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A PROOF OF THE C^1 STABILITY CONJECTURE

by Ricardo MAÑÉ

INTRODUCTION

Two continuous maps, $f_1 : X_1 \circlearrowleft$ and $f_2 : X_2 \circlearrowleft$ are *topologically equivalent* if there exists a homeomorphism $h : X_1 \to X_2$ such that $h^{-1} f_2 h = f_1$. A C^r diffeomorphism f of a closed manifold M is C^r *structurally stable* if it has a C^r neighborhood $\mathcal{U}$ such that every $g \in \mathcal{U}$ is topologically equivalent to f. This concept was introduced in the thirties by Andronov and Pontrjagin [1], in the limited (when compared with its present range) framework of flows on the two dimensional disk. The turning point of its development that connected it with much richer possibilities, came in the early sixties, through the work of Smale who, as a consequence of his improved version of a classical result of Birkhoff about homoclinic points, showed that structural stability can coexist with highly developed forms of recurrence [24].

Immediatly afterwards, the understanding of the mechanisms that grant structural stability grew substantially through the papers of Anosov [2], Smale [25] and Palis and Smale [16], that proved several new classes of dynamical systems to be structurally stable. On the light of these results, and intending to unify them, Palis and Smale conjectured in their joint paper that the two conditions known as Axiom A and the Strong Transversality Condition (whose definitions we shall recall below) are necessary and sufficient for a C^r diffeomorphism to be C^r structurally stable. Their sufficiency was proved in the well known papers of Robbin [20] for ($r \geqslant 2$) and Robinson [22] (for $r = 1$). The question of their necessity was reduced to prove that C^r structural stability implies Axiom A (Robinson [21]). This problem became known as the Stability Conjecture, and it is the objective of this paper to prove it in the C^1 case.

Theorem A. — Every C^1 *structurally stable diffeomorphism of a closed manifold satisfies Axiom A.*

In the next section we shall prove this result. The proof will be supported on six theorems. Three of them were already known; the other three will be proved in the remaining sections.

Several relevant problems closely connected with the Stability Conjecture remain open; notably the C^r case (that looks beyond the scope of the available techniques) and, even in the C^1 case, the characterization of the more flexible form of stability known as Ω-stability as well as the corresponding problems for flows, for which the methods we use here open realistic possibilities. The case of flows on compact manifolds with boundary that are tangent to the boundary pose a different type of problem. Recent examples show that Axiom A is *not* necessary for structural stability [7].

Before developing the discussion of these questions, we shall first recall the definition and main virtues of Axiom A dynamics.

From now on M will denote a closed manifold and $\mathrm{Diff}^r(M)$ will be the space of C^r diffeomorphisms of M endowed with the C^r topology. We say that Λ is a ***hyperbolic set*** of $f \in \mathrm{Diff}^r(M)$ if it is compact, invariant (i.e. $f(\Lambda) = \Lambda$) and there exists a continuous splitting $TM/\Lambda = E^s \oplus E^u$ (where TM/Λ is the tangent bundle restricted to Λ) that is invariant (i.e. $(Df)\, E^s = E^s$, $(Df)\, E^u = E^u$) and there exist constants $C > 0$, $0 < \lambda < 1$ such that

$$||(Df^n)/E^s(x)|| \leqslant C\lambda^n,$$
$$||(Df^{-n})/E^u(x)|| \leqslant C\lambda^n$$

for all $x \in \Lambda$ and $n \geqslant 0$. Expositions of the rich theory of hyperbolic sets can be found in the books of Bowen [3], Newhouse [14] and Shub [23]. Given $f \in \mathrm{Diff}^r(M)$ and $x \in M$ define the *stable* and *unstable manifolds* of x as:

$$W^s_f(x) = \{y \in M \mid \lim_{n \to +\infty} d(f^n(x), f^n(y)) = 0\}$$
$$W^u_f(x) = \{y \in M \mid \lim_{n \to +\infty} d(f^{-n}(x), f^{-n}(y)) = 0\}.$$

When dealing with only one diffeomorphism, as will be the case in this section, we shall denote these sets as $W^s(x)$ and $W^u(x)$.

When x belongs to a hyperbolic set, then $W^s(x)$ and $W^u(x)$ are C^r injectively immersed submanifolds ([25], [6]).

The nonwandering set $\Omega(f)$ of f is defined as the set of points $x \in M$ such that for every neighborhood U of x there exists $n \geqslant 1$ satisfying $f^n(U) \cap U \neq \emptyset$. When $\Omega(f)$ is a hyperbolic set and the periodic points are dense in $\Omega(f)$, we say that f satisfies *Axiom A*. In this case it is known [25] that

(1) $$M = \bigcup_{x \in \Omega(f)} W^s(x) = \bigcup_{x \in \Omega(f)} W^u(x).$$

Using this property it is easy to see that $W^s(y)$ and $W^u(y)$ are C^r injectively immersed manifolds for all $y \in M$, because by (1), for all $y \in M$, there exists $x \in \Omega(f)$ such that $y \in W^s(x)$ and then $W^s(x) = W^s(y)$. Since $W^s(x)$ is a C^r injectively immersed manifold, the property is proved.

We say that an Axiom A diffeomorphism f satisfies the *Strong Transversality Condition* when

$$T_x W^s(x) + T_x W^u(x) = T_x M$$

for all $x \in M$, or, what is equivalent by (1), if for all p and q in $\Omega(f)$, $W^s(p)$ and $W^u(q)$ intersect transversally. There are several characterizations of diffeomorphisms satisfying Axiom A and the Strong Transversality Condition. For instance, $f \in \mathrm{Diff}^r(M)$ satisfies Axiom A and the Strong Transversality Condition if and only if every tangent vector $v \in TM$ can be decomposed as $v = v^+ + v^-$, where v^+ and v^- satisfy

$$\liminf_{n\to+\infty} \|(Df^n)\, v^+\| = \liminf_{n\to-\infty} \|(Df^n)\, v^-\| = 0.$$

For this and other characterizations, see [11].

Let us now discuss the open problems related to Theorem A. The first one must be the C^r case of Theorem A with $r > 1$. Unfortunately there is little to say about this question. Not being even known whether a C^2 structurally stable diffeomorphism has at least one periodic point it seems, to say the least, difficult to prove that they are dense in the nonwandering set as the definition of Axiom A requires. Even if this density property is proved and unless the method used to achieve this feat sheds new light on these questions, other disturbingly simple unanswered questions remain (see the Introduction of [12]).

Turning to more feasible questions, we have the problem of characterizing Ω-stability, that is defined as follows: f is C^r *Ω-stable* if it has a C^r neighborhood $\mathscr{U}$ such that $g/\Omega(g)$ is topologically equivalent to $f/\Omega(f)$ for all $g \in \mathscr{U}$. Smale proved that if f satisfies Axiom A plus the so called no cycles condition then f is Ω-stable [26]. The converse problem has been reduced to proving that C^r Ω-stability implies Axiom A (Palis [15]). When $r > 1$ this problem runs into the same (or worse) stumbling blocks than the Stability Conjecture. When $r = 1$ we think, as we say above, that the techniques developed here make of it a realistic target. Similar comments hold for the corresponding problems for flows on boundaryless compact manifolds. But in the quite natural attempt to study structural stability in the space of flows on a compact manifold with boundary that are tangent to the boundary, new and different problems arise. Labarca and Pacifico [7] have found examples that show that in this framework there exist structurally stable flows that do not satisfy Axiom A. The conjecture itself, then, must be reformulated in terms that so far have not been proposed.

Returning to the case of diffeomorphisms of a closed manifold M, define $\mathscr{F}^r(M)$ as the set of diffeomorphisms $f: M \circlearrowleft$ having a C^r neighborhood $\mathscr{U}$ such that all the periodic points of every $g \in \mathscr{U}$ are hyperbolic. It is easy to see [4] that C^r structurally or Ω-stable diffeomorphisms belong to $\mathscr{F}^r(M)$. Moreover most of the steps toward proving that structural or Ω-stability imply Axiom A use only the weaker fact that such diffeomorphisms belong to $\mathscr{F}^r(M)$. For this reason we conjectured in [12] that every element of $\mathscr{F}^r(M)$ satisfies Axiom A. For the reasons we have just explained, this conjecture contains the questions of whether structural or Ω-stability imply Axiom A. Once more, and for the same reasons than in the previous problems, let us leave aside the case $r > 1$. When $\dim M = 2$ (and $r = 1$) we proved this conjecture in [12]. Even if the techniques developed here fall short of extending this result to the n-dimensional case, it is interesting, and promising, that most of the steps of the proof of Theorem A require only the hypothesis $f \in \mathscr{F}^1(M)$. It is only in the last step where we need the whole weight of the structural stability of f.

On the other hand, if we define $\mathscr{F}^1(M)$ for flows in the obvious, analogous form to that used for diffeomorphisms, it is not true that flows in $\mathscr{F}^1(M)$ satisfy Axiom A. An exemple is the Guckenheimer-Lorenz attractor [5], that also plays the key role in the construction of the example of Labarca and Pacifico mentioned above.

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I. — Proof of Theorem A

As we explained in the Introduction, in this section we shall prove Theorem A, using for this purpose six theorems that either have been already proved elsewhere or will be proved in the next sections.

Let M be a closed manifold and let $\mathscr{F}^1(M)$ be defined as in the Introduction. Let $P(f)$ denote the set of periodic points of the diffeomorphism f and, if $x \in P(f)$, let $E^s(x)$ and $E^u(x)$ be the stable and unstable subspaces of $T_x M$, i.e. the subspaces associated to the eigenvalues of $Df^n : T_x M \circlearrowleft$ (where n is the period of x) that have respectively modulus < 1 and > 1. Clearly $(Df)\, E^s(x) = E^s(f(x))$, $(Df)\, E^u(x) = E^u(f(x))$ and, if x is hyperbolic, $T_x M = E^s(x) \oplus E^u(x)$. Denote by $\overline{P}(f)$ the closure of $P(f)$.

The first step of the proof of Theorem A is the following corollary of Pugh's Closing Lemma [19] proved in the Introduction of [12].

Theorem **I.1.** — *If* $f \in \mathscr{F}^1(M)$, *then* $\Omega(f) = \overline{P}(f)$.

Now define $P_i(f)$ as the set of points $x \in P(f)$ such that $\dim E^s(x) = i$. By I.1

$$\Omega(f) = \bigcup_{i=0}^{\dim M} \overline{P}_i(f)$$

when $f \in \mathscr{F}^1(M)$. Then, if $f \in \mathscr{F}^1(M)$, it is sufficient to show that $\overline{P}_i(f)$ is a hyperbolic set for all $0 \leqslant i \leqslant \dim M$. The cases $i = 0$ and $i = \dim M$ follow from a theorem due to Pliss.

Theorem **I.2** (Pliss [18]). — *If* $f \in \mathscr{F}^1(M)$, *then* $P_0(f)$ *and* $P_{\dim M}(f)$ *are finite.*

Obviously this implies that $\overline{P}_0(f) = P_0(f)$ and $P_{\dim M}(f) = \overline{P}_{\dim M}(f)$ are hyperbolic sets when $f \in \mathscr{F}^1(M)$. To prove the hyperbolicity of the sets $\overline{P}_i(f)$ for $1 \leqslant i \leqslant \dim M$ the basic strategy is the obvious one: to start with the splittings $T_x M = E^s(x) \oplus E^u(x)$ that we have when $x \in P(f)$ to show that this splitting of $TM/P_i(f)$ extends to a splitting of $TM/\overline{P}_i(f)$ satisfying the definition of hyperbolicity. The next result provides the extension and some indications of its hyperbolicity. Its statement uses the concept of *dominated splitting*, that will appear also in several results of this section and is defined as follows. Given a compact invariant set Λ of a diffeomorphism f we say that a splitting $TM/\Lambda = E \oplus F$ is a dominated splitting if it is continuous, invariant and there exists $C > 0$ and $0 < \lambda < 1$ such that

$$||(Df^n)/E(x)||.||(Df^{-n})/F(f^n(x))|| \leqslant C\lambda^n$$

for all $x \in \Lambda$ and $n \geqslant 0$. In geometric terms this is equivalent to say that for every one-dimensional subspace $L \subset T_x M$, $x \in \Lambda$, not contained in $E(x)$, the angle between $(Df^n) L$ and $F(f^n(x))$ converge exponentially to zero as $n \to +\infty$.

Theorem **I.3.** — *If $f \in \mathscr{F}^1(M)$ there exist $C > 0$, $0 < \lambda < 1$, $m > 0$ and a C^1 neighborhood $\mathscr{U}$ of f such that for all $g \in \mathscr{U}$ and $0 < i < \dim M$ there exists a dominated splitting $TM/\overline{P}_i(g) = \widetilde{E}_i^s \oplus \widetilde{E}_i^u$ satisfying:*

a) $||(Dg^m)/\widetilde{E}_i^s(x)||.||(Dg^{-m})/\widetilde{E}_i^u(g^m(x))|| \leqslant \lambda$ for all $x \in \overline{P}_i(g)$,

b) $\widetilde{E}_i^s(x) = E^s(x)$ *and* $\widetilde{E}_i^u(x) = E^u(x)$ *if* $x \in \overline{P}_i(g)$

c) *If $x \in P_i(g)$ and has period $n > m$, then*

$$\prod_{j=0}^{[n/m]-1} ||(Dg^m)/E^s(g^{mj}(x))|| \leqslant C\lambda^{[n/m]}$$

$$\prod_{j=1}^{[n/m]} ||(Dg^{-m})/E^u(g^{mj}(x))|| \leqslant C\lambda^{[n/m]},$$

d) *For all* $x \in P_i(g)$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log||(Dg^m)/E^s(g^{mj}(x))|| \leqslant \log\lambda$$

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log||(Dg^{-m})/E^u(g^{mj}(x))|| \leqslant \log\lambda.$$

Observe that d) is interesting only when the period of x is $\leqslant m$. Otherwise it is just a corollary of c).

Theorem I.3 was independently proved in [17], [10] and [8]. The statement used above is taken from [12], where there is also a proof of I.3.

After Theorem I.3 the problem becomes to show that the splitting $TM/\overline{P}_i(f) = \widetilde{E}_i^s \oplus \widetilde{E}_i^u$ is hyperbolic for all $1 \leqslant i < \dim M$. If there is a hyperbolic splitting it must be this. The following, and fundamental, step is a theorem saying that to prove the hyperbolicity of the splitting $TM/\overline{P}_i(f) = \widetilde{E}_i^s \oplus \widetilde{E}_i^u$ it suffices to show only that Df contracts the subbundle $\widetilde{E}_i^s$. To state this result it is convenient to introduce a definition: given a compact invariant set Λ of $f: M \circlearrowleft$, we say that a subbundle $E \subset TM/\Lambda$ is *contracting* if it is continuous, invariant and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$||(Df^n)/E(x)|| \leqslant C\lambda^n$$

for all $n \geq 0$ and $x \in \Lambda$. We say that it is *expanding* if there exist $C > 0$ and $0 < \lambda < 1$ such that

$$\|(Df^{-n})/E(x)\| \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$.

Theorem **I.4.** — *If* $f \in \mathcal{F}^1(M)$, $0 < i < \dim M$ *and* $\widetilde{E}_i^s$ *is contracting, then* $\widetilde{E}_i^u$ *is expanding.*

This theorem will be proved in Section **II** as a corollary of a slightly more general result.

Now our problem is reduced to show that $f \in \mathcal{F}^1(M)$ implies that $\widetilde{E}_i^s$ is contracting for all $0 < i < \dim M$. To recognize the contracting property the following easy lemma is extremely useful because it translates this property into averages with respect to ergodic measures.

Denote by $\mathcal{M}(f/\Lambda)$ the set of invariant probabilities on the Borel σ-algebra of Λ endowed with the weak topology, i.e. the unique metrizable topology such that

$$\mu_n \to \mu \Leftrightarrow \int \varphi \, d\mu_n \to \int \varphi \, d\mu$$

for every continuous $\varphi : \Lambda \to \mathbf{R}$.

Lemma **I.5.** — *Let* Λ *be a compact invariant set of* $f \in \mathrm{Diff}^1(M)$ *and* $E \subset TM/\Lambda$ *be a continous invariant subbundle. If there exists* $m > 0$ *such that*

$$\int \log \|(Df^m)/E\| \, d\mu < 0$$

for every ergodic $\mu \in \mathcal{M}(f^m/\Lambda)$, *then* E *is contracting.*

Proof. — It is easy to see that if for each $x \in \Lambda$ there exists $n > 0$ such that

$$\|(Df^n)/E(x))\| < 1,$$

then E is contracting. Stronger than this is to say that for each $x \in \Lambda$ there exists $n > 0$ satisfying

$$\prod_{j=0}^{n-1} \|(Df^m)/E(f^{mj}(x))\| < 1.$$

Suppose this property is false. Then there exists $x \in \Lambda$ such that

$$\prod_{j=0}^{n-1} \|(Df^m)/E(f^{mj}(x))\| \geq 1$$

for all n. Hence, for all n,

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^m)/E(f^{mj}(x))\| \geq 0.$$

Define a probability μ_n by

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^{mj}(x)}$$

and let $\{ \mu_{n_k} \mid k \geqslant 0 \}$ be a convergent subsequence. Its limit μ_0 belongs to $\mathscr{M}(f^m/\Lambda)$ and

$$\int \log \|(\mathrm{D}f^m)/\mathrm{E}\| \, d\mu_0 = \lim_{k\to+\infty} \int \log \|(\mathrm{D}f^m)/\mathrm{E}\| \, d\mu_{n_k}$$

$$= \lim_{k\to+\infty} \frac{1}{n_k}\sum_{j=0}^{n_k-1} \log \|(\mathrm{D}f^m)/\mathrm{E}(f^{mj}(x))\| \geqslant 0.$$

But if the integral with respect to μ_0 of $\|(\mathrm{D}f^m)/\mathrm{E}\|$ is $\geqslant 0$, by the Ergodic Decomposition Theorem there exists an ergodic $\mu \in \mathscr{M}(f^m/\Lambda)$ with the same property and the lemma is proved.

Now suppose that $f \in \mathscr{F}^1(\mathrm{M})$ and let us try to prove that f satisfies Axiom A, which, as we explained above, is reduced to the contracting property of $\widetilde{\mathrm{E}}_i^s$ for all $0 < i < \dim \mathrm{M}$, and we shall try to do it by induction on i and using Lemma I.5. If $f \in \mathscr{F}^1(\mathrm{M})$, $\mathrm{P}_0(f)$ is hyperbolic by Theorem I.2. Now suppose that $\overline{\mathrm{P}}_k(f)$ is hyperbolic for $0 \leqslant k \leqslant j$ and let us try to prove the hyperbolicity of $\overline{\mathrm{P}}_{j+1}(f)$. For this purpose we need the following result, that was implicitly proved in [12] and will be explicitly proved in Section III.

Theorem **I.6.** — *If* $f \in \mathscr{F}^1(\mathrm{M})$ *and* $m > 0$ *is given by* I.3, *there exists* $0 < \lambda_0 < 1$ *such that if* $\overline{\mathrm{P}}_k(f)$ *is hyperbolic for all* $0 \leqslant k < i$ *and* $\mu \in \mathscr{M}(f^m/\overline{\mathrm{P}}_i(f))$ *satisfies*

(1) $$\int \log \|(\mathrm{D}f^m)/\widetilde{\mathrm{E}}_i^s\| \, d\mu \geqslant \log \lambda_0,$$

then

(2) $$\mu\Big(\bigcup_{0\leqslant k<i} \overline{\mathrm{P}}_k(f)\Big) > 0.$$

To complete the induction step, it suffices to show that

(3) $$\mu\Big(\bigcup_{0\leqslant k\leqslant j} \overline{\mathrm{P}}_k(f)\Big) = 0$$

for all $\mu \in \mathscr{M}(f^m/\overline{\mathrm{P}}_{j+1}(f))$ because, by Theorem I.6, this implies that there are no measures $\mu \in \mathscr{M}(f^m/\overline{\mathrm{P}}_{j+1}(f))$ satisfying (1). Hence

$$\int \log \|(\mathrm{D}f^m)/\widetilde{\mathrm{E}}_{j-1}^s\| \, d\mu < \log \lambda_0 < 0$$

for all $\mu \in \mathscr{M}(f^m/\overline{\mathrm{P}}_{j+1}(f))$ and then, by Lemma I.5, $\widetilde{\mathrm{E}}_{j+1}^s$ is contracting and, by I.4, $\overline{\mathrm{P}}_{j+1}(f)$ is hyperbolic. This would complete the induction step and also the proof of the Axiom A property for f. However we are not able to prove that (3) holds for every $\mu \in \mathscr{M}(f^m/\overline{\mathrm{P}}_{j+1}(f))$ using only the hypothesis $f \in \mathscr{F}^1(\mathrm{M})$. We shall do it when f is C^1 structurally stable (thus proving Theorem A). For this we need the following theorem, for whose statement we shall recall the definition of a basic set. A *basic set* of $f \in \mathrm{Diff}^1(\mathrm{M})$

is a hyperbolic set Λ that is transitive (i.e. there exists $x \in \Lambda$ whose ω-limit set is Λ) and isolated, i.e. it has a compact neighborhood U satisfying

$$\bigcap_n f^n(\mathrm{U}) = \Lambda.$$

The transitivity implies that the dimension of the fibers of the stable subspace of the hyperbolic splitting of TM/Λ is constant and we shall call it the index of Λ and denote it $\mathrm{Ind}(\Lambda)$. The stable and unstable sets of Λ are defined by

$$\mathrm{W}_f^s(\Lambda) = \{y \mid \lim_{n\to+\infty} d(f^n(y), \Lambda) = 0\}$$

$$\mathrm{W}_f^u(\Lambda) = \{y \mid \lim_{n\to+\infty} d(f^{-n}(y), \Lambda) = 0\}.$$

When it is clear with respect to which diffeomorphism we are considering $\mathrm{W}_f^s(\Lambda)$ and $\mathrm{W}_f^s(\Lambda)$, we shall denote them by $\mathrm{W}^s(\Lambda)$ and $\mathrm{W}^u(\Lambda)$. The following theorem will be proved in Section V.

Theorem **I.7.** — *Let* Λ *be a compact invariant set of* $f \in \mathrm{Diff}^1(\mathrm{M})$ *such that* $\Omega(f/\Lambda) = \Lambda$ *and having a dominated splitting* $\mathrm{TM}/\Lambda = \mathrm{E} \oplus \mathrm{F}$. *Suppose that there exist basic sets* $\Lambda_1, \ldots, \Lambda_s$ *of* f *and constants* $m > 0$, $c > 0$ *and* $0 < \lambda < 1$ *satisfying*:

I) $\mathrm{Ind}(\Lambda_i) < \dim \mathrm{E}(x)$ *for all* $1 \leqslant i \leqslant s$ *and* $x \in \Lambda$.

II) *There exists a* C^1 *neighborhood* $\mathcal{U}$ *of* f *such that if* $g \in \mathcal{U}$ *coincides with* f *in a neighborhood of* $\bigcup_1^s \Lambda_k$ *then*

$$\mathrm{W}_g^s(\Lambda_i) \cap \mathrm{W}_g^u(\Lambda_i) = \Lambda_i$$

for all $1 \leqslant i \leqslant s$.

III) *If* $\mu \in \mathcal{M}(f^m/\Lambda)$ *satisfies*

$$\int \log \|(\mathrm{D}f^m)/\mathrm{E}\| \, d\mu \geqslant -c$$

then
$$\mu(\bigcup_1^s)\Lambda_k > 0.$$

IV) $\|(\mathrm{D}f^m)/\mathrm{E}(x)\| . \|(\mathrm{D}f^{-m})/\mathrm{F}(f^m(x))\| \leqslant \lambda$ *for all* $x \in \Lambda$.

Then, given $1 \leqslant i \leqslant s$ *such that* $\Lambda - \Lambda_i$ *is not closed, there exist* $g \in \mathrm{Diff}^1(\mathrm{M})$ *arbitrarily* C^1 *near to* f, *coinciding with* f *in a neighborhood of* $\bigcup_1^s \Lambda_k$ *and* $1 \leqslant r \leqslant s$, $r \neq i$, *such that* $\Lambda - \Lambda_r$ *is not closed and*

$$\mathrm{W}_g^s(\Lambda_i) \cap \mathrm{W}_g^u(\Lambda_r) \neq \emptyset.$$

Besides this theorem, we shall need the following minor remark. If $g \in \mathcal{F}^1(\mathrm{M})$, denote by $\mathrm{N}(i, n, g)$ the number of fixed points of g^n contained in $\mathrm{P}_i(g)$.

Lemma **I.8.** — *If* $f \in \mathscr{F}^1(M)$ *there exists a* C^1 *neighborhood* $\mathscr{U}$ *of* f *such that*

a) $N(i, n, g_1) = N(i, n, g_2)$ *for all* $g_1 \in \mathscr{U}$, $g_2 \in \mathscr{U}$, $n > 0$ *and* $0 \leqslant i \leqslant \dim M$;

b) *if* $g \in \mathscr{U}$ *and* g *coincides with* f *in a neighborhood of* $\overline{P}_i(f)$ *for some* $0 \leqslant i \leqslant \dim M$, *then* $\overline{P}_i(g) = \overline{P}_i(f)$.

Proof. — Let $\mathscr{U} \subset \mathscr{F}^1(M)$ be an open connected neighborhood of f. To prove a) it suffices to show that $N(i, n, g_2) \geqslant N(i, n, g_1)$ because then, reversing the roles of g_1 and g_2, it follows that $N(i, n, g_2) \leqslant N(i, n, g_1)$ and then $N(i, n, g_1) = N(i, n, g_2)$. Let $g(t) \in \mathscr{U}$, $0 \leqslant t \leqslant 1$, be a continuous arc of diffeomorphisms with $g(0) = g_1$, $g(1) = g_2$. For every fixed point x of g_1^n there exists an arc $x(t) \in M$, $0 \leqslant t \leqslant 1$, such that $g(t)^n(x(t)) = x(t)$ and $x(0 = x$. The existence of this arc follows from the implicit function theorem recalling that, since $g(t) \in \mathscr{U} \subset \mathscr{F}^1(M)$ for all $0 \leqslant t \leqslant 1$, then if $g(t)^n(p) = p$ it follows that $D(g(t))(p) - I : T_p M \circlearrowleft$ is an isomorphism. Moreover observe that if $x \in P_i(g_1)$ then $x(t) \in P_i(g(t))$ for all $0 \leqslant t \leqslant 1$ (again because $g(t) \in \mathscr{F}^1(M)$ for all $0 \leqslant t \leqslant 1$). Then, for each fixed point x of g_1^n in $P_i(g_1)$ we have found a fixed point $x(1)$ of g_2^n in $P_i(g_2)$ and obviously the correspondance $x \mapsto x(1)$ is injective. This proves that $N(i, n, g_2) \geqslant N(i, n, g_1)$. To prove b), suppose that $g \in \mathscr{U}$ coincides with f in a neighborhood of $\overline{P}_i(f)$. Clearly every periodic point of f in $P_i(f)$ is also a periodic point of g in $P_i(g)$. Then $P_i(g) \supset P_i(f)$. But since $N(i, n, g) = N(i, n, f)$ for all $n > 0$, we have $P_i(g) = P_i(f)$ and then $\overline{P}_i(g) = \overline{P}_i(f)$ completing the proof of b).

Now let us return to the problem to which we had reduced the proof of Theorem A. The problem was to show that if f is C^1 structurally stable (and then $f \in \mathscr{F}^1(M)$) and $\overline{P}_k(f)$ is hyperbolic for all $0 \leqslant k \leqslant j$, then $\overline{P}_{j+1}(f)$ is hyperbolic. As we explained above, the hyperbolicity of $\overline{P}_{j+1}(f)$ is reduced to show that (3) holds for all $\mu \in \mathscr{M}(f^m/\overline{P}_{j+1}(f))$. Suppose that there exists $\mu_0 \in \mathscr{M}(f^m/\overline{P}_{j+1}(f))$ which does not satisfy (3), i.e.:

$$\mu_0\Big(\bigcup_{0 \leqslant k \leqslant j} \overline{P}_k(f)\Big) > 0. \tag{4}$$

To exhibit a contradiction between the existence of μ_0 and the structural stability of f we shall use I.7 and I.8. First observe that the hyperbolic set $\bigcup_{0 \leqslant k \leqslant j} \overline{P}_k(f)$ can be decomposed as

$$\bigcup_{0 \leqslant k \leqslant j} \overline{P}_k(f) = \Lambda_1 \cup \ldots \cup \Lambda_s,$$

where $\Lambda_1, \ldots, \Lambda_s$ are disjoint basic sets. This follows from a straighforward adaptation of Smale's Spectral Decomposition Theorem [25]. Moreover, let us show that there exist sets Λ_i such that $\overline{P}_{j+1}(f) - \Lambda_i$ is not closed. This will follow from the next lemma.

Lemma **I.9.** — *If* $\Lambda_i \cap \overline{P}_{j+1}(f) \neq \varnothing$ *then* $\overline{P}_{j+1}(f) - \Lambda_i$ *is not closed.*

Proof. — Suppose that there exists Λ_i such that $\Lambda_i \cap \overline{P}_{j+1}(f) \neq \varnothing$ and $\overline{P}_{j+1}(f) - \Lambda_i$ is closed. Then we can decompose $\overline{P}_{j+1}(f)$ as

$$\overline{P}_{j+1}(f) = (\overline{P}_{j+1}(f) \cap \Lambda_i) \cup (\overline{P}_{j+1}(f) - \Lambda_i)$$

and both sets in the union at right are compact and obviously disjoint. Moreover $\overline{P}_{j+1}(f) - \Lambda_i$ is not empty because if it were, then $\overline{P}_{j+1}(f)$ would be a subset of the hyperbolic set Λ_i that has index $\leqslant j$ and this is impossible because $\overline{P}_{j+1}(f)$ contains hyperbolic periodic points whose stable subspace has dimension $j+1$. Now take neighborhoods U and V of $\Lambda_i \cap \overline{P}_{j+1}(f)$ and $\overline{P}_{j+1}(f) - \Lambda_i$ respectively, such that

$$f(U) \cap V = \emptyset. \tag{5}$$

Take a sequence of points $\{x_n\} \subset \overline{P}_{j+1}(f)$ converging to a point of $\overline{P}_{j+1}(f) \cap \Lambda_i$. Let γ_n be the orbit of x_n. We claim that for n sufficiently large, $\gamma_n \subset U$. If this is false there exist arbitrarily large values of n with $\gamma_n - U \neq \emptyset$. On the other hand, since $x_n \in \gamma_n$ converges to a point in $\overline{P}_{j+1}(f) \cap \Lambda_i$, for large values of n we have $\gamma_n \cap U \neq \emptyset$. Then for infinitely many values of n the orbit γ_n contains points both in U and U^c. By (5), an orbit γ_n that intersects U cannot be contained in $U \cup V$. Then there are infinitely many values of n such that γ_n contains points in the complement of $U \cup V$. Therefore, since every γ_n is contained in $\overline{P}_{j+1}(f)$, this contradicts the fact that $\overline{P}_{j+1}(f)$ is contained in $U \cup V$ and proves the claim, i.e. that for n large, $\gamma_n \subset U$. Then

$$\gamma_n \subset \bigcap_j f^j(U). \tag{6}$$

Taking U very small, the intersection at right is a hyperbolic set close to the hyperbolic set Λ_i. Then its stable fibers have dimensions $\leqslant \operatorname{Ind}(\Lambda_i) \leqslant j$. Then by (6) the stable subspaces of the points of γ_n have dimensions $\leqslant j$, contradicting the property $\gamma_n \subset P_{j+1}(f)$. This contradiction completes the proof of the lemma.

Corollary **I.10.** — *There exist values of i such that $\overline{P}_{j+1}(f) - \Lambda_i$ is not closed.*

Proof. — If $\overline{P}_{j+1}(f) - \Lambda_i$ is closed for all $1 \leqslant i \leqslant s$ then, by Lemma I.9, the intersections $\overline{P}_{j+1}(f) \cap \Lambda_i$ are empty for all $1 \leqslant i \leqslant s$. But then

$$\mu_0\Big(\bigcup_{0 \leqslant k \leqslant j} \overline{P}_k(f)\Big) = \mu_0\Big(\bigcup_1^s \Lambda_i\Big) = 0$$

because the support of μ_0 is contained in $\overline{P}_{j+1}(f)$, thus contradicting (4).

Now let us show that we can apply Theorem I.7 to the set $\Lambda = \overline{P}_{j+1}(f)$, the dominated splitting $TM/\overline{P}_{j+1}(f) = \widetilde{E}^s_{j+1} \oplus \widetilde{E}^u_{j+1}$, the basic sets $\Lambda_1, \ldots, \Lambda_s$, $m > 0$ and $0 < \lambda < 1$ given by I.3 and $c = -\log \lambda_0$ given by Theorem I.6. Since $\operatorname{Ind}(\Lambda_i) \leqslant j$ for all i and $\dim \widetilde{E}^s_{j+1}(x) = j+1$ for all $x \in \overline{P}_{j+1}(f)$, hypothesis (I) is satisfied. Clearly $\Omega(f/\overline{P}_{j+1}(f)) = \overline{P}_{j+1}(f)$ because of the density of the periodic points in $\overline{P}_{j+1}(f)$; also IV) follows from I.3. Moreover, Theorem I.6 says that every $\mu \in \mathscr{M}(f^m/\overline{P}_{j+1}(f))$ satisfying

$$\int \log \|(Df^m)/\widetilde{E}^s_{j+1}\| \, d\mu \geqslant -c = \log \lambda_0$$

must also satisfy

$$\mu(\Lambda_1 \cup \ldots \cup \Lambda_s) = \mu\Big(\bigcup_{0 \leqslant k \leqslant j} \overline{P}_k(f)\Big) > 0,$$

thus proving hypothesis III). It remains to check hypothesis II). Suppose that a diffeomorphism g is so close to f that it belongs to the neighborhood $\mathcal{U}$ of f given by Lemma I.8. Suppose that g coincides with f in a neighborhood of $\bigcup_1^s \Lambda_k$ and by contradiction suppose that

$$W_g^s(\Lambda_q) \cap W_g^u(\Lambda_q) \neq \Lambda_q$$

for some $1 \leqslant q \leqslant s$. Without loss of generality we can assume that there exists $p \in W_g^s(\Lambda_q) \cap W_g^u(\Lambda_q) - \Lambda_q$ that is a transversal homoclinic point associated to Λ_q (perturbing g a little if necessary). Then there exist periodic points z, arbitrarily close to p, and having a stable subspace with dimension equal to $\operatorname{Ind}(\Lambda_q)$. Denote by ℓ this index. Then $z \in P_\ell(g)$. Moreover observe that p not only does not belong to Λ_q but also that $p \notin \bigcup_1^s \Lambda_k$. In fact, if $p \in \Lambda_k$ for some $1 \leqslant k \leqslant s$, it follows that its whole orbit is contained in Λ_k. Since this orbit accumulates in Λ_q, it follows that $\Lambda_k \cap \Lambda_q \neq \varnothing$ thus implying $k = q$ and contradicting $p \notin \Lambda_q$. Then we can assume that $z \notin \bigcup_1^s \Lambda_k$, because it can be taken arbitrarily close to $p \notin \bigcup_1^s \Lambda_k$. Let n be the period of z. Then $N(\ell, n, g) > N(\ell, n, f)$ because the fixed points of g^n in $P_\ell(g)$ include all the fixed points of f^n in $P_\ell(f)$ (because g coincides with f in a neighborhood of $P_\ell(f)$) and also z (that is not an element of $P_\ell(f)$ because $z \notin \bigcup_1^s \Lambda_k$, which contains $P_\ell(f)$). This contradiction with Lemma I.8 completes the proof of hypothesis II) of Theorem I.7.

Now let us apply I.7 to $\Lambda_1, \ldots, \Lambda_s$ and $\overline{P}_{j+1}(f)$. We take Λ_i such that $\overline{P}_{j+1}(f) - \Lambda_i$ is not closed (that exists by Corollary I.10), and I.7 yields a diffeomorphism g arbitrarily C^1 near to f, coinciding with f in a neighborhood of $\bigcup_1^s \Lambda_k$, and Λ_r such that the set $\overline{P}_{j+1}(f) - \Lambda_r$ is not closed, $r \neq i$ and

$$W_g^s(\Lambda_i) \cap W_g^u(\Lambda_r) \neq \varnothing. \tag{7}$$

But (7) is not enough, as far as we can see, to contradict the structural stability of f, unless we pick Λ_i with some extra properties that will yield that contradiction. Let us explain how to choose Λ_i. Let t be the minimum of the indexes of the sets Λ_k such that $\overline{P}_{j+1}(f) - \Lambda_k$ is not closed. Take Λ_i such that $\overline{P}_{j+1}(f) - \Lambda_i$ is not closed, $\operatorname{Ind}(\Lambda_i) = t$ and there do not exists sets Λ_k, with $k \neq i$, such that $\overline{P}_{j+1}(f) - \Lambda_k$ is not closed and

$$W_f^s(\Lambda_i) \cap W_f^u(\Lambda_k) \neq \varnothing.$$

Let us show that there exists such a Λ_i. If it does not exist, there is a family of different basic sets $\Lambda_{i_1}, \ldots, \Lambda_{i_p}$ such that their indexes are all t and

$$W_f^s(\Lambda_{i_n}) \cap W_f^u(\Lambda_{i_{n+1}}) \neq \varnothing$$

for $1 \leqslant n \leqslant p$ and

$$W_f^s(\Lambda_{i_p}) \cap W_f^u(\Lambda_{i_1}) \neq \varnothing.$$

Moreover all these intersections are transversal because of the structural stability of f. Then it is well known that a point z in, for instance, $W_f^s(\Lambda_{i_1}) \cap W_f^u(\Lambda_{i_2})$, belongs to $\overline{P}_t(f)$. Therefore z belongs to some set Λ_q. Hence the orbit of z is contained in Λ_q. This implies that Λ_q intersects Λ_{i_1} and Λ_{i_2} thus implying $q = i_1 = i_2$. This contradiction with the fact that all the sets $\Lambda_{i_1}, \ldots, \Lambda_{i_p}$ are different, completes the proof of the existence of Λ_i exhibiting the properties described above.

Now let $\mathcal{U}$ be the neighborhood of f given by Lemma I.8 and take $g \in \mathcal{U}$ as above, coinciding with f in a neighborhood of $\bigcup_1^s \Lambda_k$, and such that there exists Λ_r, with $r \neq i$, satisfying (7) and such that $\overline{P}_{j+1}(f) - \Lambda_r$ is not closed. Observe that since the intersection in (7) must be transversal, because of the structural stability of f (Robinson [21]), then $\mathrm{Ind}(\Lambda_i) \geqslant \mathrm{Ind}(\Lambda_r)$. But since $\mathrm{Ind}(\Lambda_i) = t$, the definition of t implies $\mathrm{Ind}(\Lambda_r) = \mathrm{Ind}(\Lambda_i) = t$. Moreover without loss of generality we can assume that g is topologically equivalent to f. Let $h : M \circlearrowleft$ be a homeomorphism such that $gh = hf$. Clearly $h(P_i(f)) = P_i(g)$ for all $0 \leqslant i \leqslant \dim M$ and then $h(\overline{P}_i(f)) = \overline{P}_i(g)$ for all $0 \leqslant i \leqslant \dim M$. Hence

$$h(\bigcup_1^s \Lambda_k) = h(\bigcup_{0 \leqslant k \leqslant j} \overline{P}_k(f)) = \bigcup_{0 \leqslant k \leqslant j} \overline{P}_k(g). \tag{8}$$

But, by part b) of Lemma I.8, $\overline{P}_k(g) = \overline{P}_k(f)$ for all $0 \leqslant k \leqslant j$ because g and f coincide in a neighborhood of the union of the sets $\overline{P}_k(f)$, $0 \leqslant k \leqslant j$. Then (8) implies

$$h(\bigcup_1^s \Lambda_k) = \bigcup_1^s \Lambda_k$$

and it is easy to check that for all $1 \leqslant k \leqslant s$, $h(\Lambda_k)$ is another set of the family $\Lambda_1, \ldots, \Lambda_s$ with the same index as Λ_k. Define $T(f)$ as the set of pairs (n, q) such that $n \neq q$, $\mathrm{Ind}(\Lambda_n) = \mathrm{Ind}(\Lambda_q) = t$ and

$$W_f^s(\Lambda_n) \cap W_f^u(\Lambda_q) \neq \emptyset. \tag{9}$$

Define $T(g)$ exactly in the same way replacing f by g. From the fact that h maps every set of the family $\Lambda_1, \ldots, \Lambda_s$ onto another set of the family with the same index, it follows that

$$\# T(g) = \# T(f). \tag{10}$$

Moreover, all the intersections in (9) are transversal by the structural stability of f. Hence, when g is sufficiently close to f, if (9) holds for certain values n and q, it holds also replacing f by g. Then $T(g) \supset T(f)$. But by (10) this implies

$$T(g) = T(f). \tag{11}$$

Now observe that $(i, r) \notin T(f)$ because by the way we chose i, no pair with i in the first entry belongs to $T(f)$. But, on the other hand, $(i, r) \in T(g)$ because of (7) and the property proved above ($\mathrm{Ind}(\Lambda_r) = t$). This contradicts (11), concluding the proof of Theorem A.

II. — Proof of Theorem I.4

Theorem I.4 will follow as a corollary of the following more general result. If Λ is a compact invariant set of $f \in \mathrm{Diff}^1(M)$, we say that a dominated splitting $TM/\Lambda = E \oplus F$ is *homogeneous* if the dimension of the subspaces $E(x)$, $x \in \Lambda$, is constant. We say that a compact neighborhood U of Λ is an *admissible neighborhood* if $TM/\bigcap_n f^n(U)$ has one and exactly one homogeneous dominated splitting $TM/\bigcap_n f^n(U) = \hat{E} \oplus \hat{F}$ extending the splitting $TM/\Lambda = E \oplus F$. It is known, and not difficult to show, that if TM/Λ has a homogeneous dominated splitting, then Λ has an admissible neighborhood U (see [6] for instance). Moreover it is clear that every compact neighborhood of Λ contained in U is another admissible neighborhood. To simplify the notation, in what follows we shall write $\bigcap_{n \in \mathbf{Z}} f^n(U) = M(f, U)$.

Theorem **II.1.** — *Let* Λ *be a compact invariant set of* $g \in \mathrm{Diff}^1(M)$ *such that* $\Omega(g/\Lambda) = \Lambda$, *let* $TM/\Lambda = E \oplus F$ *be a homogeneous dominated splitting such that* E *is contracting and suppose* $c > 0$ *is such that the inequality*

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|(Dg^{-1})/F(g^j(x))\| \leqslant -c$$

holds for a dense set of points $x \in \Lambda$. *Then either* F *is expanding (and therefore* Λ *is hyperbolic) or for every admissible neighborhood* V *of* Λ *and every* $0 < \gamma < 1$ *there exists a periodic point* $p \in M(g, V)$ *with arbitrarily large period* N *and satisfying*

$$\gamma^N \leqslant \prod_{j=1}^{N} \|(Dg^{-1})/\hat{F}(g^j(p))\| < 1,$$

where $\hat{F}$ *is given by the unique homogeneous dominated splitting* $TM/M(g, V) = \hat{E} \oplus \hat{F}$ *that extends* $TM/\Lambda = E \oplus F$.

Let us see how Theorem I.4 follows from II.1. Suppose that $f \in \mathscr{F}^1(M)$ and $\widetilde{E}_i^s$ is contracting. Let m be given by Theorem I.3 and apply II.1 to $g = f^m$, $\Lambda = \overline{P}_i(f)$ and the splitting $TM/\overline{P}_i(f) = \widetilde{E}_i^s \oplus \widetilde{E}_i^u$. Then, by Theorem II.1, either $\widetilde{E}_i^u$ is expanding (and then Theorem I.4 is proved), or, given an admissible neighborhood V and $0 < \gamma < 1$, there exists a periodic point $p \in M(f^m, V)$, with arbitrarily large period N, such that:

$$\gamma^N \leqslant \prod_{j=1}^{N} \|(Dg^{-1})/\hat{F}(g^j(p))\| < 1.$$

The fact that this product is < 1 implies that $\hat{F}(p)$ is contained in the unstable subspace $E_u(p)$. Then, by Theorem I.3

$$\prod_{j=1}^{N} \|(Dg^{-1})/\hat{F}(g^j(p))\| \leqslant \prod_{j=1}^{N} \|(Df^{-m})/E^u(f^{mj}(p))\| \leqslant C\lambda^N.$$

Thus $\gamma^N \leqslant C\lambda^N$. But if we pick γ satisfying $\lambda < \gamma < 1$ and N sufficiently large this is a contradiction.

To prove Theorem II.1 we shall use several lemmas that will be useful also in Section IV to develop the tools that in Section V will be used to prove Theorem I.7.

From now on, g, Λ and $TM/\Lambda = E \oplus F$ will be as in the statement of II.1. We shall use the following definitions. We say that a pair of points $(x, g^n(x))$ contained in Λ, $n > 0$, is a *γ-string* if

$$\prod_{j=1}^{n} \|(Dg^{-1})/F(g^j(x))\| \leqslant \gamma^n$$

and we say that it is a *uniform γ-string* if $(g^k(x), g^n(x))$ is a γ-string for all $0 \leqslant k < n$.

The first step in the proof of II.1 is the following lemma, that is a sophisticated modification of the Shadowing Lemma ([23], [14]), (or Anosov Closing Lemma) and can be proved with similar methods. An explicit proof was given by Liao [8] (only for $k = 1$, but obviously the proof applies also to the general case).

Lemma **II.2.** — *Given $0 < \hat{\gamma} < 1$ and $\delta > 0$, there exists $\varepsilon = \varepsilon(\hat{\gamma}, \delta) > 0$ such that if $(x_i, g^{n_i}(x_i))$, $i = 1, \ldots, k$, are uniform $\hat{\gamma}$-strings satisfying $d(g^{n_i}(x_i), x_{i+1}) < \varepsilon$ for all $1 \leqslant i < k$ and $d(g^{n_k}(x_k), x_1) < \varepsilon$, then there exists a periodic point x of g with period $N = n_1 + \ldots + n_k$ such that*

$$d(g^n(x), g^n(x_1)) < \delta$$

for $0 \leqslant n \leqslant n_1$ and, setting $N_i = n_1 + \ldots + n_i$,

$$d(g^{N_i+n}(x), g^n(x_{i+1})) < \delta$$

for $0 \leqslant n \leqslant n_{i+1}$, $1 \leqslant i < k$.

Before continuing with the proof of II.1, let us first give a rough outline of it. Suppose that F is not expanding. Then to prove II.1 we have to show that given $0 < \gamma < 1$ and an admissible neighborhood V of Λ, there exists a periodic point $p \in M(g, V)$, with arbitrarily large period N, such that

$$\gamma^N < \prod_{j=1}^{N} \|(Dg^{-1})/\hat{F}(g^j(p))\| < 1. \tag{1}$$

Choose $\gamma < \hat{\gamma} < 1$. The periodic point p satisfying (1) will be the point $p = x$ obtained applying II.2 to $\hat{\gamma}$ and a suitable choice of uniform $\hat{\gamma}$-strings $(x_i, g^{n_i}(x_i))$. Since by II.2 the points of the orbit of x are δ-near to the points $g^\ell(x_i)$, $1 \leqslant i \leqslant k$, $0 \leqslant \ell \leqslant n_i$, then the condition $x \in M(g, V)$ will be satisfied if we work with a sufficiently small δ. To check (1) let us analyze the product in (1). Define α by

$$\alpha^N = \prod_{j=1}^{N} \|(Dg^{-1})/\hat{F}(g^j(x))\|$$

and γ_i by

$$\gamma_i^{n_i} = \prod_{j=1}^{n_i} \|(Dg^{-1})/F(g^j(x_i))\|.$$

What we want to prove is $\gamma^N < \alpha^N < 1$. But observe that, again by the δ-condition that x satisfies, α^N is such that $\alpha/(\prod_i \gamma_i^{n_i})^{1/N}$ becomes arbitrarily close to 1 if δ is sufficiently small. For the purpose of this informal outline we shall assume

$$\alpha^N = \prod_i \gamma_i^{n_i}.$$

Then $\alpha^N < 1$ is satisfied. The problem is to check $\gamma^N < \alpha^N$. Take $\gamma < \tilde{\gamma} < \hat{\gamma}$. We shall show that we can choose $x_1, \ldots, x_k$ and $n_1, \ldots, n_k$, with a value of k, say $k = 2n + 1$, such that when i is even we have $\gamma_i > \tilde{\gamma}$. Moreover every γ_i satisfies $\gamma_i \geqslant C$ where C is the minimum of the norms $\|(Dg^{-1})/T_x M\|$, $x \in M$. Then

$$\alpha^N = \prod_j \gamma_i^{n_i} \geqslant \prod_{j=1}^{n} \tilde{\gamma}^{n_{2j}} \prod_{j=0}^{n} C^{n_{2j+1}}.$$

Let N_1 be the sum of the n_i's for even values of i and N_2 the corresponding sum for odd values of i. Then

$$\alpha^N \geqslant \tilde{\gamma}^{N_1} C^{N_2}.$$

Since $\tilde{\gamma} > \gamma$, if N_1 is sufficiently larger than N_2, then $\alpha^N > \gamma^N$. Therefore we shall choose n_i much larger than n_{i+1} for every even i. The selection of the points x_i and the integers n_i requires the hypothesis of II.1 about the existence of a dense set of points z where a subsequence of the products

$$\prod_{j=1}^{n} \|(Dg^{-1})/F(g^j(z))\|$$

converges exponentially to 0, together with the fact that since F is not expanding, there exists a value of z such that this property does not hold. Then, we shall carefully pick the points x_i in such a way that $(x_i, g^{n_i}(x_i))$ is a uniform $\hat{\gamma}$-string (thus implying $\gamma_i < \hat{\gamma}$) but with γ_i not too small (that is, $\gamma < \gamma_i$) and also satisfying all the properties that we used in our sketch.

The selection of the points x_i requires several lemmas that in Section IV will be also useful to prepare the proof of Theorem I.7.

Lemma **II.3.** — *For all* $0 < \gamma_0 < \gamma_3 < 1$ *there exist* $N(\gamma_0, \gamma_3) > 0$ *and* $0 < c(\gamma_0, \gamma_3) < 1$ *such that if* $(x, g^n(x))$ *is a* γ_0*-string and* $n \geqslant N(\gamma_0, \gamma_3)$, *then there exist* $0 < n_1 < \ldots < n_k \leqslant n$, $k > 1$, *such that* $k \geqslant nc(\gamma_0, \gamma_3)$ *and* $(x, g^{n_i}(x))$ *is a uniform* γ_3*-string for all* $1 \leqslant i \leqslant k$.

We shall not prove this lemma because it is an immediate reformulation of a result of Pliss ([17], [18]).

Let us say that $(x, g^n(x))$ is an (N, γ)*-obstruction*, $0 < \gamma < 1$, $0 \leqslant N < n$, if $(x, g^m(x))$ is not a γ-string for all $N \leqslant m \leqslant n$.

Lemma **II.4.** — *Take* $0 < \gamma_0 < \gamma_3 < 1$, $0 < \gamma_2 < \gamma_3$ *and a* γ_0*-string* $(x, g^n(x))$. *Let* $0 < n_1 < \ldots < n_k \leqslant n$ *be the set of integers such that* $(x, g^{n_i}(x))$ *is a uniform* γ_3*-string, and let* $N = N(\gamma_2, \gamma_3)$. *Then, for all* $1 \leqslant i < k$, *either* $n_{i+1} - n_i \leqslant N$ *or* $(g^{n_i}(x), g^{n_{i+1}}(x))$ *is a* (N, γ_2)*-obstruction. Moreover, either* $n_1 \leqslant N$ *or* $(x, g^{n_1}(x))$ *is* (N, γ_2)*-obstruction.*

Proof. — *If* $n_{i+1} - n_i > N$ and $(g^{n_i}(x), g^{n_{i+1}}(x))$ is not a (N, γ_2)-obstruction, there exists $n_i + N \leqslant m \leqslant n_{i+1}$ such that $(g^{n_i}(x), g^m(x))$ is a γ_2-string. Hence, by II.3, that can be applied because $m - n_i \geqslant N = N(\gamma_2, \gamma_3)$, there exists $n_i < r < m$ such that $(g^{n_i}(x), g^r(x))$ is a uniform γ_3-string. Hence $(x, g^r(x))$ is a uniform γ_3-string and r should be in the sequence $n_1 < \ldots < n_k$. But on the other hand $n_i < r < m \leqslant n_{i+1}$.

Lemma **II.5.** — *If* $0 < \gamma_0 < \gamma_3 < 1$, $0 < \gamma_1 < \gamma_2 < \gamma_3$, *and* $(x, g^n(x))$ *is a* γ_0*-string containing a* (N, γ_2)*-obstruction* $(g^r(x), g^{r+\ell}(x))$ *such that*

a) $n \geqslant N(\gamma_0, \gamma_3)$,

b) $nc(\gamma_0, \gamma_3) > r + \ell$,

c) $r + \ell \geqslant N(\gamma_1, \gamma_2)$ *and*

d) $(r + \ell)\, c(\gamma_1, \gamma_2) > r + N$,

then there exists a uniform γ_3*-string* $(x, g^m(x))$, $r + \ell \leqslant m \leqslant n$, *that is not a* γ_1*-string.*

In a more informal language these conditions require n to be large with respect to $r + \ell$ and $r + \ell$ to be large with respect to $r + N$.

Proof. — Let $0 < n_1 < \ldots < n_k \leqslant n$ be the integers such that $(x, g^{n_i}(x))$ is a uniform γ_3-string. By a), we can apply II.3 that implies

$$k \geqslant nc(\gamma_0, \gamma_3).$$

Then, by b)

$$k > r + \ell.$$

Hence $n_k > r + \ell$ because obviously $n_k \geqslant k$. Let j be the smallest integer such that

$$n_j \geqslant r + \ell. \tag{2}$$

Let us prove that $(x, g^{n_j}(x))$ (that is a uniform γ_3-string) is not a γ_1-string, thus completing the proof of the lemma. Suppose that $(x, g^{n_j}(x))$ is a γ_1-string. By c)

$$n_j \geqslant r + \ell \geqslant N(\gamma_1, \gamma_2),$$

then we can apply II.3 to the γ_1-string $(x, g^{n_j}(x))$ and $0 < \gamma_1 < \gamma_2$. It yields a family $0 < m_1 < \ldots < m_s \leqslant n_j$ such that $(x, g^{m_j}(x))$ is a uniform γ_2-string for all j (hence all the numbers m_i belong to $\{n_1, \ldots, n_j\}$, since $\gamma_2 < \gamma_3$) and

$$s \geqslant n_j\, c(\gamma_1, \gamma_2) \geqslant (r + \ell)\, c(\gamma_1, \gamma_2).$$

Applying d) we obtain

$$m_{s-1} \geqslant s - 1 \geqslant (r + \ell)\, c(\gamma_1, \gamma_2) - 1 \geqslant r + N.$$

But $m_{s-1} = n_i$ for some $1 \leqslant i < j$. Hence

$$m_{s-1} = n_i < r + \ell$$

because j was the minimum index such that (2) holds. Then $r + N \leqslant n_i < r + \ell$. Since $(g^r(x), g^{r+\ell}(x))$ is a (N, γ_2)-obstruction, $(g^r(x), g^{n_i}(x))$ is not a γ_2-string. But this contradicts the fact that $(x, g^{n_i}(x))$ is a uniform γ_2-string.

If $x \in \Lambda$, denote by $J(x, \Lambda)$ the set of points $y \in \Lambda$ that can be written as

$$y = \lim_{n \to +\infty} g^{m_n}(x_n),$$

where $\{ x_n \mid n \geqslant 0 \}$ is a sequence converging to x and $\lim_{n \to +\infty} m_n = +\infty$. Clearly to obtain $J(x, \Lambda)$, it is sufficient to use sequences $\{ x_n \mid n \geqslant 0 \}$ contained in some dense subset of Λ. Moreover the hypothesis $\Omega(g/\Lambda) = \Lambda$ implies

$$x \in J(x, \Lambda)$$

for all $x \in \Lambda$.

Let us say that a compact invariant set $\Sigma \subset \Lambda$ is a *(t, γ)-set* ($t \in \mathbf{Z}^+$, $0 < \gamma < 1$) if for every $x \in \Sigma$ there exists $-t < m < t$ such that $(g^{m-n}(x), g^m(x))$ is a γ-string for all $n > 0$. Clearly this implies that $F \mid \Sigma$ is an expanding subbundle.

Take γ_0 such that $\exp(-c) < \gamma_0 < 1$, where c is as in the statement of Theorem II.1. Then, by hypothesis, there exists a dense set $\Lambda_0 \subset \Lambda$ such that if $x \in \Lambda_0$ then there are infinitely many values of n satisfying

$$\prod_{j=1}^{n} \|(Dg^{-1})/F(g^j(x))\| < \gamma_0^n.$$

Take $\gamma_1, \gamma_2, \bar{\gamma}_2, \gamma_3$ with

(3) $$0 < \gamma_0 < \gamma_1 < \gamma_2 < \bar{\gamma}_2 < \gamma_3 < 1.$$

Lemma **II.6.** — *For every* $\varepsilon > 0$ *there exists* $N(\varepsilon)$ *such that for all* $x \in \Lambda$, *either* $J(x, \Lambda)$ *is an* $(N(\varepsilon), \gamma_3)$*-set or there exists* $y \in J(x, \Lambda)$ *such that* $(y, g^n(y))$ *is an* (N, γ_2)*-obstruction for all* $n > N(\varepsilon)$, *where* $N = N(\bar{\gamma}_2, \gamma_3)$ *is given by* II.3, *and moreover* y *satisfies one of the following properties*:

a) $d(x, y) \leqslant \varepsilon$;

b) *there exists* $z_0 \in \Lambda$ *arbitrarily near to* x *and* $m > 0$ *such that* $d(g^m(z_0), y) < \varepsilon$ *and* $(z_0, g^m(z_0))$ *is a uniform* γ_3*-string.*

Proof. — Denote by $\Lambda(N)$ the set of points $y \in \Lambda$ such that $(y, g^n(y))$ is an (N, γ_2)-obstruction for all $n > N$. It is easy to check that given $\varepsilon > 0$ there exists $N(\varepsilon) > N$ such that when $(y, g^n(y))$ is an $(N, \bar{\gamma}_2)$-obstruction and $n > N(\varepsilon)$, then $d(y, \Lambda(N)) < \varepsilon$ (here we use that $\bar{\gamma}_2 > \gamma_2$). Given $x \in \Lambda$ and $z \in J(x, \Lambda)$ there exists a sequence $\{ x_n \mid n \geqslant 0 \} \subset \Lambda_0$ converging to x and satisfying $z = \lim_{n \to +\infty} g^{m_n}(x_n)$ and $\lim_{n \to +\infty} m_n = +\infty$. For $n \geqslant 0$ define

$$\mathcal{S}(n) = \{ m > 0 \mid (x_n, g^m(x_n)) \text{ is a uniform } \gamma_3\text{-string} \} \cup \{ 0 \}.$$

By II.3 it is easy to see that $\mathscr{S}(n)$ is unbounded (since $\gamma_0 < y_3$ and $x_n \in \Lambda_0$). Set

$$k_n^+ = \min \mathscr{S}(n) \cap [m_n, +\infty)$$

and

$$k_n^- = \max \mathscr{S}(n) \cap [0, m_n).$$

Suppose that $\liminf(k_n^+ - k_n^-) \leqslant N(\varepsilon)$. Then there exists $0 \leqslant m \leqslant N(\varepsilon)$ such that $g^m(z)$ is the limit of a subsequence of $\{ g^{k_n^+}(x_n) \mid n \geqslant 0 \}$. Hence, if $r > 0$, $(g^{m-r}(z), g^m(z))$ is a γ_3-string because it is the limit of a sequence of γ_3-strings $(g^{k_n^+ - r}(x_n), g^{k_n^+}(x_n))$ (that indeed are γ_3-strings for $r \leqslant k_n^+$ because $(x_n, g^{k_n^+}(x_n))$ is a uniform γ_3-string for all n). Therefore, for some $0 \leqslant m \leqslant N(\varepsilon)$, $(g^{m-r}(z), g^m(z))$ is a γ_3-string for all $r > 0$. If this holds for all $z \in J(x, \Lambda)$ then $J(x, \Lambda)$ is a $(N(\varepsilon), \gamma_3)$-set. If it does not hold for all $z \in J(x, \Lambda)$ this argument shows that we can pick z such that for many n, $k_n^+ - k_n^- > N(\varepsilon)$. Hence $k_n^+ - k_n^- > N$ because $N(\varepsilon) > N$. Then, by Lemma II.4, $(g^{k_n^-}(x_n), g^{k_n^+}(x_n))$ is an $(N, \overline{\gamma}_2)$-obstruction. Therefore $d(g^{k_n^-}(x_n), \Lambda(N)) < \varepsilon$ for infinitely many values of n. If for an unbounded set of these we have $k_n^- > 0$, we take $y \in \Lambda(N)$ such that $d(g^{k_n^-}(x_n), y) < \varepsilon$ and then this point y, the point $z_0 = x_n$ and $m = k_n^-$ satisfy the requirements of Lemma II.6 and option b). If $k_n^- = 0$ for all sufficiently large values of n that satisfy $d(g^{k_n^-}(x_n), \Lambda(N)) < \varepsilon$, then $d(x_n, \Lambda(N)) < \varepsilon$ and since $x_n \to x$ we obtain $d(x, \Lambda(N)) \leqslant \varepsilon$. Taking $y \in \Lambda(N)$ such that $d(x, y) \leqslant \varepsilon$ it follows that y satisfies II.6 and option a).

Lemma **II.7.** — *If* F *is not expanding, for all* $\varepsilon > 0$ *there exists a compact invariant set* $\Lambda(\varepsilon) \subset \Lambda$ *such that every* $x \in \Lambda(\varepsilon)$ *has the following property: there exist* x_0 *arbitrarily near to* x, $n_0 \geqslant 0$ *and* $y \in \Lambda(\varepsilon)$ *such that* $d(g^{n_0}(x_0), y) < \varepsilon$, $(y, g^n(y))$ *is an* (N, γ_2)*-obstruction for all* $n > N(\varepsilon)$, *and, if* $n_0 > 0$, $(x_0, g^{n_0}(x_0))$ *is a uniform* γ_3*-string. Moreover* $\Lambda(\varepsilon)$ *is the closure of its interior. (One has* $N = N(\overline{\gamma}_2, \gamma_3)$ *and* $N(\varepsilon)$ *is given by* II.6).

Proof. — Let Σ be the union of all the $(N(\varepsilon), \lambda_3)$-sets. Then its closure $\overline{\Sigma}$ is an $(N(\varepsilon), \lambda_3)$-set. Since F is not expanding, $\overline{\Sigma} \neq \Lambda$. Define $\Lambda(\varepsilon)$ as the closure of the open set $\Lambda - \overline{\Sigma}$. Given $x \in \Lambda(\varepsilon)$ take $\bar{x} \in \Lambda - \overline{\Sigma}$ near to x. Since $\bar{x} \notin \Sigma$, the set $J(\bar{x}, \Lambda)$ (that contains $\bar{x}$ because $\Omega(g/\Lambda) = \Lambda$) cannot be an $(N(\varepsilon), \lambda_3)$-set. Then by II.6, there exist a point $y \in J(\bar{x}, \Lambda)$ such that $(y, g^n(y))$ is an (N, γ_2)-obstruction for all $n > N(\varepsilon)$, and x_0 arbitrarily near to $\bar{x}$ (hence near to x) and $n_0 \geqslant 0$ such that $d(g^{n_0}(x_0), y) < \varepsilon$ and, if $n_0 > 0$, $(x_0, g^n(x_0))$ is a uniform γ_3-string. To complete the requirements of Lemma II.7 we have only to show that $y \in \Lambda(\varepsilon)$. But $\bar{x} \in \Lambda - \overline{\Sigma}$. Hence there exists a neighborhood U of $\bar{x}$ with $U \cap \overline{\Sigma} = \varnothing$. Therefore if a point $z \in J(\bar{x}, \Lambda)$ is given by $z = \lim_{n \to +\infty} g^{m_n}(x_n)$, where $\{ x_n \mid n \geqslant 0 \} \subset \Lambda$ is a sequence contained in Λ converging to $\bar{x}$ and $m_n \to +\infty$, it follows that $x_n \in U$ for large values of n. This means that $x_n \in \Lambda - \overline{\Sigma} \subset \Lambda(\varepsilon)$, implying $g^{m_n}(x_n) \in \Lambda(\varepsilon)$ for n large enough; thus $z \in \Lambda(\varepsilon)$. This proves $J(\bar{x}, \Lambda) \subset \Lambda(\varepsilon)$. In particular $y \in \Lambda(\varepsilon)$ because $y \in J(\bar{x}, \Lambda)$.

Now let V be an admissible neighborhood of Λ, let $TM/M(g, V) = \hat{E} \oplus \hat{F}$ be the homogeneous dominated splitting extending $TM/\Lambda = E \oplus F$ and let $0 < \gamma < 1$

be given. Take $\gamma_0, \gamma_1, \gamma_2, \bar{\gamma}_2, \gamma_3$ such that (3) and $\gamma < \gamma_0$ and choose $0 < k_0 < 1$ such that

(4) $$\gamma < k_0^2 \gamma_1$$

(5) $$k_0^{-1} \gamma_3 < 1.$$

Take $\delta > 0$ that such if $a, b \in M(g, V)$ and $d(a, b) < \delta$ then

(6) $$||(Dg^{-1})/\hat{F}(a)|| \geqslant k_0 \, ||(Dg^{-1})/\hat{F}(b)||.$$

Let $\varepsilon = \varepsilon(\delta, \gamma_3)$ be given by Lemma II.2. We claim that if F is not expanding there exists a sequence $\{ x_i \mid i \geqslant 1 \} \subset \Lambda(\varepsilon/4)$ and a sequence of integers $n_i \geqslant 0$ such that:

I) $d(g^{n_i}(x_i), x_{i+1}) < \varepsilon/2$ for all $i \geqslant 1$.

II) If $n_i > 0$, then $(x_i, g^{n_i}(x_i))$ is a uniform γ_3-string. For all even values of i, $n_i > 0$ and $(x_i, g^{n_i}(x_i))$ is not a γ_1-string.

III) If $K = \min \{ ||(Dg^{-1})/F(x)|| \mid x \in \Lambda \}$, then

$$\gamma_1^{n_i} K^{n_{i-1}} \geqslant (k_0 \gamma_1)^{n_i + n_{i-1}}$$

for every even value of i.

We shall construct this sequence by induction. We should begin by the cases $i = 1$ and $i = 2$, but we shall proceed directly to the induction step that is sufficiently illuminating about the construction of the first two terms of the sequence. Suppose then $(x_i, g^{n_i}(x_i))$ constructed for $1 \leqslant i \leqslant j$, j even. Since $g^{n_j}(x_j) \in \Lambda(\varepsilon/4)$ we can apply II.7 that gives a point x_{j+1} arbitrarily near to $g^{n_j}(x_j)$ (in particular we can assume $d(g^{n_j}(x_j), x_{j+1}) < \varepsilon/2$) and $n_0 \geqslant 0$ such that $g^{n_0}(x_{j+1})$ is $\varepsilon/4$-near to a point $y \in \Lambda(\varepsilon/4)$ such that $(y, g^n(y))$ is an (N, γ_2)-obstruction for all $n > N(\varepsilon/4)$. Moreover, if $n_0 > 0$, then $(x_{j+1}, g^{n_0}(x_{j+1}))$ is a uniform γ_3-string. Since $\Lambda(\varepsilon/4)$ is the closure of its interior, and in its interior there is a dense set of values of x such that $(x, g^n(x))$ is a γ_0-string for infinitely many values of n, there exists $x_{j+2} \in \Lambda(\varepsilon/4)$ so near to y that

$$d(g^{n_0}(x_{j+1}), x_{j+2}) < \varepsilon/2,$$

and such that $(x_{j+2}, g^n(x_{j+2}))$ is a γ_0-string for infinitely many values of n. Take $N_1 > N(\varepsilon/4)$; taking x_{j+2} sufficiently near to y, we obtain that $(x_{j+2}, g^{N_1}(x_{j+2}))$ is an (N, γ_2)-obstruction. Taking N_1 large with respect to N, and a value of n large with respect to N_1 and such that $(x_{j+2}, g^n(x_{j+2}))$ is a γ_0-string, we can apply Lemma II.5 (with $r = 0$, $\ell = N_1$), and obtain $N_1 \leqslant n_{j+2} \leqslant n$ such that $(x_{j+2}, g^{n_{j+2}}(x_{j+2}))$ is a uniform γ_3-string but is not a γ_1-string. Then $x_{j+1}, x_{j+2}, n_{j+1} = n_0$ and n_{j+2} satisfy conditions I), II). Condition III) holds if n_{j+2} is large with respect to n_{j+1}. Then to satisfy it, take in the previous construction N_1 so large that

$$\gamma_1^n K^{n_{j+1}} \geqslant (k_0 \gamma_1)^{n + n_{j+1}}$$

for all $n \geqslant N_1$. It will hold for $n = n_{j+2}$.

Now take two points x_ℓ, x_k, with ℓ and k odd, $k > \ell$ such that $d(x_\ell, x_k) < \varepsilon/2$. To simplify the notation translate the indexes in order to have $\ell = 1$, $k = 2t + 1$. Then

$$d(g^{n_k}(x_{k-1}), x_1) = d(g^{n_k}(x_{k-1}), x_k) + d(x_k, x_1) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now we apply Lemma II.2 to $\hat{\gamma} = \gamma_3$, the sequence $x_1, \ldots, x_k$ and the sequence of integers $n_1, \ldots, n_k$. However Lemma II.2 requires every $(x_i, g^{n_i}(x_i))$ to be a uniform γ_3-string (thus, in particular, $n_i > 0$) and we have satisfied this condition only when $n_i > 0$. Since this holds for every even value of i, we can expurgate the values of i such that $n_i = 0$; in other words we apply Lemma II.2 to the set S of points x_i with $n_i > 0$. If a certain x_j is not among these points, then j must be odd, x_{j-1} and x_{j+1} are in S and

$$\begin{aligned} d(g^{n_{j-1}}(x_{j-1}), x_{j+1}) &\leqslant d(g^{n_{j-1}}(x_{j-1}), x_j) + d(x_j, x_{j+1}) \\ &= d(g^{n_{j-1}}(x_{j-1}), x_j) + d(g^{n_j}(x_j), x_{j+1}) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence, Lemma II.2 gives a periodic point x, with period $\mathrm{N} = n_1 + \ldots + n_k$, such that

$$d(g^n(x), g^n(x_1)) \leqslant \delta$$

for $0 \leqslant n \leqslant n_1$ and, setting $\mathrm{N}_i = n_1 \ldots + n_i$,

$$d(g^{\mathrm{N}_i + n}(x), g^n(x_{i+1})) < \delta$$

for $1 \leqslant i < k$ and $0 \leqslant n \leqslant n_{i+1}$. In particular, every point of the orbit of x is δ-near to Λ. If δ is taken conveniently small, the above inequality implies that the orbit of x is contained in $\mathrm{M}(g, \mathrm{V})$ as Theorem II.1 requires. Moreover it implies by (6) that

$$\|(\mathrm{D}g^{-1})/\hat{\mathrm{F}}(g^n(x))\| \geqslant k_0 \|(\mathrm{D}g^{-1})/\mathrm{F}(g^n(x_1))\|$$

for $0 \leqslant n \leqslant n_1$ and

$$\|(\mathrm{D}g^{-1})/\hat{\mathrm{F}}(g^{\mathrm{N}_i+n}(x))\| \geqslant k_0 \|(\mathrm{D}g^{-1})/\mathrm{F}(g^n(x_{i+1}))\|$$

for $1 \leqslant i < k$, $0 \leqslant n \leqslant n_{i+1}$. Hence, setting $\mathrm{N}_0 = 0$, we have

$$\prod_{n=1}^{n_{i+1}} \|(\mathrm{D}g^{-1})/\hat{\mathrm{F}}(g^{\mathrm{N}_i+n}(x))\| \geqslant k_0^{n_{i+1}} \prod_{n=1}^{n_{i+1}} \|(\mathrm{D}g^{-1})/\mathrm{F}(g^n(x_{i+1}))\|$$

for $0 \leqslant i < k$. Hence, if i is odd,

$$\prod_{n=1}^{n_{i+1}} \|(\mathrm{D}g^{-1})/\hat{\mathrm{F}}(g^{\mathrm{N}_i+n}(x))\| \geqslant k_0^{n_{i+1}} \gamma_1^{n_{i+1}}$$

and, for i even,

$$\prod_{n=1}^{n_{i+1}} \|(\mathrm{D}g^{-1})/\hat{\mathrm{F}}(g^{\mathrm{N}_i+n}(x))\| \geqslant k_0^{n_{i+1}} \mathrm{K}^{n_{i+1}}.$$

Consequently,

$$\prod_{n=1}^{\mathrm{N}} \|(\mathrm{D}g^{-1})/\hat{\mathrm{F}}(g^n(x))\| \geqslant \prod_j k_0^{n_j + n_{j+1}} \gamma_1^{n_{j+1}} \mathrm{K}^{n_j},$$

where the last product is taken over all the odd values in $[1, k-1]$. Applying property (III) we obtain

$$\prod_j k_0^{n_j+n_{j+1}} \gamma_1^{n_{j+1}} K^{n_j} \geqslant \prod_j k_0^{2(n_j+n_{j+1})} \gamma_1^{n_j+n_{j+1}} = k_0^{2N} \gamma_1^N.$$

Since $k_0^2 \gamma_1 > \gamma$ by (4), we have proved

$$\prod_{n=1}^{N} \|(Dg^{-1})/\hat{F}(g^n(x))\| > \gamma^N.$$

Finally these same methods can be used to prove the desired upper estimate for the product on the left. Recalling that $(x_i, g^{n_i}(x))$ is a γ_3-string if $n_i \neq 0$, we obtain analogously that

$$\prod_{n=1}^{N} \|(Dg^{-1})/\hat{F}(g^n(x))\| \leqslant k_0^{-N} \gamma_3^N.$$

Since γ_3 and k_0 satisfy $k_0^{-1}\gamma_3 < 1$, by (5), this implies

$$\prod_{n=1}^{N} \|(Dg^{-1})/\hat{F}(g^n(x))\| < 1.$$

III. — Proof of Theorem I.6

Let $\mathcal{M}(M)$ be the space of probabilities on the Borel σ-algebra of M endowed with the weak topology. If $f \in \mathrm{Diff}^1(M)$, let $\mathcal{M}(f)$ be the set of f-invariant elements of $\mathcal{M}(M)$ and $\mathcal{M}_e(f)$ be the set of ergodic elements of $\mathcal{M}(f)$. Specially interesting for our purpose will be the f-invariant probabilities supported on a periodic orbit of f, i.e. probabilities of the form

$$\mu = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}$$

where x satisfies $f^n(x) = x$. Denote by $\mathcal{M}_p(f)$ the set of these probabilities. The following result is a corollary of the main Theorem in [12].

Theorem **III.1.** — *Suppose that $f \in \mathrm{Diff}^1(M)$ and $\mu \in \mathcal{M}_e(f^m)$ for some $m > 0$. Then, given a neighborhood V of μ and a compact set K disjoint from the support of μ, there exists a diffeomorphism g, arbitrarily C^1 close to f and coinciding with f on K, such that there exists a probability $\mu_0 \in \mathcal{M}_p(g^m)$ contained in V whose support is disjoint from K.*

Proof. — Recall that an invariant set Λ of f is said to have total probability if $\mu(\Lambda) = 1$ for all $\mu \in \mathcal{M}(f)$. Define $\Sigma(f)$ as the set of points $x \in M$ such that for all $\varepsilon > 0$, every compact set K disjoint from the closure of the orbit of x, and every C^1 neighborhood $\mathcal{U}$ of f, there exists $g \in \mathcal{U}$ which coincides with f on K and has a periodic point y such that, if n is its period, one has $d(f^j(x), g^j(y)) \leqslant \varepsilon$ for all $0 \leqslant j \leqslant n$. Theorem A of [12] states that $\Sigma(f)$ has total probability. It is easy to see that then $\mu(\Sigma(f)) = 1$

for all $\mu \in \mathcal{M}(f^m)$ and all $m \neq 0$. Suppose that μ, m, K and V are as in the statement of Theorem III.1. We can assume that there exist continuous functions $\varphi_i : \mathrm{M} \to \mathbf{R}$, $1 \leqslant i \leqslant s$, such that V is the set of $\nu \in \mathcal{M}(\mathrm{M})$ satisfying

$$\left| \int \varphi_i \, d\nu - \int \varphi_i \, d\mu \right| \leqslant 1$$

for all $1 \leqslant i \leqslant s$. Take $x \in \Sigma(f)$ and $\mathrm{N} > 0$ such that

$$\left| \int \varphi_i \, d\mu - \frac{1}{n} \sum_{j=0}^{n-1} \varphi_i(f^{jm}(f^k(x))) \right| \leqslant 1/4 \tag{1}$$

for all $1 \leqslant i \leqslant s$, $0 \leqslant k \leqslant m-1$, $n \geqslant \mathrm{N}$. Such an x exists because $\mu \in \mathcal{M}_e(f^m)$ and $\mu(\Sigma(f)) = 1$. If x is a periodic point of f, take $n \geqslant \mathrm{N}$ such that $f^{mn}(x) = x$ and define

$$\mu_0 = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^{mj}(x)}.$$

Then, by (1), $\mu_0 \in \mathrm{V}$. Hence the theorem is proved taking $g = f$ and μ_0. Suppose that x is not periodic. Take $\varepsilon > 0$ such that if $a, b \in \mathrm{M}$ satisfy $d(a, b) \leqslant \varepsilon$ then $|\varphi_i(a) - \varphi_i(b)| \leqslant 1/4$ for all $1 \leqslant i \leqslant s$. By the definition of $\Sigma(f)$ there exists g arbitrarily C^1 near to f, coinciding with f on K and having a periodic point y such that if n is its period then $d(f^j(x), g^j(y)) \leqslant \varepsilon$ for all $0 \leqslant j \leqslant nm$. Define μ_0 as

$$\mu_0 = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{g^{mj}(y)}.$$

Observe that since x is not a periodic point of f then, taking g very near to f and ε very small, the period of y becomes arbitrarily large. Then we can assume $n \geqslant m(\mathrm{N}+1)$. Now observe that $\{jn \mid 0 \leqslant j \leqslant m-1\}$ partitions $\{jm \mid 0 \leqslant j \leqslant n-1\}$ into a disjoint union of m sets, each with approximately $[n/m]$ multiples of m. More precisely, write $\{jm \mid 0 \leqslant j \leqslant n-1\} = \bigcup_{r=1}^{m} \{\mathrm{N}_r m, \mathrm{N}_r m + m, \ldots, \mathrm{N}_r m + (n_r - 1) m\}$, where $\mathrm{N}_r m - m < (r-1) n \leqslant \mathrm{N}_r m = (r-1) n + k_r$. The integers k_r, n_r and N_r are obtained inductively by $\mathrm{N}_1 = k_1 = 0$ and, for $1 \leqslant r \leqslant m$, $n - k_r \leqslant n_r m < n - k_r + m$, $k_{r+1} = n_r m - n + k_r$ and $\mathrm{N}_{r+1} = \mathrm{N}_r + n_r$. It follows that $0 \leqslant k_r \leqslant m-1$ and $0 \leqslant k_r + (n_r - 1) m < n$, as well as $n_1 + n_2 + \ldots + n_m = \mathrm{N}_{m+1} = n$. But

$$g^{\mathrm{N}_r m + jm}(y) = g^{(r-1)n + k_r + jm}(y) = g^{k_r + jm}(y)$$

for each $1 \leqslant r \leqslant m$, $0 \leqslant j \leqslant n_r - 1$, and therefore

$$\int \varphi_i \, d\mu_0 = \frac{1}{n} \sum_{j=0}^{n-1} \varphi_i(g^{jm}(y)) = \frac{1}{n} \sum_{r=1}^{m} \sum_{j=0}^{n_r - 1} \varphi_i(g^{k_r + jm}(y))$$

for each $1 \leqslant i \leqslant s$. Moreover, since $n_r > (n-m)/m \geqslant \mathrm{N}$, (1) implies

$$\left| \int \varphi_i \, d\mu - \frac{1}{n} \sum_{r=1}^{m} \sum_{j=0}^{n_r - 1} \varphi_i(f^{k_r + jm}(x)) \right| \leqslant \frac{1}{n} \sum_{r=1}^{m} n_r \left| \int \varphi_i \, d\mu - \frac{1}{n_r} \sum_{j=0}^{n_r - 1} \varphi_i(f^{jm}(f^{k_r}(x))) \right| \leqslant 1/4.$$

Hence, for all $1 \leqslant i \leqslant s$,

$$\left|\int \varphi_i \, d\mu - \int \varphi_i \, d\mu_0\right| \leqslant \frac{1}{4} + \left|\int \varphi_i \, d\mu_0 - \frac{1}{n} \sum_{r=1}^{m} \sum_{j=0}^{n_r - 1} \varphi_i(f^{k_r + jm}(x))\right|$$

$$\leqslant \frac{1}{4} + \frac{1}{n} \sum_{r=1}^{m} \sum_{j=0}^{n_r - 1} \left| \varphi_i(g^{k_r + jm}(y)) - \varphi_i(f^{k_r + jm}(x)) \right| \leqslant 1/2.$$

Lemma **III.2.** — *Suppose that $f \in \mathscr{F}^1(\mathrm{M})$ and $\overline{\mathrm{P}}_k(f)$ is hyperbolic for $0 \leqslant k < i$. Then, for every sufficiently small neighborhood U of $\bigcup_{0 \leqslant k < i} \overline{\mathrm{P}}_k(f)$ there exists a C^1 neighborhood $\mathscr{U}_0$ of f such that*

$$\bigcup_{0 \leqslant k < i} \overline{\mathrm{P}}_k(g) = \bigcap_n g^n(\mathrm{U})$$

for all $g \in \mathscr{U}_0$.

Proof. — By the hyperbolicity of $\overline{\mathrm{P}}_k(f)$ for $0 \leqslant k < i$, there exists a neighborhood $\mathscr{U}_0$ of f, that we can and shall assume to be connected and contained in $\mathscr{F}^1(\mathrm{M})$, such that for each $g \in \mathscr{U}_0$ there exists a homeomorphism

$$h_g : \bigcup_{0 \leqslant k < i} \overline{\mathrm{P}}_k(f) \to \bigcap_n g^n(\mathrm{U})$$

such that $gh_g(x) = h_g f(x)$ for all x in the union on the left, and, for all x in that union, $h_g(x)$ depends continuously on $g \in \mathscr{U}_0$. Moreover, $h_g(\bigcup_{0 \leqslant k < i} \overline{\mathrm{P}}_k(f))$ is hyperbolic. Let us prove that

$$h_g(\bigcup_{0 \leqslant k < i} \mathrm{P}_k(f)) \subset \bigcup_{0 \leqslant k < i} \mathrm{P}_k(g). \tag{2}$$

Suppose that $x \in \mathrm{P}_k(f)$ and $g \in \mathscr{U}_0$. Take a continuous arc of diffeomorphisms $g_t \in \mathscr{U}_0$, $0 \leqslant t \leqslant 1$, with $g_0 = f$, $g_1 = g$. Then $x_t = h_{g_t}(x)$ is periodic for g_t, with the same minimum period than x. Since every $g_t \in \mathscr{U}_0 \subset \mathscr{F}^1(\mathrm{M})$, x_t is a hyperbolic periodic point of g_t for all $0 \leqslant t \leqslant 1$. Then it easy to see that the dimensions of the stable manifolds of x_t are the same for all $0 \leqslant t \leqslant 1$. Hence $x_1 = h_g(x) \in \mathrm{P}_k(g)$. This proves (2). Now let us show that

$$h_g(\bigcup_{0 \leqslant k < i} \mathrm{P}_k(f)) = \bigcup_{0 \leqslant k < i} \mathrm{P}_k(g).$$

Denote by $\mathrm{P}_{k,\ell}(g)$ the set of points in $\mathrm{P}_k(g)$ whose minimum period is ℓ. It is easy to see, using the fact that $\mathscr{U}_0$ is connected and contained in $\mathscr{F}^1(\mathrm{M})$, that, for all $0 \leqslant k \leqslant \dim \mathrm{M}$ and $\ell \geqslant 1$, $\#\mathrm{P}_{k,\ell}(g)$ is the same for all $g \in \mathscr{U}_0$. Moreover, the argument used to prove (2) shows that for all $0 \leqslant k \leqslant i$, $\ell \geqslant 1$ and $g \in \mathscr{U}_0$, $\#\mathrm{P}_{k,\ell}(f)$ is equal to the number of points in $h_g(\mathrm{P}_k(f))$ with period ℓ. Since $\#\mathrm{P}_{k,\ell}(g) = \#\mathrm{P}_{k,\ell}(f)$, this shows that $\mathrm{P}_{k,\ell}(g) \subset h_g(\mathrm{P}_k(f))$ for all $\ell \geqslant 1$. Hence

$$\mathrm{P}_k(g) = \bigcup_{\ell \geqslant 1} \mathrm{P}_{k,\ell}(g) \subset h_g(\mathrm{P}_k(f)).$$

This implies

$$\bigcup_{0\leqslant k<i} \mathrm{P}_k(g) \subset h_g(\bigcup_{0\leqslant k<i} \mathrm{P}_k(f)),$$

thus proving the equality, which clearly implies

$$\bigcap_n g^n(\mathrm{U}) = h_g(\bigcup_{0\leqslant k<i} \overline{\mathrm{P}}_k(f)) = \bigcup_{0\leqslant k<i} \overline{\mathrm{P}}_k(g).$$

Now let us prove Theorem I.6.

Given $f \in \mathscr{F}^1(\mathrm{M})$ such that $\overline{\mathrm{P}}_k(f)$ is hyperbolic for $0 \leqslant k < i$, take $m > 0$ and $0 < \lambda < 1$ given by Theorem I.3. Take any $\lambda < \lambda_0 < 1$ and suppose that $\mu \in \mathscr{M}(f^m/\overline{\mathrm{P}}_i(f))$ satisfies

$$\int \log \|(\mathrm{D}f^m)/\widetilde{\mathrm{E}}_i^s\| \, d\mu \geqslant \log \lambda_0 \tag{3}$$

and let us prove that

$$\mu(\bigcup_{0\leqslant k<i} \overline{\mathrm{P}}_k(f)) > 0. \tag{4}$$

First suppose that μ is ergodic but not supported on a periodic orbit. Then, if (4) does not hold, we have

$$\mu(\bigcup_{0\leqslant k<i} \overline{\mathrm{P}}_k(f)) = 0.$$

Take a neighborhood U of $\bigcup_{0\leqslant k<i} \overline{\mathrm{P}}_k(f)$ such that

$$\mu(\mathrm{U}) < 1/2$$

and

$$\mu(\partial \mathrm{U}) = 0.$$

These properties imply that there exists a neighborhood V of μ in $\mathscr{M}(\mathrm{M})$ such that

$$\nu(\mathrm{U}) < 1/2 \tag{5}$$

for all $\nu \in \mathrm{V}$. Take a neighborhood W of $\overline{\mathrm{P}}_i(f)$ such that there exists a dominated splitting $\mathrm{TM}/\mathrm{M}(f, \mathrm{W}) = \widetilde{\mathrm{E}}_i^s \oplus \widetilde{\mathrm{E}}_i^u$ extending $\mathrm{TM}/\overline{\mathrm{P}}_i(f) = \widetilde{\mathrm{E}}_i^s \oplus \widetilde{\mathrm{E}}_i^u$. See Section II for the definition of $\mathrm{M}(f, \mathrm{W})$ and the existence of W. Take a neighborhood W_0 of $\mathrm{M}(f, \mathrm{W})$ so small that there exists a continuous splitting $\mathrm{TM}/\mathrm{W}_0 = \hat{\mathrm{E}}^s \oplus \hat{\mathrm{E}}^u$ extending $\mathrm{TM}/\mathrm{M}(f, \mathrm{W}) = \widetilde{\mathrm{E}}_i^s \oplus \widetilde{\mathrm{E}}_i^u$. Then, by standard properties of dominated splittings ([6], [9]), there exists a neighborhood $\mathscr{U}_0$ of f such that for every $g \in \mathscr{U}_0$, $\mathrm{M}(g, \mathrm{W})$ is contained in W_0 and has a dominated splitting $\mathrm{TM}/\mathrm{M}(g, \mathrm{W}) = \widetilde{\mathrm{E}}_g^s \oplus \widetilde{\mathrm{E}}_g^u$ such that the number

$$\delta(g) = \sup\{ d(\widetilde{\mathrm{E}}_g^s(x), \hat{\mathrm{E}}^s(x)) \mid x \in \mathrm{M}(g, \mathrm{W})\}$$

converges to zero when $g \to f$. Take a continuous function $\psi : \mathrm{M} \to \mathbf{R}$ such that $\psi(x) = \log \|(\mathrm{D}f^m)/\hat{\mathrm{E}}^s(x)\|$ for $x \in \mathrm{W}_0$. Then (3) can be written as

$$\int \psi \, d\mu \geqslant \log \lambda_0.$$

Take λ_1 and $\varepsilon > 0$ such that

$$\lambda_1 e^{\varepsilon} < \lambda_0$$

and $$\lambda < \lambda_1 e^{-\varepsilon}$$

and suppose that the neighborhood V of μ is so small that

$$\int \psi \, d\nu \geqslant \log \lambda_1$$

for all $\nu \in V$. Using Theorem III.1 we can take g, arbitrarily C^1 near to f, and a periodic point x of g^m such that the probability $\mu_0 \in \mathcal{M}_p(g^m)$ given by

$$\mu_0 = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{g^{mj}(x)},$$

where n is the period of x, is in V. Then

$$\int \psi \, d\mu_0 \geqslant \log \lambda_1. \tag{6}$$

Suppose that g is so near to f that for all y in the g^m-orbit of x

$$\log \|(Dg^m)/\widetilde{E}^s(y)\| \geqslant \log \|(Df^m)/\hat{E}^s(y)\| - \varepsilon. \tag{7}$$

From (6) we obtain

$$\log \lambda_1 \leqslant \int \psi \, d\mu_0 = \frac{1}{n} \sum_{j=0}^{n-1} \psi(g^{mj}(x)) = \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^m)/\hat{E}^s(g^{mj}(x))\|.$$

Using (7):

$$\frac{1}{n} \sum_{j=0}^{n-1} \log \|(Dg^m)/\widetilde{E}^s_g(g^{mj}(x))\| \geqslant \frac{1}{n} \sum_{j=0}^{n-1} \log \|(Df^m)/\hat{E}^s(g^{mj}(x))\| - \varepsilon$$
$$\geqslant \log(\lambda_1 e^{-\varepsilon}).$$

Then

$$\prod_{j=0}^{n-1} \|(Dg^m)/\widetilde{E}^s_g(g^{mj}(x))\| \geqslant \lambda_1^n e^{-n\varepsilon}. \tag{8}$$

Property a) of I.3 says that for all j

$$\|(Dg^m)/\widetilde{E}^s_g(g^{mj}(x))\| . \|(Dg^{-m})/\widetilde{E}^u_g(g^{m(j+1)}(x))\| < \lambda.$$

Together with (8) this yields

$$\prod_{j=1}^{n} \|(Dg^{-m})/\widetilde{E}^u_g(g^{mj}(x))\| \leqslant (\lambda\lambda_1^{-1} e^{\varepsilon})^n < 1.$$

Denoting as usual by $E^u(x)$ the unstable subspace of the periodic point x, this means that

$$E^u(x) \supset \widetilde{E}^u_g(x) \tag{9}$$

because $(Dg^{mn})\, \widetilde{E}^u_g(x) = \widetilde{E}^u_g(x)$ and

$$\|(Dg^{-mn})/\widetilde{E}^u_g(x)\| \leqslant \prod_{j=1}^{n} \|(Dg^{-m})/\widetilde{E}^u_g(g^{mj}(x))\| < 1.$$

Then, by (9), $x \in P_k(g)$ for some $k \leqslant i$. Suppose that $k = i$. Observe that the hyperbolicity of the periodic point x easily implies that any subspace $E \subset T_x M$ satisfying $(Dg^{mn}) E = E$ and $E \cap E^u(x) = \{0\}$ must be contained in $E^s(x)$. When $k = i$, I.3 a) implies $E^u(x) = \widetilde{E}^u_g(x)$. Hence $\widetilde{E}^s_g(x) \cap E^u(x) = \{0\}$. Then $\widetilde{E}^s_g(x) \subset E^s(x)$. But $\widetilde{E}^s_g(x)$ and $E^s(x)$ have the same dimension, namely $\dim M - \dim E^u(x)$. Therefore $\widetilde{E}^s(x) = E^s(x)$. By c) of I.3

$$\prod_{j=0}^{n-1} \|(Dg^m)/\widetilde{E}^s(g^{mj}(x))\| \leqslant C\lambda^n.$$

This and (8) imply

$$\lambda_1^n e^{-\varepsilon n} \leqslant C\lambda^n. \tag{10}$$

But since $\lambda < \lambda_1 \exp(-\varepsilon)$, this inequality is impossible if n is very large. On the other hand, the fact that μ is not supported by a periodic orbit implies that the period of the periodic orbit that supports its aproximation μ_0 (i.e. the period n of x) is arbitrarily large if μ_0 is taken sufficiently near to μ. Hence, taking μ_0 sufficiently near to μ, (10) becomes impossible, thus showing that we cannot have $k = i$. Then $x \in P_k(g)$ for some $k < i$. But by Lemma III.2, if g is sufficiently near to f, we have

$$P_k(g) \subset U,$$

and then, by (5),

$$\mu_0(P_k(g)) \leqslant \mu_0(U) < 1/2.$$

But $x \in P_k(g)$ implies

$$\mu_0(P_k(g)) = 1.$$

This contradiction completes the proof of (4), when μ is ergodic and not supported in a periodic orbit of f^m. Observe that a μ satisfying (3) cannot be supported by a periodic point of f^m because if this were true, i.e. if μ had the form

$$\mu = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^{mj}(y)}$$

and $f^{mn}(y) = y$, then (3) would mean

$$\log \lambda < \log \lambda_0 \leqslant \int \log \|(Df^m)/\widetilde{E}^s_i\| \, d\mu = \lim_{t \to +\infty} \frac{1}{t} \sum_{j=0}^{t-1} \log \|(Df^m)/\widetilde{E}^s_i(f^{mj}(y))\|,$$

thus contradicting part d) of I.3. Finally suppose that some $\mu \in \mathcal{M}(f^m/\overline{P}_i(f))$ satisfies (3). By the Ergodic Decomposition Theorem we can write

$$\int \Big(\int \log \|(Df^m)/\widetilde{E}^s_i\| \, d\mu_x \Big) \, d\mu(x) = \int \log \|(Df^m)/\widetilde{E}^s_i\| \, d\mu \geqslant \log \lambda_0.$$

This means that the set S of points x satisfying

$$\int \log \|(Df^m)/\widetilde{E}^s_i\| \, d\mu_x > \log \lambda \tag{11}$$

has positive measure with respect to μ. Moreover

$$(12) \qquad \mu(\bigcup_{0\leq k<i} \overline{P}_k(f)) = \int \mu_x(\bigcup_{0\leq k<i} \overline{P}_k(f))\, d\mu(x) \geq \int_S \mu_x(\bigcup_{0\leq k<i} \overline{P}_k(f))\, d\mu(x).$$

But when $x \in S$, (11) holds, thus implying

$$\mu_x(\bigcup_{0\leq k<i} \overline{P}_k(f)) > 0,$$

because μ_x is ergodic. Then, since this union is invariant under f^m,

$$\mu_x(\bigcup_{0\leq k<i} \overline{P}_k(f)) = 1.$$

Then (12) implies

$$\mu(\bigcup_{0\leq k<i} \overline{P}_k(f)) \geq \mu(S) > 0.$$

IV. — Contracting sequences and attainability

In this section we shall develop a perturbation technique that will be used in the next section to prove Theorem I.7. But before entering into the somehow formally involved array of definitions and statements that form this method, we shall first expose the underlying ideas through the discussion of a simplified but closely related problem.

A loose description of the aim of Theorem I.7 is the creation of linkings between transitive hyperbolic sets that are bound together by orbits that accumulate in all of them. The problem is, using this loose linking, to create a real linking, meaning by this an intersection between a stable and an unstable manifold of these sets. A simplified version of this type of objective is the following old and still open question: Suppose that $f \in \mathrm{Diff}^k(M)$ has a hyperbolic point p such that there exists $q \in W^s(p) - \{p\}$ whose α-limit set satisfies $\alpha(q) \cap (W^u(p) - \{p\}) \neq \emptyset$; then, is it possible to find a diffeomorphism g, C^k near to f such that it coincides with f in a neighborhood of p and satisfies $q \in W^s_g(p) \cap W^u_g(p)$? Even without the requirement that g coincide with f nearby p, this question admits an obvious formulation, as open and difficult as the above one but for simplicity we shall discuss this question as we stated it.

There are at least two possible approaches to this problem. The first is the local method that consists in taking $n \geq 0$ such that the point $f^{-n}(q)$ is very close to a point $z \in W^u(p) - \{p\}$ and trying to find a diffeomorphism ψ, C^k near to the identity, such that $\psi(f^{-n}(q)) = z$ and is the identity outside a ball $B_r(z)$ that does not contain p. Then if $B_r(z)$ does not contain points of the form $f^j(q)$ for $0 \geq j \geq -(n-1)$, the sequence $\{f^j(q) \mid 0 \geq j \geq -(n-1)\} \cup \{f^j(z) \mid j \geq 0\}$ is an orbit of the diffeomorphism $g = (\psi f^{-1})^{-1}$, obviously contained in $W^s_g(p) \cap W^u_g(p)$. But if we take r so small that $f^j(q) \notin B_r(z)$ for $0 \geq j \geq -(n-1)$, and on the other hand such that $f^{-n}(q) \in B_r(z)$, then the C^k distance between g and f becomes a function of r and $d(f^{-n}(q), z)$. More specifically, ψ can be taken C^k near to the identity if $d(f^{-n}(q), z)/r^k$ is small, a condition that requires a special choice of n which cannot be always satisfied. To make the local method work we can add supplementary hypotheses, as in the results in [13] that will be stated again in the next section before using them (together with the results of this section) to prove I.7.

The other approach is more global. It consists in taking a small neighborhood U of q and a diffeomorphism ψ that is the identity outside U and then defining the diffeomorphism $g = f\psi$ with the hope of finding ψ near to the identity and also satisfying the relation $g^{-n}(q) \in W^u_g(q)$ for some $n > 0$. Stated in this form it may seem outrageously naive, but the intention is to exploit the dynamics of f in such a way that the small perturbation introduced by ψ will be amplified under iteration in such a way that the orbit of q under f (that accumulated in $W^u(p)$) will move toward $W^u(p)$ and hit it. Again, to accomplish this project, we shall, in our situation, have supplementary hypotheses that grant certain expanding behaviour of f^{-1} through which the amplification of small perturbations will be obtained. In fact *the proof of Theorem I.7 exploits an alternative: either the local method works or there are enough expanding dynamics in f to make the global method work.* What we shall do now is to prepare the techniques of the second part of the alternative.

Let Λ be a compact invariant set of $f \in \mathrm{Diff}^1(M)$ having a dominated splitting $TM/\Lambda = E \oplus F$. Given $m > 0$ and $0 < \gamma < 1$ we say that a pair of points $(x, f^{-mn}(x))$ in Λ, $n > 0$, is an *(m, γ)-string,* if

$$\prod_{j=1}^{n} \|(Df^m)/E(f^{-mj}(x))\| \leqslant \gamma^n,$$

and we say that $(x, f^{-mn}(x))$ is a *uniform (m, γ)-string* when $(f^{-mj}(x), f^{-mn}(x))$ is an (m, γ)-string for all $0 \leqslant j < n$. These two definitions are just repetitions of those introduced at the beginning of section II, applied now to $g = f^{-m}$ and the dominated splitting $TM/\Lambda = F \oplus E$.

A pair (S, ν), where $S = \{x_1, x_2, \dots\} \subset \Lambda$ is a sequence in Λ and $\nu : S \to \mathbf{Z}^+$ is a function satisfying $\lim_{n \to +\infty} \nu(x_n) = +\infty$, is an *$(m, \gamma)$-contracting sequence* if there exists $\hat{n}$ such that $(x, f^{-mj}(x))$ is an (m, γ)-string for all $\hat{n} < j \leqslant \nu(x)$ and $x \in S$. Moreover we say that (S, ν) is a *strongly (m, γ)-contracting sequence* if $(x, f^{-m\nu(x)}(x))$ is a uniform (m, γ)-string for all $x \in S$. The sequence

$$\hat{S} = \{f^{-m\nu(x_n)}(x_n) \mid n \geqslant 1\}$$

will be called the sequence of *endpoints* of (S, ν). If (S', ν') and (S'', ν'') are (m, γ)-contracting sequences, write $(S', \nu') \geqslant (S'', \nu'')$ if S'' is a subsequence of S' and $\nu'' \leqslant \nu'/S''$.

Theorem **IV.1.** — *If (S, ν) is an (m, γ)-contracting sequence then, given $\gamma < \gamma_1 < 1$, one of the following properties holds:*

a) *There exists a strongly (m, γ_1)-contracting sequence $(S', \nu') \leqslant (S, \nu)$ whose sequence of endpoints converges to a point y such that for every $0 < \gamma_2 < \gamma_1$ there exists $N > 0$ such that*

$$\prod_{j=1}^{n} \|(Df^m)/E(f^{-mj}(y))\| > \gamma_2^n$$

for all $n \geqslant N$.

b) *There exists a strongly* (m, γ_1)*-contracting sequence* $(S', \nu') \leqslant (S, \nu)$ *such that* $S - S'$ *is finite and* $\nu - \nu'$ *is bounded.*

Proof. — If $x \in S$, denote by $\mathscr{S}(x)$ the set of integers n in $(0, \nu(x)]$ such that $(x, f^{-mn}(x))$ is a uniform (m, γ_1)-string. By Lemma II.3 there exist $N_0 > 0$ and $c > 0$ such that

(1) $$\# \mathscr{S}(x) \geqslant c\nu(x)$$

when $\nu(x) \geqslant N_0$. Recalling that, by definition, if $S = \{x_1, x_2, \ldots\}$, then

$$\lim_{n \to +\infty} \nu(x_n) = +\infty,$$

it follows that (1) holds for all x_n with large n, say $n \geqslant n_0$. Define $S' = \{x_{n_0}, x_{n_0+1}, \ldots\}$ and, if $x \in S'$, let $\nu'(x)$ be the largest integer in $\mathscr{S}(x)$. Clearly (S', ν') is a strongly (m, γ_1)-contracting sequence and $S - S'$ is finite. Therefore, if $\nu - \nu'$ is bounded, (S', ν') has property b). Let us suppose $\nu - \nu'$ is unbounded and construct (S'', ν'') satisfying a). Let S'' be a subsequence of S' such that, setting $S'' = \{y_1, y_2, \ldots\}$, we have

$$\lim_{n \to +\infty} (\nu(y_n) - \nu'(y_n)) = +\infty$$

and the sequence of endpoints satisfies

$$\lim_{n \to +\infty} f^{-m\nu'(y_n)}(y_n) = y.$$

Let us prove that y satisfies the inequality required by property a), thus completing the proof of IV.1, since then the strongly (m, γ_1)-contracting sequence (S'', ν''), with $\nu'' = \nu'/S''$, has property a). Given $0 < \gamma_2 < \gamma_1$, take $\gamma_2 < \bar{\gamma}_2 < \gamma_1$. By Lemma II.3 there exists $N > 0$ such that if $(y, f^{-mn}(y))$ is a (m, γ_2)-string and $n > N$, there exists $0 < n_1 \leqslant n$ such that $(y, f^{-mn_1}(y))$ is a uniform $(m, \bar{\gamma}_2)$-string. Suppose then that for some $n > N$ the inequality in property a) does not hold. Then $(y, f^{-mn}(y))$ is a (m, γ_2)-string and since $n > N$ we have $0 < n_1 \leqslant n$ such that $(y, f^{-mn_1}(y))$ is a uniform $(m, \bar{\gamma}_2)$-string. Then there exists a neighborhood U of y such that $(z, f^{-mn_1}(z))$ is a uniform (m, γ_1)-string for every $z \in U$. Take j so large that

(2) $$f^{-m\nu'(y_j)}(y_j) \in U$$

and $\nu(y_j) = \nu'(y_j) > n_1$. Then $(y_j, f^{-m(\nu'(y_j)+n_1)}(y_j))$ is a uniform (m, γ_1)-string, because so are $(y_j, f^{-m\nu'(y_j)}(y_j))$ and (by (2)) $(f^{-m\nu'(y_j)}(y_j), f^{-m(\nu'(y_j)+n_1)}(y_j))$. Since $\nu'(y_j) + n_1 < \nu(y_j)$, it then follows that $\nu'(y_j) + n_1 \in \mathscr{S}(y_j)$, contradicting the definition of ν' and concluding the proof.

The important property of strongly contracting sequences, that in the next section we shall exploit to prove Theorem I.7, is given by the following definition and theorem.

Given a sequence $S = \{x_1, x_2, \ldots\}$ converging to a point x_0 and a set $\Sigma \subset M$, we say that Σ is *attainable* from S if given $\delta > 0$, a neighborhood U of x_0 and a C^1 neighborhood $\mathscr{U}$ of f, there exist $g \in \mathscr{U}$ and integers $k > 0$ and $\ell > 0$ such that

a) $x_k \in U$ and $g^{-\ell}(x_k) \in \Sigma$,
b) $g^{-1}(x) = f^{-1}(x)$ for all $x \notin U$, and
c) $d(f^{-n}(x_k), g^{-n}(x_k)) < \delta$ for all $0 \leqslant n \leqslant \ell$.

Recall that by [6], the domination property of the splitting $TM/\Lambda = E \oplus F$ implies that there exists a family of embedded C^1 disks $D(y)$, $y \in \Lambda$, such that:

1) $y \in D(y)$ and $T_y D(y) = F(y)$;
2) $f(D(y))$ contains a neighborhood of $f(y)$ in the disk $D(f(y))$;
3) $D(y)$ depends continuously on y.

Define $D_r(y)$ as the set of points in $D(y)$ whose distance in $D(y)$ to y is $\leq r$.

Theorem **IV.2.** — *Given $r > 0$, $m \in \mathbf{Z}^+$ and $0 < \gamma < 1$, there exists $\varepsilon = \varepsilon(r, m, \gamma)$ such that if (S, ν) is a strongly (m, γ)-contracting sequence and S converges to a non periodic point x_0, then, if $y \in \Lambda$ is ε-near to an accumulation point of the sequence of endpoints of (S, ν), $D_r(y)$ is attainable from S.*

Proof. — Take a neighborhood U_0 of $f^{-1}(x_0)$ and a C^∞ function $\psi : M \to [0, 1]$ satisfying $\psi(x) = 0$ if $x \notin U_0$ and $\psi(f^{-1}(x_0)) = 1$. For each $v \in T_{f^{-1}(x_0)} M$ define a C^∞ vector field on M setting $\xi_v(y) = 0$ if $y \notin U_0$ and

$$\xi_v(x) = \psi(x)\, \tau(f^{-1}(x_0), x)\, v$$

when $x \in U_0$, where $\tau(f^{-1}(x_0), x)$ is the linear map from $T_{f^{-1}(x_0)} M$ onto $T_x M$ given by the parallel translation along the minimizing geodesic that joins $f^{-1}(x_0)$ to x. Since this geodesic is unique when $f^{-1}(x_0)$ and x are sufficiently close, it follows that ξ_v is well defined U_0 sufficiently small. For v small, say $\|v\| < R$, define a diffeomorphism $f_v : M \circlearrowleft$ by

$$f_v^{-1}(x) = \exp_{f^{-1}(x)} \xi_v(x).$$

Observe that if $\|v\|$ is small, the map $x \mapsto \exp_{f^{-1}(x)} \xi_v(x)$ is a map C^1 close to f^{-1}. Hence it is a diffeomorphism and then f_v, that is the inverse of the map $x \mapsto \exp_{f^{-1}(x)} \xi_v(x)$, is well defined.

The idea of the proof consists in taking $\varepsilon_0 > 0$ and studying the sets $\Sigma(k)$ given for each $x_k \in S$ by

$$\Sigma(k) = \{ f_v^{-\nu(x_k)m}(x_k) \mid \|v\| < \varepsilon_0 \}.$$

We shall prove that there exists a constant $c > 0$ such that for k sufficiently large $\Sigma(k)$ contains a disk $D(k)$ tangent to $E(f^{-\nu(x_k)m}(x_k))$ at $f^{-\nu(x_k)m}(x_k)$ which (treating M as a Euclidean space) can be written as the graph of a C^1 map

$$\varphi_k : \{ w \in E(f^{-\nu(x_k)m}(x_k)) \mid \|w\| < c \} \to F(f^{-\nu(x_k)m}(x_k)).$$

This means that

$$D(k) = \{ w + \varphi_k(w) \mid w \in E(f^{-\nu(x_k)m}(x_k)), \|w\| < c \}.$$

Moreover the maps φ_k satisfy $\|(\varphi_k)'(w)\| \leq c$ for all k sufficiently large and every w in the domain of φ_k. These properties, plus a standard application of the proof of the implicit function theorem, imply that given $r > 0$, if y is near to a point $f^{-\nu(x_k)m}(x_k)$ with k sufficiently large, then $D_r(y) \cap D(k) \neq \emptyset$. This means that

$$f_v^{-\nu(x_k)m}(x_k) \in D_r(y).$$

Since ε_0 can be taken arbitrarily small and f_v^{-1} and f^{-1} coincide outside U_0, this proves two properties (a) and b)) required by the attainability of $D_r(y)$. The last property (property c)) will require more careful estimates.

To formalize this method we begin by taking a neighborhood W_0 of Λ such that there exists a continuous splitting $TM/W_0 = \hat{E} \oplus \hat{F}$ extending the splitting $TM/\Lambda = E \oplus F$. For $x \in W_0$ let $\pi_1 : T_x M \to \hat{E}(x)$ and $\pi_2 : T_x M \to \hat{F}(x)$ be the projections associated to this splitting and let $S_\varepsilon(x)$ be the cone

$$S_\varepsilon(x) = \{ v \in T_x M \mid \| \pi_2 v \| < \varepsilon \| \pi_1 v \| \}.$$

Set

$$c_0 = \frac{1}{2} \min_{x \in M} \{ \|(Df)/T_x M\|^{-1}, \|(Df^{-1})/T_x M\|^{-1} \}.$$

Using the domination condition we can choose $\gamma < \gamma_1 < 1$, an arbitrarily small $\bar{\varepsilon}_0 > 0$ and $0 < \varepsilon_1 < \varepsilon_0 < \bar{\varepsilon}_0$, $n_0 > 0$, $\ell > 0$, $A > 1$ such that for every x in a certain neighborhood $W_1 \subset W_0$ of Λ the following properties hold:

I) $(Df^{-mn_0}) \overline{S_{\bar{\varepsilon}_0}(x)} \subset S_{\varepsilon_0}(f^{-mn_0}(x))$;

II) $(Df^{-j}) \overline{S_{\varepsilon_0}(x)} \subset S_{\bar{\varepsilon}_0}(f^{-j}(x))$ for all $0 \leqslant j$;

III) $(Df^{-\ell mn_0}) \overline{S_{\bar{\varepsilon}_0}(x)} \subset S_{\varepsilon_1}(f^{-\ell mn_0}(x))$;

IV) if $x \in W_1$, $v \in S_{\varepsilon_1}(x)$ and $w \in T_x M$ satisfy $\| v \| \geqslant A \| W \|$, then $v + w \in S_{\varepsilon_0}(x)$ and $\| v + w \| \geqslant (1/2) \| v \|$;

V) ℓ is so large that

$$\left(\frac{1}{4} c_0^{4n_0 m}\right)^{k+1} \gamma_1^{-t} > A$$

when $tm > \ell k$. Moreover there exists $\beta > 1$ such that

$$\left(\frac{1}{4} c_0^{4n_0 m}\right)^{k} \gamma_1^{-t} > \beta^t$$

when $t > \ell k$.

Furthermore, recalling that x_0 is not periodic, we can choose U_0 so small that

VI) $f^{-j}(U_0) \cap U_0 = \emptyset$ for all $0 \leqslant j \leqslant 2\ell n_0 m$.

The rest of the proof consists in showing, following the method outlined above, that given $r > 0$ there exists $\varepsilon = \varepsilon(r, m, \gamma)$ such that if y is ε-near to an accumulation point of the sequence of endpoints of (S, ν), then for all $\delta > 0$ there exist an arbitrarily small $v \in T_{f^{-1}(x_0)} M$ and $x \in S \cap U_0$ such that $f_v^{-n}(x) \in D_r(y)$ for $n = m\nu(x)$ and $d(f_v^{-j}(x), f^{-j}(x)) < \delta$ for all $0 \leqslant j \leqslant n$. Clearly this suffices to prove the theorem.

Choose $R > 0$ so small that from properties I), II), III) and the definition of c_0 follows that, when $\| v \| < R$, then

I') $(Df_v^{-mn_0}) S_{\bar{\varepsilon}_0}(x) \subset S_{\varepsilon_0}(f_v^{-mn_0}(x))$ for all $x \in W_1$,

II') $(Df_v^{-j}) S_{\varepsilon_0}(x) \subset S_{\bar{\varepsilon}_0}(f_v^{-j}(x))$ for all $j \geqslant 0$ and x such that $f_v^{-i}(x) \in W_1$ for all $0 \leqslant i \leqslant j$, III') $(Df_v^{-\ell m n_0})\, S_{\bar\varepsilon_0}(x) \subset S_{\varepsilon_1}(f_v^{-\ell m n_0}(x))$ for all $x \in W_1$,

IV') $\|(Df_v^{-1})\,(x)\|^{-1} \geqslant c_0$ for all $x \in M$.

Observe that III') and I') imply:

V') $(Df_v^{-n n_0 m})\, S_{\bar\varepsilon_0}(x) \subset S_{\varepsilon_1}(f^{-n n_0 m}(x))$ for all $x \in W_1$ and $n \geqslant \ell$ such that $f_v^{-j}(x) \in W_1$ for all $0 \leqslant j \leqslant n n_0 m$.

We will use the following notation for linear maps: given $T : E \to F$, $\operatorname{Im} T = T(E)$ denotes the image of T in F and $\nu \mid T \mid = \min_{\|w\|=1} \| Tw \|$ denotes the minimum norm of T.

Given $x \in W_1$ and $n > 0$, denote by

$$D_v f_v^{-n}(x) : T_{f^{-1}(x_0)} M \to T_{f_v^{-n}(x)} M$$

the derivative with respect to v of the map $v \mapsto f_v^{-n}(x)$. Assume that $R > 0$ is so small that $\| D_v f_v^{-1}(x)\| \leqslant 2$ for all $x \in M$, $\| v \| < R$ and also that there exists a neighborhood $U_1 \subset f(U_0)$ such that

$$\nu \mid D_v f_v^{-1}(x) \mid \geqslant 1/2, \tag{3}$$

$$\operatorname{Im} D_v f_v^{-1}(x) \subset S_{\varepsilon_0}(f_v^{-1}(x)) \quad \text{and} \quad f_v^{-1}(x) \in U_0 \tag{4}$$

for all $x \in U_1$, $\| v \| < R$. Take $\delta > 0$ and for $x \in S$, define $V(x)$ as the maximal star shaped open set in $T_{f^{-1}(x_0)} M$ such that if $v \in V(x)$ then $d(f_v^{-j}(x), f^{-j}(x)) \leqslant \delta$ for all $0 \leqslant j \leqslant m\nu(x)$. Take δ so small that $d(z, \Lambda) \leqslant \delta$ implies $z \in W_1$. Moreover, given any $0 < c < 1$, we can take δ, $\bar\varepsilon_0$ and R so small that

VI') $\|(Df_v^{-k})\, w \| \geqslant c\, \|(Df^k)/E(f^{-k}(x))\|^{-1} \| w \|$ whenever $0 \leqslant k \leqslant m n_0$, $\| v \| < R$, $x \in \Lambda$, $w \in S_{\bar\varepsilon_0}(y)$, $d(y, x) \leqslant \delta$.

Take $0 < c < 1$ such that

$$c^{-1}\gamma < \gamma_1. \tag{5}$$

Lemma **IV.3.** — *Suppose that* $x \in S \cap U_1$, $v \in V(x)$ *and let* $0 < n_1 < n_2 < \ldots$ *be the sequence of integers such that* $f_v^{-n_k}(x) \in U_0$. *Then the following properties hold*:

a) $\operatorname{Im} D_v f_v^{-n}(x) \subset S_{\bar\varepsilon_0}(f_v^{-n}(x))$ *for all* $0 \leqslant n \leqslant m\nu(x)$.

b) *For all* $n_k \leqslant n < n_{k+1}$

$$\nu \mid D_v f_v^{-n}(x) \mid \geqslant \left(\frac{1}{4}\, c_0^{4 n_0 m}\right)^k \prod_{j=1}^{[n/n_0 m]} c\, \|(Df^{n_0 m})/E(f^{-n_0 m j}(x))\|^{-1}.$$

c) *For all* $k \geqslant 1$,

$$\operatorname{Im}(Df^{-1})\,(D_v f_v^{-n_k+1}(x)) \subset S_{\varepsilon_1}(f_v^{-n_k}(x)).$$

d) *For all* $k \geqslant 1$,

$$\operatorname{Im} D_v f_v^{-n_k} \subset S_{\varepsilon_0}(f^{-n_k}(x)).$$

Proof. — We shall prove it by induction. Since $x \in U_1$, then properties (3) and (4) imply that the lemma holds for n_1. Let us suppose that it holds for every integer $n \leqslant n_k$ and let us prove that it holds for every integer $n \leqslant n_{k+1}$. First we shall show that a) holds for every $n < n_{k+1}$. That it holds also for $n = n_{k+1}$ will follow from d), which we shall have to prove for n_{k+1}. Observe that if $n_k < n < n_{k+1}$ then

$$D_v f_v^{-n}(x) = (Df^{-(n-n_k)}) (D_v f_v^{-n_k}) (x) \tag{6}$$

because $f_v^{-j}(x) \notin U_0$ for $n_k < j \leqslant n$. Then (6), (V') and the induction hypothesis imply:

$$\operatorname{Im} D_v f_v^{-n}(x) \subset (Df^{-(n-n_k)}) S_{\varepsilon_0}(f_v^{-n_k}(x)) \subset S_{\bar{\varepsilon}_0}(f_v^{-n}(x)),$$

thus completing the proof of a) for $n_k < n < n_{k+1}$. By the induction hypothesis it holds for $n \leqslant n_k$. Hence it holds for $n < n_{k+1}$, as we wished to prove. Now let us prove b) for $n_k < n < n_{k+1}$ (for $n \leqslant n_k$ it follows from the induction hypothesis). Immediatly afterwards we shall prove c) and d) for $k+1$ and b) for $n = n_{k+1}$. Given $n_k < n < n_{k+1}$, write it as $n = [n/n_0\, m]\, n_0\, m + r$ with $0 \leqslant r < n_0\, m$, and write $p_1 = [n_k/n_0\, m]$, $p_2 = [n_k/n_0\, m] + 1$. Then, by the induction hypothesis applied to property c) we have

$$\begin{aligned}\operatorname{Im}(Df^{-(n_k - p_1 n_0 m)}) (D_v f_v^{-p_1 n_0 m}(x)) \\ = \operatorname{Im}(Df^{-1}) (D_v f_v^{-(n_k-1)}(x)) \subset S_{\varepsilon_1}(f^{n_k}(x)).\end{aligned} \tag{7}$$

Applying b) and (5) we get

$$\begin{aligned}&\nu\,|\,(Df^{-(n_k - p_1 n_0 m)}) (D_v f_v^{-p_1 n_0 m}(x))| \\ &\geqslant c_0^{(n_k - p_1 n_0 m)} \left(\frac{1}{4} c_0^{4 n_0 m}\right)^k \prod_{j=1}^{p_1} c\, \|(Df^{n_0 m})/E(f^{-n_0 m j}(x))\|^{-1} \\ &\geqslant \left(\frac{1}{4} c_0^{4 n_0 m}\right)^{k+1} (c^{-1}\gamma)^{-p_1 p_0} \geqslant \left(\frac{1}{4} c_0^{4 n_0 m}\right)^{k+1} \gamma_1^{-p_1 n_0}.\end{aligned}$$

Since $p_1\, n_0\, m > \ell k$ we can apply (V) to obtain

$$\nu\,|(Df^{-(n_k - p_1 n_0 m)}) (D_v f_v^{-p_1 n_0 m}(x))| \geqslant A \tag{8}$$

From IV), (3), (4), (7) and (8) it follows that

$$\begin{aligned}\nu\,|\,D_v f_v^{-n_k}(x)| &= \nu\,|(Df^{-(n_k\, p_1 n_0 m)}) (D_v f_v^{-p_1 n_0 m}(x)) \\ &+ D_v f_v^{-1}(f^{-n_k+1}(x))| \geqslant \frac{1}{2} \nu\,|(Df^{-(n_k - p_1 n_0 m)}) (D_v f_v^{-p_1 n_0 m}(x)|.\end{aligned}$$

Then

$$\nu\,|\,D_v f_v^{-n_k}(x)| \geqslant \frac{1}{2} c_0^{n_k - p_1 n_0 m} \nu\,|\,D_v f_v^{-p_1 n_0 m}(x)| \geqslant \frac{1}{2} c_0^{n_0 m} \nu\,|\,D_v f_v^{-p_1 n_0 m}(x)|.$$

Hence

$$\begin{aligned}\nu\,|\,D_v f_v^{-p_2 n_0 m}(x)| &= \nu\,|(Df^{-(p_2 n_0 m - n_k)}) (D_v f_v^{-n_k}(x))| \\ &\geqslant c_0^{n_0 m} \nu\,|\,D_v f_v^{-n_k}(x)| \geqslant \frac{1}{2} c_0^{2 n_0 m} \nu\,|\,D_v f_v^{-p_1 n_0 m}(x)|.\end{aligned} \tag{9}$$

Moreover, by V'), for all $n_k < j \leqslant n$

$$\begin{aligned}\operatorname{Im} D_v f_v^{-j}(x) &= \operatorname{Im}(Df^{-(j-n_k)}(D_v f_v^{-n_k}(x)) \\ &\subset (Df^{-(j-n_k)}) S_{\varepsilon_0}(f_v^{-n_k}(x)) \subset S_{\varepsilon_0}(f_v^{-j}(x)).\end{aligned}$$

Hence, by VI') we obtain that, for all $n_k < j < j + mn_0 \leqslant n$,

$$\begin{aligned} \nu \mid D_v f_v^{-(j+mn_0)}(x) \mid &= \nu \mid (Df^{-mn_0})\,(D_v f_v^{-j})\,(x) \mid \\ &\geqslant c \,\|(Df^{mn_0})/E(f^{-(mn_0+j)}(x))\|^{-1}\, \nu \mid D_v f_v^{-j}(x) \mid. \end{aligned}$$

Then

(10) $$\begin{aligned} &\nu \mid D_v f_v^{-[n/mn_0]\,mn_0}(x) \mid \\ &\geqslant \prod_{j=1}^{[n/mn_0]} c \,\|(Df^{mn_0})/E(f^{-(mn_0(j+p_2))}(x)\|^{-1} . \nu \mid D_v f_v^{-p_2 n_0 m}(x) \mid. \end{aligned}$$

Using (9) we get

$$\begin{aligned} \nu \mid D_v f_v^{-p_2 n_0 m}(x) \mid &\geqslant \frac{\|(Df^{mn_0})/E(f^{-p_2 mn_0}(x))\|^{-1}}{\|(Df^{mn_0})/E(f^{-p_2 mn_0}(x))\|^{-1}} \cdot \frac{1}{2} c_0^{2n_0 m}\, \nu \mid D_v f_v^{-p_1 n_0 m}(x) \mid \\ &\geqslant \frac{1}{2} c_0^{3n_0 m} \,\|(Df^{mn_0})/E(f^{-p_2 mn_0}(x))\|^{-1}\, \nu \mid D_v f_v^{-p_1 n_0 m}(x) \mid. \end{aligned}$$

Combining this with (10) we get

$$\begin{aligned} &\nu \mid D_v f_v^{-[n/mn_0]\,mn_0}(x) \mid \\ &\geqslant \frac{1}{2} c_0^{3n_0 m} \prod_{j=0}^{[n/mn_0]} c \,\|(Df^{mn_0})/E(f^{-(mn_0(j+p_2))}(x)\| . \nu \mid D_v f_v^{-p_1 n_0 m}(x) \mid. \end{aligned}$$

Using the induction hypothesis

(11) $$\begin{aligned} \nu \mid D_v f_v^{-[n/mn_0]\,mn_0}(x) \mid &\geqslant \frac{1}{2} c_0^{3n_0 m} \prod_{j=0}^{[n/mn_0]} c \,\|(Df^{mn_0})/E(f^{-(mn_0(j+p_2))}(x))\|^{-1} \\ &\quad \cdot \left(\frac{1}{4} c_0^{4n_0 m}\right)^k \prod_{j=1}^{P_1} c \,\|(Df^{n_0 m})/E(f^{-n_0 mj}(x))\|^{-1} \\ &\geqslant \frac{1}{2} c_0^{3n_0 m} \left(\frac{1}{4} c_0^{4n_0 m}\right)^k \prod_{j=1}^{[n/mn_0]} c \,\|(Df^{mn_0})/E(f^{-mn_0 j}(x))\|^{-1}. \end{aligned}$$

Then

(12) $$\begin{aligned} \nu \mid D_v f_v^{-n}(x) \mid &= \nu \mid (Df^{-r})\, D_v f_v^{-[n/n_0 m]\,n_0 m}(x) \mid \geqslant c_0^r\, \nu \mid D_v f_v^{-[n/n_0 m] n_0 m}(x) \mid \\ &\geqslant c_0^{n_0 m} \frac{1}{2} c_0^{3n_0 m} \left(\frac{1}{4} c_0^{4n_0 m}\right)^k \prod_{j=1}^{[n/mn_0]} c \,\|(Df^{mn_0})/E(f^{-mn_0 j}(x)\|^{-1}. \end{aligned}$$

This completes the proof of b) for $n < n_{k+1}$. Now observe that

(13) $$\begin{aligned} \mathrm{Im}(Df^{-1})\,(D_v f^{-n_{k+1}+1}(x)) &= \mathrm{Im}(Df^{-(n_{k+1}-n_k)})\,(D_v f_v^{-n_k}(x)) \\ &\subset (Df^{-(n_{k+1}-n_k)})\, S_{\varepsilon_0}(f_v^{-n_k}(x)) \subset S_{\varepsilon_1}(f_v^{-n_{k+1}}(x)) \end{aligned}$$

where the last inequality follows from VI) and V'). This proves c). Moreover

$$D_v f_v^{-n_{k+1}}(x) = (Df^{-1})\,(D_v f_v^{-n_{k+1}+1}(x)) + D_v f_v^{-n_{k+1}+1}(x).$$

Hence, as before, we can show that the minimum norm of the first term on the right is large, because $D_v f_v^{-n_{k+1}+1}(x)$ can be estimated using b), which we have proved to hold for $n < n_{k+1}$, and Df^{-1} contracts norms at most by a factor c_0. Then the first term on the right has a much larger minimum norm than the second term, and its image is contained in $S_{\varepsilon_1}(f_v^{-n_{k+1}}(x))$ by (13). Therefore, by IV) we have

$$\operatorname{Im} D_v f_v^{-n_{k+1}}(x) \subset S_{\varepsilon_0}(f_v^{-n_{k+1}}(x)),$$

thus proving d), and

$$\nu\,|\,D_v f_v^{-n_{k+1}}(x)| \geqslant \frac{1}{2}\,\nu\,|(Df^{-1})\,(D_v f_v^{-n_{k+1}+1}(x))|.$$

Write $n_{k+1} = nn_0 m + r$ with $0 < r \leqslant n_0 m$. Suppose $r < mn_0$. Then

$$\nu\,|\,D_v f_v^{-n_{k+1}}(x)| \geqslant \frac{1}{2}\,\nu\,|(Df^{-1})\,(D_v f_v^{-n_{k+1}+1}(x))|$$
$$= \frac{1}{2}\,\nu\,|(Df^{-r})\,(D_v f_v^{-nn_0 m}(x))| \geqslant \frac{1}{2}\,c_0^r\,\nu\,|\,D_v f_v^{-nn_0 m}(x)|.$$

Applying (11):

$$\nu\,|\,D_v f_v^{-n_{k+1}}(x)| \geqslant \frac{1}{2}\,c_0^r.\frac{1}{2}\,c_0^{3n_0 m}\left(\frac{1}{4}\,c_0^{4n_0 m}\right)^k.\prod_{j=1}^{n} c\,\|(Df^{mn_0})/E(f^{-mn_0 j}(x))\|^{-1}$$
$$\geqslant \left(\frac{1}{4}\,c_0^{4n_0 m}\right)^{k+1}\prod_{j=1}^{n} c\,\|(Df^{mn_0})/E(f^{-mn_0 j}(x))\|^{-1}.$$

From $r < mn_0$ it follows that $n = [n_{k+1}/mn_0]$ and this completes the proof of b) for $n = n_{k+1}$. When $r = mn_0$ we write

$$\nu\,|\,D_v f_v^{-n_{k+1}}(x)| \geqslant \frac{1}{2}\,\nu\,|(Df^{-1})\,(D_v f_v^{-n_{k+1}+1}(x))|$$
$$= \frac{1}{2}\,\nu\,|(Df^{-mn_0})\,(D_v f_v^{-nmn_0}(x))|.$$

But
$$\operatorname{Im} D_v f_v^{-nmn_0}(x) = \operatorname{Im}(Df^{-(nmn_0-n_k)})\,(D_v f_v^{-n_k}(x))$$
$$\subset (Df^{-(nmn_0-n_k)})\,S_{\varepsilon_0}(f_v^{-n_k}(x)).$$

Hence, by II′)

$$\operatorname{Im} D_v f_v^{-nmn_0}(x) \subset S_{\bar{\varepsilon}_0}(f_v^{-nmn_0}(x)).$$

Then, by VI′)

$$\nu\,|\,D_v f_v^{-n_{k+1}}(x)| \geqslant \frac{1}{2}\,c\,\|(Df^{mn_0})/E(f^{-n_{k+1}}(x))\|^{-1}.\nu\,|\,D_v f_v^{-mn_0 n}(x)|.$$

Using (12)

$$\nu\,|\,D_v f_v^{-n_{k+1}}(x)| \geqslant \frac{1}{2}\,c\,\|(Df^{mn_0})/E(f^{-n_{k+1}}(x))\|^{-1}$$
$$.\frac{1}{2}\,c_0^{4n_0 m}\left(\frac{1}{4}\,c_0^{4n_0 m}\right)^k\prod_{j=1}^{n} c\,\|(Df^{mn_0})/E(f^{-jmn_0}(x))^{-1}$$
$$= \left(\frac{1}{4}\,c_0^{4n_0 m}\right)^{k+1}\prod_{j=1}^{n+1} c\,\|(Df^{mn_0})/E(f^{-jmn_0}(x))\|^{-1}.$$

Since $n+1=[n_{k+1}/n_0 m]$, this concludes the proof of Lemma IV.3.

If $x \in S \cap U_1$ define, for all $0 \leqslant n \leqslant m\nu(x)$, $\Sigma_x(n)$ as the set

$$\Sigma_x(n) = \{f_v^{-n}(x) \mid v \in V(x)\}.$$

From now on we shall treat M as if it were a Euclidean space. All our arguments will be local and they can be exposed more clearly in that way instead of the formally necessary but cumbersome repeated use of local coordinates.

By IV.3 b), $\nu \mid D_v f_v^{-n}(x) \mid > 0$ for all $x \in U_1 \cap S$ and $0 \leqslant n \leqslant m\nu(x)$. By IV.3 a), $\operatorname{Im} D_v f_v^{-n}(x) \subset S_{\varepsilon_0} f_v^{-n}(x))$. Hence $\Sigma_x(n)$ is the graph of a C^1 map

$$\psi_{n,x} : D(n, x) \to \hat{F}(f^{-n}(x)) \tag{14}$$

where $D(n, x)$ is an open, simply connected subset of $\hat{E}(f_v^{-n}(x))$. Clearly the subspace

$$\{ w + (D\psi_{n,x})(y)\, w \mid w \in \hat{E}(f_v^{-n}(x))\}$$

turns out to be the tangent space of $\Sigma_x(n)$ at the point $f_v^{-n}(x) + y + \psi_{n,x}(y)$. Hence, by IV.3 a), there exists $C > 0$ such that

$$\|(D\psi_{n,x})(y)\| \leqslant C \tag{15}$$

for all $x \in U_1 \cap S$, $0 \leqslant n \leqslant m\nu(x)$, $y \in D(n, x)$.

Lemma **IV.4.** — a) *There exists* $\beta_0 > 1$ *such that*

$$\nu \mid D_v f_v^{-n}(x) \mid \geqslant \beta_0^n$$

for all $x \in S \cap U_1$, $\ell n_0 m < n \leqslant m\nu(x)$.

b) *Given* $\hat{c} > 1$ *there exists* $N(\hat{c})$ *such that*

$$\| D_v f_v^{-n}(x) \| \leqslant \prod_{j=1}^{t} \hat{c}c^{-1} \|(Df^m)/E(f^{-(n+jm)}(x))\| . \| D_v f_v^{-m\nu(x)}(x) \|$$

for all $x \in U_1 \cap S$, $N(\hat{c}) < n < n + tm \leqslant m\nu(x)$.

Proof. — By IV.3 b) and (5), if $n > 0$ we have

$$\nu \mid D_v f_v^{-n}(x) \mid \geqslant \left(\frac{1}{4} c_0^{4n_0 m}\right)^k \gamma_1^{-[n/n_0 m] n_0}$$

where k is chosen by $n_k \leqslant n < n_{k+1}$. Then

$$k \leqslant [n/\ell n_0 m].$$

and, if $n > \ell n_0 m$,

$$\frac{[n/n_0 m]\, n_0}{[n/\ell n_0 m]} \geqslant \ell n_0.$$

Then, by V)

$$\nu \mid D_v f_v^{-n}(x) \mid \geqslant \beta^{[n/n_0 m] n_0}$$

for $n > \ell n_0 m$. From here it is easy to conclude the existence of $\beta_0 > 1$ as required by part a) of the lemma. Now let us prove b) by induction on n starting at $n = m\nu(x)$. Suppose the property proved for $n+m$, $n+2m$, $\ldots$, $n+tm \leqslant m\nu(x)$ and let us prove it for n. Suppose first that $f_v^{-(n+m)}(x) \notin U_0$. Then

$$D_v f_v^{-(n+m)}(x) = (Df^{-m})\,(D_v f_v^{-n}(x)).$$

Using IV.3 a) and VI') with $k = m$ and any $w \in T_{f^{-1}(x_0)} M$,

$$\begin{aligned} \| D_v f_v^{-(n+m)}(x).w \| &= \|(Df^{-m})\,(D_v f_v^{-n}(x).w)\| \\ &\geqslant c\,\|(Df^m)/E(f^{-(n+m)}(x))\|^{-1}\,\| D_v f_v^{-n}(x).w \|. \end{aligned}$$

From this inequality and the induction hypothesis follows

$$\begin{aligned} \| D_v f_v^{-n}(x)\| &\leqslant c^{-1}\,\|(Df^m)/E(f^{-(n+m)}(x))\| \\ &\quad .\prod_{j=2}^{t} \hat{c}c^{-1}\,\|(Df^m)/E(f^{-(n+jm)}(x))\|\,\| D_v f_v^{-m\nu(x)}(x)\| \\ &\leqslant \prod_{j=1}^{t} \hat{c}c^{-1}\,\|(Df^m)/E(f^{-(n+jm)}(x))\|\,\| D_v f_v^{-m\nu(x)}(x)\|, \end{aligned}$$

thus proving b). Suppose now that $f_v^{-(n+m)}(x) \in U_0$. Then

$$D_v f_v^{-(n+m)}(x) = (Df^{-m})\,(D_v f_v^{-n}(x)) + D_v f_v^{-1}(f_v^{-(n+m)+1}(x)).$$

Since IV.4 a) says that $\nu\,|\,D_v f_v^{-(n+m)}(x)|$ is very large if n is large enough, we can write

$$\| D_v f_v^{-(n+m)}(x) - D_v f_v^{-1}(f_v^{-(n+m)+1}(x))\| \leqslant \hat{c}\,\| D_v f_v^{-(n+m)}(x)\|.$$

From IV.3 d)

$$\operatorname{Im} D_v f_v^{-(n+m)}(x) \subset S_{\varepsilon_1}(f_v^{-(n+m)}(x)).$$

Using property IV) we conclude

$$\operatorname{Im}(D_v f_v^{-(n+m)}(x) - D_v f_v^{-1}(f_v^{-(n+m)+1}(x))) \subset S_{\varepsilon_0}(f_v^{-(n+m)}(x)).$$

Hence, using VI') as above,

$$\begin{aligned} \| D_v f_v^{-n}(x)\| &= \|(Df^m)\,(D_v f_v^{-(n+m)}(x) - D_v f_v^{-1}(f_v^{-(n+m)+1}(x)))\| \\ &\leqslant c^{-1}\,\|(Df^m)/E(f^{-(n+m)}(x))\| \\ &\quad .\| D_v f_v^{-(n+m)}(x) - D_v f_v^{-1}(f_v^{-(n+m)+1}(x))\| \\ &\leqslant c^{-1}\hat{c}\,\|(Df^m)/E(f^{-(n+m)}(x))\|.\| D_v f_v^{-(n+m)}(x)\| \\ &\leqslant c^{-1}\hat{c}\,\|(Df^m)/E(f^{-(n+m)}(x))\|.\| D_v f_v^{-m\nu(x)}(x)\| \\ &\quad .\prod_{j=2}^{t} \hat{c}c^{-1}\,\|(Df^m)/E(f^{-(n+jm)}(x))\| \\ &\leqslant \prod_{j=1}^{t} \hat{c}c^{-1}\,\|(Df^m)/E(f^{-(n+jm)}(x))\|.\|(D_v f_v^{-m\nu(x)}(x))\|. \end{aligned}$$

This concludes the proof of IV.4. Take $\hat{c}$ such that $\hat{c}c^{-1}\gamma = \gamma_2 < 1$. Then b) implies

Corollary **IV.5.** — *For all* $x \in U_1 \cap S$, $N(\hat{c}) < n < n + tm \leqslant m\nu(x)$,

$$\| D_v f_v^{-m(\nu(x)-t)}(x) \| \leqslant \gamma_2^t \| D_v f_v^{-m\nu(x)}(x) \|.$$

Define $\Sigma_x(r, n)$ as the set of points in $\Sigma_x(n)$ whose distance to x in the manifold $\Sigma_x(n)$ is $\leqslant r$, and let $\Sigma_x^*(n)$ be the union of all the $\Sigma_x(r, n)$ that are complete when endowed with the metric of the submanifold $\Sigma_x(n)$. Then IV.5 implies:

Corollary **IV.6.** — *For all* $x \in U_1 \cap S$, $N(\hat{c}) < n < n + tm \leqslant m\nu(x)$,

$$\operatorname{diam} \Sigma_x^*(m\nu(x) - t) \leqslant \gamma_2^t \operatorname{diam} \Sigma_x^*(m\nu(x)).$$

Lemma **IV.7.** — *There exists* $\hat{\varepsilon}_0$ *such that*

$$\operatorname{diam} \Sigma_x^*(m\nu(x)) \geqslant \hat{\varepsilon}_0$$

for all $x \in S \cap U_1$ *with* $m\nu(x) \geqslant N(\hat{c})$.

Proof. — If the lemma is false there exist points x in $U_1 \cap S$ such that $\operatorname{diam} \Sigma_x^*(m\nu(x))$ is arbitrarily small. Set $\hat{\delta} = \operatorname{diam} \Sigma_x^*(m\nu(x))$. By the definition of $\Sigma_x^*(m\nu(x))$ there exists $p \notin \overline{\Sigma}_x(m\nu(x)) - \Sigma_x(m\nu(x))$ that is a limit of a sequence $\{ p_n \mid n \geqslant 0 \} \subset \Sigma_x^*(m\nu(x))$. Write

$$p_n = f_{v_n}^{-m\nu(x)}(x)$$

for some $v_n \in V(x)$. Then

$$f_{v_n}^{-j}(x) = f_{v_n}^{-(m\nu(x)-j)} f_{v_n}^{-m\nu(x)}(x) \in f_{v_n}^{-(m\nu(x)+j)} \Sigma_x(m\nu(x)) \subset \Sigma_x^*(j).$$

This means that

$$d(f_{v_n}^{-j}(x), f^j(x)) \leqslant \operatorname{diam} \Sigma_x^*(j).$$

Then, by IV.6, it follows that for all $0 \leqslant j \leqslant m\nu(x)$ and n, $d(f_{v_n}^{-j}(x), f^{-j}(x))$ remains arbitrarily small, say $\leqslant \hat{\varepsilon}$. Hence $d(f_v^{-j}(x), f^{-j}(x)) \leqslant \hat{\varepsilon}$ for all $0 \leqslant j \leqslant m\nu(x)$. Then, $\hat{\varepsilon} \lll \delta$ (where δ is the constant used in the definition of $V(x)$), there exists a disk D_0 in $T_{f^{-1}(x_0)} M$, centered at v such that $d(f_w^{-j}(x), f^{-j}(x)) < \delta$ for all $w \in D_0$. Then $V(x) \cup D_0$ is obviously open and, decreasing D_0 if necessary, star shaped because the center of D_0 is in the boundary of $V(x)$. Then $V(x) \cup D_0$ is open, star shaped and satisfies the condition required by the definition of $V(x)$. Moreover it contains v that is not in $V(x)$. This contradicts the maximality of $V(x)$ and completes the proof.

Using (14) we can now take an open subset $D^*(n, x) \subset D(n, x)$ such that

$$\Sigma_x^*(n) = \{ w + \psi_{n,x}(w) \mid w \in D^*(n, x) \}.$$

From the definition of $\Sigma_x^*(n)$, property (15) and Lemma IV.7 there exists $\rho > 0$ such that the disk $B_\rho(f^{-n}(x)) = \{ w \in \hat{E}(f^{-n}(x)) \mid \| w \| < \rho \}$ is contained in $D^*(n, x)$ for all $x \in S \cap U_1$, $0 < n \leqslant m\nu(x)$. To conclude the proof of Theorem IV.2 we shall use the following easy lemma.

Lemma **IV.8.** — *Let* E_1, E_2 *be Banach spaces and let* $B^i_r(p)$ *be the ball of radius* r *in* E_i *centered at* p. *Let* $C > 0$, $\rho_0 > 0$ *and* $\varepsilon > 0$ *be constants such that* ε *is so small that*

$$\varepsilon C < 1, \quad \frac{\varepsilon + \varepsilon^2}{1 - \varepsilon C} < \min\{(\rho_0 - \varepsilon)/C, \rho_0\}.$$

Suppose that

$$\varphi : B^2_{\rho_0}(0) \to E^1, \quad \psi : B^1_{\rho_0}(p) \to E^2$$

satisfy:

a) $\varphi(0) = 0$, $\| \varphi(w_2) - \varphi(w_1)\| \leqslant \varepsilon \| w_2 - w_1 \|$ *for all* $w_1, w_2 \in B^2_{\rho_0}(0)$;
b) $\| \psi(p)\| \leqslant \varepsilon$ *and* $\| \psi(w_2) - \psi(w_1)\| \leqslant C \| w_2 - w_1 \|$ *for all* $w_1, w_2 \in B^1_{\rho_0}(0)$;
c) $\| p \| \leqslant \varepsilon$.

Then $\quad \operatorname{graph}(\varphi) \cap \operatorname{graph}(\psi) \neq \emptyset.$

Proof. — We have to find $x \in B^2_{\rho_0}(0)$ and $y \in B^1_{\rho_0}(p)$ such that

$$(x, \varphi(x)) = (\psi(y), y).$$

This is equivalent to finding $y \in B^1_{\rho_0}(p)$ such that

$$\psi(y) \in B^2_{\rho_0}(0)$$
$$\varphi\psi(y) = y.$$

Observe that

$$\| \psi(y)\| \leqslant \| \psi(p)\| + C \|y - p\| \leqslant \varepsilon + C \|y - p\|.$$

Then, for every $0 < \rho_1 < \min\{\rho_0, (\rho_0 - \varepsilon)/C\}$,

$$\psi(B^1_{\rho_1}(p)) \subset B^2_{\rho_2}(0).$$

Now we can consider

$$\varphi\psi : B^1_{\rho_1}(p) \to E^1$$

that satisfies

$$\| \varphi\psi(p)\| \leqslant \varepsilon \| \psi(p)\| \leqslant \varepsilon^2$$

and

(16) $$\| \varphi\psi(w_1) - \varphi\psi(w_2)\| \leqslant \varepsilon C \| w_1 - w_2 \|.$$

Hence

(17) $$\| \varphi\psi(w) - p \| \leqslant \| p \| + \varepsilon^2 + \varepsilon C \| w - p \| \leqslant \varepsilon + \varepsilon^2 + \varepsilon C \| w - p \|.$$

Since by hypothesis

$$\frac{\varepsilon + \varepsilon^2}{1 - \varepsilon C} < \min\{\rho_0, (\rho_0 - \varepsilon)/C\},$$

there exists

(18) $$\frac{\varepsilon + \varepsilon^2}{1 - \varepsilon C} < \rho_2 < \min\{\rho_0, (\rho_0 - \varepsilon)/C\}.$$

Thus $\rho_2 < \min\{\rho_0, (\rho_0 - \varepsilon)/C\}$ implies

$$\psi(B^1_{\rho_2}(p)) \subset B^2_{\rho_0}(0)$$

and then from (17) follows

$$(\varphi\psi)\ (B^1_{\rho_2}(p)) \subset B^1_{\rho_2}(p),$$

because $\varepsilon + \varepsilon^2 + \varepsilon C \rho_2 < \rho_2$ by (18).

Moreover, since $\varepsilon C < 1$, (16) implies that $\varphi\psi$ is a contraction $B^1_{\rho_2}(p)$ into itself. The fixed point of this contraction satisfies the required properties.

To complete the proof of Theorem IV.2, we shall apply Lemma IV.8 to the maps

$$\psi_{n,x} : B_\rho(f^{-n}(x)) \to \hat{F}(f^{-n}(x)),$$
$$\varphi : B_{r_1}(y) \to \hat{E}(y)$$

where $B_{r_1}(y)$ is the disk of radius r_1 centered at 0 of the fiber $\hat{F}(y)$ and φ is a C^1 map such that

$$\operatorname{graph}(\varphi) = D_r(y).$$

Then $\varphi(0) = 0$ and $(D\varphi)\ (0) = 0$. Diminishing r_1 we can satisfy the condition:

$$\|(D\varphi)\ (x)\| \leqslant \varepsilon$$

for all $x \in B_{r_1}(y)$, for any given $\varepsilon > 0$. Choose ε so small that $\varepsilon C \ll 1$. Then IV.8 says that if $f^{-n}(x)$ and y are ε-near, then

$$\operatorname{graph}(\varphi) \cap \operatorname{graph}(\psi_{n,x}) \neq \emptyset.$$

This means that

$$D_r(y) \cap \Sigma^s_x(n) \neq \emptyset$$

and then there exists $v \in V(x)$ such that

$$f_v^{-n}(x) \in D_r(y).$$

V. — Proof of Theorem I.7

We shall begin by recalling the statements of three theorems, proved in [13], about the creation of homoclinic points.

If $x \in M$ and $f \in \operatorname{Diff}^1(M)$, let $\mu_x(f, n)$ be the probability

$$\mu_x(f, n) = \frac{1}{n} \sum_{j=1}^{n} \delta_{f^{-j}(x)}.$$

Denote by $\mathcal{M}(f, x)$ the set of accumulation points of the sequence $\{\mu_x(f, n) \mid n > 0\}$. Clearly $\mathcal{M}(f, x) \subset \mathcal{M}(f)$.

Given a basic set Λ of a diffeomorphism f (see Section I for the definition of a basic set) we say that p is a homoclinic point associated to Λ if $p \in W^s(\Lambda) \cap W^u(\Lambda) - \Lambda$.

Recall that every hyperbolic set Σ of f that is isolated (i.e. $\bigcup_n f^n(\mathrm{U}) = \Sigma$ for some compact neighborhood of Λ) and satisfies $\Omega(f/\Sigma) = \Sigma$, can be decomposed in a unique way in a union of disjoint basic sets

Theorem **V.1** ([13]). — *If Σ is an isolated hyperbolic set of $f \in \mathrm{Diff}^1(\mathrm{M})$, with $\Omega(f/\Sigma) = \Sigma$, and there exist $x \in \mathrm{W}^s(\Sigma) - \Sigma$ and $\mu \in \mathscr{M}(f, x)$ such that $\mu(\Sigma) > 0$, then there exists $g \in \mathrm{Diff}^1(\mathrm{M})$ arbitrarily C^1 near to f, coinciding with f in a neighborhood of $\Sigma \cup \{f^n(x) \mid n \geqslant 0\}$ and such that either g has homoclinic points associated to a basic set of Σ or else,*

$$x \in \mathrm{W}^s_g(\Sigma) \cap \mathrm{W}^u_g(\Sigma).$$

Theorem **V.2** ([13]). — *If Σ is an isolated hyperbolic set of $f \in \mathrm{Diff}^1(\mathrm{M})$, with $\Omega(f/\Sigma) = \Sigma$, and there exists $x \notin \mathrm{W}^u(\Sigma)$ such that $\mu(\Sigma) > 0$ for all $\mu \in \mathscr{M}(f, x)$, then there exists $g \in \mathrm{Diff}^1(\mathrm{M})$, arbitrarily near to f and coinciding with f in a neighborhood of Σ, having a homoclinic point associated to a basic set of Σ.*

Theorem **V.3** ([13]). — *Let Σ be an isolated hyperbolic set of $f \in \mathrm{Diff}^1(\mathrm{M})$ such that $\Omega(f/\Sigma) = \Sigma$. Suppose that $\{x_j\} \subset \mathrm{M}$ is a sequence converging to a point $x \notin \Sigma$ and $n_1 < n_2 < \ldots$ is a sequence of integers such that the probabilities $\mu_{x_j}(f, n_j)$ converge to a probability μ with $\mu(\Sigma) > 0$. Then, given a C^1-neighborhood $\mathscr{U}$ of f, one of the following properties holds:*

a) *There exists $g \in \mathscr{U}$ coinciding with f in a neighborhood of Σ and having a homoclinic point associated to a basic set of Σ.*

b) *For every neighborhood U of Σ, there exists another neighborhood $\mathrm{V} \subset \mathrm{U}$ of Σ and $g \in \mathscr{U}$, coinciding with f in $\mathrm{V} \cup \mathrm{U}^c$, such that for some $j > 0$ and $0 < n \leqslant n_j$, g satisfies*

$$g^{-i}(x_j) = f^{-i}(x_j)$$

for $0 \leqslant i \leqslant n - 2$ and

$$g^{-t}(x_j) \in \mathrm{V}$$

for all $t \geqslant n$.

Observe that the last condition, together with the fact that g and f coincide in V, implies that

$$\mathrm{V} \ni g^{-i}(g^{-n}(x_j)) = f^{-i}(g^{-n}(x_j))$$

for all $i \geqslant 0$. If U (and then V) is small enough, this implies

$$g^{-n}(x_j) \in \mathrm{W}^u(\Sigma).$$

The proof of the following easy lemma is left to the reader.

Lemma **V.4**. — a) *If $\mathrm{A} \subset \mathrm{M}$ is a compact set and a sequence of probabilities $\mu_{x_j}(f^m, n_j)$, $j = 1, 2, \ldots$, converges to $\mu \in \mathscr{M}(f^m)$ such that $\mu(\mathrm{A}) > 0$, then every accumulation point ν of the sequence $\{\mu_{x_j}(f, mn_j) \mid j > 0\}$ satisfies $\nu(\mathrm{A}) > 0$.*

b) *If $\mathrm{A} \subset \mathrm{M}$ is a compact set and $x \in \mathrm{M}$ satisfies $\mu(\mathrm{A}) > 0$ for all $\mu \in \mathscr{M}(f^m, x)$, then $\nu(\mathrm{A}) > 0$ for all $\nu \in \mathscr{M}(f, x)$.*

Now let us prove Theorem I.7. We shall find a diffeomorphism g arbitrarily C^1 near to f, coinciding with f in a neighborhood of $\bigcup_1^s \Lambda_k$, and such that if $\Lambda - \Lambda_i$ is not closed then

$$W^s_g(\Lambda_i) \cap W^u_g(\Lambda_1 \cup \ldots \cup \Lambda_s) - (\Lambda_1 \cup \ldots \cup \Lambda_s) \neq \emptyset.$$

This proves I.7, because it implies the existence of $1 \leqslant r \leqslant s$ such that

$$W^s_g(\Lambda_i) \cap W^u_g(\Lambda_r) - (\Lambda_i \cup \Lambda_r) \neq \emptyset$$

and then, from hypothesis II) of I.7, it follows that $i \neq r$. Finally, it will be obvious from the construction of g that, as required by Theorem I.7, $\Lambda - \Lambda_r$ is not closed.

From now on, let $f, \Lambda, \Lambda_1, \ldots, \Lambda_s$, $m > 0$, $c > 0$ and $0 < \lambda < 1$ be given by the hypotheses of Theorem I.7. Choose $\gamma, \gamma_1, \gamma_2$ such that

$$\lambda < \gamma_2$$

and $$\exp(-c) < \gamma_2 < \gamma < \gamma_1 < 1.$$

Lemma **V.5.** — *If* $x \in \Lambda$ *satisfies*

$$\prod_{j=1}^{n} \|(Df^m)/E(f^{-mj}(x))\| \geqslant \gamma_2^n$$

for all n *sufficiently large, then* $x \in W^u(\Lambda_1 \cup \ldots \cup \Lambda_s)$.

Proof. — Suppose that the inequality holds for all $n \geqslant N$. Then, if $\mu \in \mathcal{M}(f^m, x)$ is the limit of a sequence $\mu_x(f^m, n_j)$, $j \geqslant 0$, we obtain

$$\begin{aligned}\int \log \|(Df^m)/E\| \, d\mu &= \lim_{j\to+\infty} \int \log \|(Df^m)/E\| \, d\mu_x(f^m, n_j) \\ &= \lim_{j\to+\infty} \frac{1}{n_j} \sum_{i=j}^{n_j} \log \|(Df^m)/E(f^{-mi}(x))\| \\ &= \lim_{j\to+\infty} \log\Big(\prod_{i=1}^{n_j} \|(Df^m)/E(f^{-mi}(x))\|\Big)^{1/n_j} \geqslant \log \gamma_2 > -c.\end{aligned}$$

Hence, by hypothesis III), for all $\mu \in \mathcal{M}(f^m, x)$ we have

$$\mu(\Lambda_1 \cup \ldots \cup \Lambda_s) > 0.$$

Using V.4 b), this inequality holds for all $\mu \in \mathcal{M}(f, x)$. Applying V.2 we conclude that either $x \in W^u(\Lambda_1 \cup \ldots \cup \Lambda_s)$ or that there exists $g \in \mathrm{Diff}^1(M)$ arbitrarily C^1 near to f and coinciding with f in a neighborhood of $\Lambda_1 \cup \ldots \cup \Lambda_s$ such that there exists a homoclinic point $p \notin \Lambda_1 \cup \ldots \cup \Lambda_s$ associated to one of the basic sets Λ_i. By hypothesis II) of I.7 this is impossible. Hence $x \in W^u(\Lambda_1 \cup \ldots \cup \Lambda_s)$, proving the lemma.

To construct g, let now Λ_i be such that $\Lambda - \Lambda_i$ is not closed. Suppose, to simplity the notation, that $i = 1$. Take an isolating block W of Λ_1 (i.e. $\bigcap_n f^n(W) = \Lambda_1$) and set

$$\Lambda_1^s = \bigcap_{n\geqslant 0} f^{-n}(W),$$

$$\Lambda_1^u = \bigcap_{n\geqslant 0} f^{n}(W).$$

Let us show that $(\Lambda_1^s - \Lambda_1) \cap \Lambda \neq \emptyset$. Suppose that $(\Lambda_1^s - \Lambda_1) \cap \Lambda = \emptyset$. Then

$$\emptyset = (\Lambda_1^s - \Lambda_1) \cap \Lambda = (\bigcap_{n \geqslant 0} f^{-n}(W) - \Lambda_1) \cap \Lambda = \bigcap_{n \geqslant 0} f^{-n}(W \cap \Lambda) - \Lambda_1.$$

Hence
$$\Lambda_1 = \bigcap_{n \geqslant 0} f^{-n}(W \cap \Lambda).$$

This implies ([26]) that there exists a compact neighborhood W_0 of Λ_1 in Λ such that $f^{-1}(W_0) \subset \operatorname{Int} W_0$ and

$$\bigcap_{n \geqslant 0} f^{-n}(W_0) = \Lambda_1.$$

When $\operatorname{Int} W_0 - f^{-1}(W_0) = \emptyset$, or, what is the same, if $W_0 = f^{-1}(W_0)$, this equality implies $W_0 = \Lambda_1$ because $f^{-n}(W_0) = W_0$ for all n. But this is impossible because a neighborhood of Λ_1 in Λ must contain points not in Λ_1, since $\Lambda - \Lambda_1$ is not closed. Suppose now that $\operatorname{Int} W_0 - f^{-1}(W_0) \neq \emptyset$. Clearly

$$f^{-n}(\operatorname{Int} W_0 - f^{-1}(W_0)) \subset f^{-n}(\operatorname{Int} W_0) = \operatorname{Int} f^{-n}(W_0) \subset f^{-n}(W_0) \subset f^{-1}(W_0).$$

Hence
$$f^{-n}(\operatorname{Int} W_0 - f^{-1}(W_0)) \cap (\operatorname{Int} W_0 - f^{-1}(W_0)) = \emptyset.$$

Therefore $\operatorname{Int} W_0 - f^{-1}(W_0)$ contains no nonwandering points of f/Λ. This contradicts $\Omega(f/\Lambda) = \Lambda$ and completes the proof of $(\Lambda_1^s - \Lambda_1) \cap \Lambda \neq \emptyset$.

Now, unfortunately, the proof divides in two cases. The first case is when there exists $p \in (\Lambda_1^s - \Lambda_1) \cap \Lambda$ such that $p \in \alpha(p)$. Then we can take a sequence of integers $n_1 < n_2 < \ldots$ such that the sequence $\{f^{-n_j m}(p) \mid j \geqslant 1\}$ converges to a point in Λ_1.

Lemma **V.6.** — *If* $(\{p\}, \{n_1, n_2, \ldots\})$ *is not an* (m, γ)*-contracting sequence, then Theorem I.7 is true.*

Proof. — By hypothesis, there exist $j_1 < j_2 < \ldots$ such that, setting $\bar{n}_i = n_{j_i}$, we have

$$\prod_{j=1}^{\bar{n}_i} \|(Df^m)/E(f^{-mj}(p))\| \geqslant \gamma^{\bar{n}_i}.$$

Hence
$$\int \log \|(Df^m)/E\| \, d\mu_p(f^m, \bar{n}_i) \geqslant \log \gamma > -c.$$

Then, if $\mu \in \mathscr{M}(f^m)$ is an accumulation point of the sequence $\{\mu_p(f^m, \bar{n}_i) \mid i \geqslant 1\}$,

$$\int \log \|(Df^m)/E\| \, d\mu \geqslant \log \gamma - c.$$

Hence, by hypothesis,

$$\mu(\Lambda_1 \cup \ldots \cup \Lambda_s) > 0.$$

Then, V.4 implies that there exists $\mu_0 \in \mathscr{M}(f, p)$ satisfying

$$\mu_0(\Lambda_1 \cup \ldots \cup \Lambda_s) > 0.$$

Applying V.1 to $\Sigma = \Lambda_1 \cup \ldots \cup \Lambda_s$ we obtain a diffeomorphism g, arbitrarily C^1 near to f, coinciding with f in a neighborhood of $\Sigma \cup \{f^n(p) \mid n \geqslant 0\}$ and having p as a homoclinic point associated to Σ, the other option being ruled out by hypothesis II) of I.7. Since $g^n(p) = f^n(p)$ for all $n \geqslant 0$, it follows that

$$p \in W^s_g(\Lambda_1) \cap W^u_g(\Lambda_1 \cup \ldots \cup \Lambda_s).$$

This proves I.7 and then also Lemma V.6.

By Lemma V.6 we can continue the proof of I.7 assuming that $(\{p\}, \{n_1, n_2, \ldots\})$ is an (m, γ)-contracting sequence. Let us apply IV.1 to $(\{p\}, \{n_1, \ldots\})$ and our choice of $\gamma < \gamma_1 < 1$. If property a) of IV.1 holds, there exists a subsequence $\{\bar{n}_1, \bar{n}_2, \ldots\}$ of $\{n_1, n_2, \ldots\}$ such that $(\{p\}, \{\bar{n}_1, \bar{n}_2, \ldots\})$ is strongly (m, γ)-contracting and $f^{-\bar{n}_j m}(p)$ converges to a point $y \in \Lambda$ satisfying

$$\prod_{i=1}^{n} \|(Df^m)/E(f^{-mi}(y))\| \geqslant \gamma_2^n \tag{1}$$

for all n larger than a certain N. By Lemma V.5,

$$y \in W^u(\Lambda_1 \cup \ldots \cup \Lambda_s). \tag{2}$$

Now hypothesis IV) of Theorem I.7 says that

$$\|(Df^m)/E(x)\| . \|(Df^{-m})/F(f^m(x))\| < \lambda$$

for all $x \in \Lambda$. Hence, this inequality and (1) imply

$$\prod_{i=0}^{n-1} \|(Df^{-m})/F(f^{-mi}(y))\|$$
$$= \prod_{i=0}^{n-1} \|(Df^m)/E(f^{-m(i+1)}(y)\| . \|(Df^{-m})/F(f^{-mi}(y))\|$$
$$. \prod_{i=0}^{n-1} \|(Df^m)/E(f^{-m(i+1)}(y))\|^{-1} \leqslant \lambda^n \gamma_2^{-n}.$$

Since we chose γ_2 satisfying $\lambda\gamma_2^{-1} = \lambda_0 < 1$, we have

$$\prod_{i=0}^{n-1} \|(Df^{-m})/F(f^{-mi}(y))\| \leqslant \lambda_0^n \tag{3}$$

for all $n \geqslant N$. Let $D_r(x)$, $x \in \Lambda$, be the family of disks tangent at x to $F(x)$, associated to the splitting $TM/\Lambda = E \oplus F$, as we explained in Section IV. From (3) it is easy to deduce

$$\lim_{n \to +\infty} \operatorname{diam} f^{-n}(D_r(y)) = 0 \tag{4}$$

when r is small enough; this, together with (2), implies

$$D_r(y) \subset W^u(\Lambda_1 \cup \ldots \cup \Lambda_s). \tag{5}$$

Now let us apply the Attainability Theorem IV.2 to the strongly (m, γ_1)-contracting sequence $(\{p\}, \{\bar{n}_1, \bar{n}_2, \ldots\})$ and y. It yields a diffeomorphism g, arbitrarily C^1 near to f and such that g^{-1} and f^{-1} differ only in an arbitrarily small neighborhood U of p, and moreover satisfying

$$(6) \qquad g^{-n}(p) \in D_r(y)$$

for some $n > 0$. Suppose that U was chosen so small that

$$(7) \qquad U \cap (\{f^n(p) \mid n \geqslant 1\} \cup (\bigcup_{n \geqslant 0} f^{-n}(D_r(y))) \cup \Lambda_1) = \emptyset.$$

This can be done because

$$p \notin \bigcup_{n \geqslant 0} f^{-n}(D_r(y));$$

otherwise (5) would imply

$$p \in W^u(\Lambda_1 \cup \ldots \cup \Lambda_s)$$

and since $p \in W^s(\Lambda_1)$, Theorem I.7 would follow just taking $f = g$. Then p does not belong to the set

$$\{f^n(p) \mid n \geqslant 1\} \cup (\bigcup_{n \geqslant 0} f^{-n}(D_r(y)) \cup \Lambda_1,$$

that (using (5)) is easily seen to be closed. Then U can be chosen satisfying (7). From (7) it follows easily that

$$(8) \qquad D_r(y) \subset W^u_g(\Lambda_1 \cup \ldots \cup \Lambda_s).$$

Also from (7) follows that

$$g^k(p) = f^k(p)$$

for all $k \geqslant 0$. Hence

$$p \in W^s_g(\Lambda_1),$$

and this together with (6) and (8), implies

$$p \in W^s_g(\Lambda_1) \cap W^u_g(\Lambda_1 \cup \ldots \cup \Lambda_s)$$

which once more, proves Theorem I.7. Now consider the case when applying IV.1 to $(\{p\}, \{n_1, n_2, \ldots\})$ it is property b) that holds. Then there exists a sequence of positive integers $0 < \bar{n}_j \leqslant n_j$ such that $(\{p\}, \{\bar{n}_1, \bar{n}_2, \ldots\})$ is a strongly (m, γ_1)-contracting sequence and

$$\sup_j (n_j - \bar{n}_j) < \infty.$$

This last relation implies that without loss of generality we can assume (recalling that $f^{-n_j m}(p)$ converges to a point in Λ_1) that

$$\lim_{j \to +\infty} f^{-\bar{n}_j m}(p) = p_0 \in \Lambda_1.$$

Let $TM/\Lambda_1 = E^s \oplus E^u$ be the hyperbolic splitting of Λ_1. By hypothesis, $\dim E^u(x) > \dim F(x)$ for all $x \in \Lambda_1 \cap \Lambda$. Then we have two dominated splittings of $TM/\Lambda_1 \cap \Lambda$, namely $TM/\Lambda_1 = E^s \oplus E^u$ and $TM/\Lambda = E \oplus F$. Using that f/Λ_1 is transitive and well known (and easy) properties of dominated splittings (see [9], for instance) we obtain that $F(x) \subset E^u(x)$ for all $x \in \Lambda_1$. Then, by definition of hyperbolic splitting, $F(x)$ satisfies property (3), for all $x \in \Lambda_1$ (with y obviously replaced by x and suitable values of $0 < \lambda_0 < 1$ and $m > 0$). Take $x = p_0$. Then, as before, we have that

$$\lim_{n \to +\infty} \operatorname{diam} f^{-n}(D_r(p_0)) = 0$$

when r is small enough and then

$$D_r(p_0) \subset W^u(p_0) \subset W^u(\Lambda_1).$$

Now, arguing as before, take a neighborhood U of p such that

$$U \cap (\{f^n(p) \mid n > 0\} \cup (\bigcup_{n \geqslant 0} f^{-n}(D_r(p_0)))) = \emptyset.$$

Applying again the Attainability Theorem, now to the strongly (m, γ_1)-contracting sequence, we can find $g \in \operatorname{Diff}^1(M)$ arbitrarily C^1 near to f, such that g^{-1} and f^{-1} differ only in U, and satisfying for some $N > 0$

$$g^{-N}(p) \in D_r(p_0).$$

Since g^{-1} and f^{-1} coincide in the set

$$\{f^n(p) \mid n > 0\} \cup (\bigcup_{n \geqslant 0} f^{-n}(D_r(p_0))),$$

we have

$$f^n(p) = g^n(p)$$

for all $n \geqslant 0$ and

$$\lim_{n \to +\infty} \operatorname{diam} g^{-n}(D_r(p_0)) = 0.$$

Hence $\quad p \in W^s_g(\Lambda_1)$

and $\quad g^{-N}(p) \in D_r(p_0) \subset W^u_g(\Lambda_1).$

Therefore $\quad p \in W^s_g(\Lambda_1) \cap W^u_g(\Lambda_1) - \Lambda_1,$

a contradiction with hypothesis II) of I.7.

This completes the proof of Theorem I.7 when there exists $p \in (\Lambda_1^s - \Lambda_1) \cap \Lambda$ such that $p \in \alpha(p)$. Now let us suppose that

(9) $\quad p \notin \alpha(p)$ for all $p \in (\Lambda_1^s - \Lambda_1) \cap \Lambda$.

Take $q \in (\Lambda_1^s - \Lambda_1) \cap \Lambda$. Since $q \notin \alpha(q)$ there exists a ball $B_\rho(q)$ such that

(10) $\quad f^{-n}(q) \notin B_\rho(q)$

for all $n \geqslant 1$. Since $q \in \Lambda = \Omega(f/\Lambda)$, there exist a sequence of points $\{q_j \mid j \geqslant 1\} \subset \Lambda$ converging to q and integers $\bar{n}_j \to +\infty$ such that $\lim_{j \to +\infty} f^{-\bar{n}_j}(q_j) = q$. Given any $\varepsilon > 0$, we can take $\rho > 0$ so small that, using property (10), there exists a sequence of integers $0 < n_j < \bar{n}_j/m$ satisfying $\lim_{j \to +\infty} n_j = +\infty$ and for, all j,

(i) $f^{-n}(q_j) \notin B_\rho(q)$ for all $0 < n \leqslant mn_j$,
(ii) $d(f^{-mn_j}(q_j), \Lambda_1) < \varepsilon$.

The next lemma parallels Lemma V.6, in such a way that the whole proof, under assumption (9), parallels the proof of the case when we had $p \in (\Lambda_1^s - \Lambda_1) \cap \Lambda$ satisfying $p \in \alpha(p)$.

Lemma **V.7.** — *If* $(\{q_j\}, \{n_j\})$ *is not an* (m, γ)*-contracting sequence, then Theorem* I.7 *is true.*

Proof. — By hypothesis, there exist $j_1 < j_2 < \ldots$ such that, denoting $\hat{n}_i = n_{j_i}$ and $\hat{q}_i = q_{j_i}$, we have

$$\prod_{j=1}^{\hat{n}_j} \|(Df^m)/E(f^{-mj}(\hat{q}_i))\| \geqslant \gamma^{\hat{n}_i}.$$

Hence $$\int \log \|(Df^m)/E\| \, d\mu_{\hat{q}_i}(f^m, \hat{n}_i) \geqslant \log \gamma > -c.$$

Then, if $\mu \in \mathcal{M}(f^m)$ is an accumulation point of the sequence $\{\mu_{\hat{q}_i}(f^m, \hat{n}_i) \mid i \geqslant 1\}$,

$$\int \log \|(Df^m)/E\| \, d\mu \geqslant \log \gamma > -c.$$

Hence, by hypothesis

$$\mu(\Lambda_1 \cup \ldots \cup \Lambda_s) > 0.$$

Then V.4 implies that if μ_0 is an accumulation point of the sequence $\{\mu_{\hat{q}_i}(f, m\hat{n}_i) \mid i \geqslant 1\}$, one has

$$\mu_0(\Lambda_1 \cup \ldots \cup \Lambda_s) > 0.$$

Now we can apply V.3 to obtain a diffeomorphism g arbitrarily C^1 near to f and coinciding with f in a neighborhood of $\Lambda_1 \cup \ldots \cup \Lambda_s$, such that (since option a) of V.3 is ruled out by hypothesis II) of I.7) there exists $\hat{q}_i$, with i arbitrarily large, such that

$$g^{-n}(\hat{q}_i) = f^{-n}(\hat{q}_i) \tag{11}$$

for all $n \geqslant 0$ less than a certain $N \leqslant m\hat{n}_j$, and

$$g^{-t}(\hat{q}_i) \in V \tag{12}$$

for all $t \geqslant N$, where V is an arbitrarily small neighborhood of $\Lambda_1 \cup \ldots \cup \Lambda_s$ where g and f coincide. Then, if V is taken being an isolating block of $\Lambda_1 \cup \ldots \cup \Lambda_s$, we conclude that

$$g^{-N}(\hat{q}_i) \in W^u(\Lambda_1 \cup \ldots \cup \Lambda_s).$$

Moreover (11) and (12) imply that

$$g^{-n}(\hat{q}_i) \notin B_\rho(q)$$

for all $n \geqslant 1$. Now observe that the forward orbit of q converges to Λ_1 and therefore, without loss of generality, we can assume that is does not intersect $B_\rho(q)$. By the local stability of the hyperbolic set Λ_1, there exists $\bar{q}$ nearby q whose forward orbit converges to Λ_1 without intersecting $B_{\rho/2}(q)$. Moreover the distance $d(q, \bar{q})$ is arbitrarily small if g is sufficiently C^1 near to f. Therefore the quotient $d(\hat{q}_i, \bar{q})/(\rho/2)$ can be obtained arbitrarily small. Then there exists a diffeomorphism $h \in \mathrm{Diff}^1(M)$ such that

$$h(\hat{q}_i) = \bar{q},$$

and $$h(x) = x \quad \text{if } x \notin B_{\rho/2}(q)$$

and whose C^1 distance to the identity goes to zero together with $d(\hat{q}_i, \bar{q})/(\rho/2)$. Consider $\bar{g} \in \mathrm{Diff}^1(M)$ defined by $\bar{g} = hg$. It is easy to check that

$$\{ g^n(\bar{q}) \mid n \geqslant 0 \} \cup \{ g^{-n}(\hat{q}_i) \mid n \geqslant 1 \}$$

has the property of being an orbit of $\bar{g}$, and clearly an orbit in

$$W^s_{\bar{g}}(\Lambda_1) \cap W^u_{\bar{g}}(\Lambda_1 \cup \ldots \cup \Lambda_s) - (\Lambda_1 \cup \ldots \cup \Lambda_s),$$

thus proving I.7 and also Lemma V.7.

As before, Lemma V.7 means that we have only to complete the proof of Theorem I.7 assuming that $(\{ q_j \}, \{ n_j \})$ is an (m, γ)-contracting sequence. Observe that we reached this conclusion independently of the ε used in the construction of the sequence $(\{ q_j \}, \{ n_j \})$. On the other hand we have already shown that if $r > 0$ is small enough then

$$D_r(x) \subset W^u(\Lambda_1)$$

for all $x \in \Lambda_1$. Let $0 < \gamma_2 < \gamma < \gamma_1 < 1$ be as chosen and let $\varepsilon > 0$ be smaller than the $\varepsilon(r, m, \gamma_1)$ given by Attainability Theorem IV.2. Now let us apply Theorem IV.1 to the (m, γ)-contracting sequence $(\{ q_j \}, \{ n_j \})$. If property a) of IV.1 holds, there exists $(\{ q'_j \}, \{ n'_j \}) \leqslant (\{ q_j \}, \{ n_j \})$ that is strongly (m, γ_1)-contracting and such that

$$\lim_{j \to +\infty} f^{-n'_j m}(q'_j) = y \in \Lambda$$

where for the same reasons as in the proof of (5), y satisfies

$$D_r(y) \subset W^u(\Lambda_1 \cup \ldots \cup \Lambda_s).$$

Take, as before, $0 < \rho_1 < \rho$ such that

$$B_{\rho_1}(q) \cap (\{ f^n(q) \mid n \geqslant 1 \} \cup (\bigcup_{n \geqslant 0} f^{-n}(D_r(y)))) = \varnothing.$$

Now, applying the Attainability Theorem to $(\{ q'_j \}, \{ n'_j \})$, we can take a diffeomorphism g, arbitrarily C^1 near to f, such that $g^{-1} = f^{-1}$ in the complement of $B_{\rho_1}(q)$ and such that for an arbitrarily large j there exists $N > 0$ satisfying

(13) $$g^{-N}(q'_j) \in D_r(y).$$

The arguments used in the previous case now show that

$$D_r(y) \subset W^u_g(\Lambda_1 \cup \ldots \cup \Lambda_s)$$

and

(14) $$f^n(q) = g^n(q) \quad \text{for all } n \geqslant 0.$$

Moreover, the Attainability Theorem also grants that g can be taken satisfying:

$$d(g^{-n}(q'_j), f^{-n}(q'_j)) \leqslant \rho - \rho_1$$

for all $0 \leqslant n \leqslant N$. Hence

$$\{ g^{-n}(q'_j) \mid 1 \leqslant n \leqslant N \} \cap B_{\rho_1}(q) = \varnothing.$$

Moreover by (13)

$$\{ g^{-n}(q'_j) \mid n \geqslant N \} \cap B_{\rho_1}(q) \subset (\bigcup_{n \geqslant 0} g^{-n}(D_r(y))) \cap B_{\rho_1}(q)$$
$$= (\bigcup_{n \geqslant 0} f^{-n}(D_r(y))) \cap B_{\rho_1}(q) = \varnothing.$$

Hence $\{ g^{-n}(q'_j) \mid n \geqslant 1 \} \cap B_{\rho_1}(q) = \varnothing.$

and from (14)

$$\{ g^n(q) \mid n > 0 \} \cap B_{\rho_1}(q) = \varnothing.$$

Now take $h \in \mathrm{Diff}^1(M)$ such that

$$h(q'_j) = q,$$

and $h(x) = x$ if $x \notin B_{\rho_1}(q)$.

Then the set

$$\{ g^n(q) \mid n \geqslant 0 \} \cup \{ g^{-n}(q'_j) \mid n \geqslant 1 \}$$

has the property of being an orbit of $\bar{g} = hg$, and moreover, by (13) and (14), it is an orbit in

$$W^s_{\bar{g}}(\Lambda_1) \cap W^u_{\bar{g}}(\Lambda_1 \cup \ldots \cup \Lambda_s).$$

Hence

(15) $$q \in W^s_{\bar{g}}(\Lambda_1) \cap W^u_{\bar{g}}(\Lambda_1 \cup \ldots \cup \Lambda_s) - (\Lambda_1 \cup \ldots \cup \Lambda_s),$$

thus proving Theorem I.7. Finally, let us consider the case when in the application of Theorem IV.1 to $(\{ q_j \}, \{ n_j \})$ it is property b) that holds. This property means that without loss of generality we can suppose that $(\{ q_j \}, \{ n_j \})$ is a strongly (m, γ_1)-contracting sequence. Since $d(f^{-mn_j}(q_j), \Lambda_1) < \varepsilon$, there exists $y \in \Lambda_1$ ε-near to an accumulation point of the sequence of endpoints of $(\{ q_j \}, \{ n_j \})$. Then, given an arbitrarily small neighborhood U of q, we can apply the Attainability Theorem (observing that for this purpose we take ε smaller than the $\varepsilon(r, m, \gamma_1)$ of this theorem) and obtain g, arbitrarily C^1 near to f, such that $g^{-1} = f^{-1}$ in U^c, and satisfying

$$g^{-N}(q_j) \in D_r(y)$$

for some $N > 0$ and $j \geqslant 1$ that can be taken arbitrarily large. Now, repeating the method of the proof in the previous case (i.e. when it was option a) of IV.1 that held) we take a diffeomorphism h, C^1 close to the identity, satisfying $h(q_j) = q$ and $h(x) = x$ if $x \notin U$, and we define $\bar{g} = hg$ and show that q satisfies (15).

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