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A PROCESS MODEL FOR INTERACTION AND MATHEMATICAL LEVEL RAISING

ABSTRACT. In this article we present a process model we have developed for interaction and mathematical level raising. In the process model the focus is on the individual learning process. The model is based on our own research experience and our common interest in individual learning processes. We relate it to other research. The model is meant to show how level raising can be realised by letting students work in small groups on a mathematical problem.

1. INTRODUCTION

'Language plays an essential role in the formation and expression of mathematical ideas. School children should be encouraged to discuss and explain the mathematics which they are doing.' In referring to this quotation from the Cockroft Report (1982) Celia Hoyles has made a plea in this journal to open up the discussion on group discussion, especially pupil-pupil discussion in mathematics (Hoyles, 1985). As a first step she suggests that mathematical understanding be considered from a social interaction perspective and perhaps be defined as the ability to:

- form a view of the mathematical idea;
- step back and reflect upon it;
- use it appropriately and flexibly;
- communicate it effectively to another; and
- reflect on another's perspective from one's own framework or challenge and logically reject this alternative view.

In initiating this discussion, Hoyles explicitly asks for criticism and comment on her suggestions. However, only in one instance have authors in this journal reacted to her ideas. In an article by Pirie and Schwarzenberger, both authors support Hoyles's distinction between the cognitive function of talk and the communicative function of talk, but criticise her suggested definition of mathematical understanding as the ability to act in all the ways mentioned above. 'Our own perspective,' they write, 'is that such actions are evidence of mathematical understanding but not part of understanding itself' (Pirie and Schwarzenberger, 1988, p. 463).

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The lack of explicit reaction in this journal does not mean that there is no interest in research on pupil-pupil discussion. On the contrary, now, during the intervening twelve years, a great deal of research on pupil-pupil discussion in mathematics has occurred. Some of this research has been published in this journal (see Dekker, 1987; and Cobb et al., 1992) and will be referred to in Section 3. To reopen the discussion on interaction and mathematical understanding and to challenge all those colleagues who undertake interesting research in the field of small group work in mathematics, we will present in this paper a process model for interaction and mathematical level raising.

2. THE PROCESS MODEL

2.1. *Framework, prior research*

In the Netherlands the interest in independent and self-regulated learning has grown considerably and is related to a reform of the upper secondary school system. Interest in self-regulated learning has been accompanied by interest in learning in small groups. Learning in small groups is perceived as promoting the independence of the students and stimulating all kinds of metacognitive activities. Many research outcomes support the idea that small group work promotes mathematical understanding. Webb (1991), for instance, has shown that giving explanations has a positive effect on an individual's own learning (see also Chi et al., 1994). At the moment many researchers report case studies on student-student discussions (see Section 3). Wonderful examples of how student-student discussions increase opportunities for learning are given by Yackel et al. (1991) and Wood (1996).

However, much of the research on learning in groups lacks a focus on the progress in the learning of the individual student. It remains unclear which processes are considered necessary for the realisation of mathematical level raising. In the process model developed by us the focus lies precisely on the individual learning process. The model is based on our own research experience and our common interest in individual learning processes and is related to other research. It is meant to show how level raising can be realised by allowing students to work in small groups on mathematical problems. The main prior research we refer to is that undertaken by Dekker (1991). Her research analyses the interaction taking place in small groups between students working on problems in graphs. In her analysis, Dekker examines the relation between the quality of the interaction and the process of level raising (see also Section 3). Our process model will make this relation more explicit.

According to Dekker (1991, 1994), the problems for the small groups should be *realistic* or meaningful for the students, *complex* in the sense that several abilities are needed to solve the problem, and *constructive* (producing a graph, table, drawing, story) in order that differences between students become visible and subject to discussion. The problems should also aim at *level raising.* An example of such a problem is:

Show in a distance/time graph how you travel to school in the morning.

The problem is *realistic* and evidently demands *construction*. The task is *complex*, because several abilities are necessary to fulfil the task: imagining the situation, relating differences in speed to changes in distance increase and translating this into a graph. For students with no prior experience in constructing distance-time graphs and with inadequate notions about the relation between distance and time, the task aims at *level raising*. For example, some students will have the notion that the relation between distance and time is always linear, or that when speed goes up and down, the distance-time graph goes up and down as well. Level raising involves discovering that this mathematical understanding is too simple to successfully construct a distance-time graph and also develops new mathematical understanding.

2.2. *Key activities*

The situation described in the process model is that of a small group of students working together on the same mathematical problem, such as the one mentioned above. The work of the students is assumed to be different. We have constructed the process model around four key activities:

- to show one's work,
- to explain one's work,
- to justify one's work,
- to reconstruct one's work.

When level raising occurs, this will be revealed in the reconstruction of one's work. The result of the reconstruction can be shown or told again, which completes the cycle of these four activities.

The four key activities which are all described in literature on mathematics learning are mental activities that have the following characteristics:

(a) they can be demonstrated by students who work individually, but more so by students who communicate with each other during their work;
(b) they can be observed very well;
(c) they have a function in the learning process and contribute to mathematical understanding;

(d) they can be influenced by didactic factors, such as the nature of problems and the coaching of the teacher.

We will illustrate the four key activities by the student Anna in the following reconstructed example. The example is based on observations published in Dekker (1991). It is reconstructed in the sense that it combines some of the discussion of a small group of students working on the problem and leaves out irrelevant parts. The example is meant to place the key activities and the conceptual issues we deal with in an integrative context:

Anna and Ben are working on the problem mentioned above. After a while Ben asks Anna: ‘What are you doing? What have you got?’
Anna *shows* her work and *explains*: ‘Look, I have done it like this, because in the beginning I go slowly, so the graph starts low and goes up slowly. Did you start your graph that way too?’
Ben shows how he has started his graph: ‘I have done it differently. I thought you have to start at zero and then just a straight line. In the beginning I bike slowly too, but I don’t know how... But what you have got is wrong, isn’t it?’
Anna tries to *justify* her work: ‘Yes, but if you bike slowly, the distance grows slowly, so the graph goes up slowly... Wait a minute... that isn’t completely right. Indeed I have to start at zero, because in the beginning I haven’t travelled anything. But it is not a straight line, because I don’t bike at the same speed all the time. So I have to...’
Anna *reconstructs* her own work and improves her graph.

In principle a student who works alone can perform all the key activities, but it takes a lot of self-regulation. In communication with other students the key activities will take place in a more natural way. That is why the process model also contains interactive or communicative activities. We call them regulating activities. The process model is presented in Figure 1. The key activities are indicated in bold and the regulating activities in italic print.

2.3. *Regulating activities*

A fellow-student can stimulate the showing of work by asking questions such as: ‘What have you got? What are you doing?’ He can evoke explanations by asking: ‘How did you get that? I don’t understand it,’ and he can stimulate the process of justifying by voicing criticism: ‘I think it’s wrong.’ If justification fails, it can lead to the reconstruction of the work. The presence of the other student who has voiced criticism and has noticed how justification has failed can ensure that the criticism will not be ignored. Every fellow-student or small group of students or a teacher in the role of a coach can stimulate Anna to perform the four key activities. In a situation where Anna works in a small group these regulating activities can come naturally when the work of the students varies. Comparing different work

A and B are working on the same mathematical problem. Their work is different.

A is working		B is working
A asks B to show his work	*what are you doing?* *what have you got?*	*B asks A to show her work*
A becomes aware of her own work		B becomes aware of his own work
A shows her own work	**I am doing this...** **I have got this...**	**B shows his own work**
A becomes aware of B's work		B becomes aware of A's work
A asks B to explain his work	*why are you doing that?* *how did you get that?*	*B asks A to explain her work*
A thinks about her own work		B thinks about his own work
A explains her own work	**I'm doing this, because...** **I have got this, because...**	**B explains his own work**
A thinks about B's work		B thinks about A's work
A criticises B's work	*but that's wrong, because...*	*B criticises A's work*
A thinks about B's criticism		B thinks about A's criticism
A justifies her own work	**I thought it was right, because...**	**B justifies his own work**
A thinks about her justification		B thinks about his justification
A criticises her own work	oh no, it isn't right, because...	B criticises his own work
A reconstructs her own work	**I'll better do it like this...**	**B reconstructs his own work**

bold: key activities
standard: mental activities
italic: regulating activities

Figure 1. Process model (Rijkje Dekker & Marianne Elshout-Mohr, 1996).

can form the starting point of questions such as: 'Why do you have that?' or even 'Why do I have this?'

2.4. *Mental activities*

While key activities and regulating activities can all be observed, mental activities take place which are not, or only partly, observable. Mental activities involve those activities which 'go along' with the key activities. For instance, *to show one's work* 'includes' the mental activity of becoming aware of one's own work, and *to explain one's work* means that one has to think about one's own work. A prerequisite *to reconstruct one's work* is to criticise one's own work and that may happen internally too. In the process model these mental activities are displayed in standard print.

The process model described in Figure 1 contains the key activities, the regulating activities, and the mental activities of individual students in a situation in which they work together and their work is different. This model can be used as a basic model in researching students' mathematical level raising processes in different situations that elicit different regulating activities. Figure 2 shows how the model can be used to represent A's level raising process.

The model can also be used to represent particular situations as in the instance in which the work of the students is not different. In that case there is no stimulus to ask or give explanations. Another situation arises when a student works in a group of three or more students. In that case A's activities are regulated by two or more B's. Even the situation in which a student works alone can be represented. In that case we must assume that the italic components remain 'empty' (unless the student fulfils them herself). According to the model, this will affect the mental activities and the key activities.

Hypotheses about how this might affect the other components of the learning process can be made explicit by using the model. The model can give rise to new research questions. In addition, we think that it will be a valuable conceptual framework to summarise and integrate research findings that have already been reported in separate case studies.

3. RELATION WITH RESEARCH

The process model has relations with research findings and notions from a variety of research on the learning of mathematics in small groups. The main relations are:

(a) the supposition that the four key activities form the basis of mathematical level raising;

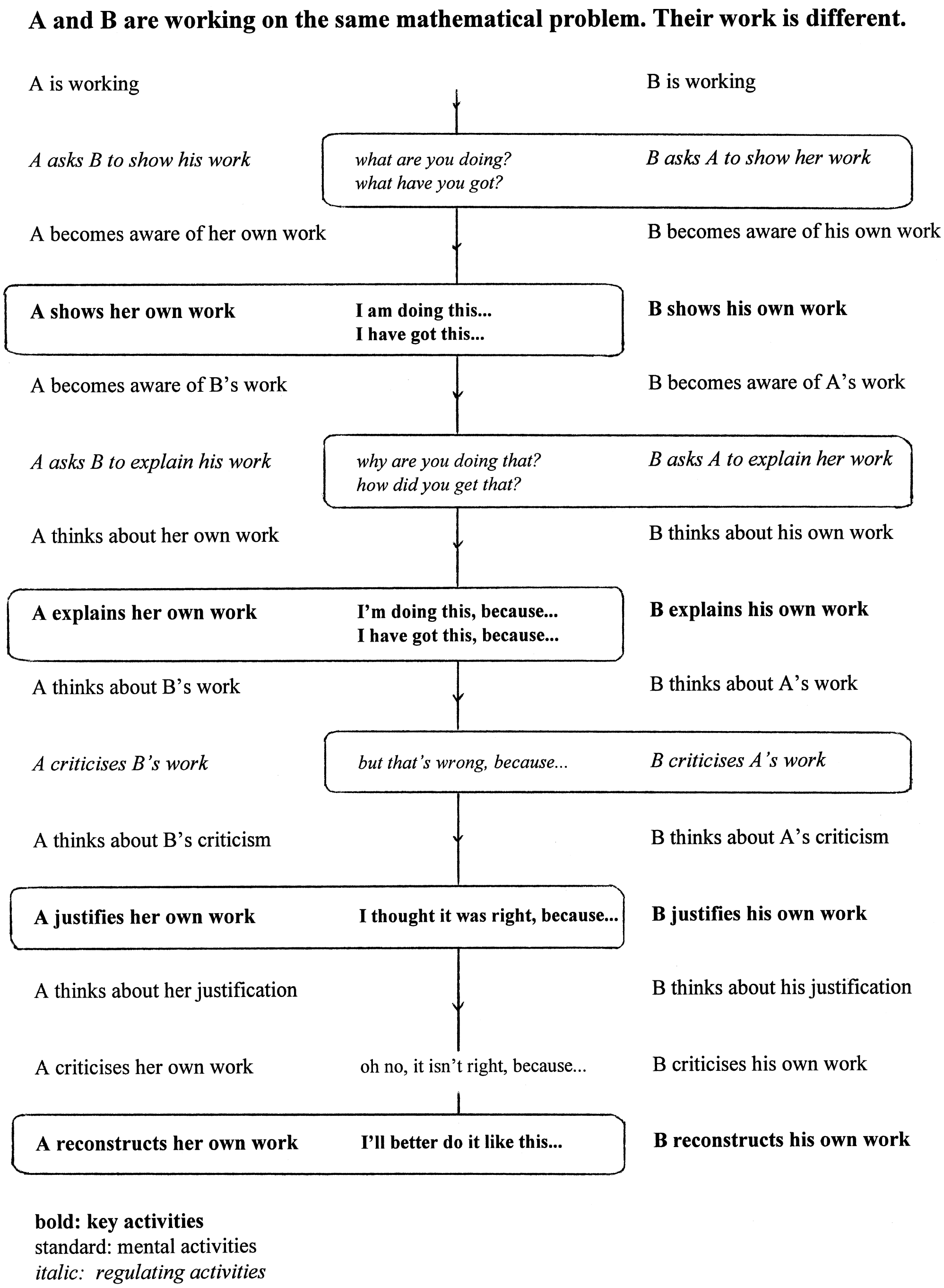

Figure 2. Process model (Rijkje Dekker & Marianne Elshout-Mohr, 1996).
Frames and arrows indicate the observable parts of the level raising process for A.

(b) the supposition that the interaction between students stimulates the key activities;

(c) the supposition that the comparison of different work stimulates the learning process.

The first supposition can be seen as an elaboration of the notion that reflection plays a crucial role in the learning of mathematics. In his plenary lecture at the Fifth International Congress for Mathematics Education, Kilpatrick illuminated the role of reflection in mathematics education and linked this to the renewed interest in reflection or metacognition in cognitive psychology (Kilpatrick, 1985). He points at the radical view of Freudenthal who considers mathematics itself as a reflection on one's own activities and those of others. For Freudenthal the main issue in mathematics education is how reflection on one's own activities can be stimulated. Freudenthal himself suggests that children be allowed to work together on mathematical problems in small heterogeneous groups so that they can observe each other's activities and start thinking about their own activities by giving explanations to each other. Heterogeneity relates to the level of the students which becomes apparent when they jointly approach a mathematical subject. The learning materials should be appropriate for this purpose, namely, not pre-structured but problematic (Freudenthal, 1978, pp. 60–63).

Dekker has examined Freudenthal's ideas on learning mathematics in small heterogeneous groups in her dissertation (Dekker, 1991). A central feature of her study, related to the second supposition, is the analysis of the interaction taking place in small heterogeneous groups in which students aged 12 to 13 are working with specific learning materials. Besides providing explanations which reveal reflection on one's own activities, she considers giving criticism as an activity which reveals reflection on the activities of other students (see also Dekker, 1995). Both explaining and criticising are considered as a means to start reasoning: 'I have this because...', or 'but that is wrong, because...' The third supposition that the comparison of different work stimulates the learning process is strongly manifested in the work of Dekker as well.

The process model has a cyclic character as is usual for models in which reflection is seen as crucial for level raising. An analysis of several case studies has clarified the character of these cycles. Wistedt (1994) has shown the cycles in the interpretation by students of the task. In his study Wistedt has indicated the importance of showing, explaining and discussing one's own way of working in a case study involving four 11 year olds working together on an open mathematical problem. The problem has a relation with their own knowledge and is challenging at the same time, because the children have no knowledge of the mathematics they will encounter. In the beginning all students interpret the problem in their own way, depending on their own knowledge and experience. Then each of them (except for one) gets an idea of what can be done and discovered. Sometimes the students go their own way, sometimes they go along for a while with another student. In this process it is important for them to express and explain their thoughts to the group. This leads to articulation of and reflection on their own thoughts and the development of their ideas.

More than Wistedt, Balacheff (1991) stresses the importance of moments of contradictions between students (aged 13 to 14) in their method or solution of a mathematical problem. When a student notices that her own method or solution is being doubted, justification of her work is needed. Students also have to negotiate on examples and counter-examples. By letting students work together on mathematical problems there is a good chance that the contradictions that arise are very close to their own work and method. That is a prerequisite for the solving of the contradictions and for the restructuring of conceptual knowledge (see also Chinn and Brewer, 1993).

From several case studies Yackel, Cobb and Wood (1991) show that the explanation of one's own work and the comparison of different ways of working also works for young 7 to 8 year olds. In one case student A explains to student B why B's solution to a problem is wrong and during that explanation A reconstructs a better solution. In a second case, two students C and D successfully analyse how their working methods differ and why their methods do not lead to the same answer.

From these case studies it is clear that all sorts of cycles are possible. They can be small and limited to a part of the process model, but students can also go through the whole sequence of the four key activities several times. Based on the process model we could predict that the more key activities a student performs and the more this happens in line with the process model, the more optimal the learning process will be for the student. This prediction forms the starting point for research which we are currently undertaken.

Additional comments on the relevance of research into the relation between cognitive development (that is, mathematical level raising) and classroom social interaction patterns are to be found in Cobb (1996), Bauersfeld (1988), and Bartolini-Bussi (1991). Fruitful critical comments on this relation can be found in Vollmer & Krummheuer (in print). Webb & Palincsar (1996) should be consulted for an excellent overview of research findings on group processes in the classroom, including solving problems in small groups.

4. CONCLUSION

We shall conclude our article in the same manner we began, namely by referring to Celia Hoyles. We want to comment on her list of abilities in defining mathematical understanding from a social interactive perspective by using our process model. We will repeat the abilities in italic and comment on them in standard print:

- *to form a view of the mathematical idea*

This ability goes further than what is described in our process model. In our model the students are working on a mathematical problem and their understanding can be implicit in the beginning.

- *to step back and reflect upon it*

This ability is very much in line with our model as described and concretised by the key activities of showing and explaining one's work.

- *to use it appropriately and flexibly*

Again, this ability goes much further than the context of our model. Our model is much more local and aims at the reconstruction of one's work. In the long term one can expect this ability as the outcome of many cycles of the process model.

- *to communicate it effectively to another*

This is again very much in line with our process model. We have described and concretised this ability in the key activities of showing, explaining, and justifying one's work.

- *to reflect on another's perspective from one's own framework or challenge and logically reject this alternative view.*

This ability is described by us in the mental activities of becoming aware of and thinking about another's work and in the regulating activity of criticising another's work.

At last we have commented on Hoyles's suggestions of twelve years ago, and our process model has helped us to do so. In turn we hope to elicit comments on our work and to continue the discussion in this journal on how student-student interaction can increase mathematical understanding.

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