A Proof of the Stability of the Spectral Difference Method for All Orders of Accuracy

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1 Introduction

While second order methods for computational simulations of fluid flow are by now quite mature and reliable, providing the basis of widely used commercial software, the development of higher order methods that could potentially yield better accuracy or reduced computational costs is a subject of ongoing research. One of the most promising approaches is the Discontinuous Galerkin (DG) method, for which the theoretical basis has been provided in a series of papers by Cockburn and Shu [1, 2, 3, 4]. The rapid growth of the computational complexity of DG methods with increasing order has spurred the search for more efficient variants or alternatives. One approach is the nodal DG scheme in which the solution is represented by Lagrange interpolation at a set of collocation points in each element, and the quadratures required by the DG method are pre-integrated to produce local mass and stiffness matrices. An exposition of the nodal DG method can be found in the recent book by Hesthaven and Warburton [5].

The spectral difference (SD) method has recently emerged as a promising alternative. The basic idea of the SD method was first put forward by Kopriva and Kolias [6] under the name "staggered grid Chebyshev multidomain" method. In order to discretize the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \tag{1.1}$$

they proposed to represent the solution by polynomials of degree p in each element and the flux by polynomials of degree p+1 with interlocking collocation points for the solution and the flux. The flux collocation points include the element boundaries, where a single valued numerical flux is imposed which is common to each element and its neighbors on the left or right. Then the value of $\frac{\partial u}{\partial t}$ at each solution point is obtained directly as the derivative of the flux polynomial. Kopriva and Kolias used Chebyshev and Chebyshev-Lobatto points as the solution and flux collocation points, and it remains unclear whether the SD scheme is stable with this choice, although they did prove the scheme to be conservative. Some years later Liu, Wang and Vinokur presented a general formulation of SD methods on both quadrilateral and triangular elements [7]. The SD method seems considerably simpler to implement than the local DG method, and it has been observed by May [8] that it may be regarded as a variant of a pre-integrated nodal DG method. While the SD method has proved robust and productive in a variety of applications [9, 10, 11, 12, 13, 14, 15], doubts have been raised about its stability. In particular it has been suggested that the SD scheme

is not stable for higher order triangular elements [16], and sometimes weakly unstable in one dimension depending on the choice of flux collocation points.

The purpose of this note is to present a proof of the stability of the SD method for the one dimensional linear advection equation for all orders of accuracy in an energy norm of Sobolev type. Specifically using solution polynomials of degree p, which are expected to yield accuracy of order p+1. The norm is

$$||u|| = \int (u^2 + cu^{(p)^2})dx$$

where the coefficient c must be determined to cause a cancellation of terms as will appear in the derivation. Because the proof rests on a comparison of the SD method with the nodal DG method, the proof that the DG method is stable is reviewed in Section 2, as a precursor to the proof that the SD method is stable, which is presented in Section 3.

2 Stability of the nodal discontinuous Galerkin method

Consider a scalar conservation law of the form (1.1). We shall generally restrict our attention to the linear advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{2.1}$$

for which the flux is $f = au$, and we assume $a > 0$, corresponding to a right running wave. In the local DG method (1.1) or (2.1) is discretized by representing the discrete solution u_h in each element as an expansion in a set of basis functions Φ_j defined within the element.

$$u_h = \sum_{j=1}^{n} u_j \Phi_j \tag{2.2}$$

Then we require the residual

$$R_h = \frac{\partial u_h}{\partial t} + \frac{\partial}{\partial x} f(u_h) = 0$$

to be orthogonal to a set of test functions which are taken to be the basis functions. Thus, integrating by parts, we require

$$\int_{x_L}^{x_R} \frac{\partial u_h^k}{\partial t}\Phi_j dx - \int_{x_L}^{x_R} f(u_h^k)\frac{\partial \Phi_j}{\partial x} dx + \hat{f}\Phi_j\Big|_{x_L}^{x_R} = 0 \tag{2.3}$$

to hold for j =1 to n, where u_h^k is the discrete solution in element k and x_L and x_R are the left and right boundaries of the element, while $\hat{f}$ is the single valued numerical flux at the interface which is also used in corresponding equations for the neighboring elements. The weak form (2.3) can be converted to the corresponding strong form by integrating the middle term back by parts.

$$\int_{x_L}^{x_R} \frac{\partial u_h^k}{\partial t}\Phi_j dx + \int_{x_L}^{x_R} \Phi_j \frac{\partial}{\partial x} f(u_h^k) dx + (\hat{f} - f(u_h^k))\Phi_j\Big|_{x_L}^{x_R} = 0 \tag{2.4}$$

Inserting the expansion (2.2) into equations (2.3) and (2.4) we obtain the equations

$$\mathbf{M}\frac{d\mathbf{u}}{dt} - \mathbf{S}^T\mathbf{f} + \hat{f}\mathbf{\Phi}\Big|_{x_L}^{x_R} = 0 \tag{2.5}$$

and

$$\mathbf{M}\frac{d\mathbf{u}}{dt} + \mathbf{S}\mathbf{f} + (\hat{f} - f)\mathbf{\Phi}\Big|_{x_L}^{x_R} = 0 \tag{2.6}$$

in weak and strong form for the local solution vectors

$$\mathbf{u}^T = [u_1, ...u_n]$$

$$\mathbf{f} = a\mathbf{u} \tag{2.7}$$

$$\mathbf{\Phi}^T = [\Phi_1, ...\Phi_n]$$

where $\mathbf{M}$ and $\mathbf{S}$ are the local mass and stiffness matrices

$$\mathbf{M}_{ij} = \int_{x_L}^{x_R} \Phi_i \Phi_j dx \tag{2.8}$$

$$\mathbf{S}_{ij} = \int_{x_L}^{x_R} \Phi_i \Phi_j' dx$$

In the nonlinear case, if we replace $f(u_h^k)$ by a local expansion in the basis functions

$$f = \sum_{j=1}^{n} f_j \Phi_j$$

we obtain the same equations with

$$\mathbf{f} = [f_1, ...f_n]$$

In this case, however, $\frac{\partial}{\partial x} f(u_h^k)$ is not exactly equal to $\sum_{j=1}^{n} f_j \Phi_j'$, so equations (2.5) and (2.6) involve a further approximation.

In the foregoing the coefficients u_j do not correspond to the value of the solution at any particular location. In the nodal DG method we introduce collocation points x_j in each element and define the local solution by the Lagrange polynomial of degree $p = n - 1$.

$$u_h^k = \sum_{j=1}^{n} u_j l_j(x)$$

where

$$l_j(x_i) = 0, i \neq j$$

$$l_j(x_j) = 1$$

and u_j is now the solution value at x_j

$$u_j = u_h^k(x_j)$$

It is also convenient to make a local transformation of each element to a reference element covering the interval [-1,1]. Then the mass and stiffness matrices in the k^{th} element become

$$\mathbf{M}_{ij}^k = (x_R - x_L)\mathbf{M}_{ij}, \tag{2.9}$$

$$\mathbf{S}_{ij}^k = \mathbf{S}_{ij}$$

where M and S are the reference mass and stiffness matrices

$$\mathbf{M}_{ij} = \int_{-1}^{1} l_i l_j dx, \tag{2.10}$$

$$\mathbf{S}_{ij} = \int_{-1}^{1} l_i l_j' dx$$

and $\Phi_j = l_j$ in the weak and strong equations (2.5) and (2.6). Also, multiplying by $\mathbf{M}^{-1}$, the strong form (2.6) can be expressed as

$$\frac{d\mathbf{u}}{dt} + D\mathbf{u} + \mathbf{M}^{-1}\mathbf{l}(\hat{f} - f)\Big|_{-1}^{1} = 0 \tag{2.11}$$

where D is the differentiation matrix

$$\mathbf{D} = \mathbf{M}^{-1}\mathbf{S} \tag{2.12}$$

Multiplying the linear advection equation (2.1) by u and integrating over x,

$$\int_a^b u\frac{\partial u}{\partial t}dx = -a\int_a^b u\frac{\partial u}{\partial x}dx = -a\int_a^b \frac{\partial(\frac{u^2}{2})}{\partial x}dx$$

Thus it satisfies the energy estimate

$$\frac{d}{dt}\int_a^b \frac{u^2}{2}dx = \frac{1}{2}a(u_a^2 - u_b^2) \tag{2.13}$$

In order to prove the stability of the nodal DG scheme we wish to prove that the discrete solution satisfies a similar estimate. For this purpose we can take the local solution, which is a linear combination of the basis polynomials, as the test function. Using the strong form, this yields

$$\mathbf{u}^T\mathbf{M}\frac{d\mathbf{u}}{dt} + \mathbf{u}^T\mathbf{S}\mathbf{f} + \mathbf{u}^T\mathbf{l}(\hat{f} - f)\Big|_{x_L}^{x_R} = 0 \tag{2.14}$$

Using the fact that $\mathbf{M}$ and $\mathbf{S}$ have been pre-integrated exactly, this is equivalent to

$$\frac{d}{dt}\int_{x_L}^{x_R} \frac{u_h^2}{2}dx + a\int_{x_L}^{x_R} u_h\frac{\partial u_h}{\partial x}dx + u_h(\hat{f} - au_h))\Big|_{x_L}^{x_R} = 0 \tag{2.15}$$

where the middle term can be integrated and combined with the last term to give

$$\frac{d}{dt}\int_{x_L}^{x_R} \frac{u_h^2}{2}dx = -(u_h\hat{f} - a\frac{u_h^2}{2}))\Big|_{x_L}^{x_R} \tag{2.16}$$

Let u_L and u_R be values of u_h on the left and right sides of a cell interface. For the numerical flux we now take

$$\hat{f} = \frac{1}{2}a(u_R + u_L) - \frac{1}{2}\alpha|a|(u_R - u_L), 0 \leq \alpha \leq 1$$

where if $\alpha = 0$ we have a central flux, and if $\alpha = 1$ we have the upwind flux. Now on summing (2.16) over the elements, the left side yields $\frac{d}{dt}\int_a^b \frac{u_h^2}{2}dx$, while at each interior interface, collecting the contributions from the elements on the left and right sides, there is a total contribution

$$\begin{aligned} & u_R\hat{f} - a\frac{u_R^2}{2} - (u_L\hat{f} - a\frac{u_L^2}{2}) \\ = & \frac{1}{2}a(u_R^2 - u_L^2) - \frac{1}{2}\alpha|a|(u_R - u_L)^2 - \frac{1}{2}a(u_R^2 - u_L^2) \\ = & -\frac{1}{2}\alpha|a|(u_R - u_L)^2 \end{aligned}$$

If we set the numerical flux to the true value au_a at the inflow boundary, and to the extrapolated upwind value au_h at the outflow boundary, it now follows that there is a negative contribution at every element boundary except the inflow boundary, where the contribution is

$$au_au_h - \frac{1}{2}au_h^2 = \frac{1}{2}au_a^2 - \frac{1}{2}a(u_a - u_h)^2$$

which is strictly less than the boundary contribution $a\frac{u_a^2}{2}$ in the true solution. This completes the proof that the DG scheme is energy stable for the linear advection equation.

3 Stability of the spectral difference method

As in the case of the DG scheme, it is convenient to represent the SD scheme in terms of a reference element covering [-1,1]. Then the discrete solution is locally represented by Lagrange interpolation on the solution collocation points x_j as

$$u_h(x) = \sum_{j=1}^{n} u_j l_j(x) \tag{3.1}$$

where for polynomials of degree p, $n = p + 1$. Correspondingly the flux is represented by a polynomial of degree $p + 1$,

$$f_h(x) = \sum_{j=1}^{n+1} f_j \hat{l}_j(x) \tag{3.2}$$

where $\hat{l}_j(x)$ are the Lagrange polynomials defined by the $n + 1$ flux collocation points $\hat{x}_j$, which include the element boundaries. At the interior flux collocation points f_j is set equal to $f(u_h(\hat{x}_j))$ where $u_h(\hat{x}_j)$ is interpolated from $u_h(x)$. At the element boundaries $f(-1)$ and $f(1)$ are defined to be the single valued numerical flux $\hat{f}$ which is common to the element and its left or right neighbor. Then we differentiate the flux polynomial at the solution collocation points to obtain

$$\frac{du_i}{dt} = -\sum_{j=1}^{n+1} f_j \hat{l}'_j(x_i) \tag{3.3}$$

Restricting our attention to the case of linear advection, $f = au$, the first step is to rewrite the flux at each boundary as

$$\hat{f}(-1) = au_h(-1) + f_{CL}, \;\; \hat{f}(1) = au_h(1) + f_{CR}$$

where f_{CL} and f_{CR} are boundary corrections

$$f_{CL} = \hat{f}(-1) - au_h(-1), \;\; f_{CR} = \hat{f}(1) - au_h(1) \tag{3.4}$$

This follows the flux reconstruction procedure proposed by Huynh [17]. Now

$$f_h(x) = f_{CL}\hat{l}_1(x) + f_{CR}\hat{l}_{n+1}(x) + a\sum_{j=1}^{n+1} u_h(\hat{x}_j)\hat{l}_j(x)$$

But since $u_h(x)$ is a polynomial of degree p, it is exactly represented by the sum. Hence

$$f_h(x) = f_{CL}\hat{l}_1(x) + f_{CR}\hat{l}_{n+1}(x) + au_h(x) \tag{3.5}$$

We can now rewrite the SD scheme as

$$\frac{\partial u_h}{\partial t} = -a\frac{\partial u_h}{\partial x} - f_{CL}\hat{l}_1' - f_{CR}\hat{l}_{n+1}'$$

Evaluating this at the solution points

$$\frac{du_i}{dt} = -a\sum_{j=1}^{n}\mathbf{D}_{ij}u_j - f_{CL}\hat{l}_1'(x_i) - f_{CR}\hat{l}_{n+1}'(x_i) \tag{3.6}$$

where $\mathbf{D}$ is the differentiation matrix associated with the solution collocation points, and is uniquely determined by the location of these points and the polynomial degree p. Thus $\mathbf{D}$ is represented by equation (2.12). In the case of an upwind numerical flux there will only be a correction from the left boundary, and in order to simplify the analysis this will now be assumed.

Equation (3.6) can be converted to a form which resembles the nodal DG method by multiplying it by the mass matrix to produce

$$\sum_j \mathbf{M}_{ij}\frac{du_j}{dt} + a\sum_j \mathbf{S}_{ij}u_j = -f_{CL}\sum_j \mathbf{M}_{ij}\hat{l}_1'(x_j) \tag{3.7}$$

Now since $\hat{l}_1(-1) = 1$ and $\hat{l}_1(1) = 0$,

$$\sum_{j=1}^{n}\mathbf{M}_{ij}\hat{l}_1'(x_j) = \int_{-1}^{1} l_i(x)\sum_{j=1}^{n}\hat{l}_1'(x_j)l_j(x)dx = \int_{-1}^{1} l_i(x)\hat{l}_1'(x)dx$$

$$= \hat{l}_1 l_i\Big|_{-1}^{1} - \int_{-1}^{1} l_i'(x)\hat{l}_1(x)dx$$

$$= -l_i(-1) - \int_{-1}^{1} l_i'(x)\hat{l}_1(x)dx$$

Thus

$$\sum_j \mathbf{M}_{ij}\frac{du_j}{dt} + a\sum_j \mathbf{S}_{ij}u_j = f_{CL}\left(l_i(-1) + \int_{-1}^{1} l_i'(x)\hat{l}_1(x)dx\right) \tag{3.8}$$

This differs from the corresponding nodal DG equation only in the last term. In order to compensate for this we can replace the mass matrix $\mathbf{M}$ by a matrix $\mathbf{Q} > 0$ such that

$$\mathbf{QD} = \mathbf{S} \tag{3.9}$$

$\mathbf{Q}$ must have the form $\mathbf{M} + \mathbf{C}$ where

$$\mathbf{CD} = 0$$

Thus each row of $\mathbf{C}$ must be orthogonal to every column of $\mathbf{D}$. Because $\mathbf{D}R_p = R_p'$ for any polynomial $R_p(x)$ of degree p, the coefficients of each row of $\mathbf{D}$ must sum to zero, so the rank of $\mathbf{D}$ is no greater than $n-1$. In order to find a row vector which is orthogonal to every column of $\mathbf{D}$, consider the p^{th} difference operator d^T which gives

$$\sum_{j=1}^{n} d_j R_p(x_j) = R_p^{(p)}$$

for any polynomial of degree p. Then

$$\sum_i d_i \sum_j D_{ij} R_p(x_j) = R_p^{(p+1)} = 0$$

Thus the matrix

$$Q = M + cdd^T$$

where c is an arbitrary parameter, satisfies equation (3.9). Also since any polynomial $R_p(x)$ of degree p can be represented exactly as

$$R_p = \sum_i R_p(x_i) l_i(x)$$

it follows that if $l_i(x)$ is expanded as

$$l_i(x) = a_i x^p + ...$$

where a_i is the leading coefficient, then

$$d_i = l_i^{(p)} = p! a_i$$

Now on multiplying equation (3.6) by $\mathbf{Q}$ instead of $\mathbf{M}$ we obtain the extra term

$$-cf_{CL}d_i\sum_j d_j\hat{l}_1'(x_j) = -cf_{CL}\hat{l}_1^{(p+1)}l_i^{(p)}$$

on the right, so that equation (3.8) is replaced by

$$\sum_j \mathbf{Q}_{ij}\frac{du_j}{dt} + a\sum_j \mathbf{S}_{ij}u_j = f_{CL}\left(l_i(-1) + \int_{-1}^{1} l_i'(x)\hat{l}_1(x)dx - c\hat{l}_1^{(p+1)}l_i^{(p)}\right) \tag{3.10}$$

Now if we can choose c so that the last two terms on the right cancel, we can attain an energy estimate with the norm $\mathbf{u}^T\mathbf{Q}\mathbf{u}$ replacing $\mathbf{u}^T\mathbf{M}\mathbf{u}$ in each element. For this purpose we can choose the interior flux collocation points as the zeros of the Legendre polynomial $L_p(x)$ of degree p. Then

$$\hat{l}_1(x) = (-1)^p\frac{1}{2}(1-x)L_p(x)$$

and

$$\int_{-1}^{1}\hat{l}_1(x)l_i'(x)dx = (-1)^{p+1}\frac{1}{2}\int_{-1}^{1} xL_p(x)l_i'(x)dx$$

since $l_i'(x)$ is a polynomial of degree $p-1$ and $L_p(x)$ is orthogonal to all polynomials of degree $< p$. Moreover only the leading term in $xl_i'(x)$ contributes to the integral for the same reason. Let

$$L_p(x) = c_p x^p + ...$$

where the leading coefficient c_p can be evaluated from Rodrigues formula

$$\begin{aligned}
L_p(x) &= \frac{1}{2^p}\frac{1}{p!}\frac{d^p}{dx^p}(x^2-1)^p \\
&= \frac{1}{2^p}\frac{1}{p!}\frac{d^p}{dx^p}(x^{2p} + ...) \\
&= \frac{1}{2^p}\frac{1}{p!}2p(2p-1)...(p+1)x^p + ... \\
&= \frac{1}{2^p}\frac{(2p)!}{(p!)^2}x^p + ... \\
&= \frac{1\cdot 3\cdot 5...\cdot(2p-1)}{p!}x^p + ...
\end{aligned}$$

Also

$$xl_i'(x) = pa_i x^p + ...$$

where a_i is the leading coefficient in $l_i(x)$. Noting that

$$\int_{-1}^{1} L_p^2 dx = \frac{2}{2p+1}$$

we obtain

$$\int_{-1}^{1} x L_p(x) l_i'(x) dx = \frac{2p}{2p+1}\frac{a_i}{c_p}$$

Also

$$\hat{l}^{(p+1)} = (-1)^{p+1}\frac{1}{2}(p+1)!c_p$$

and

$$l_i{}^{(p)} = p!a_i$$

Thus the desired cancellation is obtained by setting

$$c = \frac{2p}{2p+1}\frac{1}{c_p^2}\frac{1}{p!(p+1)!} > 0$$

In the case that the interface flux is not fully upwind, a similar calculation shows that the convection from the right boundary is correspondingly reduced, so that finally

$$\sum_j \mathbf{Q}_{ij}\frac{du_j}{dt} + a\sum_j \mathbf{S}_{ij}u_j = f_{CL}l_i(-1) - f_{CR}l_i(1) \tag{3.11}$$

Since u_h is a polynomial of degree p

$$\sum_i d_i u_i = u_h^{(p)}$$

and in each element

$$\sum_i \sum_j u_i \mathbf{Q}_{ij} u_j = \int_{xL}^{xR} (u_h^2 + c u_h^{(p)^2}) dx$$

Now the same argument that was used to prove the energy stability of the nodal DG scheme establishes the energy stability of the SD scheme with the norm

$$\int_a^b (u_h^2 + c u_h^{(p)^2}) dx$$

for the case of solution polynomials of degree p, provided that the interior flux collocation

points are the zeros of $L_p(x)$. It can be seen that c decreases very rapidly with increasing p, as is illustrated by the following table of values of c.

p	c
1	$\frac{1}{3}$
2	$\frac{4}{135}$
3	$\frac{1}{1050}$

4 Conclusion

The result is consistent with the conclusion of Van den Abeele, Lacor and Wang [16] that the stability of the spectral difference method depends only on the location of the flux collocation points. While it establishes the stability of the SD scheme when the interior flux collocation points are the zeros of the Legendre polynomial $L_p(x)$, it does not preclude the stability of the SD scheme with other choices of the flux collocation points, possibly in a different norm. However, extensive calculations for the second, third and fourth order cases (not included here) have indicated that the conditions for the cancellation of the last two terms of the boundary correction in equation (3.10) can only be satisfied by choosing the interior flux collocation points as the zeros of $L_p(x)$.

It is also interesting that as the order of accuracy is increased the norm in which the SD method is stable asymptotically approaches the usual energy norm. Whether a similar proof of stability for all orders of accuracy can be established for the multidimensional case with either tensor product or simplex elements remains a subject for future research.

References

[1] Cockburn,B., Shu, C.-W, "TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws II: General framework", Math. Comput, 52, 1989, 411-435.

[2] Cockburn,B., Shu, C.-W, "TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws III: One-dimensional systems", J. Comput. Phy., 84, 1989, 90-113.

[3] Cockburn,B., Shu, C.-W, "TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: The multidimensional case", Math. Comput, 54, 1990, 545-581.

[4] Cockburn,B., Shu, C.-W, "TVB Runge-Kutta local projection discontinuous Galerkin finite element method for conservation laws V: Multidimensional system", J. Comput. Phy., 141, 1998, 199-224.

[5] Hesthaven,J.-S, Warburton, T., "Nodal Discontinuous Galerkin Methods - Algorithms, Analysis, and Applications", Springer Verlog, 141, 2008.

[6] Kopriva,D.A, Kolias, J.H, "A conservative stagered-grid Chebyshev multidomain method for compressible flows", J. Comput. Phy., 125(1), 1996, 244-261.

[7] Liu,Y., Vinokur, M., Wang, Z.J., "Spectral difference method for unstructured grids I: Basic formulation", J. Comput. Phy., 216, 2006, 780-801.

[8] May, G., "The spectral difference scheme as a quadrature-free discontinuous Galerkin method", AICES, 11, 2008

[9] Z.J. Wang, Y. Liu, G. May and A. Jameson, "Spectral Difference Method for Unstructured Grids II: Extension to the Euler Equations, Journal of Scientific Computing, Vol. 32, No. 1, pp. 45-71 (2007).

[10] C. Liang, S. Premasuthan, A. Jameson, "High-order accurate simulation of flow past two side-by-side cylinders with Spectral Difference method", Journal of Computers and Structures, 2009, vol 87, pp. 812-817.

[11] C. Liang, S. Premasuthan, A. Jameson, Z. J. Wang, "Large Eddy Simulation of compressible turbulent channel flow with Spectral Difference method", AIAA paper, AIAA-2009-402, Orlando, FL.

[12] C. Liang, A. Jameson, Z. J. Wang, "Spectral Difference method for two-dimensional compressible flow on unstructured grids with mixed elements", Journal of Computational Physics, vol 228, pp 2847-2858, 2009.

[13] S. Premasuthan, C. Liang, A. Jameson, Z. J. Wang, "A P-multigrid Spectral Difference method for viscous flow", AIAA paper, AIAA-2009-950, Orlando, FL.

[14] K. Ou, C. Liang, S. Premasuthan, A. Jameson, "High-Order Spectral Difference Simulation of Laminar Compressible Flow Over Two Counter-Rotating Cylinders", AIAA-2009-3956, 27th AIAA AA Conference, San Antonio, June 2009.

[15] A. H. Mohammad, Z. J. Wang and C. Liang, "LES of turbulent flow past a cylinder using spectral difference method", AIAA applied aerodynamics meeting, 2008, Hawaii. AIAA-2008-7184.

[16] Van den Abeele, K., Lacor, C., Wang, Z.J., "On the Stability and Accuracy of the Spectral Difference Method", J. Sci. Comput., 37, 2008, 162-188.

[17] Huynh, H.T., "A flux reconstruction approach to high-order schemes including discontinuous Galerkin methods", AIAA 2007-4079,18th AIAA CFD Conference, Miami, June 2008.