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A proof of the Hilbert-Smith conjecture for actions by Lipschitz maps

Dušan Repovš [1,★], **Evgenij V. Ščepin** [2,★★]

[1] Institute for Mathematics, Physics and Mechanics, Jadranska St. 19, SI-1001 Ljubljana, Slovenia (e-mail: dusan.repovs@uni-lj.si)

[2] Steklov Mathematical Institute, Vavilova St. 42, 117966 Moscow, Russia (e-mail: scepin@class.mian.su)

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1 Introduction

The classical Hilbert 5^{th} problem [14] asks whether every (finite-dimensional) locally Euclidean topological group is necessarily a Lie group. It was solved, in the affirmative, by von Neumann [23] for compact groups in 1933, and by Gleason [11] and by Montgomery and Zippin [20] for locally compact groups in 1952. A more general version of the Hilbert 5^{th} problem, called the *Hilbert-Smith Conjecture*, asserts that among all locally compact groups only Lie groups G can act *effectively* on (finite-dimensional) manifolds M (i.e. each $g \in G \setminus \{e\}$ moves at least one point of M) [28]. It follows from the work of Newman [24] and Smith [29] that this conjecture is equivalent to its special case when the acting group G is the group of p-adic integers A_p.

In 1946 Bochner and Montgomery [3] proved the Hilbert-Smith Conjecture for groups G acting effectively on a manifold M by *diffeomorphisms*. A simpler, geometrical proof was obtained by Skopenkov and the authors [25] using the idea of smooth homogeneity: a compact subset $K \subset M$ of a smooth manifold M is said to be *smoothly ambiently homogeneous*, i.e. for each $x, y \in K$ there exists a diffeomorphism $h\colon (M,K,x) \to (M,K,y)$. It was shown that this property implies that K is a smooth submanifold of M (therefore $G \cong K$ is a Lie group). The proof reveals a close relationship between homogeneity and *taming theory* for compact subsets of $\mathbb{R}^n$, which are pinched by tangent balls (the latter problem was investigated in the past by various authors [6,10,12,16,17]). See also a very interesting paper by Hahn [13].

An interesting approach to the Hilbert-Smith conjecture is via wild Cantor sets in $\mathbb{R}^n$ with strong homogeneity properties. Note that the Antoine necklace

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[1] is an ambiently homogeneous Cantor set in $\mathbb{R}^3$. For further examples of this type see [5, 26, 27, 30]. However, neither one of these examples can be extended to effective actions of A_p on $\mathbb{R}^n$ (see also [2, 8]).

Malešič proved in 1994 that the standard Cantor set in $\mathbb{R}^2$ is Lipschitz ambient homogeneous. He also constructed Antoine's necklace in $\mathbb{R}^3$ which is also Lipschitz ambiently homogeneous [18]. Intersections of self-similar objects like those in Malešič's construction are of fractal nature. This was our motivation to apply the Hausdorff dimension to prove the *Lipschitz* case of the Hilbert-Smith conjecture:

Theorem (1.1). *The group of p-adic integers A_p (p any prime) cannot act effectively by Lipschitz homeomorphisms on any (finite-dimensional) Riemannian manifold.*

2 The proof of Theorem 1.1

Suppose, to the contrary, that for some prime p, the group $G = A_p$ acted effectively on some Riemannian n-manifold M, with a Riemannian metric ρ on M, considered embedded in some Euclidean space $\mathbb{R}^k$. Then the classical Lebesgue (covering) dimension and the (fractal) Hausdorff dimension (with respect to this metric ρ) of M agree: $\dim M = \dim_\rho M$ (cf. e.g. [9] and [22]). Without losing generality we may assume M to be closed.

Suppose further, that the action is Lipschitz, i.e. that for every autohomeomorphism $g \in G$ of M, there exists $l_g > 1$ such that

$$\frac{1}{l_g} \leqq \frac{\rho(g(x), g(y))}{\rho(x, y)} \leqq l_g \,.$$

Apply now the Baire Category theorem to the following countable family of closed sets (whose union is obviously the entire group G):

$$E_n = \left\{ g \in G \;\middle|\; \frac{1}{n} \leqq \frac{\rho(g(x), g(y))}{\rho(x, y)} \leqq n \text{ for every } x \neq y \in M \right\} .$$

We can conclude that there must exist $L > 1$ and a nonempty open set N in G, such that $l_g \leqq L$ for each $g \in N$. Since the p-adic integers $G = A_p$ are of "fractal" nature, N will always contain a copy of the entire group A_p. So without losing generality, we may assume that $l_g \leqq L$ for each $g \in G$.

The above argument can actually be generalized (avoiding the use of the fractal nature of the p-adic integers) to arbitrary compact groups G – by using a finite covering of G by the sets $g_1 N, \ldots, g_s N$ and by simply invoking the obvious inequality $l_{gh} \leqq l_g l_h$.

There is a Haar measure on the group G. We can thus define an equivariant metric ρ_G on the manifold M as follows:

$$\rho_G(x, y) = \int_G \rho(g(x), g(y))\, dg \,.$$

Let $p: M \to M/G$ be the canonical projection onto the orbit space. Consider the induced metric on M/G given by

$$\delta_G(p(x), p(y)) = \min_{g \in G} \{\rho_G(x, g(y))\} .$$

The key argument now follows from the following sequence of (in)equalities:

$$n = \dim M \underset{(1)}{=} \dim_\rho M \underset{(2)}{=} \dim_{\rho_G} M$$

$$\underset{(3)}{\geqq} \dim_{\delta_G}(M/G) \underset{(4)}{\geqq} \dim(M/G) \underset{(5)}{\geqq} \dim_{\mathbb{Z}}(M/G) \underset{(6)}{=} n + 2 .$$

Here, (1) follows by our choice of the metric ρ above. Since the action is by hypothesis Lipschitz, metrics ρ and ρ_G are equivalent, and the equality (2) follows. Since the projection p: $M \to M/G$ does not increase distance between points, the inequality (3) follows. The inequality (4) follows e.g. by [15, Theorem 7.3], whereas (5) is a classical theorem of cohomological dimension theory [7]. Finally, the equality (6) follows by a well-known theorem of Yang [31] (see also [4]) since by hypothesis the action of G is effective and $G = A_p$. □

3 Epilogue

Analogously to [25] one can prove that a locally compact C^n-smoothly ambiently homogeneous subset of a C^n-manifold M is a C^n-submanifold of M. We conjecture the following:

Conjecture (3.2). *A locally compact, analytically ambiently homogeneous subset of C^n (or analytic) n-manifold M is an analytic submanifold of M.*

Added in proof. The authors wish to acknowledge comments from F. Raymond, S. Illman, and the referee.

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