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A PROOF FROM 'FIRST PRINCIPLES' OF KESTEN'S RESULT FOR THE PROBABILITIES WITH WHICH A SUBORDINATOR HITS POINTS.

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Abstract
We give a simpler and shorter proof of Kesten's result for the probabilities with which a subordinator hits points.

1 Introduction

A subordinator is an increasing Lévy process. That is, a subordinator $X = (X_t, t \geq 0)$ is defined to to be an increasing right-continuous random process with stationary independent increments. We may write (for more background we refer the reader to [1])

$$X_t = \mathrm{d}t + \sum_{s\in[0,t]} \Delta X_s \tag{1}$$

where $\mathrm{d} \geq 0$ is the drift and $\Delta X = (\Delta X, t \geq 0)$ is a Poisson point process with characteristic measure Π. Clearly $X_0 = 0$ and Π is supported on $[0, \infty)$ while if $\Pi(0, \infty) < \infty$ then the jump process $\sum_{s\in[0,t]} \Delta X_s$ is compound Poisson. Henceforth we will only consider strictly increasing subordinators. That is, we exclude the pure (driftless) compound Poisson case where the probability the subordinator hits points is the same as for the corresponding Random Walk/Renewal process.
For all $x \geq 0$ we define

$$\tau_x := \inf\{t > 0 : X_t > x\},$$

the time at which X passes above the point x.
For any open or closed interval A we let

$$\mathbb{P}\{X \text{ visits } A\} := \mathbb{P}\left\{\sum_{t\geq 0} \mathbf{1}_{\{X_t\in A\}} > 0\right\}.$$

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For points $x \geq 0$ we use the shorthand

$$p_x := \mathbb{P}\{X \text{ visits } x\} = \mathbb{P}(X_{\tau_x} = x),$$

where the a.s. equivalence of $\{X \text{ visits } x\}$ and $\{X_{\tau_x} = x\}$ holds by the a.s. strict monotonicity of X. Note that $p_0 = 1$.
To express the result we will also need the potential measure U,

$$U(A) = \mathbb{E}\left(\int_0^\infty \mathbf{1}_{\{X_t \in A\}} dt\right)$$

where $A \in \mathcal{B}[0, \infty)$.
We give a new proof of the following Theorem.

Theorem 1 *For any a.s. strictly increasing subordinator X:*

(i) *If* $\mathrm{d} = 0$ *then* $\mathbb{P}(X_{\tau_x} = x) = 0$ *for all* $x > 0$*;*

(ii) *If* $\mathrm{d} > 0$ *then* $x \mapsto p_x$ *is strictly positive and continuous on* $[0, \infty)$ *and* $U(dx) \equiv d^{-1} p_x dx$.

This result was first proved by Kesten [3] in a much more general setting. Namely, he determined when the probability a general Lévy process hit given points was positive. His proofs, however, used relatively involved potential theory and were very lengthy. In [2] Bretagnolle gave a streamlined proof of Kesten's results, again based on potential theory. (See also [1] for a version of the proof for the subordinator case.) Here, albeit in our simpler monotonic setting, we give a still shorter, less demanding proof relying principally upon pathwise arguments.

2 Proof of Theorem 1

We define for any $x > 0$

$$\{X \text{ jumps onto } x\} := \{X_{\tau_x} = x > X_{\tau_x -}\}$$

and

$$\{X \text{ jumps from } x\} := \{X_{\tau_x} > x = X_{\tau_x -}\}.$$

Lemma 2 *For any a.s. strictly increasing subordinator X:*

(i) *For any* $x > 0$, $p_x = \lim_{n\uparrow\infty} \mathbb{P}\{X \text{ visits } (x - 1/n, x)\}$*;*

(ii) *If* $\exists\, x > 0$ *such that* $p_x > 0$, *then* $\limsup_{\varepsilon\downarrow 0} p_\varepsilon = 1$.

Proof. (i) For an arbitrary integer m, let $X_t = X_t^1 + X_t^2$ where X^2 consists of the jumps of X larger than $1/m$. ie.

$$X_t^2 = \sum_{s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s > 1/m\}}.$$

Thus X^2 is a compound Poisson process with jumps at say times $\{t_n\}$ and of size $\{j_n\}$ (where $0 < t_1 < t_2 < \ldots$ and $\Delta X_{t_n} = j_n$).
For any $x > 0$, consider for given integer n

$$\begin{aligned} V_n &:= \{X \text{ jumps onto } x \text{ with the } n\text{th jump of } X^2\} \\ &= \{\tau_x = t_n, X_{\tau_x} = x = X_{\tau_x -} + j_n\} \end{aligned}$$

and

$$W_n := \{X \text{ jumps from } x \text{ with the } n\text{th jump of } X^2\}.$$

Now since X^1, $\{t_n\}$ and $\{j_n\}$ are independent we may first determine X^1 and $\{j_n\}$ to find $G_n := \{t : X^1_t + \sum_{k=1}^{n-1} j_k = x - j_n\}$, and then notice

$$\mathbb{P}(V_n | X^1, \{j_n\}) = \mathbb{P}(t_n \in G_n) = 0$$

since G_n contains at most one point (by strict monotonicity) while t_n has an absolutely continuous (Gamma) distribution.
Hence $\mathbb{P}(V_n) = 0$, while by an analogous argument $\mathbb{P}(W_n) = 0$. Thus, summing over n and then m, we have

$$\mathbb{P}\{X \text{ jumps onto } x\} = \mathbb{P}\{X \text{ jumps from } x\} = 0, \tag{2}$$

which implies

$$\{X \text{ visits } x\} \overset{\text{a.s.}}{=} \bigcap_{n \geq 1} \{X \text{ visits } (x - 1/n, x)\}$$

and the claim follows by monotone convergence.
(ii) From (2), we see by stopping X on entry into $(x - 1/n, x)$ that

$$p_x \leq \mathbb{P}\{X \text{ visits } (x - 1/n, x)\} \sup_{y \leq 1/n} p_y.$$

Letting n go to ∞ and applying part (i) then gives the result. ∎
Henceforth we shall make extensive use of the following two inequalities which both follow readily from the strong Markov property applied at τ_y. For $y < x$ we have

$$p_x \geq p_y p_{x-y} \tag{3}$$

and

$$p_x \leq p_y p_{x-y} + 1 - p_y. \tag{4}$$

Lemma 3 *If for some $x > 0$, we have $p_x > 3/4$ then for all $y \leq x$*

$$p_y \geq 1/2 + \sqrt{p_x - 3/4}.$$

We first show that $p_x > 3/4$ implies that

$$\forall y \leq x,\ p_y \in [0, 1/2 - \sqrt{p_x - 3/4}] \cup [1/2 + \sqrt{p_x - 3/4}, 1]. \tag{5}$$

Note that the most intuitive way to see why a statement similar to (5) (which is perhaps the key step in our version of the proof) should hold is to argue as follows: assume p_x is large (ie. 'close' to 1) and then further assume for contradiction that there exists $y < x$ such that p_y is 'close' to 1/2. Then, considering the reverse process started from x, this would imply p_{x-y} was small (i.e. replacing y with $x - y$ in (4), $p_x \leq p_{x-y} p_y + 1 - p_{x-y} \approx 1 - p_{x-y}/2$). But then most times X visited y it would fail to hit x (ie. $0 \approx 1 - p_x \geq p_y(1 - p_{x-y}) \approx p_y \approx 1/2$), giving the required contradiction. Hence if p_x is close to 1 there are no $y < x$ such that p_y is close to 1/2.

Proof. (of (5)) Suppose that we are given $y < x$. We may assume without loss of generality that $p_y < p_x$ (since otherwise the claim is automatic). Replacing y by $x - y$ in (4) and rearranging, we have

$$p_{x-y} \leq \frac{1-p_x}{1-p_y}. \tag{6}$$

Hence $p_{x-y} < 1$ and so (replacing $x - y$ by y in (6))

$$p_y \leq \frac{1-p_x}{1-p_{x-y}},$$

while by rearranging (6)

$$1 - p_{x-y} \geq \frac{p_x - p_y}{1-p_y}.$$

Combining the last two inequalities we therefore have

$$p_y \leq \frac{(1-p_x)(1-p_y)}{p_x - p_y}$$

and hence the quadratic inequality $p_y^2 - p_y + 1 - p_x \geq 0$. Solving for p_y gives (5). ∎

We have thus established that if p_x is large then there is an interval of values around 1/2 which no p_y may take for $y < x$. Since from Lemma 2 (i) we have

$$p_y = \lim_{n\uparrow\infty} \mathbb{P}\{X \text{ visits } (y - 1/n, y)\} \geq \limsup_{z\uparrow y} p_z, \tag{7}$$

we may show that p_y cannot 'jump over' this gap.

Proof. (of Lemma 3) Suppose that we have $x > 0$ such that $p_x > 3/4$ (and hence that (5) holds). Now assume for contradiction that $\exists\, y < x$ such that $p_y \leq 1/2 - \sqrt{p_x - 3/4}$. Define

$$g := \sup\{z \in [0, y) : p_z \geq 1/2 + \sqrt{p_x - 3/4}\}$$

which is well-defined since at least $p_0 = 1$. From (7) we see that $p_g \geq 1/2 + \sqrt{p_x - 3/4}$ and hence in particular that $g < y$. But then by (3) and Lemma 2 (ii)

$$\limsup_{\varepsilon\downarrow 0} p_{g+\varepsilon} \geq p_g \limsup_{\varepsilon\downarrow 0} p_\varepsilon \geq p_g$$

which implies (by (5)) that there must exist $g' \in (g, y)$ such that $p_{g'} \geq 1/2 + \sqrt{p_x - 3/4}$. Hence we have the required contradiction and the Lemma holds. ∎

Lemma 4 *Suppose there exists an $x > 0$ such that $p_x > 0$. Then*

(i) $\lim_{\varepsilon\downarrow 0} p_\varepsilon = 1$;

(ii) *$y \mapsto p_y$ is strictly positive and continuous on $[0, \infty)$.*

Proof. By Lemma 2 (ii) and Lemma 3 we have immediately that $\lim_{\varepsilon\downarrow 0} p_\varepsilon = 1$. Positivity then follows from the strong Markov property as $p_y \geq (p_{y/n})^n$ for all $n \in \mathbb{N}$. To prove continuity on

$(0,\infty)$ we reason as follows (using once again (3) and (4)):

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0} p_{y+\varepsilon} &\leq \limsup_{\varepsilon\downarrow 0}(p_\varepsilon p_y + 1 - p_\varepsilon) = p_y,\\
\liminf_{\varepsilon\downarrow 0} p_{y+\varepsilon} &\geq \liminf_{\varepsilon\downarrow 0} p_y p_\varepsilon = p_y,\\
\limsup_{\varepsilon\downarrow 0} p_{y-\varepsilon} &\leq \limsup_{\varepsilon\downarrow 0} \frac{p_y}{p_\varepsilon} = p_y,\\
\liminf_{\varepsilon\downarrow 0} p_{y-\varepsilon} &\geq \liminf_{\varepsilon\downarrow 0} \frac{p_y + p_\varepsilon - 1}{p_\varepsilon} = p_y.
\end{aligned}$$

■

Proof. (of Theorem 1) Consider the function

$$M(a) := \mathbb{E}\left(\int_0^a \mathbf{1}_{\{X \text{ visits } x\}} dx\right) = \int_0^a p_x dx$$

for all $a > 0$, where the equality holds by Fubini's theorem.
Now since X fails to visit precisely those points which it jumps over (or jumps from) we also have

$$M(a) = \mathbb{E}\left(X_{\tau_a} - \sum_{t\leq\tau_a} \Delta X_t\right).$$

Thus using (1) and the fact that τ_a is the Lebesgue measure of the set of times for which X is below a we have

$$M(a) = \mathbb{E}(\mathrm{d}\tau_a) = \mathrm{d}U(0,a]. \tag{8}$$

Hence if $\mathrm{d} = 0$ then $M(a) = 0$ for all a. Lemma 4 then implies that $p_x = 0$ for all $x > 0$.
If on the other hand $\mathrm{d} > 0$ then there must exist an $x > 0$ such that $p_x > 0$. Hence, from Lemma 4, $x \mapsto p_x$ is strictly positive and continuous. Moreover, for all $a > 0$ we have

$$\int_0^a p_x dx = \mathrm{d}\int_0^a U(dx)$$

and thus $p_x dx$ is equivalent to $\mathrm{d}U(dx)$ on $[0,\infty)$. ■

NOTES

(i) If X is composed of a (positive) drift and a compound Poisson jump process then as may be expected $x \mapsto p_x$ can be shown to be everywhere infinitely differentiable. Moreover, for all X with positive drift (say for convenience $\mathrm{d} = 1$) we have

$$\lim_{\varepsilon\downarrow 0} \frac{1 - p_\varepsilon}{\varepsilon} = \Pi(0,\infty).$$

To see this first assume that $\Pi(0,\infty) < \infty$ and note that
$\lim_{\varepsilon\downarrow 0} \mathbb{P}\{\text{'The first jump of } X\text{'} > \varepsilon\} = 1$. Hence

$$\begin{aligned}
\lim_{\varepsilon\downarrow 0} \frac{1 - p_\varepsilon}{\varepsilon} &= \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon}\mathbb{P}\left\{\sum_{t\leq\varepsilon} \Delta X_t > 0\right\}\\
&= \lim_{\varepsilon\downarrow 0} \varepsilon^{-1}(1 - e^{-\varepsilon\Pi(\varepsilon,\infty)})\\
&= \Pi(0,\infty).
\end{aligned}$$

Now assume $\Pi(0,\infty)=\infty$ and then for any $\delta>0$ let $X_t = X^1_t + X^2_t$ where $X^2_t = \sum_{s\le t}\Delta X_s \mathbf{1}_{\{\Delta X_s<\delta\}}$, the sum of the jumps smaller than δ by time t. We have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{1-p_\varepsilon}{\varepsilon} &\ge \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathbb{P}(\Delta X_{T_\varepsilon}>\delta)\\ &= \lim_{\varepsilon\downarrow 0}\varepsilon^{-1}\mathbb{P}(\Delta X^1_{T^1_\varepsilon}>\delta)\\ &= \Pi(\delta,\infty),\end{aligned}$$

where the second line follows since $X_t t^{-1} \xrightarrow{a.s.} d$. Thus as δ is arbitrary we have

$$\lim_{\varepsilon\downarrow 0}\frac{1-p_\varepsilon}{\varepsilon}=\infty.$$

(ii) That $\lim_{\varepsilon\downarrow 0}p_\varepsilon = 1$ when X has positive drift reflects the dominance of the drift process over the jump process at small times. ie. It is related to the fact that $X_t t^{-1} \xrightarrow{a.s.} \mathrm{d}$ as $t\to 0$ (see for example [1] p.84). Indeed, from here we may argue

$$\begin{aligned}\limsup_{x\downarrow 0} p_x &\ge \limsup_{a\downarrow 0} a^{-1}M(a)\\ &= \limsup_{a\downarrow 0} a^{-1}\mathbb{E}\left\{X_{\tau_a -} - \sum_{t<\tau_a}\Delta X_t\right\}\\ &\ge 1-\limsup_{a\downarrow 0}\ a^{-1}\mathbb{E}\left\{\sum_{t<\tau_a}\Delta X_t\right\}\\ &\ge 1-\limsup_{a\downarrow 0} a^{-1}\mathbb{E}\left\{\sum_{t\le a\mathrm{d}^{-1}}\Delta X_t \mathbf{1}_{\{\Delta X_t<1\}}\right\}\\ &\ge 1 \quad \text{(by dominated convergence).}\end{aligned}$$

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