# A Proof of Gromov's Algebraic Lemma

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**Abstract :** Following the analysis of differentiable mappings of Y. Yomdin, M. Gromov has stated a very elegant "Algebraic Lemma" which says that the "differentiable size" of an algebraic subset may be bounded in terms only of its dimension, degree and diameter - regardless of the size of the underlying coefficients. We give a complete and elementary proof of Gromov's result using the ideas presented in his Bourbaki talk as well as other necessary ingredients.

## 1 Introduction

Several problems in, e.g., Analysis and Dynamical Systems, require estimating the differentiable size of semi-algebraic subsets. Y. Yomdin developped many tools to this end [7]. M. Gromov observed that one of these tools could be refined to give the following very elegant statement :

**Theorem 1** *For all integers $r \geq 1$, $d \geq 0$, $\delta \geq 0$, there exists $M < \infty$ with the following properties. For any semi-algebraic compact subset $A \subset ]0,1[^d$ of maximum dimension $l$ and of degree $\leq \delta$, there exist an integer $N$ and maps $\phi_1, ..., \phi_N : [0,1]^l \mapsto ]0,1[^d$ satisfying $\bigcup_{i=0}^N \phi_i([0,1]^l) = A$, such that :*

- $\|\phi_{i/]0,1[^l}\|_r := \max_{\beta:|\beta|\leq r} \|\partial^\beta \phi_i\|_\infty \leq 1$ ;
- $N \leq M$ ;
- $deg(\phi_i) \leq M$.

In his Séminaire Bourbaki [11], M. Gromov gives many ideas but stops short of a complete proof. On the other hand, this result has been put to much use, especially in Dynamical System Theory. Y. Yomdin [14],[15] used it to compare the topological entropy and the "homological size" for $\mathcal{C}^r$ maps (in particular, Y. Yomdin proves in [14] Shub's conjecture in the case of $\mathcal{C}^\infty$ maps). S. Newhouse [12] then showed, using Pesin's theory, how this gives, for $\mathcal{C}^\infty$ smooth maps, upper-semicontinuity of the metric entropy and therefore the existence of invariant measures with maximum entropy. J. Buzzi [5] observed that in fact Y. Yomdin's estimates give a more uniform result called asymptotic h-expansiveness, which was in turn used by M. Boyle, D. Fiebig and U. Fiebig [3] to prove existence of principal symbolic extensions. The dynamical consequences of the above theorem are still developping in the works of M. Boyle, T. Downarowicz, S. Newhouse and others [10],[2].

The proof of this theorem is trivial in dimension 1 and easy in dimension 2 (see part 6). To prove the theorem in higher dimensions, we introduce the notion of triangular $(\mathcal{C}^\alpha, K)$-Nash maps : it is the subject of the part 3. Part 4 is devoted to the structure of semi-algebraic sets. In part 5, by taking the limit of "good" parametrizations, we reduce the main theorem to a proposition about the parametrization of semi-algebraic "smooth" maps (thus avoiding the singularities). The other difficulties are dealt with as suggested by M. Gromov. The proof by induction of this proposition is done in the last section. Describe briefly the structure of this proof. We distinguish three independent steps :

- we consider a semi-algebraic map defined on a subset of higher dimension and we bound the first derivative in the first coordinate.
- we bound the derivative of higher order in the first coordinate.
- fixing the dimension of the semi-algebraic set and the order of derivation, we bound the next derivative for the order defined on $\mathbb{N}^d$ in part 3.

As I was completing the submission of this paper, I learnt that A. Wilkie had written a proof of the same theorem [13]. I am grateful to M. Coste for this reference. In the first version of this article, M. Coste also pointed out a mistake corrected here by Remark 3.

## 2 Semi-algebraic sets and maps

First recall some basic results concerning semi-algebraic sets. We borrow from [8]. For completeness, other references are [1],[6],[7].

**Definition 1** *$A \subset \mathbb{R}^d$ is a semi-algebraic set if it can be written as a finite union of sets of the form $\{x \in \mathbb{R}^d \mid P_1(x) > 0, ...P_r(x) > 0, P_{r+1}(x) = 0, ..., P_{r+s}(x) = 0\}$, where $r, s \in \mathbb{N}$ and $P_1, ..., P_{r+s} \in \mathbb{R}[X_1...X_d]$ . Such a formula is called a presentation of $A$.*

*The degree of a presentation is the sum of the total degrees of the polynomials involved (with multiplicities). The degree of a semi-algebraic set is the minimum degree of its presentations.*

**Definition 2** *$f : A \subset \mathbb{R}^d \to \mathbb{R}^n$ is a semi-algebraic map if the graph of $f$ is a semi-algebraic set.*

**Definition 3** *A Nash manifold is an analytic submanifold of $\mathbb{R}^d$, which is a semi-algebraic set.*

*A Nash map is a map defined on a Nash manifold, which is analytic and semi-algebraic.*

We have the following description of a semi-algebraic set (See [8], Prop. 3.5 p 124 and see [7] Prop. 4.4 p 48) :

**Theorem 2** *(stratification) Let $A \subset \mathbb{R}^n$ be a semi-algebraic set. There exist an integer $N$ (bounded in terms of deg(A)) and connected Nash manifolds $A_1, ..., A_N$ such that $A = \coprod_{i=1}^N A_i$ and $\forall j \neq i$ $(A_i \bigcap adh(A_j) \neq \emptyset) \Rightarrow (A_i \subset adh(A_j)$ et $dim(A_i) < dim(A_j))$. ($\coprod$ : disjoint union).*

**Definition 4** *In the notations of the previous proposition, the maximum dimension of $A$ is the maximum dimension of the Nash manifolds $A_1, ...A_N$.*

## 3 $(\mathcal{C}^\alpha, K)$-Nash maps and triangular maps

**Definition 5** *$\mathbb{N}^d$ is provided with the order $\preceq$, defined as follows :*
*for $\alpha = (\alpha_1, ...\alpha_d)$, $\beta = (\beta_1, ...\beta_d) \in \mathbb{N}^d$*
*$\alpha \preceq \beta$ iff ($|\alpha| := \sum_i \alpha_i < |\beta|$) or ($|\alpha| = |\beta|$ et $\alpha_k \leq \beta_k$, where $k := \max\{l \leq n \ : \ \alpha_l \neq \beta_l\}$)*

**Notations 1** *The order $\preceq$ is a total order. Hence, for $\alpha \in \mathbb{N}^d$, we can set :*

$$\alpha + 1 := \min\{\beta \in \mathbb{N}^d : \alpha \preceq \beta \ \text{ and } \alpha \neq \beta\}$$

.

**Definition 6** *Let $K \in \mathbb{R}^+$, $d \in \mathbb{N}$, $\alpha \in \mathbb{N}^d - \{0\}$. Let $A \subset ]0,1[^d$ be a semi-algebraic open set.*
*A map $f : A \to ]0,1[^d$ is a <u>$\mathcal{C}^0$-Nash map</u>, if $f := (f_1, ...f_d)$ is a Nash map, which can be continuously extended to $adh(A)$. We call again $f$ this unique extension.*
*A map $f : A \to ]0,1[^d$ is a <u>$(\mathcal{C}^\alpha, K)$-Nash map</u>, if $f$ is a $\mathcal{C}^0$-Nash map and if $\|f\|_\alpha := max_{\beta \preceq \alpha, 1 \leq i \leq d} \|\partial^\beta f_i\|_\infty \leq K$. If $\alpha = (0, 0..., r)$, we write $(\mathcal{C}^r, K), \|.\|_r$ instead of $(\mathcal{C}^\alpha, K), \|.\|_\alpha$.*

The two following lemmas deal with the composition of $(\mathcal{C}^\alpha, 1)$-Nash maps.

**Lemma 1** *For all $d, r \in \mathbb{N}^*$, there exists $K < +\infty$, such that if $\psi, \phi : ]0,1[^d \to ]0,1[^d$ are two $(\mathcal{C}^r, 1)$-Nash maps, then $\psi \circ \phi$ is a $(\mathcal{C}^r, K)$-Nash map.*

PROOF : Immediate. □

One of the key points of the proof of Gromov's lemma is to control the derivatives one after one. This is made possible by the folllowing definition.

**Definition 7** *We say that a map $\psi : ]0,1[^l \to ]0,1[^d$ is <u>triangular</u> if $l \leq d$ and if there exists a family of maps $(\psi_i : ]0,1[^{min(l,d+1-i)} \to ]0,1[)_{i=1...d}$, such that*

$$\psi = (\psi_1(x_1...x_l), ..., \psi_{d-l+1}(x_1...x_l), \psi_{d-l+2}(x_2...x_l), ..., \psi_{d-l+k}(x_k...x_l), ..., \psi_d(x_l))$$

**Remark 1** *If $\psi : ]0,1[^n \to ]0,1[^m$ and $\phi : ]0,1[^m \to ]0,1[^p$ are triangular, then so is $\phi \circ \psi : ]0,1[^n \to ]0,1[^p$.*

In the case of triangular maps, we give the following version of the lemma 1. This result allows an induction on $\alpha \in \mathbb{N}^d$ rather than $r \in \mathbb{N}$, in the proof of the proposition 4.

**Lemma 2** *For all $d, r \in \mathbb{N}^*$, there exists $K < +\infty$ such that if $\psi, \phi : ]0,1[^d \to ]0,1[^d$ are two triangular $(\mathcal{C}^\alpha, 1)$-Nash maps with $|\alpha| = r$, then $\psi \circ \phi$ is a $(\mathcal{C}^\alpha, K)$-Nash map.*

PROOF : Immediate. □

**Definition 8** *<u>(resolution of a semi-algebraic set)</u> Let $M : \mathbb{N}^3 \to \mathbb{R}^+$ and let $K \in \mathbb{R}^+$, $d \in \mathbb{N}^*$. Let $A \subset [0,1]^d$ be a semi-algebraic set of maximum dimension $l$ and let $\alpha \in \mathbb{N}^d - \{0\}$. The family of maps $(\phi_i : ]0,1[^l \to ]0,1[^d)_{i=1...N}$ is a $(M)$-resolution [resp. $(\mathcal{C}^\alpha, K, M)$-resolution] of $A$ if :*

- *each $\phi_i$ is triangular ;*

- *each $\phi_i$ is a Nash map* [1] *[resp. a $(\mathcal{C}^\alpha, K)$-Nash map] ;*
- $A = \bigcup_{i=1}^N \phi_i(]0,1[^l)$ [2] *[resp. $adh(A) = \bigcup_{i=1}^N \phi_i([0,1]^l)$]*
- *$N$, $deg(\phi_i)$ are less than $M(0, d, deg(A))$ [resp. $M(|\alpha|, d, deg(A))$].*

**Definition 9** *(resolution of a family of maps) Let $M : \mathbb{N}^4 \to \mathbb{R}^+$, $K \in \mathbb{R}^+$, $d \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^d - \{0\}$ and let $f_1, ..., f_k : A \to ]0,1[$ be semi-algebraic maps, where $A \subset ]0,1[^d$ is a semi-algebraic set of maximum dimension $l$. The family of maps $(\phi_i : ]0,1[^l \to ]0,1[^d)_{i=1,...,N}$ is a $(\mathcal{C}^\alpha, K, M)$-resolution of $(f_i)_{i=1,...,k}$ if :*

- *each $\phi_i$ is triangular ;*
- *each $\phi_i$, $f_j \circ \phi_i$ is a $(\mathcal{C}^\alpha, K)$-Nash map ;*
- *$adh(A) = \bigcup_{i=1}^N \phi_i([0,1^l])$ ;*
- *$N$, $deg(\phi_i)$, $deg(f_j \circ \phi_i)$ are less than $M(|\alpha|, d, k, max_j(deg(f_j)))$.*

We shall consider only functions $M$ in the above setting that are independent of the algebraic datas (*i.e.* the functions $f_1, ..., f_k$ or the set $A$). Such function can be called "universal". By a $(\mathcal{C}^\alpha, K)$-resolution, we mean a $(\mathcal{C}^\alpha, K, M)$-resolution with a universal function $M$.

The following remark is very useful later on :

**Lemma 3** *For all $M : \mathbb{N}^2 \to \mathbb{R}^+$, there exists $M' : \mathbb{N}^2 \to \mathbb{R}^+$ such that we have the following property.*

*Let $d \in \mathbb{N}^*$, $\alpha \in \mathbb{N}^d - \{0\}$. If $f : ]0,1[^d \to ]0,1[$ is a $(\mathcal{C}^\alpha, M(|\alpha|, d))$-Nash map, then $f$ admits a $(\mathcal{C}^\alpha, 1, M')$-resolution.*

Proof : Linear reparametrizations. □

## 4 Tarski's Principle

**Proposition 1** *(Tarski's principle) Let $A$ a semi-algebraic set of $\mathbb{R}^{d+1}$ and $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ the projection defined by $\pi(x_1, ..., x_{d+1}) = (x_1, ..., x_d)$ then $\pi(A)$ is a semi-algebraic set and $deg(\pi(A))$ is bounded by a function of $deg(A)$ and $d$.*

Proof : See [6] Thm 2.2.1, p 26 and [7] Prop 4.3 p 48 □

**Corollary 1** *Any formula combining sign conditions on semi-algebraic functions by conjonction, disjunction, negation and universal and existential quantifiers defines a semi-algebraic set.*

**Corollary 2** *Let $f : A \subset \mathbb{R}^k \to \mathbb{R}^l$ be a semi-algebraic function, then $A$ is a semi-algebraic set and $deg(A)$ is bounded by a function of $deg(f)$, $k$ and $l$.*

[1] not necessarily $\mathcal{C}^0$-Nash map

[2] by convention $]0,1[^0 = \{0\}$

**Corollary 3** *If $\phi$ and $\psi$ are two semi-algebraic maps, such that the composition $\phi \circ \psi$ is well defined, then $\phi \circ \psi$ is a semi-algebraic map and its degree is bounded by a function of $deg(\phi)$ and $deg(\psi)$.*

Proof : See [6] Prop 2.2.6 p 28 □

**Proposition 2** *For all $P_1, ..., P_s \in \mathbb{R}[X_1, ..., X_{d+1}]$, there exist a partition of $]0,1[^d$ into Nash manifolds $\{A_1, ..., A_m\}$ and a finite family of Nash maps, $\zeta_{i,1} < ... < \zeta_{i,q_i} : A_i \to ]0,1[$, for all $1 \leq i \leq m$, such that :*

- *for each $i$ and each $k$, the sign $P_k(x_1, y)$, with $x_1 \in ]0,1[$ et $y := (x_2, ..., x_{d+1}) \in A_i$, only depends on the signs of $x_1 - \zeta_{i,j}(y)$, $j = 1, ..., q_i$ ;*
- *the zero set of $P_k$ coincide with the graphs of $\zeta_{i,j}$ ;*
- *the integers $m$, $q_i$, $deg(A_i)$, $deg(\zeta_{i,j})$ are bounded by a function of $\sum_k deg(P_k)$ and $d$.*

Proof : [8] Thm 2.3 p 112. □

From the above we deduce easily the following proposition :

**Proposition 3** *For all semi-algebraic subsets $A \subset ]0,1[^{d+1}$ , there exist integers $m$, $q_1, ..., q_m$, a partition of $]0,1[^d$ into Nash manifolds $A_1, ..., A_m$ and Nash maps, $\zeta_{i,1} < ... < \zeta_{i,q_i} : A_i \to ]0,1[$, for all $1 \leq i \leq m$, such that :*

- *$A$ coincide with a union of slices of the two following forms $\{(x_1, y) \in ]0,1[ \times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\}$ and $\{(\zeta_{i,k}(y), y) : y \in A_i\}$;*
- *the integers $m$, $q_i$, $deg(A_i)$, $deg(\zeta_{i,j})$ are bounded by a function of $deg(A)$ and $d$.*

For open semi-algebraic sets, we have the following result :

**Corollary 4** *For all semi-algebraic open subsets $A \subset ]0,1[^{d+1}$, there exist integers $m$, $q_1, ..., q_m$, disjoint semi-algebraic open sets $A_1, ..., A_m$ and Nash maps, $\zeta_{i,1} < ... < \zeta_{i,q_i} : A_i \to ]0,1[$, for all $1 \leq i \leq m$, such that :*

- *$adh(A)$ coincide with a union of "slices" of the following form $adh(\{(x_1, y) \in ]0,1[ \times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\})$ ;*
- *the integers $m$, $q_i$, $deg(A_i)$, $deg(\zeta_{i,j})$ are bounded by a function of $deg(A)$ and $d$.*

In the following corollary, we reparametrize a semi-algebraic set with Nash maps of bounded degree.

**Corollary 5** *(decomposition into cells) There exists $M : \mathbb{N}^3 \to \mathbb{R}^+$, such that any semi-algebraic set $A \subset ]0,1[^d$ admits a $(M)$-resolution.*

Proof : We argue by induction on $d$. We note $P(d)$ the claim of the above corollary. $P(0)$ is trivial. Assume $P(d)$.

Let $A \subset ]0,1[^{d+1}$ be a semi-algebraic set of maximum dimension $l$. Proposition 3 gives us integers $m$, $q_1, ..., q_m$, Nash manifolds $A_1, ..., A_m \subset ]0,1[^d$ and Nash maps, $\zeta_{i,1} < ... < \zeta_{i,q_i} : A_i \to ]0,1[$ such that :

- $A$ coincides with an union of slices of the two following forms $\{(x_1, y) \in ]0,1[\times A_i : \zeta_{i,k}(y) < x_1 < \zeta_{i,k+1}(y)\}$ and $\{(\zeta_{i,k}(y), y) : y \in A_i\}$;
- $m$, $q_i$, $deg(A_i)$ are bounded by a function of $deg(A)$ and $d$.

Let $1 \leq i \leq m$. We apply the induction hypothesis to $A_i \subset ]0,1[^d$ : there exist a resolution of $A_i$, *i.e.* an integer $N_i$ (bounded by a function of $deg(A_i)$ and $d$, therefore by a function of $deg(A)$ and $d$) and Nash maps $\phi_{i,1}, ..., \phi_{i,N_i} : ]0,1[^l \rightarrow ]0,1[^d$, such that $A_i = \bigcup_{p=1}^{N_i} \phi_{i,p}(]0,1[^l)$. Then we define $\psi_{i,k,p} : ]0,1[^l \rightarrow ]0,1[^{d+1}$ as follows : $\psi_{i,k,p}(x_1, y) := (x_1(\zeta_{i,k+1}(y) - \zeta_{i,k}(y)) \circ \phi_{i,p}(x_1, ..., x_l) + \zeta_{i,k} \circ \phi_{i,p}(x_1, ..., x_l), \phi_{i,p}(x_1, ..., x_l))$ for the first form and $\psi_{i,k,p}(x_1, y) := (\zeta_{i,k} \circ \phi_{i,p}(x_1, ..., x_l), \phi_{i,p}(x_1, ..., x_l))$ for the second form.

The $\psi_{i,k,p}$ are Nash triangular maps, such that

- the number of these parametrizations is bounded by $3\sum_{i=1}^{m} q_i N_i$
- $A = \bigcup_{i,k,p} \psi_{i,k,p}(]0,1[^l)$
- $deg(\psi_{i,k,p})$ is bounded by a function of $deg(A)$ and $d$ (See Corollary 3)

Thus these maps form a resolution of $A$. □

The following lemma is another application of the Tarski's principle :

**Lemma 4** *Let $A \subset \mathbb{R}^d$ be a semi-algebraic open set, $f : A \rightarrow \mathbb{R}^n$ a Nash map defined on $A$. The partial derivatives of $f$ of all orders are also semi-algebraic maps of degree bounded by a function of $deg(f)$, $d$ and $n$.*

Proof : Apply corollary 1. See [6] p 29. □

## 5 Proof of the Yomdin-Gromov Theorem

First we show the following technical proposition, in which we work with "smooth" functions. Finally we explain how we reduce the proof of the main theorem to this proposition.

**Proposition 4** *Let $A \subset ]0,1[^d$ be a semi-algebraic open set. Let $f_1, ..., f_k : A \rightarrow ]0,1[$ be Nash maps and let $\alpha \in \mathbb{N}^d - \{0\}$. There exists a sequence $(A_n)_{n\in\mathbb{N}} \subset A^{\mathbb{N}}$ of semi-algebraic sets, such that*

- $a_n := \sup_{x\in A} d(x, A_n) \xrightarrow[n\rightarrow+\infty]{} 0$ (#), *where $d(x, A_n)$ is the distance between $x$ and $A_n$ ;*
- *$deg(A_n)$ is bounded by a function of $\max_i(deg(f_i))$, $|\alpha|$ and $d$ ;*
- *$(f_{i/A_n})_{i=1,...,k}$ admits a $(\mathcal{C}^\alpha, 1)$-resolution.*

*We will say that such a sequence $(A_n)_{n\in\mathbb{N}}$ is $\alpha$-adapted to $(f_i)_{i=1,...,k}$.*

The following corollary follows from the above proposition :

**Corollary 6** *There exists $M : \mathbb{N}^4 \to \mathbb{R}^+$, such that for all integers $k \geq 1$, $d \geq 1$, multiindices $\alpha \in \mathbb{N}^d - \{0\}$, any family $(f_i : A \to ]0,1[)_{i=1,...,k}$ of Nash maps, where $A \subset ]0,1[^d$ is a semi-algebraic open set, admits a $(\mathcal{C}^\alpha, 1, M)$-resolution.*

Now we show how Proposition 4, Corollary 6 and the Yomdin-Gromov theorem follow from the case $k = 1$ of the proposition 4. In fact we show stronger results, which are used in the induction in the last section.

**Notations 2** *We consider the set $E$ of pairs $(\alpha, d)$, where $d \in \mathbb{N}^*$ and $\alpha \in \mathbb{N}^d - \{0\}$. The set $E$ is provided with the following order $\ll$ :*

*$(\beta, e) \ll (\alpha, d)$ iff $(e < d)$ or $(e = d$ and $\beta \preceq \alpha)$*

*We will write :*
*$P4(\alpha, d)$ the claim of the proposition 4 for all pairs $(\beta, e)$ with $(\beta, e) \ll (\alpha, d)$.*
*$C6(\alpha, d)$ the claim of the corollary 6 for all pairs $(\beta, e)$ with $(\beta, e) \ll (\alpha, d)$.*
*$YG(\alpha, d)$ the existence of a $(\mathcal{C}^\beta, 1)$ resolution for all Nash manifolds $A \subset [0,1]^e$, and for all pairs $(\beta, e) \ll (\alpha, d)$.*

**Remark 2** *With the above notations, we have : theorem 1 $\Longleftrightarrow YG(\alpha, d) \quad \forall(\alpha, d) \in E$.*

**Lemma 5** *The claim $C6(\alpha, d)$ [resp. $P4(\alpha, d)$] for $k = 1$ implies the claim $C6(\alpha, d)$ [resp. $P4(\alpha, d)$] for all $k \in \mathbb{N}^*$.*

PROOF OF LEMMA 5 (CASE OF $C6(\alpha, d)$) : We argue by induction on $k$.

Assume that for $k$-families $g_1, ..., g_k : B \to ]0,1[$ of Nash maps, with $B \subset ]0,1[^d$ a semi-algebraic open set, admit a $(\mathcal{C}^\alpha, 1)$-resolution.

Let $f_1, ..., f_{k+1} : A \to ]0,1[$ be Nash maps, with $A \subset ]0,1[^d$ a semi-algebraic open set. According to the induction hypothesis, there exists $(\phi_i)_{i=1,...,N}$ a $(\mathcal{C}^\alpha, 1)$-resolution of $(f_1, ..., f_k)$. According to $C6(\alpha, d)$ for $k = 1$, for each $i$ we can find $(\psi_{i,j})_{j=1,...,N_i}$ a $(\mathcal{C}^\alpha, 1)$-resolution of $f_{k+1} \circ \phi_i$. According to Lemma 2, the maps $\phi_i \circ \psi_{i,j}$, of which the number is $\sum_{i=1}^N N_i$, are $(\mathcal{C}^\alpha, K)$-Nash triangular maps, as well as the maps $f_p \circ \phi_i \circ \psi_{i,j}$ for all $1 \leq p \leq k$ (with $K = K(|\alpha|, d)$). Finally, for each $i$, $(\psi_{i,j})_{j=1,...,N_i}$ being a $(\mathcal{C}^\alpha, 1)$-resolution of $f_{k+1} \circ \phi_i$, the maps $f_{k+1} \circ \phi_i \circ \psi_{i,j}$ are $(\mathcal{C}^\alpha, 1)$-Nash maps. Moreover, we have in a trivial way : $adh(A) = \bigcup_{i,j} \phi_i \circ \psi_j([0,1]^d)$. We conclude the proof for $C6(\alpha, d)$ thanks to Lemma 3. □

PROOF OF LEMMA 5 (CASE OF $P4(\alpha, d)$) :

We adapt the above proof for $P4(\alpha, d)$ as follows. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence $\alpha$-adapted to $(f_i)_{i=1,...,k}$. Hence, for all $n \in \mathbb{N}$, there exists $(\phi_j^n)_{j=1,...,N_n}$ a $(\mathcal{C}^\alpha, 1)$ resolution of $(f_{i/A_n})_{i=1,...,k}$. For $n$, $j$, let $(A_p^{n,j})_{p\in\mathbb{N}}$ be a sequence $\alpha$-adapted to $f_{k+1} \circ \phi_j^n$. We use the following remark, which is an easy consequence of the compactness of $[0,1]^d$ :

**Remark 3** *If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of $]0,1[^l$ satisfying $\sup_{x\in]0,1[^l} d(x, A_n) \xrightarrow[n\to+\infty]{} 0$ and $\phi : ]0,1[^l \to ]0,1[^d$ is a continuous map [3], then $\sup_{x\in\phi(]0,1[^l)} d(x, \phi(A_n)) \xrightarrow[n\to+\infty]{} 0$.*

[3] possibly not uniformly continuous

According to the above remark, we can choose an integer $p_{j,n}$ for each $n \in \mathbb{N}$ and each $1 \leq j \leq N_n$, such that $\sup_{x\in\phi_j^n(]0,1[^l)} d(x, \phi_j^n(A_{p_{j,n}}^{n,j})) < 1/n$. Now, let us show that $B_n := \bigcup_{j=1}^{N_n} \phi_j^n(A_{p_{j,n}}^{n,j})$ defines a sequence $\alpha$-adapted to $(f_i)_{i=1,...,k+1}$.

Observe that $B_n$ is a semi-algebraic set because each $\phi_j^n$ is a semi-algebraic map and each $A_p^{n,j}$ are semi-algebraic sets. Moreover $N_n$, $deg(\phi_j^n)$ and $deg(A_{p_{j,n}}^{n,j})$ and therefore $deg(B_n)$ are bounded by a function of $\max_i(deg(f_i))$, $|\alpha|$ and $d$. Finally, we check the "density condition" (#).

$\sup_{x\in A} d(x, B_n) \leq \sup_{x\in A} d(x, A_n) + \max_{j=1,...,N_n}(\sup_{x\in\phi_j^n(]0,1[^l)} d(x, \phi_j^n(A_{p_{j,n}}^{n,j})) \leq a_n + 1/n \xrightarrow[n\to+\infty]{} 0.$ □

PROOF OF COROLLARY 6 ($P4((0,...,0,r+1),d) \Rightarrow C6((0,...,0,r),d).$) : According to Lemma 5, it is enough to consider a single Nash map $f : A \to ]0,1[$, where $A \subset ]0,1[^d$ is a semi-algebraic open set. According to $P4((0,...,0,r+1),d)$, there exists a $(0,...,0,r+1)$-adapted sequence $(A_n)_{n\in\mathbb{N}}$ to $f$. Let $(\phi_i^k)_{i\leq N_k}$ be a $(\mathcal{C}^{r+1},1)$-resolution of $f_{/A_k}$. By hypothesis, $N_k$ is bounded by a function of $deg(A_k)$ and $r$ and thus by a function of $deg(A)$ and $r$ ; consequently $(N_k)_{k\in\mathbb{N}}$ is a bounded sequence. By extracting a subsequence, we can assume $N_k = N$, for all $k \in \mathbb{N}$. According to the Ascoli theorem, $B(r+1)^{2N}$ is a compact set in $B(r)^{2N}$, where $B(r)$ is the closed unit ball of $\mathcal{C}^r(]0,1[^d)$ (set of $\mathcal{C}^r$ maps on $]0,1[^d$ onto $\mathbb{R}$). By extracting again a subsequence, we can assume that for each $i = 1,...,N$ $(\phi_i^n)_{n\in\mathbb{N}}$ converge in $\|.\|_r$ norm to a $(\mathcal{C}^r,1)$-Nash map, $\psi_i$. Obviously $f \circ \psi_i$ is a $(\mathcal{C}^r,1)$-Nash map. One only needs to see $\bigcup_{i=1,...,N} \psi_i([0,1]^d) = adh(A)$. It is enough to show that $A \subset \bigcup_{i=1,...,N} \psi_i([0,1]^d)$. We have $\psi_i([0,1]^d) \subset adh(A)$, for all $i$, by convergence of $\phi_i^n$ to $\psi_i$. Let $x \in A$. According to the "density condition" (#), there exists a sequence $x_n \in A_n$, such that $x_n \to x$. By extracting a subsequence, we can assume that there exist $1 \leq i \leq N$ and a sequence $(y_n \in [0,1]^d)_{n\in\mathbb{N}}$ such that $x_n = \phi_i^n(y_n)$. By the uniform convergence of $\phi_i^n$ to $\psi_i$, we have $\psi_i(y_n) \to x$. We easily conclude that $\bigcup_{i=1,...,N} \psi_i([0,1]^d) = adh(A)$. Finally $(\psi_i)_{i\leq N}$ is a $(\mathcal{C}^r,1)$-resolution of $f$. □

PROOF OF THEOREM 1 ( $C6(\alpha,d) \Rightarrow YG(\alpha,d+1)$) :

Under Proposition 3, it is enough to consider the two following special cases :

1. $A \subset ]0,1[^{d+1}$ is a semi-algebraic set of the form : $\{(x_1,y) \in ]0,1[\times A' : \eta(y) < x_1 < \zeta(y)\}$, where $A' \subset ]0,1[^d$ is a semi-algebraic set of maximum dimension $e$ and $\eta, \zeta : A' \to ]0,1[$ Nash maps, such that $deg(\eta), deg(\zeta), deg(A')$ depend only on $deg(A)$ and $d$. By using a $\alpha$-resolution of $A'$ $(\phi_i : ]0,1[^e \to ]0,1[^d)_{i=1,...,N}$ and by considering $\eta \circ \phi_i$ and $\zeta \circ \phi_i$, we can assume that $A' = ]0,1[^e$, with $e \leq d$.

   Applying $C6(\alpha,d)$ to $(\zeta,\eta)$, there exists $(\phi_i)_{i=1,...,N}$ a $(\mathcal{C}^\alpha,1)$-resolution of $(\zeta,\eta)$.

   For each $i$, we define $\psi_i : ]0,1[\times]0,1[^e \to ]0,1[^{d+1}$ in the following way : $\psi_i(x,y) = (x(\zeta\circ\phi_i - \eta\circ\phi_i)(y) + \eta\circ\phi_i(y), \phi_i(y))$. Then $(\psi_i)_{i=1,...,N}$ is a $(\mathcal{C}^\alpha,2)$-resolution of $A$. We conclude the proof using Lemma 3.

2. $A$ is a semi-algebraic set of the form $\{(\zeta_{i,k}(y), y) : y \in A'\}$. The decomposition into cells gives (see Corollary 5) us a resolution of $A$, $(\phi_i : ]0,1[^l \to ]0,1[^{d+1})_{i=1,...,N}$, with $l < d+1$. We conclude the proof, by applying for each $i$, $C6(\alpha,d)$ to the coordinates of $\phi_i$.

□

## 6 Proof of Corollary 6 in dimension 1

First we study the case of dimension 1, where we can prove right away Corollary 6. The case of dimension 1 allows us to introduce simple ideas of parametrizations, which will be adapted in higher dimensions.

The semi-algebraic sets of $]0,1[$ are the finite unions of open intervals and points. So it's enough to prove the Corollary 6 for $A$ of the form $]a,b[\subset]0,1[$.

Proof of $C6(1,1)$ (Case of the first derivative) : Let $f:]a,b[\to]0,1[$ be a $\mathcal{C}^0$-Nash map. [4] We cut the interval $]a,b[$ into a minimal number $N$ of subintervals $(J_k)_{k=1,\dots,N}$, such thatfor each $k$, $\forall x\in J_k$, $|f'(x)|\geq 1$ or $\forall x\in J_k$, $|f'(x)|\leq 1$.

The required bound on $N$ results from Tarski's principle.
On each interval $J_k$, we consider the following parametrization $\phi$ of $adh(J_k)=[c,d]\subset[0,1]$ :

- $\phi(t)=c+t(d-c)$ if $|f'|\leq 1$ and then we have $deg(\phi)=1$, $deg(f\circ\phi)=deg(f)$.
- $\phi(t)=f^{-1}_{|[c,d]}(f(c)+t(f(d)-f(c)))$ if $|f'|\geq 1$ and then we have $deg(\phi)=deg(f)$ (indeed $deg(f^{-1})=deg(f)$) and $deg(f\circ\phi)=1$.

□

Proof of $C6(r,1)$ (Case of higher derivatives) : We argue by induction on $r$ : assume $C6(r,1)$, with $r\geq 1$ and prove $C6(r+1,1)$.

Let $f:]a,b[\subset]0,1[\to]0,1[$ be a $\mathcal{C}^0$- Nash map. By considering $(f\circ\phi_i)_{i=1,\dots,N}$, where $(\phi_i)_{i=1\dots N}$ is a $(\mathcal{C}^r,1)$-resolution of $f$ given by $C6(r,1)$, we can assume that $f$ is a $(C^r,1)$-Nash map.

We divide the interval $]a,b[$ into a minimal number $n_i$ of subintervals on which $|f^{(r+1)}|$ is either increasing or decreasing, *ie*, the sign of $f^{(r+1)}f^{(r+2)}$ is constant. Consider the case where $|f^{(r+1)}|$ is decreasing, the increasing case being similar. We reparametrize those intervals from $[0,1]$ with linear increasing maps $\widetilde{\phi}_i$. We define $f_i=f\circ\widetilde{\phi}_i$. Obviously $f_i$ is $C^r,1)$-Nash map and $|f_i^{(r+1)}|$ is decreasing. In the following computations, we note $f$ instead of $f_i$.

Setting $h(x)=x^2$, we have :

$$(f\circ h)^{(r+1)}(x)=(2x)^{r+1}f^{(r+1)}(x^2)+R(x,f(x),...f^{(r)}(x))$$

where $R$ is a polynomial depending only on $r$. Therefore

$$\forall x\in[0,1]\quad |(f\circ h)^{(r+1)}(x)|\leq|(2x)^{r+1}f^{(r+1)}(x^2)|+C(r),$$

where $C(r)$ is a function of $r$.
Furthermore, we have

$$x|f^{(r+1)}(x)|=\int_0^x|f^{(r+1)}(x)|dt\leq|\int_0^x f^{(r+1)}(t)dt|=|f^{(r)}(x)-f^{(r)}(0)|\leq 2 \tag{1}$$

[4]In dimension 1, a bounded Nash map (defined on a bounded intervall) is a $\mathcal{C}^0$-Nash map (See [6] p 30)

thus

$$|(f\circ h)^{(r+1)}(x)| \leq C(r) + 2\frac{(2x)^{r+1}}{x^2} \leq C(r) + 2^{r+2}$$

Enfin $deg(\widetilde{\phi_i}\circ h) = 2$ and $deg(f\circ h) = 2deg(f)$. The claim concerning the integers $n_i$ results from the Tarski's principle. We conclude the proof of $C6(r+1,d)$ thanks to the lemma 3.

□

## 7 Proof of Proposition 4

Let us fix two integers $r \geq 2, c \geq 1$. In this section we show $P4((0,...,0,r-1),c)$ for $k=1$, as this implies the general case by Lemma 5.

We argue by induction on the set $E_{rc}$ of pairs $(\alpha, d)$, where $d \in \mathbb{N}^*, d \leq c$ and $\alpha \in \mathbb{N}^d, |\alpha| \leq r+c-d$. $E_{rc}$ is provided with the order $\ll$.

We assume now that $P4(\alpha,d)$ is checked and we distinguish three cases depending on the values of the pair $(\alpha, d)$ :

*Increase of the dimension :* $P4((0,...,0,r+c-d),d) \Rightarrow P4((1,0,...,0),d+1)$

Proof :

**Claim 1** *It is enough to show the result for Nash maps* $f : ]0,1[^{d+1} \to ]0,1[$.

Proof of Claim 1 :

Let $f : A \subset ]0,1[^{d+1} \to ]0,1[$ a Nash map, defined on a semi-algebraic open set of $\mathbb{R}^{d+1}$.

Consider a resolution $(\phi_i : [0,1]^{d+1} \to [0,1]^{d+1})_{i=1,...,N}$ of $A$ given by Lemma 5. If $(A_n)_{n\in\mathbb{N}}$ is an adapted sequence to $(f\circ\phi_i, \phi_i)$ and $(\psi_j^{i,n})_{j=1,...,N_{i,n}}$ a $C^{(1,0,...,0)}$ resolution of $(f\circ\phi_{i/A_n}, \phi_{i/A_n})$, then under Remark 3, the sequence $(B_n)_{n\in\mathbb{N}}$, defined as follows $B_n = \bigcup_{i=1,...,N}\phi_i(A_n)$, is an adapted sequence to $f$ with $(\phi_i\circ\psi_j^{i,n})_{i,j}$ as a resolution of $f_{/B_n}$. □

We work on $A_n = ]1/n, 1-1/n[^{d+1}$ in order to ensure that $f$ extends continuously on $adh(A_n)$. For simplicity, we note $A$ instead of $A_n$.

We consider the following semi-algebraic open sets : $A_+ = int(\{x \in A, |\partial_{x_1} f(x)| > 1\})$ and $A_- = int(\{x\in A, |\partial_{x_1} f(x)| \leq 1\})$. We have $adh(A) = adh(A_+)\bigcup adh(A_-)$. Obviously $adh(A_+)\bigcup adh(A_-) \subset adh(A)$. Let show $A \subset adh(A_+)\bigcup adh(A_-)$. Let $x \in A$. If $d(x, A_n^+) = 0$, then $x \in adh(A_+)$ ; if not, as $A$ is an open set, there exists $r > 0$, such that the ball $B(x,r) \subset A \bigcap A_+^c \subset \{x \in A, |\partial_{x_1} f(x)| \leq 1\}$ and thus $x \in A_-$.

According to $P4((0,...,0,r+c-d),d) \Rightarrow P4((0,...,0,2),d) \Rightarrow C6((0,...,0,1,d)) \Rightarrow YG((0,...,0,1),d+1)$, there exist $(\mathcal{C}^{(1,0,...0)},1)$-Nash triangular maps $(\phi_j)_{1\leq j\leq N}$ such that $adh(A_-) = \bigcup_{1\leq j\leq N_-}\phi_j([0,1]^d)$ and such that $N_-$, $deg(\phi_j)$ are bounded by a function of $deg(A_-)$, and thus by a function of $deg(f)$ (according to the lemma 4 and the corollary 6). We have $|\partial_{x_1}(f\circ\phi_j)| \leq 1$, so the maps $\phi_i$ can be used to build a resolution of $f$.

For $A_+$, we consider the inverse of $f$. Observe first, that according to the corollary 4, we can assume that $A_+$ is a slice of the following form $\{(x_1, y) \in ]0, 1[\times A'_+ \ : \zeta(y) < x_1 < \eta(y)\}$, where $A'_+ \subset ]0, 1[^d$ is a semi-algebraic open set of $\mathbb{R}^d$ and $\zeta, \eta : A'_+ \to ]0, 1[$ are Nash maps.

Define $D_+ = \{(f(x_1, y), y) : (x_1, y) \in A_+\}$. We define $g : A_+ \to D_+$, $g(x_1, y_1) = (f(x_1, y), y))$. This map $g$ is a local diffeomorphism, by the local inversion theorem. Moreover, $g$ is one to one, because $g(x_1, y) = g(x'_1, y')$ implies $y = y'$, and $f(x_1, y) = f(x'_1, y)$ implies $x_1 = x'_1$, because $|\partial_{x_1} f(x)| \geq 1$ for $x \in A_+$. The map $g$ extends to $g : adh(A_+) \to adh(D_+)$, a homeomorphism, since $f$ is continuous on $adh(A)$ (Recall that we note $A := A_n$).

Observe that $D_+$ is a semi-algebraic open set of $\mathbb{R}^{d+1}$. On $D_+$ we define $\phi : \phi(t, u) := g^{-1}(t, u) = (f(., u)^{-1}(t), u)$. The Nash map $\phi : D_+ \to A_+$ is triangular and $deg(\phi) = deg(f)$. Define $\phi(t, u) = (x_1, y)$. We compute :

$$D\phi(t, u) = \begin{pmatrix} \frac{1}{\partial_{x_1} f(x_1,y)} & -\frac{1}{\partial_{x_1} f} \nabla_y f(x_1, y) \\ 0 & Id \end{pmatrix}$$

As $(x_1, y) \in A_+$, we have $|\partial_{x_1} \phi| \leq 1$. Furthermore, we check

$$f \circ \phi(t, u) = t.$$

Therefore, $\phi$ and $f \circ \phi$ are $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$-Nash triangular maps. In order to obtain a resolution, we apply again $YG((0, 0, ..., 0, 1), d + 1)$ to $adh(D_+)$. That gives a $(C^{(1,0,\ldots,0)}, 1)$ -Nash triangular parametrization $\psi_j : [0, 1]^{d+1} \to adh(D_+)$, $j \leq N_+$, such that $N_+$, $deg(\psi_j)$ are bounded by a function of $deg(D_+)$, thus by a function of $deg(f)$. Moreover

$$|\partial_{x_1}(\phi \circ \psi_j)| = |\partial_{x_1}(\phi)|.|\partial_{x_1}(\psi^1_j)| \leq 1$$

because $\psi_j$ is triangular and

$$|\partial_{x_1}(f \circ \phi \circ \psi_j)| = |\partial_{x_1} \psi^1_j| \leq 1,$$

where $\psi_j := (\psi^1_j, ..., \psi^{d+1}_j)$. The following parametrizations $\phi \circ \psi_j : [0, 1]^{d+1} \mapsto [0, 1]^{d+1}$ are therefore $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$-Nash triangular maps such that :

- $adh(A_+) = \bigcup_{j=1}^{N_+} \phi \circ \psi_j([0, 1]^{d+1})$ ;
- each $f \circ \phi \circ \psi_j$ is a $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$-Nash map ;
- $deg(\phi \circ \psi_j)$, $deg(f \circ \phi \circ \psi_j)$ are bounded by a function of $|\alpha|, d$, and $deg(f)$ (See Corollary 3).

Finally, we combine the maps $\phi_1, ..., \phi_{N_-}$ with the maps $\phi \circ \psi_1, ..., \phi \circ \psi_{N_+}$, so that we obtain a $(\mathcal{C}^{(1,0,\ldots,0)}, 1)$-resolution of $f$. The bound on the number of parametrizations is the result of the bounds on $N_-$ and $N_+$ from the Yomdin-Gromov theorem and of the bounds from the proposition 2.

□

*Increase of the derivation order :* $P4((0, ..., 0, s), d) \Rightarrow P4((s + 1, 0, ..., 0), d)$ pour $s < r + c - d$

Until the end, $C(|\alpha|, d)$ are functions of $|\alpha|$ and $d$.

Proof :

In this case, we adapt the proof in dimension 1. We begin with a remark similar to the previous Claim 1.

**Claim 2** *It is enough to show the result for $(\mathcal{C}^s, 1)$-Nash maps $f : A = ]0,1[^d \rightarrow ]0,1[$.*

Proof of claim 2 : Let $f : A \subset ]0,1[^{d+1} \rightarrow ]0,1[$ a Nash map, defined on a semi-algebraic open set of $\mathbb{R}^{d+1}$. By applying $P4((0,...,0,s),d)$ to $f$, we obtain a $(\mathcal{C}^s,1)$-resolution $(\phi_i^n)_{i=1,...,N_n}$ of $f_{/A_n}$, with $A_n$ an adapted sequence. We conclude by applying $P4((s+1,0,...,0),d)$ to the family of $(\mathcal{C}^s,1)$-Nash maps $(f \circ \phi_i^n, \phi_i^n)$, and by applying remark 3. □

Let $f : A = ]0,1[^d \rightarrow ]0,1[$ be a $(\mathcal{C}^s,1)$-Nash map.

We cut up $]0,1[^d$ according to the sign $\frac{\partial^{s+1} f}{\partial x_1^{s+1}} \frac{\partial^{s+2} f}{\partial x_1^{s+2}}$, and we assume (See corollary 4) that $A$ is a slice of the following form $\{(x_1,y) \in ]0,1[ \times A' \ : \zeta(y) < x_1 < \eta(y)\}$, where $A' \subset ]0,1[^{d-1}$ is a semi-algebraic open set and $\zeta, \eta : A' \rightarrow ]0,1[$ are Nash maps.

Applying the estimate (1) obtained in section 6 to the function $x_1 \mapsto \frac{\partial^{s+1} f}{\partial x_1^{s+1}}(x_1,y)$ (we fix $y$), we get

$$|\frac{\partial^{s+1} f}{\partial x_1^{s+1}}(x_1,y)| \leq \frac{2}{|x_1 - \zeta(y)|} \tag{2}$$

or

$$|\frac{\partial^{s+1} f}{\partial x_1^{s+1}}(x_1,y)| \leq \frac{2}{|x_1 - \eta(y)}|, \tag{3}$$

according to the sign of $\frac{\partial^{s+1} f}{\partial x_1^{s+1}} \frac{\partial^{s+2} f}{\partial x_1^{s+2}}$.

The induction hypothesis $P4((0,...,0,s),d)$ implies $P4((0,...,0,s+2),d-1)$, because $(0...0,s+2),d-1) \ll ((0,...,0,s),d))$ and $P4((0,...,0,s+2),d-1)$ implies $C6((0,...,0,s+1),d-1)$. Apply $C6((0,...,0,s+1),d-1)$ to $(\zeta,\eta)$ : there exist $(\mathcal{C}^{s+1},d-1)$-Nash triangular maps $h : [0,1]^{d-1} \rightarrow [0,1]^{d-1}$, of which the images cover $adh(A')$, such that $\zeta \circ h$ and $\eta \circ h$ are $(\mathcal{C}^{s+1},d-1)$-Nash maps. Define $\psi : [0,1] \times [0,1]^{d-1} \rightarrow adh(A)$,

$$\psi(v_1,w) = (\zeta \circ h(w).(1-v_1^2) + \eta \circ h(w).v_1^2, h(w))$$

The map $\psi$ is triangular and $\|\psi\|_{s+1} \leq 2$.

In the new coordinates $(v_1,v_2,...,v_d)$, the previous estimates (2) and (3) become, with $w = (v_2,...,v_d)$ :

$$|\frac{\partial^{s+1} f}{\partial x_1^{s+1}}(\psi(v_1,w))| \leq \frac{2}{v_1^2|\eta \circ h(w) - \zeta \circ h(w)|}$$

Moreover, $\frac{\partial^{s+1}(f\circ\psi)}{\partial v_1^{s+1}}(v_1,w) = (2v_1)^{s+1}(\eta \circ h(w) - \zeta \circ h(w))\frac{\partial^{s+1} f}{\partial x_1^{s+1}}(\psi(v_1,w)) + R(\eta \circ h(w) - \zeta \circ h(w), v_1, (\frac{\partial^k f}{\partial x_1^k}(\psi(v_1,w)))_{k\leq s})$, where $R$ is a polynomial, which depends only on $s$. The first part is less than $2^{s-1}$. Consider the second part. The map $f$ is a $(\mathcal{C}^s,1)$-Nash map, therefore $|\frac{\partial^k f}{\partial x_1^k}| \leq 1$, for $k \leq s$ ; thus $|R(\eta \circ h(w) - \zeta \circ h(w), v_1, (\frac{\partial^k f}{\partial x_1^k}(\psi(v_1,w)))_{k\leq s})|$ is bounded by a function of $s$, and therefore $|\frac{\partial^{s+1}(f\circ\psi)}{\partial v_1^{s+1}}| \leq C(s,d)$. According to lemma 1, the derivatives of lower order than $s$ of $f \circ \psi$ are also bounded by a function of $s$. Using Lemma 3, we can assume that $\psi$ is a $(\mathcal{C}^{s+1}, 1)$-Nash map and $f \circ \psi$ is a $(\mathcal{C}^{(s+1,0,...,0)}, 1)$-Nash map.

By Lemma 4, Proposition 2 and $C6((0,...,0,s+1), d-1)$ the number of parametrizations $h$ and their degree are also bounded by such a function. It follows that the total number of parametrizations $\psi$ is bounded by a function of $d$ and of $deg(f)$. We conclude using Corollary 3, that the same holds for the degree of the parametrizations $\psi$.

□

*Control of the following derivative :* $P4(\alpha, d) \Rightarrow P4(\alpha+1, d)$ with $\alpha \neq (0,...,0,s+1)$

Proof : According to the Claim 2, we can assume that $f : ]0,1[^d \to ]0,1[$ is a $(\mathcal{C}^{\alpha}, 1)$-Nash map.

Define $A_n = ]1/n, 1-1/n[^{d-1}$ and $b_n = 1 - 2/n$. According to the Tarski's principle (See Corollary 1), $B = \{(x_1, y) \in adh(A_n) \ : \ |\frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(x_1, y)| = sup_{t \in [1/n, 1-1/n]}(|\frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(t, y)|)\}$ is a semi-algebraic set of degree bounded by a function of $deg(f)$ and $s$. . We have introduced the concept of adapted sequence, so that the $sup$ above is bounded (recall that $f$ is not supposed analytic in a neighbourhood of $A$). According to Proposition 3, $B$ is covered by sets $(B_i)_{i=1,...,N}$, $B_i = \{(x_1, y) \in ]0,1[ \times B'_i : \gamma_i(y) < x_1 < \Delta_i(y)\}$ or $B_i = \{(\sigma(y), y) \in B'_i\}$ , where $B'_i \subset ]1/n, 1-1/n[^{d-1}$ are semi-algebraic sets of $\mathbb{R}^{d-1}$, such that $\bigcup_{i=1}^{N} B'_i = ]1/n, 1-1/n[^{d-1}$ and where $\sigma_i, \gamma_i, \Delta_i : B'_i \to ]0,1[$ are Nash maps. In the first case, we set $\sigma_i := 1/2(\Delta_i + \gamma_i)$. Afterwards, we consider only the open sets $B'_i$. Observe that for these sets we have $\bigcup adh(B'_i) = [1/n, 1-1/n]^{d-1}$.

We check thanks to the Tarski's principle and the proposition 4 that $N$ and the degree of $\sigma_i$ are bounded by a function of $deg(f)$ and $s$. Define $g_i(y) = \frac{\partial^{(\alpha+1)_1} f}{\partial x_1^{(\alpha+1)_1}}(\sigma_i(y), y)$ with $y \in adh(B'_i)$, where $(\alpha+1)_i$ represent the $i^{th}$ coordinate of $\alpha+1$. The induction hypothesis $P4(\alpha, d)$ implies $P4((0,...,0,|\alpha|+1), d-1)$ and thus $C6((0,...,0,|\alpha|), d-1)$, which applied to $\sigma_i$ et $g_i$ gives $(\mathcal{C}^{|\alpha|}, 1)$-Nash triangular maps $h_{i,k} : [0,1]^{d-1} \to [0,1]^{d-1}$, such that $g_i \circ h_{i,k}$ and $\sigma_i \circ h_{i,k}$ are $(\mathcal{C}^{|\alpha|}, 1)$-Nash and such that $\bigcup_k h_{i,k}([0,1]^{d-1}) = adh(B'_i)$.

Then,

$$\frac{\partial^{((\alpha+1)_2,...,(\alpha+1)_d)}(g_i \circ h_{i,k})}{\partial x^{((\alpha+1)_2,...,(\alpha+1)_d)}}(y) = \frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y)) \times (\frac{\partial h_{i,k}}{\partial_{x_2}})^{(\alpha+1)_2} ... (\frac{\partial h_{i,k}}{\partial x_d})^{(\alpha+1)_d} + R$$

where $R$ is a polynomial of derivatives of order $\preceq \alpha$, and of the derivatives of $h_{i,k}$ and $\sigma_i \circ h_{i,k}$ of order less than $|\alpha|$, $R$ depending only on $\alpha$. The map $h_{i,k}$ is a $(\mathcal{C}^{|\alpha|}, 1)$-Nash map and by hypothesis $f$ is a $(\mathcal{C}^{\alpha}, 1)$-Nash map, so that we have $|R| < C(|\alpha|, d)$.
After all $g_i \circ h_{i,k}$ is a $(\mathcal{C}^{|\alpha|}, 1)$-Nash map. Hence we have

$$|\frac{\partial^{\alpha+1} f}{\partial x^{\alpha+1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y))(\frac{\partial h_{i,k}}{\partial_{x_2}})^{(\alpha+1)_2} ... (\frac{\partial h_{i,k}}{\partial x_d})^{(\alpha+1)_d}| \leq |\frac{\partial^{((\alpha+1)_2,...,(\alpha+1)_d)}(g_i \circ h_{i,k})}{\partial x^{((\alpha+1)_2,...,(\alpha+1)_d}}| + |R| < C(|\alpha|, d)$$

Define $\phi_{i,k} : [0,1]^d \to [0,1]^d$ by :

$$\phi_{i,k}(x_1, y) = (1/n + b_n x_1, h_{i,k}(y))$$

$\phi_{i,k}$ is a $(\mathcal{C}^{\alpha+1}, 1)$-Nash triangular map. We check the two following points :

- $\frac{\partial^{\alpha+1}(f\circ\phi_{i,k})}{\partial x^{\alpha+1}} = \frac{\partial^{\alpha+1}f}{\partial x^{\alpha+1}}(1/n + b_n x_1, h_{i,k}(y)) \times (b_n)^{(\alpha+1)_1}(\frac{\partial h_{i,k}}{\partial_{x_2}})^{(\alpha+1)_2}...(\frac{\partial h_{i,k}}{\partial x_d})^{(\alpha+1)_d} + S$, where $S$ is a polynomial of the derivatives of $f$ of order $\beta \preceq \alpha$ (because $h_{i,k}$ is triangular) and of the derivatives of $h_{i,k}$ of order less than $|\alpha|$, $S$ depending only on $\alpha$. From above we deduce that $|S| < C(|\alpha|, d)$.

  Moreover by definition of $\sigma_i$, $|\frac{\partial^{\alpha+1}f}{\partial x^{\alpha+1}}(1/n + b_n x_1, h_{i,k}(y)) \times (\frac{\partial h_{i,k}}{\partial_{x_2}})^{(\alpha+1)_2}...(\frac{\partial h_{i,k}}{\partial x_d})^{(\alpha+1)_d}| \leq |\frac{\partial^{\alpha+1}f}{\partial x^{\alpha+1}}(\sigma_i \circ h_{i,k}(y), h_{i,k}(y)) \times (\frac{\partial h_{i,k}}{\partial_{x_2}})^{(\alpha+1)_2}...(\frac{\partial h_{i,k}}{\partial x_d})^{(\alpha+1)_d}| < C(|\alpha|, d)$,
  thus $|\frac{\partial^{\alpha+1}(f\circ\phi_{i,k})}{\partial x^{\alpha+1}}| \leq |\frac{\partial^{\alpha+1}f}{\partial x^{\alpha+1}}(1/n + b_n x_1, h_{i,k}(y)) \times (\frac{\partial h_{i,k}}{\partial_{x_2}})^{(\alpha+1)_2}...(\frac{\partial h_{i,k}}{\partial x_d})^{(\alpha+1)_d}| + |S| < C(|\alpha|, d)$

- finally for $\beta \preceq \alpha$, in the expression $\frac{\partial^{\beta}(f\circ\phi_{i,k})}{\partial x^{\beta}}$ take part only the derivatives of $f$ of order $\preceq \alpha$, again because of the triangularity of $h_{i,k}$. Hence $|\frac{\partial^{\beta}(f\circ\phi_{i,k})}{\partial x^{\beta}}| < C(|\alpha|, d)$.

The lemma 3 gives us a $(\mathcal{C}^{\alpha}, 1)$-resolution of $f_{/A_n}$.

□

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