-group is a PE-group; for if $G \in [\mathrm{FIZ}]$ and $|G: Z(G)| = n$, then every set of $n+1$ elements of G must contain two that are congruent modulo the centre $Z(G)$, and thus commute. The theorem thus follows.

4. Odds and ends

We have seen that if the centre of the group G has index n in G, then the graph $\Gamma(G)$ contains no complete subgraph of order greater than n; and this can be immediately improved to $n-1$ in general, and a little further if the arithmetical structure of n is taken into account. Thus $n-1=3$ is attained for the quaternion group and also for the dihedral group of order 8 [these two groups have isomorphic graphs Γ]; and I owe to Dr M. F. Newman the remark that when $n=2^d$, then $n-1$ is attained by the free group of rank d of the variety generated by the quaternion group; but for the symmetric group of degree 3 the index of the centre is $n=6$, while the biggest complete subgraph has order 4. This immediately raises the question of a bound for the index of the centre, $|G\colon Z(G)|=n$, given that the graph $\Gamma(G)$ contains no complete subgraph of order greater than m. One can obtain such a bound from the proof of Theorem 6, namely

$$\log n = O(m^2),$$

but this bound is very crude, and my guess, based on no evidence, is

$$n = O(m^2).$$

A similar question might be asked with respect to Lemma 2: if the FC-group G has an abelian subgroup A, what estimate in terms of $|G\colon A|$ can be given for the index n of the centre of G? The answer is: none. On only needs to take A as an abelian group of finite odd order and make G the splitting extension of A by a cyclic group of order 2 whose generator inverts all elements of A. Then $|G\colon A|=2$, but the centre of G is trivial, hence $n=|G|$, and this can be made as large as you please.

If m is taken as the precise maximum order of a complete subgraph of $\Gamma(G)$, then not all integers occur as values of m. Paul Erdös remarked that 2 does not occur, for to any two elements a, b that do not commute with each other there is a third, namely ab, that commutes with neither of them. It is then natural to ask what (finite) values of m are possible. The answer is that 2 is the only exception. The dihedral group of order $4(m-1)$ is an example of a group whose graph contains a complete subgraph of order m, but none of greater order. The (easy) verification is omitted.

References

B. H. Neumann (1955), 'Groups with finite classes of conjugate subgroups', *Math. Z.* **63**, 76–96.

Derek J. S. Robinson (1972), *Finiteness conditions and generalized soluble groups, Part* 1 (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 62. Springer-Verlag, Berlin, Heidelberg, New York 1972).

The Australian National University
and
Commonwealth Scientific and Industrial Research Organization,
Division of Mathematics and Statistics,
Canberra, Australia

Added 18 *July* 1975. I have just heard that the results in this paper were obtained also, by much the same methods, by Professor Ralph N. McKenzie, two or three months before I had them. He and Dr. Vance Faber hope to publish in due course some extensions of these results to higher cardinal numbers and to cancellation semigroups.